# On the computation and inversion of the Normal Inverse Gaussian cumulative distribution function

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April 28, 2024

## Contents

# 1 Introduction

Variance-mean mixture distribution

$$Z \sim \mathcal{IG}(\delta\gamma, \gamma^2), \quad X \sim \mathcal{N}(\mu + \beta Z, Z), \tag{1}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. The domain of the parameters is

$$0 \le |\beta| < \alpha, \quad \mu \in \mathbb{R}, \quad \delta > 0. \tag{2}$$

The density function is given by

$$f(x;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta}{\pi} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}} e^{\delta\gamma + \beta(x-\mu)} \tag{3}$$

The cumulative distribution function is given by

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi} \int_{-\infty}^{x} \frac{K_1\left(\alpha\sqrt{\delta^2 + (t-\mu)^2}\right)}{\sqrt{\delta^2 + (t-\mu)^2}} e^{\beta(t-\mu)}\, dt \tag{4}$$

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta}{\sqrt{2\pi}} \int_{0}^{\infty} \Phi\left(\frac{x - (\mu + \beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{(\delta - \gamma t)^2}{2t}}\, dt \tag{5}$$

**Standard case $\mu = 0$ and $\delta = 1$**

**Case $\mu = 0$**

**Case $\delta = 0$**

**Case $x = \mu$** If $x = \mu$ and $\beta = 0$ then $F(x;\alpha,0,\mu,\delta) = \frac{1}{2}$.

# 2 Preliminaries

In this section, we present some results to be used throughout this work.

## 2.1 The function $\Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$

The function $F(t;a,b) = \Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$ is part of the integrand of the integral representation in (5). Given its relevance throughout this work, we introduce here some results that shall be used subsequently. $F(t;a,b)$ has the following integral representation [1, §7.7.6]

$$F(t;a,b) = \frac{1}{2}\operatorname{erfc}\left(-\frac{\frac{a}{\sqrt{t}} + b\sqrt{t}}{\sqrt{2}}\right) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \int_{-b/\sqrt{2}}^{\infty} e^{-(tu^2 - \sqrt{2}au)}\, du \tag{6}$$

### 2.1.1 Expansion $t \to 0$

Let us consider the case $a < 0$, since we can use the mirror property $\Phi(z) = 1 - \Phi(-z)$ otherwise. To obtain an expansion for $t \to 0$, we expand $e^{-tu^2}$ and interchange summation and integration obtaining

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au} u^{2k}\, du\,.$$

For $a < 0$ the integral can be expressed in closed-form in terms of the incomplete gamma function, $\Gamma(a,x)$

$$\int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au} u^{2k}\, du = \frac{\Gamma(2k+1, -ab)}{(\sqrt{2}a)^{2k+1}},$$

and for the special case $b = 0$, it reduces to

$$\int_0^\infty e^{\sqrt{2}au} u^{2k}\, du = \frac{\Gamma(2k+1)}{(\sqrt{2}a)^{2k+1}}.$$

Then, we obtain the series expansion valid for $t \to 0$, $a \to -\infty$ and fixed $b$

$$F(t; a, b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} t^k}{k!} \frac{\Gamma(2k+1, ab)}{(\sqrt{2}a)^{2k+1}}. \tag{7}$$

Moreover, another expansion valid for large values of $a > 0$ and $b$ can be obtained after expanding $F(t; a, b)$ at $t = 0$. The first coefficients are

$$c_0 = \frac{1}{a}, \quad c_1 = \frac{ab+1}{a^3}, \quad c_2 = \frac{a^2b + 3ab + 3}{a^5}, \quad c_3 = \frac{a^3b^3 + 6a^2b^3 + 15ab + 15}{a^7} \tag{8}$$

and the expansion reads

$$F(t; a, b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)^2}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} c_k t^{k+\frac{1}{2}}. \tag{9}$$

The coefficients are expressible in terms of Bessel polynomials $y_k(x)$ [2, §A001498], and it follows that

$$F(t; a, b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)^2}}{a\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k y_k\left(\frac{1}{ab}\right) t^{k+\frac{1}{2}}, \tag{10}$$

where $y_k(x)$ has an explicit formula

$$y_k(x) = \sum_{m=0}^{k} \binom{k}{m} (k+1)_m \left(\frac{x}{2}\right)^m. \tag{11}$$

Using the connection of the Bessel polynomials with the modified Bessel function of the second kind $K_k(x)$ given by [3, §33.1.3]

$$y_k(x) = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{k+\frac{1}{2}}\left(\frac{1}{x}\right), \tag{12}$$

the resulting expansion is represented as a Bessel-type expansion

$$F(t; a, b) = 1 + \frac{e^{-\frac{a^2}{2t} - \frac{b^2}{2}t}}{\pi} \sqrt{\frac{b}{a}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k K_{k+\frac{1}{2}}(ab) t^{k+\frac{1}{2}}. \tag{13}$$

The expansion is convergent for $t < 1$. The convergence follows from the asymptotic estimate of $(b/a)^k K_k(ab) \sim (b/a)^k \sqrt{\frac{\pi}{2ab}} e^{-ab}$ as $|ab| \to \infty$. The expansion can be seen as an asymptotic expansion for large $a$, or as a uniform asymptotic expansion for $a \sim b$. The coefficients can be computed by using a recurrence relation for the modified Bessel function.

### 2.1.2 Expansion $t \to \infty$

Let us focus on the case $t \to \infty$. We can develop an asymptotic expansion after expanding the term $e^{\sqrt{2}au}$ in (6), which yields

$$F(t; a, b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k\, du\,.$$

Considering the case $b < 0$ (again, we can use the mirror property), the integral has a closed-form

$$\int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k\, du = \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{2t^{\frac{k+1}{2}}}.$$

Thus,

$$F(t; a, b) = \sqrt{\frac{t}{\pi}} \frac{e^{-\frac{a^2}{2t}}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}}. \tag{14}$$

The asymptotic behaviour of the terms in the series is

$$\frac{\Gamma\left(\frac{k+1}{2},\frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}}\sim\left(\frac{b^2}{2}\right)^{\frac{k+1}{2}}e^{-\frac{b^2}{2}t},\quad t\to\infty.$$

In fact this series is convergent, as can be observed taking the asymptotic estimate of $\Gamma(k,x)$ as $k\to\infty$. A simpler convergent expansion can be obtained transforming the integral in (6)

$$\sqrt{\frac{t}{\pi}}e^{-\frac{a^2}{2t}}\int_{-b/\sqrt{2}}^{\infty}e^{-(tu^2-\sqrt{2}au)}\,du=\sqrt{\frac{t}{\pi}}e^{-\frac{a^2}{2t}-ab-\frac{b^2}{2}t}\int_0^{\infty}e^{\sqrt{2}(a+bt)u}e^{-tu^2}\,dt,$$

and expanding $e^{\sqrt{2}(a+bt)u}$ obtaining

$$F(t;a,b)=\sqrt{\frac{t}{\pi}}\frac{e^{-\frac{a^2}{2t}-ab-\frac{b^2}{2}t}}{2}\sum_{k=0}^{\infty}\frac{(\sqrt{2}(a+bt))^k}{k!}\frac{\Gamma\left(\frac{k+1}{2}\right)}{t^{\frac{k+1}{2}}}.\tag{15}$$

Similarly to the expansion at $t\to 0$, we can obtain an asymptotic expansion expanding $F(t;a,b)$ at $t\to\infty$. For $b>0$, the first terms of the expansion are

$$c_0=1,\quad c_1=2+2ab+a^2b^2,\quad c_2=24+24ab+12a^2b^2+4a^3b^3+a^4b^4,\tag{16}$$

$$F(t;a,b)=1+\frac{e^{-ab-\frac{b^2}{2}t}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2^k k!}\frac{c_k}{b^{2k+1}}\left(\frac{1}{t}\right)^{k+\frac{1}{2}}.\tag{17}$$

The coefficients $c_k$ are expressible in terms of the incomplete gamma function, since

$$c_k=\sum_{j=0}^{2k}\frac{(2k)!}{j!}(ab)^j=e^{ab}\Gamma(2k+1,ab).$$

Rearranging terms, we get

$$F(t;a,b)=1+\frac{e^{-\frac{b^2}{2}t}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2^k k!}\frac{\Gamma(2k+1,ab)}{b^{2k+1}}\left(\frac{1}{t}\right)^{k+\frac{1}{2}}.\tag{18}$$

### 2.1.3 Expansion $t\to u$

Lastly, we study the expansion of $F(t;a,b)$ at $t=u$. This expansion shall be crucial when developing various Bessel-type asymptotic expansions later on. The first coefficients of the Taylor series are

$$c_0=\Phi\left(\frac{a}{\sqrt{u}}+b\sqrt{u}\right)d_0,\quad c_1=\phi\left(\frac{a}{\sqrt{u}}+b\sqrt{u}\right)d_1,\quad c_2=-\phi\left(\frac{a}{\sqrt{u}}+b\sqrt{u}\right)d_2,\quad c_3=\phi\left(\frac{a}{\sqrt{u}}+b\sqrt{u}\right)d_3,$$

where

$$\begin{aligned}
d_0&=1\\
d_1&=\frac{-a+bu}{2u^{3/2}}\\
d_2&=\frac{a^3-3au-a^2bu+bu^2-ab^2u^2+b^3u^3}{8u^{7/2}}\\
d_3&=\frac{-a^5+10a^3u+a^4bu-15au^2-6a^2bu^2+2a^3b^2u^2+3bu^3-6ab^2u^3-2a^2b^3u^3+2b^3u^4-ab^4u^4+b^5u^5}{48u^{11/2}},
\end{aligned}$$

and $\phi(x)=\frac{e^{-x^2/2}}{\sqrt{2\pi}}$ is the probability density function of the standard normal distribution. Thus, we have

$$F(t;a,b)=\sum_{k=0}^{\infty}c_k(t-u)^k.\tag{19}$$

Additional terms satisfy the following recurrence

$$c_{k+4}=\frac{f_0(k)c_k+f_1(k)c_{k+1}+f_2(k)c_{k+2}+f_3(k)c_{k+3}}{f_4(k)},\quad k\geq 0\tag{20}$$

where

$$\begin{aligned}
f_0(k) &= -kb^3 \\
f_1(k) &= -(1+k)b(1+2k-ab+3b^3u) \\
f_2(k) &= (2+k)(5a+2ka+a^2b-8bu-6kbu+2ab^2u-3b^3u^2) \\
f_3(k) &= -(3+k)(a^3-11au-4kau-a^2bu+13bu^2+6kbu^2-ab^2u^2+b^3u^3) \\
f_4(k) &= 2(3+k)(4+k)u^2(-a+bu)
\end{aligned}$$

### 2.2 Bessel-type expansions

## 3 Expansions: case $\beta = 0$

### 3.1 Expansions $|x-\mu| \to 0$

We use the integral (5). For $|x-\mu| \to 0$, we use the series expansion of $\Phi(x)$ given by

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k k!(2k+1)}$$

$$\Phi(x) = \frac{1}{2} + \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!!}$$

Use closed-form in terms of the modified Bessel function. Alternating series

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (x-\mu)^{2k+1}}{2^k k!(2k+1)} \left(\frac{\alpha}{\delta}\right)^{k+1} K_{k+1}(\alpha\delta). \tag{21}$$

Asymptotic analysis of $K_{k+1}(\alpha\delta)$. To obtain a series with positive term for $x-\mu > 0$, use alternative expansion of $\Phi(x)$

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi} \sum_{k=0}^{\infty} \frac{(x-\mu)^{2k+1}}{(2k+1)!!} \left(\frac{\alpha}{\omega}\right)^{k+1} K_{k+1}(\alpha\omega), \quad \omega = \sqrt{\delta^2+(x-\mu)^2}. \tag{22}$$

### 3.2 Expansion $|x-\mu| \to \infty$

For $x-\mu < 0$

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\Gamma(2k+1)}{2^k k!(x-\mu)^{2k+1}} \left(\frac{\omega}{\alpha}\right)^{k} K_k(\alpha\omega), \quad \omega = \sqrt{(x-\mu)^2+\delta^2}. \tag{23}$$

### 3.3 Expansion $\alpha \to \infty$

## 4 Expansions: general case

### 4.1 Expansions $|x-\mu| \to 0$ (option 1)

### 4.2 Expansions $|x-\mu| \to 0$ (option 2)

The starting point is the integral representation in (5) after expanding the exponential

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \int_0^{\infty} \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt,$$

and replacing $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ by the expansion in (14)

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \int_0^{\infty} \Gamma\left(\frac{k+1}{2}, \frac{\beta^2}{2}t\right) t^{-3/2-k/2} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\, dt,$$

where $\omega = \sqrt{\delta^2 + (x-\mu)^2}$. Consider the ascending series of the incomplete gamma function given by [1, §8.7]

$$\Gamma(a,x) = \Gamma(a) - \sum_{j=0}^{\infty} \frac{(-1)^j x^{a+j}}{j!(a+j)}. \tag{24}$$

We proceed splitting the inner integral into two terms

$$T_1 = \Gamma\left(\frac{k+1}{2}\right) \int_0^\infty t^{-3/2-k/2} e^{-\frac{\omega^2}{2t} - \frac{\gamma^2}{2}t}\, dt \tag{25}$$

$$T_2 = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)} \int_0^\infty t^{j-1} e^{-\frac{\omega^2}{2t} - \frac{\gamma^2}{2}t}\, dt, \tag{26}$$

and observe that both integrals are expressible in terms of modified Bessel function, resulting in the sums $S_1$ and $S_2$, such that $F(x;\alpha,\beta,\mu,\delta) = C(S_1 - S2)$, defined as follows

$$S_1 = \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) 2K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}} \tag{27}$$

$$S_2 = \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)} 2K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j. \tag{28}$$

Interchanging the order of summation in $S_2$, we observe that the sum in $k$ is convergent an expressible in terms of the lower incomplete gamma function $\gamma(a,x)$. Assuming $\beta > 0$

$$\sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!(\frac{k+1}{2}+j)} \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}} = -\frac{\sqrt{2}}{(x-\mu)^{2j+1}\beta^{2j}} \gamma\left(2j+1, -(x-\mu)\beta\right). \tag{29}$$

Thus,

$$S_2 = -2\sqrt{2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\gamma\left(2j+1, -(x-\mu)\beta\right)}{(x-\mu)^{2j+1}\beta^{2j}} \left(\frac{\beta^2}{2}\right)^j K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j$$

Rearranging terms

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\pi\sqrt{2}} \Bigg[ & \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}} \\ & + \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\gamma\left(2k+1, -(x-\mu)\beta\right)}{(x-\mu)^{2k+1}} K_k(\omega\gamma) \left(\frac{\omega}{2\gamma}\right)^k \Bigg] \end{aligned} \tag{30}$$

If $\beta < 0$ then $F(x;\alpha,\beta,\mu,\delta) = 1 - F(-x;\alpha,-\beta,-\mu,\delta)$. The expansion is convergent for small $x-\mu$ and fixed values of the rest of parameters. Moreover, the convergence improves when $\gamma \sim \omega$, also valid for large values for these two parameters.

### 4.3 Expansion $\alpha \to \infty, \delta \to \infty$

### 4.4 Expansion $|x-\mu| \to \infty$

The expansion of $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ at $t \to 0$ is given in terms of the Bessel polynomial $y_k(x)$. To simplify notation, we take $a = x - \mu$ and $b = -\beta$. Then,

$$\Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right) = 1 + \frac{e^{-a^2/(2t) - ab - b^2/2t}}{a\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k y_k\left(\frac{1}{ab}\right) t^{\frac{1}{2}+k} \tag{31}$$

Using the connection of the Bessel polynomials with the modified Bessel function of the second kind $K_n(x)$ given by

$$y_n(x) = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{n+\frac{1}{2}}\left(\frac{1}{x}\right), \tag{32}$$

we replace in the integral, we obtain

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \delta e^{\delta\gamma}\sqrt{\frac{b}{2a\pi^3}}\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{b}{a}\right)^k K_{k+\frac{1}{2}}(ab)\int_0^{\infty} t^{k-1}e^{-\frac{\delta^2+a^2}{2t}-\frac{\gamma^2+b^2}{2}t}\,dt \tag{33}$$

$$\int_0^{\infty} t^{k-1}e^{-\frac{\delta^2+a^2}{2t}-\frac{\gamma^2+b^2}{2}t}\,dt = 2K_k(\omega\alpha)\left(\frac{\omega}{\alpha}\right)^k, \tag{34}$$

where $\omega = \sqrt{\delta^2 + (x-\mu)^2}$. Rearranging terms

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \delta e^{\delta\gamma}\sqrt{\frac{2\beta}{(\mu - x)\pi^3}}\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{\beta\omega}{\alpha(\mu-x)}\right)^k K_{k+\frac{1}{2}}((\mu-x)\beta)K_k(\omega\alpha) \tag{35}$$

# 5 Numerical integration

For cases do not covered by the described expansions, we need to resort to numerical integration. The Laplace-type integral (5), whose integrand includes the complementary error function, should be faster to evaluate than the Bessel integral in (4).

To use numerical integration methods requiring a finite interval, we truncate the integral (5) at some point $N$, such that

$$I = \int_0^N \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)t^{-3/2}e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\,dt + \int_N^{\infty}\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)t^{-3/2}e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\,dt,$$

and $F(x;\alpha,\beta,\mu,\delta) = CI$, where $C = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}$. The truncation error can be bounded by

$$\int_N^{\infty}\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)t^{-3/2}e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\,dt \le \frac{e^{-\frac{\delta^2}{2N}}}{N^{3/2}}\int_N^{\infty}e^{-\frac{\gamma^2}{2}t}\,dt \le \frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2}.$$

We can select $N$ for a desired absolute tolerance $\epsilon$ via a bisection procedure or by solving using root-finding methods the equation

$$\frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C}. \tag{36}$$

Moreover, a slightly lesser sharper bound allows a closed-form solution of the above equation in terms of the principal branch of the Lambert $W$ function [1, §4.13]

$$\frac{2e^{-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C} \longrightarrow N = \frac{3}{\gamma^2}W_0\left(\frac{\gamma^2}{3u}\right), \quad u = \left(\frac{\gamma^2\epsilon}{2C}\right)^{2/3}. \tag{37}$$

To accurately estimate $N$ to achieve a relative tolerance, we need an estimate of the order of magnitude of $I$. First, we rewrite the integrand as $e^{g(t)}$, where

$$g(t) = -\frac{\delta^2}{2t} - \frac{\gamma^2}{2}t - \frac{3}{2}\log(t) + \log\left(\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)\right),$$

and

$$g'(t) = \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} - \varphi(x;\beta,\mu), \quad \varphi(x;\beta,\mu) = \frac{1}{2}\left(\frac{x-\mu}{t^{3/2}} + \frac{\beta}{\sqrt{t}}\right)\frac{\phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}{\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}$$

The saddle point $t_0$ and maximum contribution $e^{g(t_0)}$ of the integrand is obtained as the solution of the equation $g'(t) = 0$. Thus, $N$ for relative tolerance can be estimated after replacing $\epsilon$ with $\epsilon e^{g(t_0)}$ in (37). For the case where $\gamma$ and $\delta$ are both large and $\beta$ and $x - \mu$ are fixed, the last term in $g'(t)$ can be neglected, obtaining the quadratic equation

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + (\gamma\delta)^2}}{\gamma^2}, \tag{38}$$

taking the positive internal saddle point $t_0$. The case where $\gamma$ and $\delta$ are small requires further analysis. If $x-\mu>0$, as $x-\mu$ increases the contribution of $\varphi(x;\beta,\mu)$ vanishes and (38) is valid. Contrarily, if $x-\mu<0$ and $\beta\to 0$ (since $|\beta|<\gamma<\alpha$), $\varphi(x;\beta,\mu)$ can be approximated as follows

$$\frac{\phi\left(\frac{x-\mu}{\sqrt{t}}\right)}{\Phi\left(\frac{x-\mu}{\sqrt{t}}\right)} \approx -\frac{x-\mu}{\sqrt{t}}, \quad \varphi(x;\beta,\mu) \approx -\frac{(x-\mu)^2}{2t^2},$$

then, we have another quadratic equation

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} + \frac{(x-\mu)^2}{2t^2}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + \gamma^2\left((x-\mu)^2+\delta^2\right)}}{\gamma^2}.$$

If $\beta<0$, a better approximation is

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} + \frac{(x-\mu)^2}{2t^2} + \frac{\beta(x-\mu)}{2t}, \quad t_0 = \frac{h + \sqrt{h^2 + \gamma^2\left((x-\mu)^2+\delta^2\right)}}{\gamma^2}, \quad h = \frac{\beta(x-\mu)-3}{2}.$$

The saddle point estimates $t_0$ can also be used as a starting point for root-finding, however, for the purpose of approximating the order of magnitude of $I$, the approximations are sufficient.

- Gauss-Legendre
- Double-exponential tanh-sinh numerical integration

A double-exponential integration arises as follows. Because $|\beta|<\alpha$, we can write $\beta=\alpha\tanh(\theta)$. Substituting in (4) $x-\mu=\delta\sinh(\theta+u)$ we obtain

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{-\infty}^{\tau} K_1\left(\alpha\delta\cosh(\theta+u)\right) e^{\beta\delta\sinh(\theta+u)}\,du, \tag{39}$$

where

$$\tau = \operatorname{arcsinh}\left(\frac{x-\mu}{\delta}\right) - \theta. \tag{40}$$

**Proposition 5.1** *For $x-\mu<0$, an incomplete Laplace-type integral representation in terms of modified Bessel function $K_0(x)$ is given by*

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\sqrt{2}\delta e^{\delta\gamma}}{\pi}\int_{\beta/\sqrt{2}}^{\infty} e^{\sqrt{2}(x-\mu)t} K_0\left(\sqrt{2\left((x-\mu)^2+\delta^2\right)}\sqrt{\frac{\gamma^2}{2}+t^2}\right) dt, \tag{41}$$

*and $F(x;\alpha,\beta,\mu,\delta) = 1-F(-x;\alpha,\beta,-\mu,\delta)$, otherwise.*

**Proof:** Consider the integral representation of the function $\Phi\left(\frac{x-\mu}{\sqrt{t}}-\beta\sqrt{t}\right)$

$$\Phi\left(\frac{x-\mu}{\sqrt{t}}-\beta\sqrt{t}\right) = \sqrt{\frac{t}{\pi}}e^{-\frac{(x-\mu)^2}{2t}}\int_{\beta/\sqrt{2}}^{\infty} e^{-(tu^2-\sqrt{2}(x-\mu)u)}\,du\,.$$

Replacing in (5) and interchanging the order of integration we obtain

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2}\pi}\int_{\beta/\sqrt{2}}^{\infty} e^{\sqrt{2}(x-\mu)u}\int_0^{\infty} t^{-1} e^{-\frac{\left((x-\mu)^2+\delta^2\right)}{2t}-\left(\frac{\gamma^2}{2}+u^2\right)t}\,dt\,du\,.$$

The observation that the inner integral can be represented in terms of the modified Bessel function $K_0(x)$

$$\int_0^{\infty} t^{-1} e^{-\frac{\left((x-\mu)^2+\delta^2\right)}{2t}-\left(\frac{\gamma^2}{2}+u^2\right)t}\,dt = 2K_0\left(2\sqrt{\frac{(x-\mu)^2+\delta^2}{2}}\sqrt{\frac{\gamma^2}{2}+t^2}\right).$$

$\square$

**Proposition 5.2** *For $x - \mu > 0$, a Fourier cosine transform integral representation in terms of the exponential integral $E_1(x)$ is given by*

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{\delta e^{\delta\gamma}}{\pi}\int_0^\infty E_1\left((x-\mu)(\sqrt{t^2+\alpha^2}-\beta)\right)\cos(\delta t)\,dt\,. \tag{42}$$

*A Fourier sine transform integral representation in terms of basic functions is*

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{e^{\delta\gamma}}{\pi}\int_0^\infty \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)}\sin(\delta t)\,dt\,. \tag{43}$$

**Proof:**

$$\frac{K_1\left(\alpha\sqrt{\delta^2+z^2}\right)}{\sqrt{\delta^2+z^2}} = \frac{1}{\alpha z}\int_0^\infty e^{-z\sqrt{t^2+\alpha^2}}\cos(\delta t)\,dt\,.$$

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{x-\mu}^\infty \frac{e^{\beta z}}{\alpha z}\int_0^\infty e^{-z\sqrt{t^2+\alpha^2}}\cos(\delta t)\,dt\,dz$$

$$\int_{x-\mu}^\infty \frac{1}{z}e^{-(\sqrt{t^2+\alpha^2}-\beta)z}\,dz = E_1\left((x-\mu)(\sqrt{t^2+\alpha^2}-\beta)\right).$$

□

# 6 Inversion methods

Ideas:

- Central region
  1. The moment generating function is simple. The computation of its central moments is easy.
  2. Use multiple central moments to estimate the quantile using a Cornish-Fisher expansion.
- Tails (asymptotic methods) [3, §42]
  1. Direct application using the standard form integral representation (4).
- Root-finding: Halley's or Schwarzian-Newton method.

# 7 Numerical experiments

# References


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[2] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences.

[3] N. M. Temme. *Asymptotic Methods for Integrals*, volume 6 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.