ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/277025321

The logistics of choice

Article *in* Journal of the Experimental Analysis of Behavior · May 2015

Impact Factor: 1.87 · DOI: 10.1002/jeab.156

CITATION
1

READS
189

1 author:

Peter R Killeen
Arizona State University Tempe

220 PUBLICATIONS **6,062** CITATIONS

SEE PROFILE

All in-text references underlined in blue are linked to publications on ResearchGate, letting you access and read them immediately.

Available from: Peter R Killeen
Retrieved on: 04 June 2016

The Logistics of Choice

Peter R. Killeen

Arizona State University

This is the pre-peer reviewed version of:

Killeen, P. R. (2015). The Logistics of Choice. *Journal of the Experimental Analysis of Behavior, 104, 74-92. doi: 10.1002/jeab.156*

which has been published in final form at http://is.gd/lwIIIH.

Author Note

I thank Billy Baum, Charlie Catania, Michael Davison, Doug Elliffe, Randy Grace, Greg Jensen, Jim Mazur, Dave MacEwen, Greg Madden, Fed Sanabria, John Staddon, and Geoff White for comments on earlier draughts. Please don't blame them for the results.

Killeen@asu.edu

Abstract

The generalized matching law (GML) is reconstructed as a logistic regression equation that privileges no particular value of the sensitivity parameter, *a*. That value will often approach 1 due to the feedback that drives switching intrinsic to most concurrent schedules. A model of that feedback reproduced some features of concurrent data. The GML is a law only in the strained sense that any equation that maps data is a law. The machine under the hood of matching is in all likelihood the very law that was displaced by the Matching Law. It is now time to return the Law of Effect to centrality in our science.

Keywords: matching, generalized matching, logistic function, law of effect, concurrent schedules, Hooke's Law, negative feedback, conditioned reinforcement

The Logistics of Choice

After the Law of Effect, the Matching Law, formulated[1] by R. J. Herrnstein (1961) and developed by his students and colleagues, has taken central place in behavior analysis. Indeed, during the last decade of the 20th century of all the laws mentioned in psychological journals in general, the Matching Law (hereafter, ML) is most commonly cited, followed by Weber's Law, with the Law of Effect moving down from #1 to #8 (Teigen, 2002). So central to our core assumptions is the ML, that it has been said that it "can be neither attacked nor defended on empirical grounds" because it governs not equality between measured variables such as response counts and reinforcer amounts, but rather between transformations of those variables. "The more our results approximate [matching], the surer we are that we eliminated or balanced extraneous reinforcers in the situation" (Rachlin, 1971, p. 251). This perspective resonates with Smedslund's: "Psychologists do not analyze the conceptual relations between their independent and dependent variables. … the plausibility of their hypotheses stems from the conceptual relatedness of the variables. The outcome is research that appears to test hypotheses but really tests only procedures, because the hypotheses involve conceptually related variables and are necessarily true." (Smedslund, 2002, p. 51). *Ensuite*: "[Equation 1 below] is an identity rather than an empirical finding…interest in matching lies in not whether organisms match (that they do is our underlying assumption) but in what the parameters of matching are" (Rachlin & Locey, 2010, p. 365); including unforeseen parameters: "If we found an organism choosing A over B 2:1, but reinforcements delivered were only 1.5:1 we should have to invent other reinforcers" (Rachlin, 1971, p. 250). In the history of science, such maneuvers were said to "save the appearance" of the phenomena—that is, they tune the model to make it consistent with the data.

It is a credible approach for scientists to seek variables that rest in lawful balance on either side of an equality sign. If acceleration does not equal applied force divided by mass, it is prudent to discover or invent other forces such as friction to restore balance. "Herrnstein, and

---

[1] The name "Matching Law" may have been borrowed from Estes's (1957) underappreciated article in which he reported a matching law based on stimulus sampling theory "of which 'probability matching' is simply a special case. …The general matching law requires no restriction whatever on the schedule of reinforcement. …" (p. 612).

many others before and after, have followed the same strategy: tweaking their experimental procedures until they produce orderly results" (Staddon, 2014, p. 570). It is not against this approach that this brief note is aimed; it is against the particular "conceptual relatedness" assumed for the variables in the ML. That relationship is stated verbally as "the proportion of behavior allocated to an alternative equals the proportion of reinforcement received from that alternative", and rendered mathematically as:

$$\frac{B_i}{B_i + B_j} = \frac{R_i}{R_i + R_j} \tag{1}$$

This may be rearranged as ratios, and then in search of equality, further transformed:

$$\frac{B_i}{B_j} = b\left(\frac{R_i}{R_j}\right)^a \tag{2}$$

Equation 2 is known as the Generalized Matching Law (GML: Baum, 1974). It has been applied to hundreds if not thousands of data sets (Davison & McCarthy, 1988), and generally provides a very good fit to the data. This is the case even though values of the *bias* parameter *b* or *sensitivity* parameter *a* reliably different than 1 falsify the original (unparameterized) matching law (Equation 1). Equation 2 has precedent in psychophysics (Stevens, 1986/1975) and in behavior analysis (Lander & Irwin, 1968; Staddon, 1968). In all its varieties the ML holds interest for applied behavior analysts (see, e.g., Reed & Kaplan, 2011).

In a thorough and thoughtful review of the literature to that date, Baum (1979) found that typical values for a were around 0.8 for responses, and somewhat higher using time allocation as the dependent variable. He summarized that values between 0.9 and 1.1 could "be considered good approximations to matching" (p. 269). Since the mean for response allocations fell reliably below 1, in accord with Rachlin's perspective Baum then spent considerable time discussing the possible causes of deviation from matching—testing, as it were, the procedures. Taylor and Davison (1983) furthered the analysis by noting that sensitivities for response allocations were systematically lower when the schedules comprised arithmetic progressions (mean $a = .79 \pm .17$) rather than random (exponential) progressions (mean $a = .96 \pm .16$).

What is so Special about Matching?

Why should we expect *a* to fall between 0.9 and 1.1, and why should we search for extraneous sources of influence when it disappoints that expectation? Why do we expect animals to allocate their behavior in proportion to the reinforcers for that behavior? Why should we agree to that particular "conceptual relatedness between variables", however successful our identification of extraneous reinforcers and transformations to render that relationship? Should a value of 2 for *a* (*overmatching*) or a value of 0.5 for *a* (*undermatching*) be taken as a validation or invalidation of the ML? of the GML? Should matching be considered optimal or should one or another deviation from it be considered optimal?

Consider the following experiment involving the logistics of obtaining a preferred outcome: You are offered $1 for responses on one lever, and 50¢ for responses on the other. Will you match? What would your mother say if you did? Consider three scenarios. 1) Each response is reinforced (a concurrent Fixed-Ratio 1, Fixed Ratio 1 schedule). Then whatever your preference, even if a perverse one for the lesser amount, the feedback of the program would give you more of it; the program would match its payoff proportion to your preference. The program matches, not you. And it matches not the amount tendered, but the proportion of payoffs to your proportion of choices, even if that is random. You need not be conscious of the consequences of your behavior, as the program will handily match nonetheless. Equation 1 will *a priori* predict nothing about your next *choice* without knowing your behavior up to that point; but it can nonetheless predict *matching* knowing nothing, as the program guarantees it. 2) On each trial the program flips a fair coin to decide which side (the $1 side or the 50¢ side) to pay off, and will not flip again until that is collected. (This corresponds to a "corrections" procedure in which the correct side varies randomly). In this case you may always hopefully choose the $1 lever on the first choice, and if it doesn't pay off, your choice of the second lever is forced. Then you will make an average of 2 responses on the $1 lever for each time it pays off, and 1 response on the 50¢ lever—2-to-1—matching! But it would be the same 2-to-1 if the payoffs were $10 and 50¢: Invariant, and far from matching. 3) On each trial the program flips a fair coin to decide which side to pay off, and will not flip again until that is collected. It then rolls a many-sided die with

each response, to decide whether it will be paid off (if to the blessed side). (This corresponds to a "single-tape concurrent random ratio (RR)" schedule; Killeen, Palombo, Gottlob, & Beam, 1996). In this case you may start responding on the side arranging the larger reward amount, but eventually come to suspect that the other side is primed, and make one or two responses to it to check out that possibility before switching back to the preferred side. It is the argument of this paper that (1) tells us nothing about preference for the outcomes, and matching there is a forced triviality; (2) tells us little about degree of preference, but might suggest the direction of it. It tells us more about the arbitrary contingencies that force a choice of the dispreferred to get unstuck from a hung system. (3) Tells us more about preference by balancing the tendency to persevere in responding for the preferred outcome against the increasing probability of (an inferior) payoff on the alternative. It tells us how long the weight of the preferred outcome will pull against the force of increasing certainty that the alternative, and not it, will pay off on this trial. It is the thesis of this paper that concurrent schedules, like the examples above, vary in the strength with which they pull the subject from always choosing the better; that they are measurement tools, properly analyzed with logistic regressions, and when successful, they yield accurate predictions, not laws.

A version of Scenario 3 was done with pigeons, who acted as wisely as your mother might have wished for them. Crowley and Donahoe (2004) gave pigeons extensive training on multiple schedules comprising Variable Interval schedules (VIs) of 30 s, 90 s, and 270 s. At the end of this training, brief unreinforced probe exposures to pairs of the stimuli permitted the authors to measure the degree of preference for one or the other stimulus. In a subsequent condition, extensive training on concurrent schedules with equal-valued VIs (e.g., 30 s with 30 s, and so on) was conducted to instruct the pigeons about the feedback contingencies on concurrent schedules, followed by additional unreinforced probes under the multiple-schedule-correlated stimuli, and finally by traditional concurrent schedules and probes. In the initial probes, would the pigeons match, or would they choose the stimuli signaling the higher rate of reinforcement most of the time? The discs in Figure 1 show their results averaged over subjects: "choice approximated exclusive preference for the alternative associated with the higher reinforcement frequency" (Crowley & Donahoe, 2004, p. 143). Training on equal concurrents flattened the functions during the probes (squares in Figure 1): The pigeons learned something about proper

behavior under concurrent scheduling (or perhaps they just embodied the cumulative effects model of Davis, Staddon, Machado, & Palmer, 1993). In the final condition of regular concurrent schedules they obtained near-classic matching. Thus, the rational behavior of exclusive preference for the preferred alternative is tempered by the contingencies operative in concurrent schedules, which cause choices to be less than exclusive. Just how much they are tempered depends on the precise contingencies.

Some time later Mazur (2010) extended the result by showing that on discrete trial choice experiments pigeons show exclusive preferences for the preferred outcome—the one with shorter delay to food or higher probability of food—unless the parameters of the outcomes were too close for the pigeons to call. He ruled out a number of theoretical explanations for matching and non-exclusive preference in his experiment, and concluded that "exclusive preference is an animal's 'default option' in discrete trial choice experiments" (p. 334); but that preference may become non-exclusive when the outcomes are difficult to discriminate, or when a number of dimensions control choice (e.g., key bias along with differential outcomes), or when the procedure requires the occasional choice of the lesser alternative.

Figure 1 about here

Crowley and Donahoe (2004) fit logistic functions (Equation 3) to their data. Logistic functions generate ogives that mimic cumulative normal distributions, as seen in Figure 1. In Equation 3 the steepness parameter *a* is the slope of the curve when it crosses 0.5. It is inversely proportional to the standard deviation of the density. For large values of *a* the ogive is almost a step function and for smaller values a 45-degree line or flatter. The parameter *m* plays the role of bias, or offset; when it is 0, the ogive passes through the point 0.5 when the variables are equal; for other values it is shifted to the left or right. The functions *f(x)* allows transformation of the independent variables before entering in the logistic. Crowley and Donahoe logarithmically transformed reinforcement rates before taking a relative measure of them, making *f(x) = ln(x)*.

$$p(B_i) = \frac{1}{1 + e^{-a\left(f(x_i) - f(x_j) - m\right)}} \tag{3}$$

Logistic functions are often used to describe behavior in psychophysical discrimination

tasks. We start with logistic functions because they bridge the gap between traditional psychophysics and the GML. Equation 3 would provide a fine descriptions of the results of Mazur (2010), both for exclusive and near-exclusive performance, and when those were flatter, being confounded by key biases. It is clear that the data from the probe multiple schedules shown by filled circles in Figure 1 were very steep, as expected. The pigeons chose what they liked best almost exclusively (gross "overmatching"). The other conditions shown by open symbols come closer to matching. This is not because they could not discriminate which alternative had the higher reinforcement rate, as they started the second and third phases with a fine discrimination in place. It was because the contingencies of reinforcement on concurrent schedules either undermined that discrimination, or changed the conditions of reinforcement to favor matching. Herrnstein and Loveland (1976) found similar results in an analogous experiment, and concluded that: "matching requires an ongoing interaction with the conditions of reinforcement, and what is learned about individual alternatives bears an as-yet-unspecified relationship to frequency or probability of reinforcement" (p. 153). Given the data cited above, it appears that what is unlearned about individual alternatives also bears an as-yet-unspecified relationship to the contingencies of concurrent reinforcement.

Crowley and Donahoe (2004) discussed the many possible reasons for the devolution of choice from exclusive toward matching, just as Baum (1979) had analyzed many of the possible reasons for the deviation of matching toward undermatching. The formers' data (Figure 1) show that matching clearly had more to do with the contingencies operative under concurrent schedules than it did with the preference for, say, a VI 30 s schedule over a VI 90 s schedule. That was clear-cut enough, mother-rational enough. Matching is not the animals' default option when confronting a choice between two outcomes. It is what they do when the outcomes are obtained by concurrent schedules, in which elapsing time (VI schedules) or responses (Jensen & Neuringer, 2008; Killeen et al., 1996; MacDonall, 1988) increase the posterior likelihood that the alternative will pay off.

This is not the first psychophysical rapprochement with choice data in our literature. Going in the reverse direction as Crowley and Donahoe (2004), Davison and Tustin (1978) showed how the GML could be applied to discriminations and yield classical measures of detectability and bias. Davison and Nevin (1999) extended that work for a full-fledged modeling

of the potential confusions/generalizations between stimuli controlling behavior and the behavior that they controlled, and between reinforcers and that behavior (the latter sometimes called the "allocation of credit" problem). Versions of the GML were used as core models, with their relation to logistic psychometric functions a major theme of the research. The current note amounts to little more than a reminder of the work of the many behavior analysts cited here.

Roots of the GML

In this section it is shown that the GML is a special case of Equation 3, with *f(x) = ln(x)*. This is done to bridge from the psychometric data of Mazur (2010), Crowley and Donahoe (2004), and the vast literature on delay discounting (see, e.g., the section on preference in Killeen, 2015), and to wave in passing at the standard technique of logistic regression. Write the complement of Equation 3, the probability of not emitting the target response:

$$1-p(B_i)=1-\frac{1}{1+e^{-a\left(f(x_i)-f(x_j)-m\right)}} \tag{4}$$

Rearrange the right-hand-side to be a single term:

$$1-p(B_i)=\frac{e^{-a\left(f(x_i)-f(x_j)-m\right)}}{1+e^{-a\left(f(x_i)-f(x_j)-m\right)}} \tag{4'}$$

Notice that the denominator of Equation 3 is the same as that of Equation 4'. In dividing Equation 3 by Equation 4' those denominators cancel to yield:

$$\frac{p(B_i)}{1-p(B_i)}=e^{a\left(f(x_i)-f(x_j)-m\right)} \tag{5}$$

Take natural logarithms of each side:

$$\ln\left[\frac{p(B_i)}{1-p(B_i)}\right]=a\left(f(x_i)-f(x_j)-m\right) \tag{6}$$

Equation 6 is a special case of the *logistic regression equation* (Equation 7). The quantity on the left of the equation is called a *logit*. This equation is a standard form of regression when

the outcome is binary-valued (viz., the probability of a left or right response) and the predictors are continuous variables:

$$\ln\left[\frac{p}{1-p}\right] = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots \tag{7}$$

If we take logarithms of each side of Equation 2, the GML, we may write it as:

$$\ln\left[\frac{p(B_i)}{p(B_j)}\right] = b + a\ln(R_i) - a\ln(R_j) \tag{8}$$

or rearrange for:

$$\ln\left[\frac{p(B_i)}{p(B_j)}\right] = \ln[b'] + a\ln\left[\frac{R_i}{R_j}\right] \tag{9}$$

Equation 8 is a logistic regression, under certain assumptions a version of Equations 6 and 7; Equation 9 is the way that the GML is usually graphed. It is often extended to include predictors based on other aspects of reinforcement, such as amount and delay (Davison, 1983; Kyonka & Grace, 2008). It is clear that the GML (Equation 2) is a special case of logistic regression, instantiated in it by two assumptions: (1) The outcome variable $p(B_i)$ may be estimated as the number of responses on one operandum divided by the number of responses on both operanda (i.e., $p(B_j) = 1 - p(B_i)$). That is, that this outcome variable will be invariant over whether the response rate is low or high in real time, and whatever else the animal may be doing during that time. This latter assumption is also called "independence from irrelevant alternatives". Davison (1982) discussed the assumption and felt that it was viable for concurrent schedules. (2) The predictor variable is properly constituted as the difference of logarithms of reinforcement rates (or other predictor variables) available to the two alternatives (i.e., $f(x) = ln(x)$; the use of natural logarithms is standard in logistic regression, but in the case of Equation 9 the choice of a base is irrelevant, as others merely introduce a constant that cancels out of both sides). We expect complementary functions whichever outcome we place in the numerator of

Equation 8—that is, that choice of *i* and *j* is arbitrary. This forces the coefficients of the log reinforcement rates to be equal—here to both equal *a* (Allen, 1981; Marr, Fryling, & Ortega-González, 2013). Davison (1982), however, found it necessary to use different values of *a* for log reinforcement rates from VI and FR schedules when those were concurrently presented.

Under the above constraints, the logistic regression (Equations 6 and 7), derived from the logistic distribution (Equation 3) becomes the GML (Equations 2 and 9). In turn, when $a = 1$ and $b = 0$, these reduce to the ML, Equation 1. The GML is not, however, consistent with Herrnstein's "single key matching law" (Herrnstein, 1970, 1979)—an equally robust rate equation that stands on its own legs (Herrnstein, Rachlin, & Laibson, 1997).

Matching *Law*?

Equation 7 is often used to predict the log-odds of binary outcomes from continuous variables (Logistic regression, 2015), but outside our field the terms "matching" and "law" are seldom applied to it. Equation 1 deserved the term had it been true. Do regression equations such as 7 or 8 deserve the term? It seems that Rachlin (1971) was making an act of faith that some transformations of reinforcement variables could be found that would predict some aspect of behavior—the time spent at one operandum or another, or the responses made on one or another—to make the logit of response probabilities a linear combination of some function of the reinforcement ratios. That is a faith that I share. It is less certain to me that Equation 8 deserves the name *law* if both parameters in it are freely adjustable. Nor is it clear that it should be called "matching", unless we mean by that that *any* equation that balances and describes data is a matching law. Nor is it clear that we should rush to proclaim our proudest law unfalsifiable without taking care to defend against the reasonable inference that that assertion thereby makes it intrinsically unscientific (Popper, 2002; cf. Stove, 1982).

Newton's law of universal gravitation predicts the force between two bodies of masses m_1 and m_2 as $F = Gm_1m_2 / r^2$. Big G is an empirical physical constant that was not measured until nearly 100 years after the formulation of the law. The exponent 2 is not an adjustable parameter. It was derived from theory. Kepler was curve-fitting, Newton was not. If the exponent were systematically off then the law would be rejected. If curves were fit to the orbits of celestial bodies, the law would be validated ± epsilon—that is, the best fitting value for the exponent

might be slightly different than 2: $2 \pm \varepsilon$. Then the Bureau of Standards would analyze the experiments for flaws—for "missing reinforcers" in Rachlin's terms—just as Baum analyzed the studies he reviewed for ways in which the exponent might have fallen wide of 1.0. Indeed, the orbit of the moon does not conform precisely to simple Newtonian expressions, as it is perturbed by its interaction with both the earth and the sun. But this set of three-body interactions can be computed and is consistent with an exponent of 2. Baum's matching exponent of 1 was not, like Newton's law, derived from (and forced by) theory. There are many ways an exponent could deviate from 1. If, for instance a changeover delay (COD)—a delay between switching from one key to the other— is not employed then animals would tend to undermatch. If Crowley and Donahoe (2004) had not used a COD of 2 s then, with continued exposure to the concurrent contingencies, the loci of their post concurrent-schedule data would have rotated from vertical (the discs) through 45° (the circles) toward undermatching. The value of a in Equation 8 increases with the value of the COD, approaching 1 as the COD is increased (Shahan & Lattal, 1998; Shull & Pliskoff, 1967), but once trained with a COD, its value seems to become unimportant, as sensitivity may remain higher when it is removed (i.e., a will be larger than previously: Krägeloh & Davison, 2003). Conversely, Baum, Schwendiman, and Bell (1999) found that after several hundred sessions of concurrent schedule training without a COD, its addition did not elevate sensitives above $a = 0.8$. How is that captured in the model? The value of a also varies with how long the reinforcer will remain available for collection—the *hold* interval (Jensen & Neuringer, 2008). How should that be represented in a generalized relative rate equation such as a GML?

The Matching Paradigm is a Contraption

Concurrent schedules are complex feedback machines designed to moderate the allocation of behavior among alternatives. Without constant-probability VI (viz., RI) schedules, allocation becomes inhomogeneous in time (Baum, 1979). Without the unlimited hold in typical schedules (Jensen & Neuringer, 2008; Killeen & Shumway, 1971) allocation deviates from perfect matching. In concurrent variable ratio (VR) schedules, response allocation favors the richest, and that often drives reinforcement and behavior to exclusivity. Without the COD or a change-over response technique, allocation could become too indiscriminate. Despite these deviations from Equation 1, the patterns of behavior will often satisfy Equation 2. This is

because when time or response ratios are to be predicted from reinforcement ratios, a power function on the latter is privileged (Allen, 1981)–*without* also privileging 1.0 as a special value for its exponent.

As contraptions, concurrent schedules as we study them in the laboratory are unnatural—they are artifacts. They are heavier-than-air flying machines that rely on tuning and feedback gadgets to keep them stable. They are tinkered so that "research that appears to test hypotheses … really tests only procedures" (Smedslund, 2002, p. 51). Such "tuning" has been present *ab ovo* (Staddon, 2014; Timberlake, 1988). The matching paradigm makes functional behavior dysfunctional: Organisms naturally choose what they prefer in order to get more of it, but getting more of it—beyond the values scheduled by the experimenter—is seldom permitted on concurrent interval schedules. Like airplanes, however, concurrent schedules have their purpose. Catania (1963) was the first to note the ability of concurrent schedules to magnify otherwise small effects, with amounts of reinforcement that had no regular effects on a single-key schedule showing very orderly and substantial effects on concurrent schedules. The effect was conceptually replicated by Neuringer (1967), who employed a unique concurrent-chain schedule in which the first link was Fixed-Ratio (FR) 1 and the second links Fixed Interval (FI) 5 s with probabilistic payoffs determined by VI 1 schedules. He varied the amount of reinforcement available from one alternative, holding the other constant at 2 s. There were only moderate changes in response rates on the FI as a function of reinforcement magnitude, but substantial changes in the proportion of choices as a function of the magnitude of the variable reinforcer.

Although a contraption, concurrent VI schedules with COD mimic some important contingencies in the extra-laboratory world. If one restricts the infinite "hold" on reinforcer availability (Jensen & Neuringer, 2008) it emulates foraging in which movement between patches both incurs a delay, and relies on a limited replenishment of the resources sought. Think of bees foraging in a field for nectar in a context of other bees. It is good to find such consilience. It is even better to take on the project of such emulation seriously (Fantino, 2004; Fantino, Gaitan, Kennelly, & Stolarz-Fantino, 2007; Kacelnik & Houston, 1984; Navarro & Fantino, 2009) and to test the consequences of lab models in the field (Cialdini, 1995).

Two ways to weigh preference.

It is heuristic to place the measurement of the utility of options in the context of other techniques for measuring magnitude. Take weight, for instance, the force that the earth exerts on massive bodies. We can measure the mass of an object by placing it in one pan of a beam balance, and then adding and subtracting weights of known masses to the other pan until the bar connecting them levels to a horizontal equilibrium. Transposing the objects in the pans is a way to check for bias in the scales. The method of adjustment is also used in doctors' scales, where the force of small weights in balancing our large body weight is given leverage by an off-centered fulcrum. A quicker and easier method of weighing an object is with a spring scale, in which an object extends a spring a distance proportional to its weight, according to Hooke's law. This method of scaling is also used with modern single-balance scales such as the bathroom scale that rely on calibrated load-cells whose resistance or voltage is proportional to the force being measured.

The adjustment procedure of the beam balance is familiar to psychophysicists as a "nulling" technique. Its seminal proponent in the field of behavior analysis is Jim Mazur (e.g., 1987, 1989). In numerous studies by him and others using his technique, one of the attributes of an option is varied until the subject is indifferent between it and a standard. Perhaps the most common use of this technique today is in the study of how individuals balance the utility of a delayed income against the utility of a smaller magnitude immediate income (e.g., Green & Myerson, 2004; Madden & Bickel, 2010). All of these are scaling procedures.

The scaling procedure is familiar to psychophysicists as two paradigms—one yielding *psychometric* functions such as those shown in Figure 1, and the other yielding *psychophysical* functions. Psycho*metric* functions measure how discriminable two stimuli are from one another as a function of their distance on some dimension such as weight. The probability of saying "heavier" to comparison stimulus contrasted with a standard one as the comparison increases from a light to a heavy will rise from 0 toward 1 following a sigmoidal function such as seen in Figure 1. The logistic generally provides a good description of the data, with its mean called the "point of subjective equality", or PSE. That is the point that is directly determined by adjustment procedures. The standard deviation of the sigmoid is one measure of the *jnd*—the just noticeable

difference. Its reciprocal is the slope at the PSE, and gives a measure of sensitivity. Frequently a cumulative normal distribution is used to draw the curve, one deriving from the subtraction of two Gaussian distributions. This was the foundation of Thurstone's (1927) model of comparative judgment, and was subsequently imported into signal detection theory. The mechanics of its use in scaling preference for grains is illustrated by Killeen, Cate, and Tran (1993).The logistic distribution has the advantage of a closed form, and is not discriminable from the normal within our experimental sampling error. It has been used in analyses of concurrent performance with a signal detection model (e.g., Davison & Tustin, 1978).

Psycho*physical* functions measure how loud or bright or heavy an object seems as a function of physical variables. Novices often confuse these two functions[2]. One of the points of this article is that the same confusion between psychometric and psychophysical can be made in interpreting results from the matching paradigm: Behavior ratios that are inferred to measure response strength (a psychophysical property) might instead be reflecting confusion concerning the allocation of credit for reinforcement to one or the other response (Catania et al., 2015; Davison & Nevin, 1999), a psychometric issue. You may like carrot cake twice as much as meatloaf (psychophysics), but you may still pull the wrong container of brown food out of the back of the refrigerator by mistake (psychometrics).

A key difference between nulling techniques and scaling techniques is seen in the contrast between balance-beam and spring weighing machines. The former measure mass. They would give the same reading of your mass on the moon as on earth. The weaker gravity on the moon affects the objects in both balance pans equally. Your bathroom scale would weigh you as much lighter on the moon where the force of gravity is less, and affects only you, not the tension in the spring. Concurrent VI schedules are spring devices, and the spring is the tension toward the alternate side that increases as time elapses and probability of reinforcement on that side increases.

There is negative feedback inherent in most concurrent schedules—the longer the subject

---

[2] That is not helped by the fact that Fechner used psychometric measurements to infer an underlying psychophysical function, based on the premise that all jnds are subjectively equal. He then integrated over Weber's law ($jnd = k/I$) for his resulting logarithmic scale.

spends on one alternative, the more likely the other alternative is to pay off. Even though one payoff may be of less utility to the subject, that lesser utility at some point is balanced by its greater immediacy or probability. Concurrent schedules thus constitute dynamic delay/probability discounting experiments in miniature. The more time spent on one side, the more probable the outcome on the other. This negative feedback constitutes the spring in the scale. How hard subjects will pull against it to get the preferred outcome may provide a measure of the strength of their preference with respect to the countervailing strength of that spring. Thus, concurrent schedules may conflate two factors: the psychometric one of allocation of credit (did this outcome arise from the last response to the left, or from the prior 5 to the right followed by the leftie?), and the psychophysical one of the tradeoff between probability (or delay) vs. utility. Inferences concerning the latter can be strengthened by minimizing the confusions of the former (e.g., using separate hoppers for outcomes, stimuli salient in the situation, marking the response, and so on).

The strength of the spring in the concurrent scale depends on the length of the VI schedules. The longer the stay on the preferred side, the stronger is the tension pulling them to the dispreferred side. A simple model of this tension is given in Appendix A, and a picture of its action on behavior in Silberberg, Hamilton, Ziriax, and Casey's (1978) Figure 3. Proposed is a kind of Hooke's law for concurrents, but instead of the restoring force being proportional to the distance the spring is stretched, it is proportional to the time away from the alternative. The greater the mean value of the initial link VIs, the less sensitive the paradigm is to differences in terminal link outcomes (Fantino & Davison, 1983)—more of the behavior is distant from the outcomes, and they are less effective in exerting a differential effect on that behavior. More specific and powerful models have been generated for incorporating the initial and terminal link effects in unified models of choice (Grace, Berg, & Kyonka, 2006; Nevin & Grace, 2001; O'Daly, Meyer, & Fantino, 2005) that cover both simple concurrent schedules and concurrent-chain schedules. These models provide another argument for not taking the absolute values of the slopes in the GML too seriously: Sensitivity—the value of *a* in all of the above equations, depends on the particular scale used to measure preference, and the strength and balance of the springs in it (Grace & Reid, 2007). Detailed analyses of these dynamics are provided by (e.g., Baum, 2010; MacDonall, 2000, 2005) and others, in most cases echoing the conclusions of Silberberg and associates that the major features of choice behavior issue from two processes:

response perseveration (cf. Smith, McLean, Shull, Hughes, & Pitts, 2014) and reinforcement maximization (Silberberg et al., 1978, p. 368). And that maximization is primarily a function of the reinforcement of the behavior of changing between alternatives, and secondarily for persisting on them.

Applications

So what? How does this change what we do or say? We shall see that on the one hand it opens new doors to the study of mechanism (e.g., Appendix A) and on the other encourages us to experiment with other predictors than the log reinforcement ratio of the GML. But it is time to reground a too-theoretical discussion up to now with a stabilizing dose of old data. In one of my early experiments I gave 4 pigeons a choice between a 20 s Fixed Interval (FI) schedule and an FI *x* schedule, with *x* taking values of 5, 14, 30 or 60 s. These were offered as the terminal links of concurrent independent VI 1 schedules. The top panel of Figure 2 shows the average proportion of choices for the FI *x* key as a function of its length. The middle shows a typical GML plot of logit probability vs. log ratio of delays. The third panel shows the predictions from a logistic regression of preference based on conditioned reinforcement strength for the two alternatives.

The top panel is framed by coordinates that plot proportion of choices to the varied FI component as a function of its FI. The curve through it is given by Equation 3, with the function on the predictor variables being not the logarithms of the delays of reinforcement, but the conditioned reinforcement strength of the stimuli signaling those delays:

$$f\left(t_i\right) = s\frac{1-e^{-kt_i}}{kt_i} \tag{10}$$

This is a periodically resuscitated model of conditioned reinforcement (Killeen, 2001b, 2005, 2011). It is a convenient alternative to the default logarithmic transformation of the GML, and suggested because we know that conditioned reinforcement is an important factor governing concurrent chain performance (Mazur, 1997; Williams & Dunn, 1991), even though not the only factor (Killeen & Fantino, 1990). It is derived by assuming an exponential delay of reinforcement gradient (justified in Killeen, 2011) having a rate constant *k*, typically ranging

between 0.03 and 0.3. Because the stimulus extends through the delay interval *t*, the average area under the gradient is computed (the fraction on the right of the equation) and assigned as the reinforcing strength of the stimulus. As *t* goes to 0, the function goes to *s* in the limit. As the interval *t* is extended this strength decreases as a hyperbolic function of that delay. This explains the relative impotence of stimuli signaling long delays of reinforcement, as often employed in early studies of conditioned reinforcement in chain schedules.

The coefficient *s* measures the salience of the stimulus. That may be varied by marking the stimulus or emphasizing it in other ways. It corresponds to the ability of one stimulus to overshadow another in associative conditioning. Most concurrent studies attempt to keep the salience the same for each stimulus, and distracting stimuli that might increase the value of *k* are also controlled (Johansen et al., 2009), so we may assign a common value $a = s/k$ for the stimuli associated with each outcome (Equation 10'). This ratio, *a*, gives the area under the delay-of reinforcement gradient, the total associability of the stimulus if extended indefinitely. Equations 10 and 10' give its *average* reinforcing potential if it habitually signals a delay of *t* s. This symmetry permits us to use *a* as the common sensitivity parameter in Equations 3 and 6, where it takes a value of 1.42 for the data in Figure 2, and where *k* takes a value of 0.08. The bias parameter *m* in Equation 3 and 6 is set to 0 in this paper, but in some applications must take non-zero values.

$$f(t_i) = a\left(1 - e^{-kt_i}\right)/t_i \qquad (10')$$

Figure 2 about here

The middle panel gives a standard rendition of the GML showing the logit of choice probability against the log ratio of delays in the terminal link: 20/60, 20/30 etc. The slope for these variables is 2.1. The bottom panel shows the regression of the outcome variable, the logit of the probability of choosing FI *x*, against the predictor, as in Equation 6. The predictor is the difference in conditioned reinforcement strengths (Equation 10') for the variable and fixed (20 s) schedules. In this application, t_i is the value of the varied FI, and t_j is 20. Note that the line of slope 1 in the bottom panel is not a regression, but it is the line showing accurate prediction, with slope necessarily 1. For all panels, the bias is set to 0.

All three panels provide useful windows on data, but the scenes through them are quite different. The top panel is useful for when there is only one manipulated variable, and versions of it are increasingly seen in delay discounting studies where per cent choice is plotted against delay to the preferred outcome. The middle is classic GML, with the exponent of 2.1 telling us about the sensitivity to delays. The bottom is a display of a logistic regression of the outcome—logit preference—against the predictors—the difference in conditioned reinforcement strengths of the stimuli associated with the two delays. The middle panel could also be plotted as in the bottom panel, with $\beta_1 = \beta_2 = a = 2.1$ and its argument $ln(20) - ln(x)$. In that reframing its meaning would change, from a Generalized Matching Law to a regression equation scattergram. Which of these representations one chooses is a matter of preference, although the parsimony of the GML (one parameter rather than two) favors it. That is not the case in the next analysis.

Turn now to the data of David MacEwen (1972), and analyze them the same way. Four pigeons chose between FI schedules in terminal links, always in 2-to-1 ratio: 5 s vs. 10 s; 10-20; 20-40; and 40-80. For the initial links MacEwen employed a "single tape" procedure, priming a reinforcer for one of the alternatives every 30 s. The top panel of Figure 3 shows that as the values of the FIs increased, preference for the shorter FI became more extreme. The middle panel shows that, because the ratio of the two terminal link delays is always 2-to-1, the GML is unable to capture the data. The bottom panel plots the logistic regression, Equation 6, using Equation 10' as the functional transformation. The parameters are $a = 2.5$ and $k = .033$.

Figure 3 about here

Discussion

It is not the brief of this article to propose a new theory of choice of delayed reinforcers on concurrent schedules. There are many well-honed models of that. Among them are Hinson and Staddon's, which extends to paradigms that yield exclusive preference: "In simple situations, animals consistently choose the better of two alternatives. On concurrent [(VI, VI), and (VI, VR)] schedules they approximately match aggregate choice and reinforcement ratios. The matching law attempts to explain the latter result but does not address the former. Hill-climbing

rules such as momentary maximizing can account for both" (Hinson & Staddon, 1983b, p. 321. The spring model in Appendix A is a hill-climbing rule.).

Alternatives to the GML (e.g., Equations 6 and 10') were considered to make these points: The GML has uncertain status as a law; a better representation of the relation between predictors and outcome variables is a logistic function (when the outcome data are presented as proportional choice: Equation 3) or a logistic regression equation (when the outcome data are presented as logits: Equation 6). Those more general representations encourage experimentation with different functions on the predictor variables. These include but certainly are not limited to GML's logarithm of predictors, or alternatives such as Equation 10. Additional alternatives are considered in Appendix B. The constraints of symmetry—here meaning that the value of *a* must be the same for each symmetric predictor—should be the default in these exercises. Such opening up of the fundamental equations facilitates the perspective of Rachlin, one that might otherwise be misconstrued (Rachlin, personal communication, 2015). It clarifies our goals. The GML is not an unfalsifiable law—that would be problematic—and in fact it has withstood tests for systematic deviations (Cording, McLean, & Grace, 2011; Sutton, Grace, McLean, & Baum, 2008). The goal of predictions falling along a 45 degree line when plotted against observations (the bottom panels of Figs. 2 & 3) is, however, unfalsifiably good. It exemplifies the oldest law of science: Predictions from theoretical models should account for the data that they are about; it is necessary (but not sufficient) that they "save their appearances".

A change of coordinates as in the middle to the bottom panels of the above two figures may inspire different visions of the processes. This was the case at the dawn of modern science. Galileo championed the Copernican heliocentric model of the solar system over the Ptolemaic geocentric one more for philosophical than empirical reasons, as few extant data compelled that change of origin; —and phenomena such as the sun's course through the sky, and the absence of prevailing wind or other signs of movement by the earth, suggested otherwise (Hofstadter, 2009). Galileo's inquisitor, Cardinal Bellarmine, wrote:

> If there were a true demonstration that the sun was in the center of the universe … and that the sun does not circle the earth but the earth circles the sun, then it would be necessary to [reanalyze] Scriptures that appear contrary…. But I will not believe that

> there is such a demonstration, none has been shown to me. Nor is it the same to demonstrate that by assuming the sun to be at the center and the earth in heaven one can save the appearances, as it is to demonstrate that in truth the sun is at the center and the earth in heaven; for I believe the first demonstration may be available, but I have very great doubts about the second. (Finnocchiaro, 2008) cited in (Bellarmine, 2014; cf. Halsall, 1999).

Bellarmine understood quite clearly (as did Ptolemy) that fitting curves to data, or changing origins of the coordinate system, are not the same as providing scientific proof; that awaited the discovery of stellar parallax, and the larger theoretical context provided by the understanding of the relation of the tides to the motion of sun and moon, due to Newton. To "save the appearance of data" is to provide a formal description; one does not have an explanation until the other three causes of the phenomena—its triggers, mechanisms, and function—are addressed (Killeen, 2001a, 2013). Progress in understanding the mechanisms of choice has been advancing, with one of the more astute students of the underlying short to medium-range forces ironically being one of the strongest critics of molecular analyses (Baum, 2012; Baum & Davison, 2009). The multi-scaled molecular/molar forces (Baum, 2015) that drive the GML may in fact originate in the operation of the first law of behavior analysis.

Herrnstein (1970) dethroned Thorndike's Law of Effect (LE) for his ML on tenuous grounds—its failure to predict response rates on ratio schedules, and the discordant behavior of two pigeons when shifted from interval to ratio schedules—without noting that his new ML could predict these things no better than did the old LE. The ML tells us nothing about acquisition or stimulus control, as Thorndike's original law did. The LE has greater generality than the ML (see, e.g., Nevin, 1999; Staddon, 2014). The ML is derivative from moment-to-moment dynamic changes (e.g., Baum & Davison, 2009), and numerous theoreticians have derived it in various ways from the LE. Thorndike defined reinforcers in terms of approach: As a "state of affairs … which the animal does nothing to avoid, often doing such things as attain and preserve" (Thorndike, 1911, p. 244). Thorndike's LE holds that responses followed by such states of affairs will, in similar contexts be more likely to recur. This is essentially Skinner's "discriminated operant". Many of the models of the machinery of behavior on concurrent schedules are principled on the organism's approaching "states of affairs"--patches or

alternatives that are momentarily better, or that maximize the local probability of reinforcement. Donahoe (2012) provides a fascinating thumbnail of the history of the tension between molar and molecular accounts of conditioning and of choice, and comes to a similar conclusion as this paper: The LE drives the molecular forces that underlie the GML.

The GML has its roots and strength in concurrent schedules; but given the flexibility of adjusting parameters to save appearances, it is best construed as a research program for understanding how to translate physical variables into ones that are relevant to the behaving organism (Killeen, 1992). "The matching law is a descriptive statistic, not a psychological principle of choice allocation. … Choice is controlled…by prior choices and local reinforcement probabilities" (Silberberg et al., 1978, p. 396; also Lau & Glimcher, 2005). When more than two alternatives are available, a whole new world of analyses opens (Baum, 2012), such as Jensen's barycentric extension of the GML (Jensen, 2014; Jensen & Neuringer, 2009), providing greater leverage by restriction of the class of applicable models. Even the father of the ML continued to champion the LE while grooming his ML for succession: "certain writers … have, in one way or another, called for full or partial repeal of the law of effect. But that may be premature, for the law of effect promises no more than to account for behavior in terms of its consequences (however conceived), which should never have been taken as a guarantee that the account must be simple or short" (Herrnstein & Loveland, 1972, p. 383).

References

Allen, C. M. (1981). On the exponent in the "generalized" matching equation. *Journal of the Experimental Analysis of Behavior, 35*(1), 125-127. doi: 10.1901/jeab.1981.35-125

Aparicio, C. F., & Baum, W. M. (2006). Fix and sample with rats in the dynamics of choice. *Journal of the Experimental Analysis of Behavior, 86*(1), 43-63. doi: 10.1901/jeab.2006.57-05

Baum, W. M. (1974). On two types of deviation from the matching law: Bias and undermatching. *Journal of the Experimental Analysis of Behavior, 22*, 231-242. doi: 10.1901/jeab.1974.22-231

Baum, W. M. (1979). Matching, undermatching, and overmatching in studies of choice. *Journal of the Experimental Analysis of Behavior, 32*(2), 269-281. doi: 10.1901/jeab.1979.32-269

Baum, W. M. (2010). Dynamics of choice: A tutorial. *Journal of the Experimental Analysis of Behavior, 94*(2), 161-174. doi: 10.1901/jeab.2010.94-161

Baum, W. M. (2012). Rethinking reinforcement: allocation, induction, and contingency. *Journal of the Experimental Analysis of Behavior, 97*(1), 101-124. doi: 10.1901/jeab.2012.97-101

Baum, W. M. (2015). Driven by consequences: The Multiscale molar View of choice. *Managerial and decision economics*(in press).

Baum, W. M., & Davison, M. (2004). Choice in a variable environment: Visit patterns in the dynamics of choice. *Journal of the Experimental Analysis of Behavior, 81*(1), 85-127. doi: 10.1901/jeab.2004.81-85

Baum, W. M., & Davison, M. (2009). Modeling the dynamics of choice. *Behavioural Processes, 81*(2), 189-194. doi: 10.1016/j.beproc.2009.01.005

Baum, W. M., Schwendiman, J. W., & Bell, K. E. (1999). Choice, contingency discrimination, and foraging theory. *Journal of the Experimental Analysis of Behavior, 71*(3), 355-373. doi: 10.1901/jeab.1999.71-355

Bellarmine. (2014). Robert Bellarmine. (2014, December 8). In Wikipedia, The Free Encyclopedia. Retrieved 00:14, December 21, 2014, from http://en.wikipedia.org/w/index.php?title=Robert_Bellarmine&oldid=637164525.

Catania, A. C. (1963). Concurrent performances: A baesline for the study of reinforcement magnitude. *Journal of the Experimental Analysis of Behavior, 6*(2), 299-300. doi: 10.1901/jeab.1963.6-299

Catania, A. C., Reilly, M. P., Hand, D., Kehle, L. K., Valentine, L., & Shimoff, E. (2015). A quantitative analysis of the behavior maintained by delayed reinforcers. *Journal of the Experimental Analysis of Behavior, 103*(2), 288-331. doi: 10.1002/jeab.138

Cialdini, R. B. (1995). A full-cycle approach to social psychology. In G. C. Brannigan & M. R. Merrens (Eds.), *The Social Psychologists* (pp. 52-72). New York: McGraw-Hill, Inc.

Cording, J. R., McLean, A. P., & Grace, R. C. (2011). Testing the linearity and independence assumptions of the generalized matching law for reinforcer magnitude: a residual meta-analysis. *Behavioural Processes, 87*(1), 64-70. doi: 10.1016/j.beproc.2011.02.011

Crowley, M. A., & Donahoe, J. W. (2004). Matching: its acquisition and generalization. *Journal of the Experimental Analysis of Behavior, 82*(2), 143-159. doi: http://dx.doi.org/10.1901/jeab.2004.82-143

Davis, D. G. S., Staddon, J. E. R., Machado, A., & Palmer, R. G. (1993). The process of recurrent choice. *Psychological Review, 100*(2), 320. doi: http://dx.doi.org/10.1037/0033-295X.100.2.320

Davison, M. (1982). Preference in concurrent variable-interval fixed-ratio schedules. *Journal of the Experimental Analysis of Behavior, 37*(1), 81-96. doi: 10.1901/jeab.1982.37-81

Davison, M. (1983). Bias and sensitivity to reinforcement in a concurrent-chain schedule. *Journal of the Experimental Analysis of Behavior, 40*(1), 15. doi: 10.1901/jeab.1983.40-15

Davison, M., & Baum, W. M. (2002). Choice in a variable environment: effects of blackout duration and extinction between components. *Journal of the Experimental Analysis of Behavior, 77*(1), 65-89.

Davison, M., & McCarthy, D. (1988). *The matching law: A research review*. Hillsdale, NJ: Erlbaum.

Davison, M., & Nevin, J. A. (1999). Stimuli, reinforcers and behavior: An integration. *Journal of the Experimental Analysis of Behavior, 71*, 439-482. doi: 10.1901/jeab.1999.71-439

Davison, M., & Tustin, R. D. (1978). The relation between the generalized matching law and signal-detection theory. *Journal of the Experimental Analysis of Behavior, 29*(2), 331-336. doi: 10.1901/jeab.1978.29-331

Donahoe, J. W. (2012). Origins of the molar-molecular divide. *European Journal of Behavior Analysis, 19*, 196-200.

Dreyfus, L. R., Dorman, L. G., Fetterman, J. G., & Stubbs, D. A. (1982). An invariant relation between changing over and reinforcement. *Journal of the Experimental Analysis of Behavior, 38*(3), 327-338. doi: 10.1901/jeab.1982.38-327

Elliffe, D., & Alsop, B. (1996). Concurrent choice: Effects of overall reinforcer rate and the temporal distribution of reinforcers. *Journal of the Experimental Analysis of Behavior, 65*(2), 445-463. doi: 10.1901/jeab.1996.65-445

Estes, W. K. (1957). Of models and men. *American Psychologist, 12*(10), 609-617. doi: http://dx.doi.org/10.1037/h0046778

Fantino, E. (2004). Behavior-analytic approaches to decision making. *Behavioural Processes, 66*(3), 279-288.

Fantino, E., & Davison, M. (1983). Choice: Some quantitative relations. *Journal of the Experimental Analysis of Behavior, 40*(1), 1-13. doi: 10.1901/jeab.1983.40-1

Fantino, E., Gaitan, S., Kennelly, A., & Stolarz-Fantino, S. (2007). How reinforcer type affects choice in economic games. *Behavioural Processes, 75*(2), 107-114. doi: 10.1016/j.beproc.2007.02.001

Fantino, E., Preston, R. A., & Dunn, R. (1993). Delay reduction: Current status. *Journal of the Experimental Analysis of Behavior, 60*, 159-169. doi: 10.1901/jeab.1993.60-159

Finnocchiaro, M. A. (2008). *The essential Galileo*. Indianapolis, IN, USA: Hackett Publishing Co.

Grace, R. C., Berg, M. E., & Kyonka, E. G. E. (2006). Choice and timing in concurrent chains: Effects of initial-link duration. *Behavioural Processes, 71*(2-3), 188-200. doi: 10.1016/j.beproc.2005.11.002

Grace, R. C., & Reid, A. K. (2007). SQAB 2006: "It's the Non-Arbitrary Metrics, Stupid!". *Behavioural Processes, 75*(2), 91-96.

Green, L., & Myerson, J. (2004). A discounting framework for choice with delayed and probabilistic rewards. *Psychological Bulletin, 130*(5), 769-792. doi: 10.1037/0033-2909.130.5.769

Hachiga, Y., Sakagami, T., & Silberberg, A. (2014). Preference pulses induced by reinforcement. *Journal of the Experimental Analysis of Behavior, 102*(3), 335-345. doi: 10.1002/jeab.108

Halsall, P. (1999). Robert Bellarmine: Letter on Galileo's Theories, 1615. *Modern History Sourcebook.* Retrieved 12 April, 2015, from http://legacy.fordham.edu/halsall/mod/1615bellarmine-letter.asp

Herrnstein, R. J. (1961). Relative and absolute strength of response as a function of frequency of reinforcement. *Journal of the Experimental Analysis of Behavior, 4*, 267-272. doi: http://dx.doi.org/10.1901/jeab.1961.4-267

Herrnstein, R. J. (1970). On the law of effect. *Journal of the Experimental Analysis of Behavior, 13*, 243-266. doi: http://dx.doi.org/10.1901/jeab.1970.13-243

Herrnstein, R. J. (1979). Derivatives of matching. *Psychological Review, 86*, 486-495.

Herrnstein, R. J., & Loveland, D. H. (1972). Food-avoidance in hungry pigeons, and other perplexities. *Journal of the Experimental Analysis of Behavior, 18*(3), 369-383.

Herrnstein, R. J., & Loveland, D. H. (1976). Matching in a network. *Journal of the Experimental Analysis of Behavior, 26*(2), 143-153. doi: 10.1901/jeab.1976.26-143

Herrnstein, R. J., Rachlin, H., & Laibson, D. I. (Eds.). (1997). *The matching law*. Cambridge, MA: Harvard University Press.

Hinson, J. M., & Staddon, J. E. R. (1983a). Hill-climbing by pigeons. *Journal of the Experimental Analysis of Behavior, 39*(1), 25-47. doi: 10.1901/jeab.1983.39-25

Hinson, J. M., & Staddon, J. E. R. (1983b). Matching, maximizing, and hill-climbing. *Journal of the Experimental Analysis of Behavior, 40*, 321-331. doi: 10.1901/jeab.1983.40-321

Hofstadter, D. (2009). *The Earth Moves: Galileo and the Roman Inquisition*. New York: W. W. Norton & Co.

Jensen, G. (2014). Compositions and their application to the analysis of choice. *Journal of the Experimental Analysis of Behavior*. doi: 10.1002/jeab.89

Jensen, G., & Neuringer, A. J. (2008). Choice as a function of reinforcer "hold": From probability learning to concurrent reinforcement. *Journal of Experimental Psychology: Animal Behavior Processes, 34*(4), 437-460. doi: http://dx.doi.org/10.1037/0097-7403.34.4.437

Jensen, G., & Neuringer, A. J. (2009). Barycentric extension of generalized matching. *Journal of the Experimental Analysis of Behavior, 92*(2), 139-159. doi: 10.1901/jeab.2009.92-139

Johansen, E. B., Killeen, P. R., Russell, V. A., Tripp, G., Wickens, J. R., Tannock, R., . . . Sagvolden, T. (2009). Origins of altered reinforcement effects in ADHD. Behavioral and Brain Functions, 5, 7. doi: http://dx.doi.org/10.1186/1744-9081-5-7

Kacelnik, A., & Houston, A. I. (1984). Some effects of energy costs on foraging strategies. Animal Behaviour, 32(2), 609-614.

Killeen, P. R. (1970). Preference for fixed-interval schedules of reinforcement. *Journal of the Experimental Analysis of Behavior, 14*, 127-131. doi: 10.1901/jeab.1970.14-127

Killeen, P. R. (1981). Averaging theory. In C. M. Bradshaw, E. Szabadi, & C. F. Lowe (Eds.), *Quantification of steady-state operant behaviour* (pp. 21-34). Amsterdam: Elsevier.

Killeen, P. R. (1992). Mechanics of the animate. *Journal of the Experimental Analysis of Behavior, 57*(3), 429-463. doi: 10.1901/jeab.1992.57-429

Killeen, P. R. (1998). The first principle of reinforcement. In C. D. L. Wynne & J. E. R. Staddon (Eds.), *Models of Action: Mechanisms for Adaptive Behavior* (pp. 127-156). Mahwah, NJ: Lawrence Erlbaum Associates.

Killeen, P. R. (2001a). The four causes of behavior. *Current Directions in Psychological Science, 10*(4), 136-140. doi: 10.1111/1467-8721.00134

Killeen, P. R. (2001b). Modeling games from the 20th century. *Behavioural Processes, 54*, 33-52.

Killeen, P. R. (2005). *Gradus ad parnassum:* Ascending strength gradients or descending memory traces? *Behavioral and Brain Sciences, 28*, 432-434.

Killeen, P. R. (2011). Models of trace decay, eligibility for reinforcement, and delay of reinforcement gradients, from exponential to hyperboloid. *Behavioural Processes, 8*(1), 57-63. doi: http://dx.doi.org/10.1016/j.beproc.2010.12.016

Killeen, P. R. (2013). The structure of scientific evolution. *The Behavior Analyst, 36*, 325–344.

Killeen, P. R. (2015). The arithmetic of discounting. *Journal of the Experimental Analysis of Behavior, 103*(1), 249-259. doi: 10.1002/jeab.130

Killeen, P. R., Cate, H., & Tran, T. (1993). Scaling pigeons' choice of feeds: Bigger is better. *Journal of the Experimental Analysis of Behavior, 60*, 203-217.

Killeen, P. R., & Fantino, E. (1990). A unified theory of choice. *Journal of the Experimental Analysis of Behavior, 53*, 189-200.

Killeen, P. R., Palombo, G.-M., Gottlob, L. R., & Beam, J. J. (1996). Bayesian analysis of foraging by pigeons (*Columba livia*). *Journal of Experimental Psychology: Animal Behavior Processes, 22*, 480-496. doi: http://dx.doi.org/10.1037/0097-7403.22.4.480

Killeen, P. R., & Shumway, G. (1971). Concurrent random-interval schedules of reinforcement. *Psychonomic Science, 22*, 23-24.

Krägeloh, C. U., & Davison, M. (2003). Concurrent-schedule performance in transition: changeover delays and signaled reinforcer ratios. *Journal of the Experimental Analysis of Behavior, 79*(1), 87-109.

Kyonka, E. G. E., & Grace, R. C. (2008). Rapid Acquisition of Preference in Concurrent Chains When Alternatives Differ on Multiple Dimensions of Reinforcement. *Journal of the Experimental Analysis of Behavior, 89*(1), 49-69. doi: 10.1901/jeab.2008.89-49

Lander, D. G., & Irwin, R. J. (1968). Multiple schedules: effects of the distribution of reinforcements between component on the distribution of responses between conponents. *Journal of the Experimental Analysis of Behavior, 11*(5), 517-524. doi: 10.1901/jeab.1968.11-517

Lau, B., & Glimcher, P. W. (2005). Dynamic response-by-response models of matching behavior in Rhesus monkeys. *Journal of the Experimental Analysis of Behavior, 84*(3), 555-579. doi: 10.1901/jeab.2005.110-04

Logistic regression. (2015, March 24). In *Wikipedia, The Free Encyclopedia*. Retrieved April 5, 2015, from http://en.wikipedia.org/w/index.php?title=Logistic_regression&oldid=653361299

MacDonall, J. S. (1988). Concurrent variable-ratio schedules: Implications for the generalized matching law. *Journal of the Experimental Analysis of Behavior, 50*(1), 55-64. doi: 10.1901/jeab.1988.50-55

MacDonall, J. S. (2000). Synthesizing concurrent interval performances. *Journal of the Experimental Analysis of Behavior, 74*(2), 189-206. doi: 10.1901/jeab.2000.74-189

MacDonall, J. S. (2005). Earning and obtaining reinforcers under concurrent interval scheduling. *Journal of the Experimental Analysis of Behavior, 84*(2), 167-183. doi: 10.1901/jeab.2005.76-04

MacEwen, D. (1972). The effects of terminal-link fixed-interval and variable-interval schedules on responding under concurrent chained schedules. *Journal of the Experimental Analysis of Behavior, 18*, 253-261. doi: http://dx.doi.org/10.1901/jeab.1972.18-253

Madden, G. J., & Bickel, W. K. (Eds.). (2010). *Impulsivity: The behavioral and neurological science of discounting*. Washington, DC: American Psychological Association.

Marr, M. J., Fryling, M., & Ortega-González, M. (2013). Tweedledum and tweedledee: Symmetry in behavior analysis. *Conductual, 1*(1), 16-35.

Mazur, J. E. (1987). An adjusting procedure for studying delayed reinforcement. *Quantitative analyses of behavior, 5*, 55-73.

Mazur, J. E. (1989). Theories of probabilistic reinforcement. *Journal of the Experimental Analysis of Behavior, 51*(1), 87-99. doi: 10.1901/jeab.1989.51-87

Mazur, J. E. (1997). Choice, delay, probability, and conditioned reinforcement. *Animal Learning & Behavior, 25*(2), 131-147.

Mazur, J. E. (2005). Exploring a concurrent-chains paradox: decreasing preference as an initial link is shortened. *Journal of Experimental Psychology: Animal Behavior Processes, 31*(1), 3. doi: http://dx.doi.org/10.1037/0097-7403.31.1.3

Mazur, J. E. (2010). Distributed versus exclusive preference in discrete-trial choice. *Journal of Experimental Psychology: Animal Behavior Processes, 36*(3), 321-333. doi: 10.1037/a0017588

Myerson, J., & Hale, S. (1988). Choice in transition: A comparison of melioration and the kinetic model. *Journal of the Experimental Analysis of Behavior, 49*, 291-302. doi: 10.1901/jeab.1988.49-291

Myerson, J., & Miezin, F. M. (1980). The kinetics of choice: An operant systems analysis. *Psychological Review, 87*, 160-174. doi: http://dx.doi.org/10.1037/0033-295X.87.2.160

Navarro, A. D., & Fantino, E. (2009). The Sunk-Time Effect: An Exploration. *J Behav Decis Mak, 22*(3), 252-270. doi: 10.1002/bdm.624

Neuringer, A. J. (1967). Effects of reinforcer magnitude on choice and rate of responding. *Journal of the Experimental Analysis of Behavior, 10*, 417-424. doi: 10.1901/jeab.1967.10-417

Nevin, J. A. (1999). Analyzing Thorndike's law of effect: The question of stimulus—response bonds. *Journal of the Experimental Analysis of Behavior, 72*(3), 447-450.

Nevin, J. A., & Grace, R. C. (2001). Behavioral momentum and the Law of Effect. *Behavioral and Brain Sciences, 23*, 73-90.

O'Daly, M., Meyer, S., & Fantino, E. (2005). Value of conditioned reinforcers as a function of temporal context. *Learning and Motivation, 36*(1), 42-59. doi: http://dx.doi.org/10.1016/j.lmot.2004.08.001

Popper, K. R. (2002). *The logic of scientific discovery*. London: Routledge.

Preston, R. A., & Fantino, E. (1991). Conditioned reinforcement value and choice. *Journal of the Experimental Analysis of Behavior, 55*(2), 155-175.

Rachlin, H. (1971). On the tautology of the matching law. *Journal of the Experimental Analysis of Behavior, 15*(2), 249-251. doi: 10.1901/jeab.1971.15-249

Reed, D. D., & Kaplan, B. A. (2011). The Matching Law: A Tutorial for Practitioners. *Behavior analysis in practice, 4*(2), 15-24.

Shahan, T. A., & Cunningham, P. (2015). Conditioned reinforcement and information theory reconsidered. *Journal of the Experimental Analysis of Behavior, 103*(2), 405-418. doi: 10.1002/jeab.142

Shahan, T. A., & Lattal, K. A. (1998). On the functions of the changeover delay. *Journal of the Experimental Analysis of Behavior, 69*(2), 141-160. doi: 10.1901/jeab.1998.69-141

Shull, R. L., & Pliskoff, S. S. (1967). Changeover delay and concurrent schedules: some effects on relative performance measures. *Journal of the Experimental Analysis of Behavior, 10*(6), 517-527. doi: 10.1901/jeab.1967.10-517

Silberberg, A., Hamilton, B., Ziriax, J. M., & Casey, J. (1978). The structure of choice. *Journal of Experimental Psychology: Animal Behavior Processes, 4*(4), 368-398. doi: 10.1037/0097-7403.4.4.368

Smedslund, J. (2002). From hypothesis-testing psychology to procedure-testing psychologic. *Review of General Psychology, 6*(1), 51-72. doi: http://dx.doi.org/10.1037/1089-2680.6.1.51

Smith, T. T., McLean, A. P., Shull, R. L., Hughes, C. E., & Pitts, R. C. (2014). Concurrent performance as bouts of behavior. *Journal of the Experimental Analysis of Behavior, 102*, 102-125. doi: 10.1002/jeab.90

Staddon, J. E. R. (1968). Spaced responding and choice: a preliminary analysis. *Journal of the Experimental Analysis of Behavior, 11*(6), 669-682. doi: 10.1901/jeab.1968.11-669

Staddon, J. E. R. (2014). On choice and the law of effect. *International Journal of Comparative Psychology, 27*(4), 569-584. doi: https://escholarship.org/uc/item/1tn9q5ng

Stevens, S. S. (1986/1975). *Psychophysics: Introduction to its perceptual, neural, and social prospects* (2nd ed.). New Brunswick, NJ: Transaction.

Stove, D. C. (1982). *Popper and After: Four Modern Irrationalists*. Oxford: Pergamon.

Stubbs, D. A., & Pliskoff, S. S. (1969). Concurrent responding with fixed relative rate of reinforcement. *Journal of the Experimental Analysis of Behavior, 12*, 887-895. doi: 10.1901/jeab.1969.12-887

Stubbs, D. A., Pliskoff, S. S., & Reid, H. M. (1977). Concurrent schedules: A quantitative relation between changeover behavior and its consequences. *Journal of the Experimental Analysis of Behavior, 27*(1), 85-96. doi: 10.1901/jeab.1977.27-85

Sutton, N. P., Grace, R. C., McLean, A. P., & Baum, W. M. (2008). Comparing the generalized matching law and contingency discriminability model as accounts of concurrent schedule performance using residual meta-analysis. *Behavioural Processes, 78*(2), 224-230. doi: 10.1016/j.beproc.2008.02.012

Tanno, T., Silberberg, A., & Sakagami, T. (2010). Concurrent VR VI schedules: Primacy of molar control of preference and molecular control of response rates. *Learning & Behavior*, 38(4), 382-393. doi: 10.3758/LB.38.4.382

Taylor, R., & Davison, M. (1983). Sensitivity to reinforcement in concurrent arithmetic and exponential schedules. *Journal of the Experimental Analysis of Behavior,* 39(1), 191-198. doi: 10.1901/jeab.1983.39-191

Teigen, K. H. (2002). One hundred years of laws in psychology. *American Journal of Psychology*, 115, 103–118. doi: http://dx.doi.org/10.2307/1423676

Thorndike, E. L. (1911). *Animal intelligence*. New York: Macmillan.

Thurstone, L. L. (1927). A law of comparative judgment. *Psychological Review, 34*, 273-286. doi: http://dx.doi.org/10.1037//0033-295X.101.2.266

Timberlake, W. (1988). The behavior of organisms: Purposive behavior as a type of reflex. *Journal of the Experimental Analysis of Behavior, 50*(2), 305-317. doi: 10.1901/jeab.1988.50-305

Williams, B. A., & Dunn, R. (1991). Preference for conditioned reinforcement. *Journal of the Experimental Analysis of Behavior, 55*, 37-46.

For EndNote footnote: (Estes, 1957)

Appendices

Appendix A: Hooke's Law for Concurrents

In traditional ("independent tape") concurrent VI schedules, as time elapses on either schedule the objective probability of reinforcement for the next response increases for both alternatives. In interdependent schedules a single reinforcer is randomly allocated to either alternative. As time elapses on one alternative the posterior probability that reinforcement has been primed on the alternative increases (Killeen et al., 1996). When concurrent VR schedules are programmed in the fashion of Jensen and Neuringer (2008), each response on either switch tests the probability of reinforcement for both. If one primes, it is held until the animal responds on the appropriate key. This parallels the situation for typical concurrent VI schedules. With some loss of nuance, these cases will be treated alike, with the constant-probability VR schedule (a Random Ratio schedule, RR) being the paragon. The following simple hypothetical mechanism redraws the clock-space model of Hinson and Staddon (1983a).

The probability of reinforcement.

On a RR schedule with mean requirement N, the probability of reinforcement for any response is $p = 1/N$ and the cumulative probability of reinforcement by the nth response in a sequence is:

$$p_{\text{Reinf}}(n,p) = 1-(1-p)^n \tag{A1}$$

As long as n is small relative to the mean of the schedule (say, less than one third of the mean), this may be approximated as:

$$p_{\text{Reinf}}(n,p) \approx np \tag{A2}$$

Use the subscripts s for *same,* or *stay,* and a for alternate. The force keeping the animal on a side is the utility of the payoff on that side (u_s, some function of the amount or delay of reinforcement on that side, such as Equation 10) times its probability of being reinforced (its

"expected utility"). In the case of the stay side, each response after the first to that side has p_s = $1/N_s$ of being reinforced. Then the force keeping the animal on the stay side is $u_s\, p_s$. At the same time, each response on the *stay* side increments the cumulative probability of reinforcement for the first response to the *alternate* side, as $u_a n_s p_a$. This is the force to switch back. The net force to stay is the difference of these:

$$F_S = u_s p_s - u_a n_s p_a \tag{A3}$$

The reader should remember that while the probability for reinforcement on the stay side for each additional response after the changeover is $p_s = 1/N_s$, at the same time responses on the stay side accumulate the potential for reinforcement on the alternate side according to Equation A2, $p_a = n_s p_a$. Equation A3 has affinity to Myerson and colleagues' (Myerson & Hale, 1988; Myerson & Miezin, 1980) governing equation in their kinetics of choice. In equilibrium, these forces balance. That occurs when the net force is 0:

$$u_a n_s p_a = u_s p_s \tag{A4}$$

at that point

$$n_s = \frac{u_s}{u_a}\frac{p_s}{p_a} \tag{A5}$$

If this is the rich side (that is, if the expected utilities are $u_s p_s > u_a p_a$), it predicts a dwell $n_s > 1$. If it is the lean side, under the same assumptions it predicts a dwell of < 1. Since this is impossible, the predicted dwell on the lean side becomes $n = 1$ response (or a minimal unit of time).

In general (and assuming no punishment of switching by a COD, which reduces the utility of the alternate side),

$$n_2 = \max\left(1,\ \frac{u_2}{u_1}\frac{p_2}{p_1}\right) \tag{A6}$$

If the stay side is the lean side, then after the changeover response the numerator in A6 is generally less than 1, and the animal should revert. If it is the rich side, the animal should stay there until $\frac{u_a}{u_s}\frac{p_a}{p_s} \geq 1$. This has the animal staying on the lean side for a single response, and with n_a then equaling 1, the relative number of responses on the rich, stay side matches the base expected utility ratios:

$$\frac{n_s}{n_a} = \frac{u_s}{u_a}\frac{p_s}{p_a} \tag{A7}$$

This force law predicts a nominal time on the dispreferred alternative—a pattern noted by Baum and associates (Aparicio & Baum, 2006; Baum & Davison, 2004; Baum, Schwendiman, & Bell, 1999; and Silberberg et al., 1978; also see Smith et al., 2014), and called "fix [on the preferred side] and sample [the dispreferred side]". Note that the relative expected utilities from both alternatives nonetheless affect the mean time on the preferred side (the ratio on the right of A7). When the schedules are interval rather than ratio, a relatively constant response rate delivers the same results. The situation changes with the introduction of a COD, as noted below.

In the case of arithmetic progressions of ratios or intervals with mean *M,* the probability is a staircase function that, as the number of discrete intervals increase, approximates:

$$p_{\text{Reinf}}(n, M) \approx \min\left(1,\ \frac{n}{2M}\right) \tag{A8}$$

This is because the largest ratio (or interval) in such schedules is twice the mean of the schedule (2*M*). Note that the probability for a random (RR or RI) schedule with $M = 1/p$ starts increasing twice as fast as that of an arithmetic progression (as *n/M*, rather than as *n/2M*), eventually falling below it as the latter continues linearly to its ceiling. This will generate divergent dynamic forces on these two kinds of schedules (Elliffe & Alsop, 1996). In particular, the restoring force in the realm of interest (small *n*) is less than for random schedules, predicting a lower switching rate; and because the restoring force (Equation A3) is less, a lower sensitivity when oscillation is perturbed by any factor (Taylor and Davison, 1983).

Just as with the scales of a doctor's office, the fulcrum of the concurrent scale may be moved off center. If one of the initial links is shorter than the other, relative response rates shifts to favor it (Mazur, 2005; Preston & Fantino, 1991). This is because the shorter link tightens the spring asymmetrically in favor of that link—the probability of a payoff, even of an inferior good, increases, and carries behavior with it.

Winding the spring.

How do animals estimate the value of *p?* The simplest scheme is that they update their estimates with an exponentially-weighted moving average (or "linear average", e.g., Killeen, 1981; Killeen, 1998). If so, then there should be an increment in the value of *p* after each reinforcer from that schedule. Such "preference pulses" have been demonstrated in numerous studies (e.g., Davison & Baum, 2002), and Baum and Davison (2009) showed that a linear average would in fact account for most of the variance in those traces. Smith and associates (Smith et al., 2014) observed that, because behavior occurs in bouts, some of the that variance may be due to simple persistence (Silberberg et al., 1978), rather than reinforcement. Recent critical research is reviewed and resolution proposed by Hachiga and colleagues (Hachiga, Sakagami, & Silberberg, 2014). These authors developed a model of preference pulses based on an earlier model of arousal pulses in which *arousal* was the state variable corresponding to the organism's estimate of *p.* A similar deployment of *arousal* was invoked in the kinetic model of choice by (Myerson & Miezin, 1980).

The role of the change-over delay (COD).

Stubbs and associates (Stubbs, Pliskoff, & Reid, 1977) noted a strong regularity in the effects of COD duration on inter-changeover times: That time increases as a power function of the delay, with exponents scattered around 0.9. An example of this relationship is shown in Figure A1, with data from Stubbs and Pliskoff (1969) who studied concurrent VI 2, VI 6 schedules. The COD affects the utility of the side the animal is contemplating switching to, whether that is the rich or lean one. To keep this treatment at least somewhat coherent, let us assume it does so according to Equation 10, which decreases the utility of the alternate by

multiplying the left side of Equation A4: That is, that switching is under the control of the conditioned reinforcing value of approaching the alternative. Solving A4 for n_s then gives:

$$n_s = \max\left(t,\ c\frac{u_s}{u_a}\frac{p_s}{p_a}\frac{t}{1-e^{-kt}} \right) \tag{A9}$$

The animals should stay at least as long as the change-over requirement, *t*. The free parameter *c* both estimates response rate and absorbs the constant *a* from Equation 10'. Inspection of Equation A9 shows that number of responses and associated time on the alternate should increase as a concave, almost-proportional function of *t,* linearizing at large values of *t* where the exponential term gets quite small.

Figure A1 displays the trace of this equation alongside the classic COD data of Stubbs and Pliskoff (1969) as reported in Stubbs and associates (1977). Assume that the intrinsic utilities of reinforcers on each side are the same (e.g., equal amounts, etc.) so the values of *u* are equal and cancel out. Response rates were relatively constant at 1 per second. The ratio of probabilities of food are either 3/1 or 1/3, depending on whether the alternative is the rich or lean side. Assign a value of 15 to *c* and 0.26 for *k*, and see the lines through the data.

Figure A1 about here

It can also be seen that power functions with slopes of about 0.7 would also provide a good fit to these data, as noted above by Stubbs and associates (1977). The offset of these curves from one another is determined by which VI probabilities are in the numerator and denominator of A9. When the VI schedules are equal, the curves should superimpose with a common intercept just above *c*. The goodness-of-fit for this very small set of data suggests that the transition to the alternate link in concurrents may be mediated by the conditioned reinforcement value of that stimulus, determined by the time to collect a reinforcer at the end of the changeover delay. (This is the time where most of the reinforcers on concurrent schedules are received; Dreyfus, Dorman, Fetterman, & Stubbs, 1982). Whether it can adequately address the rich diversity of other data in this field without modification remains to be seen.

Time or responses? For convenience the above mini-model was developed in terms of responses, but the negative feedback in concurrent interval schedules evolves as a function of time on the alternatives. It is likely that time allocation measures will be a more robust and reliable measure of choice for that reason, with behavior while on each alternative controlled by the contingencies operative on each (Tanno, Silberberg, & Sakagami, 2010).

But what is choice? It is the organism's response to how heavily the reinforcers on one side or another pull against the springs of the concurrent schedules. When the utilities are equal but the springs have different strengths, as in traditional concurrent schedules, choice is a measure of the sensitivity to those strengths. When the utilities differ but the springs are balanced, as in typical concurrent chain schedules, choice measures how differentially strong the outcomes pull against the springs. When there are no springs, choice tells us which the animal prefers, but not by how much, for in those simple situations animals are usually rational enough to always choose what they prefer.

Appendix B: Other Models

The data shown in Figure 3 constitute a challenge for other models of concurrent chain performance. Fantino's delay reduction theory (e.g., Fantino, Preston, & Dunn, 1993) predicts a smooth increase in preference with increases in the duration of the terminal link FI schedules, but its predictions lie below the obtained data, rising from 0.54 to 0.83 (compare with 0.70 to 0.92).

Figure Captions

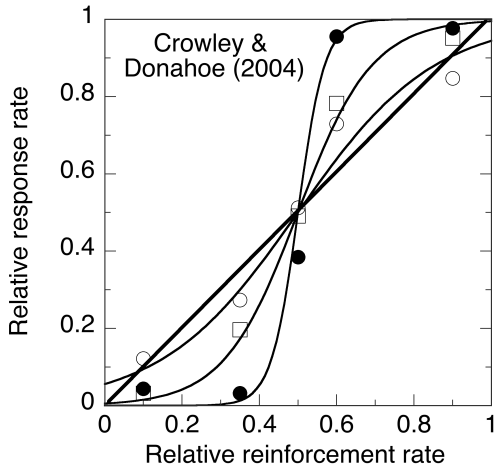


Fig 1. Disks show proportion of choices for the richer schedule in probe concurrent tests after multiple training. Squares show choices after training on equal concurrent schedules, and circles in the case of full concurrent exposure. Data from Crowley and Donahoe (2004), curves from Equation 3.

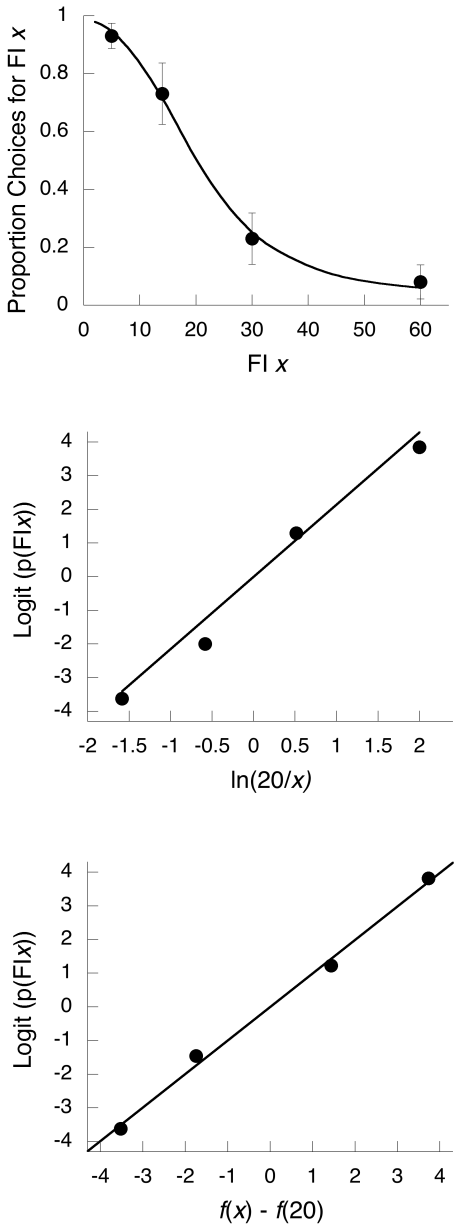
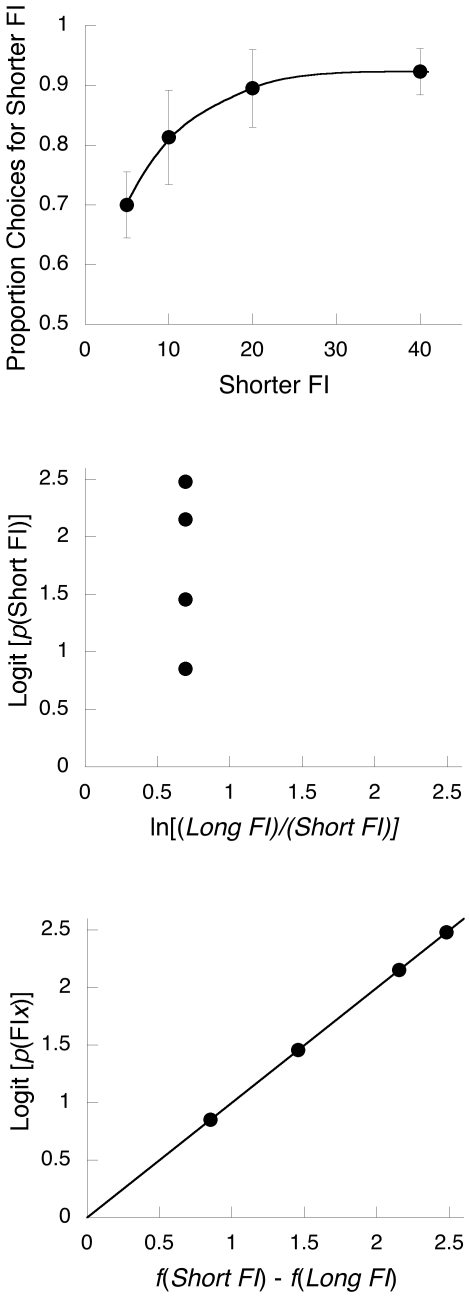


Fig 2. *Top panel:* The relative rate of choosing the varied FI schedule as a function of its length. The data are the mean performance of four pigeons, from Killeen (1970). Error bars show the semi-interquartile range. The curve is from Eqs. 3 and 10'. *Middle panel:* The GML, the logit of the probability of choosing the varied FI schedule, $p(\text{FI}|x)$, as a function of the logarithm of the delay ratios, $20/x$. *Bottom panel:* The logit of the probability of choosing the varied FI schedule as a function of the differences in conditioned reinforcement strength, given by Eqs. 6 and 10'.

Fig 3. *Top panel:* The relative rate of choosing the shorter FI schedule as a function of its length. The longer FI is always twice the value of the shorter FI. The data are the mean performance of four pigeons, from MacEwen (1972). Error bars show the semi-interquartile range. The curve is from Equations 3 and 10'. *Middle panel:* The logit of the probability of choosing the shorter FI schedule as a function of the logarithm of the delay ratios. *Bottom panel:* The logit of the probability of choosing the varied FI schedule as a function of the predictors, Equations 6 and 10'.



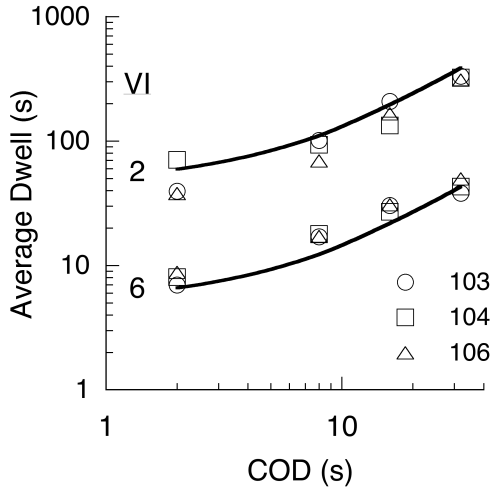


Fig. A1. The symbols give the average times spent on one VI schedule before switching to the other as a function of COD value (2, 8, 16 and 32 s; Stubbs and Pliskoff, 1969). The curves are drawn by Equation A9 with the scale factor $c = 15$ and the rate constant $k = 0.26$.