On the computation of the cumulative distribution function of the Normal Inverse Gaussian distribution

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Abstract

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1 Introduction

The normal inverse Gaussian distribution (NIG) is a flexible four-parameter distribution introduced by Barndorff-Nielsen in 1977 [3], which arises as a particular case of the generalized hyperbolic distribution. In the last two decades, it has gained significant popularity in various fields due to its desirable properties and the ability to model a wide range of shapes, particularly suited for skewed and heavy-tailed data.

The NIG distribution was initially introduced in the context of financial data modelling, with applications in option pricing to capture non-Gaussian asset returns [12], credit risk management and credit derivatives for pricing collateralized default obligations [34, 22], stochastic modelling of energy derivatives [7] and stochastic volatility modelling [4]. Besides the numerous applications in financial modelling, it has found applications in wireless communication and remote sensing [17, 38], turbulence and fluid dynamics [18], astrophysics [24] and reliability analysis [45].

In this work, we focus on deriving series and asymptotic expansions to efficiently compute the cumulative distribution function of the normal inverse Gaussian. The new expansions, derived from the Laplace-type integral representation, are proposed to complement the direct use of numerical integration methods, the latter being the method used in all the implementations publicly available.

The outline of the paper is as follows. In Section 2, we revisit the distribution properties of the NIG distribution and introduce the derivation of alternative integral representation for the cumulative distribution function. Then, in Section 3, we derive and analyse a variety of series expansions and asymptotic expansions for two special cases and the general case, providing a rigorous calculation of error bounds. In addition, we provided a detailed treatment of the strategies to compute two possible integral representations using different quadrature methods. Section 4 presents an extensive discussion on algorithmic aspects to ease its implementation. After that, in Section 5, we assess the performance and accuracy of the proposed algorithms. Finally, Section 6 presents our conclusions and several possible enhancements.

2 Distribution properties

As the NIG distribution is a special case of the GH distribution, distributional properties can be inferred from results for the GH distribution. Convolution and infinity divisible.

REVIEW: [32]. A special type of mixture, called the variance-mean mixture of normals. It is defined as a mixture of normal distributions such that the mean and variance of the mixed distribution is in a special (affine-linear) relation. Consider Z is a positive random variable following an incomplete Gaussian distribution, and μ and β are constants. The variance-mean mixture distribution

$$Z \sim \mathcal{IG}(\delta\gamma, \gamma^2), \quad X \mid Z \sim \mathcal{N}(\mu + \beta Z, Z), \tag{2.1}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. The domain of the parameters is

$$0 \leq |\beta| < \alpha, \quad \mu \in \mathbb{R}, \quad \delta > 0. \tag{2.2}$$

Define $\omega = \sqrt{(x-\mu)^2 + \delta^2}$.

NOTES

- μ: location parameter

- Scale-invariant parameterization
- Moment generating function
- First four moments and reference to the paper of generalized hyperbolic distribution moments.
- The family of generalized hyperbolic distributions is obtained by mixing normals using the generalized inverse Gaussian family of weights.
- Parameterization options. Use of the parameterization dominating the literature.

2.1 Density function

The normal inverse Gaussian (NIG) distribution denote by NIG(alpha, beta, mu, delta) has probability density function (PDF)...

The density function is given as

$$f(x;\alpha,\beta,\mu,\delta)=\frac{\alpha\delta}{\pi}\frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}}e^{\delta\gamma+\beta(x-\mu)} \tag{2.3}$$

Parameterization: Standard case $\mu=0$ and $\delta=1$. The parameters have the following interpretation: α is the tail heaviness, β is the asymmetry or skewness, μ is the location parameter and δ the scale parameter. Where μ is the location of the density, β is the skewness parameter, α measures the heaviness of the tails.

If the parameter $\beta\neq 0$, then the distribution will usually be skewed. The use of the modified Bessel function of the second kind, here simply called modified Bessel function, will be omnipresent.

2.2 Cumulative distribution function

The cumulative distribution function is given by

$$F(x;\alpha,\beta,\mu,\delta)=\frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{-\infty}^{x}\frac{K_1\left(\alpha\sqrt{\delta^2+(t-\mu)^2}\right)}{\sqrt{\delta^2+(t-\mu)^2}}e^{\beta(t-\mu)}\,dt \tag{2.4}$$

$$F(x;\alpha,\beta,\mu,\delta)=\frac{\delta}{\sqrt{2\pi}}\int_{0}^{\infty}\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)t^{-3/2}e^{-\frac{(\delta-\gamma t)^2}{2t}}\,dt \tag{2.5}$$

Also denote the reflection formula

$$\tilde{F}(x;\alpha,\beta,\mu,\delta)=1-F(-x;\alpha,-\beta,-\mu,\delta), \tag{2.6}$$

which follows from the reflection formula of $\Phi(x)=1-\Phi(-x)$.

2.2.1 Alternative integral representations

We obtain the following alternative integral representations in terms other special functions. More interestingly, we derive an integral representation in terms of elementary functions. Although these representations have restrictions on the sign of $x-\mu$, the use of the reflection formula (2.6) permits the computation on the whole domain.

Proposition 2.1 *An incomplete Laplace-type integral representation in terms of modified Bessel function $K_0(x)$ is given by*

$$F(x;\alpha,\beta,\mu,\delta)=\frac{\sqrt{2}\delta e^{\delta\gamma}}{\pi}\int_{\beta/\sqrt{2}}^{\infty}e^{\sqrt{2}(x-\mu)t}K_0\left(\sqrt{2\left((x-\mu)^2+\delta^2\right)}\sqrt{\frac{\gamma^2}{2}+t^2}\right)dt,\quad x-\mu<0. \tag{2.7}$$

Proof: Consider the integral representation of the function $\Phi\left(\frac{x-\mu}{\sqrt{t}}-\beta\sqrt{t}\right)$

$$\Phi\left(\frac{x-\mu}{\sqrt{t}}-\beta\sqrt{t}\right)=\sqrt{\frac{t}{\pi}}e^{-\frac{(x-\mu)^2}{2t}}\int_{\beta/\sqrt{2}}^{\infty}e^{-(tu^2-\sqrt{2}(x-\mu)u)}\,du\,.$$

Replacing in (2.5) and interchanging the order of integration, we obtain

$$F(x;\alpha,\beta,\mu,\delta)=\frac{\delta e^{\delta\gamma}}{\sqrt{2}\pi}\int_{\beta/\sqrt{2}}^{\infty}e^{\sqrt{2}(x-\mu)u}\int_{0}^{\infty}t^{-1}e^{-\frac{\left((x-\mu)^2+\delta^2\right)}{2t}-\left(\frac{\gamma^2}{2}+u^2\right)t}\,dt\,du\,.$$

The observation that the inner integral can be represented in terms of the modified Bessel function $K_0(x)$

$$\int_{0}^{\infty}t^{-1}e^{-\frac{\left((x-\mu)^2+\delta^2\right)}{2t}-\left(\frac{\gamma^2}{2}+u^2\right)t}\,dt=2K_0\left(2\sqrt{\frac{(x-\mu)^2+\delta^2}{2}}\sqrt{\frac{\gamma^2}{2}+t^2}\right).$$

□

Proposition 2.2 *An integral representation of the cumulative distribution function in terms of elementary functions is given by*

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{e^{\delta\gamma}}{\pi}\int_0^\infty \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)}\sin(\delta t)\,dt, \quad x-\mu>0. \tag{2.8}$$

Proof: Consider the integral representation (2.4). Then, we replace the term involving $K_1(x)$ by its sine transform [6, §2.4]

$$\frac{K_1(\alpha\sqrt{t^2+\delta^2})}{\sqrt{t^2+\delta^2}} = \frac{1}{\alpha\delta}\int_0^\infty \frac{ze^{-t\sqrt{z^2+\alpha^2}}}{\sqrt{z^2+\alpha^2}}\sin(\delta z)\,dz,$$

valid for $t \geq 0$. Replacing in (2.4) and interchanging the order of integration, we have

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) &= 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi}\frac{1}{\alpha\delta}\int_0^\infty \frac{z\sin(\delta z)}{\sqrt{z^2+\alpha^2}}\int_{x-\mu}^\infty e^{-t\sqrt{z^2+\alpha^2}+\beta t}\,dt\,dz \\ &= 1 - \frac{e^{\delta\gamma}}{\pi}\int_0^\infty \frac{z\sin(\delta z)}{\sqrt{z^2+\alpha^2}}\frac{e^{-(x-\mu)\left(\sqrt{z^2+\alpha^2}-\beta\right)}}{\sqrt{z^2+\alpha^2}-\beta}\,dz \end{aligned}$$

where the inner integral converges for $x-\mu>0$. □

Analogously, we obtain an integral representation in terms of the exponential integral $E_1(x)$ using the cosine transform of the modified Bessel function.

Proposition 2.3 *An integral representation of the cumulative distribution function in terms of the exponential integral $E_1(x)$ is given by*

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{\delta e^{\delta\gamma}}{\pi}\int_0^\infty E_1\left((x-\mu)(\sqrt{t^2+\alpha^2}-\beta)\right)\cos(\delta t)\,dt, \quad x-\mu>0. \tag{2.9}$$

Proof: This result can be derived from the sine transform in (2.8) using integration by parts. Instead, we can use the cosine transform [6, §1.4]

$$\frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}} = \frac{1}{\alpha t}\int_0^\infty e^{-t\sqrt{z^2+\alpha^2}}\cos(\delta z)\,dz\,.$$

Thus,

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{x-\mu}^\infty \frac{e^{\beta t}}{\alpha t}\int_0^\infty e^{-t\sqrt{z^2+\alpha^2}}\cos(\delta z)\,dz\,dt,$$

where the inner integral is expressible in closed form in terms of the exponential integral $E_1(x)$

$$\int_{x-\mu}^\infty \frac{1}{t}e^{-(\sqrt{z^2+\alpha^2}-\beta)t}\,dt = E_1\left((x-\mu)(\sqrt{z^2+\alpha^2}-\beta)\right).$$

□

Moreover, an integral representation whose integrand decays double-exponentially is obtained using the transformation described in [42, §42.4]. First, we write $\beta = \alpha\tanh(\theta)$, valid since $|\beta|<\alpha$. Substituting in (2.4) the term $x-\mu = \delta\sinh(\theta+u)$, we obtain

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{-\infty}^\tau K_1\left(\alpha\delta\cosh(\theta+u)\right)e^{\beta\delta\sinh(\theta+u)}\,du, \tag{2.10}$$

where

$$\tau = \operatorname{arcsinh}\left(\frac{x-\mu}{\delta}\right) - \theta. \tag{2.11}$$

To conclude, a discussion on the numerical integration methods for effective computation of the previous integrals is presented in Section 3.4.

3 Methods of computation

In this Section, we describe the methods used for efficient computation of the CDF for the general case and various special cases, $\beta = 0$ and $x = \mu$, respectively. In particular, we explore series expansions, asymptotic expansions, uniform asymptotic expansions and numerical integration. We emphasize that many of the techniques and numerical methods introduced in the sections devoted to the two special cases shall also be used to develop expansions of the general case in Section 3.3.

3.1 Special case $\beta = 0$

The case $\beta = 0$ corresponds to the symmetric case. The symmetric NIG distribution has been widely used in financial modelling, for example for pricing collateralized default obligations [22] and market risk [34] where symmetric distributions are employed for modelling macro-economical factors as an fat-tailed alternative to the normal distribution.

3.1.1 Expansions $|x-\mu| \to 0$

For developing a series expansion for the case $|x-\mu| \to 0$, we start from the integral representation in (2.5), after expanding the square

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\sqrt{2\pi}} \int_0^\infty \Phi\left(\frac{x-\mu}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\alpha^2}{2}t}\, dt\,. \tag{3.1}$$

We proceed expanding the term $\Phi\left(\frac{x-\mu}{\sqrt{t}}\right)$, by using the two well-known absolutely convergent series expansions of $\Phi(x)$ [23, §2]

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{2^k k!(2k+1)}, \tag{3.2}$$

and

$$\Phi(x) = \frac{1}{2} + \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{x^{2k+1}}{(2k+1)!!}. \tag{3.3}$$

If we choose the expansion (3.2) and interchange the order of integration and summation, the resulting integral has the form

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k (x-\mu)^{2k+1}}{2^k k!(2k+1)} \int_0^\infty t^{-k-2} e^{-\frac{\delta^2}{2t}-\frac{\alpha^2}{2}t}\, dt, \tag{3.4}$$

where the integral has a closed-form in terms of the modified Bessel function. In general,

$$\int_0^\infty t^{\lambda-1} e^{-a/t-zt}\, dt = 2\left(\frac{\alpha}{z}\right)^{\lambda/2} K_\lambda(2\sqrt{\alpha z}). \tag{3.5}$$

Inserting (3.5) in (3.4) and rearranging terms, we obtain the the alternating series

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \sum_{k=0}^\infty \frac{(-1)^k (x-\mu)^{2k+1}}{2^k k!(2k+1)} \left(\frac{\alpha}{\delta}\right)^{k+1} K_{k+1}(\alpha\delta). \tag{3.6}$$

Moreover, to obtain a similar series with positive terms, we choose the expansion of $\Phi(x)$ in (3.3), yielding

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \sum_{k=0}^\infty \frac{(x-\mu)^{2k+1}}{(2k+1)!!} \left(\frac{\alpha}{\omega}\right)^{k+1} K_{k+1}(\alpha\omega). \tag{3.7}$$

Note that the series expansions (3.6) and (3.7) can be written as a truncated series with the corresponding remainder term. For example, truncating the series (3.7) at N, we can write

$$\sum_{k=0}^\infty T_k = \sum_{k=0}^{N-1} T_k + \sum_{k=N}^\infty T_k, \quad T_k = \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^k \frac{K_{k+1}(\alpha\omega)}{(2k+1)!!}. \tag{3.8}$$

Thus, one has the series with remainder term $R_N = \sum_{k=N}^\infty T_k$

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega}\left(\sum_{k=0}^{N-1} T_k + R_N\right). \tag{3.9}$$

For a rigorous evaluation of error bounds given a number of terms N, it is convenient to calculate an upper bound for R_N in (3.8). We can, for example, bound R_N by comparison with a geometric series

$$|R_N| \le \frac{|T_N|}{1-C}, \quad C = \left|\frac{T_{N+1}}{T_N}\right| \tag{3.10}$$

iff $C < 1$, where T_N is the first omitted term in the expansion. The following lemma provides an upper bound for $K_{\nu+1}(x)$, required for a posterior derivation of an upper bound for the term T_N.

Lemma 3.1 *For $x \ge 0$ and $\nu \ge -\frac{1}{2}$ we have*

$$K_{\nu+1}(x) < \frac{\Gamma(\nu+1)2^\nu}{x^{\nu+1}}. \tag{3.11}$$

Proof: The proof reduces to combining the uniform bound in [14]

$$xK_{\nu+1}(x)I_\nu(x) \leq 1. \tag{3.12}$$

with the lower bound [25]

$$\left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} < I_\nu(x). \tag{3.13}$$

Then, it follows that

$$K_{\nu+1}(x) \leq \frac{1}{xI_\nu(x)} < \frac{\Gamma(\nu+1)2^\nu}{x^{\nu+1}}. \tag{3.14}$$

□

Theorem 3.2 *Given $\alpha > 0$, $\omega > 0$, $x - \mu \in \mathbb{R}$ and $N \in \mathbb{N}$, the remainder term in* (3.9) *satisfies*

$$R_N \leq \frac{|T_N|}{1-C}, \tag{3.15}$$

where

$$T_N < \frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.16}$$

and

$$C < \left(\frac{x-\mu}{\omega}\right)^2 \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{2N+3}. \tag{3.17}$$

Proof: The use of Lemma 3.1 gives

$$\begin{aligned}
T_N &= \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^N \frac{K_{N+1}(\alpha\omega)}{(2N+1)!!} \\
&< \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^N \frac{\Gamma(N+1)2^N}{(\alpha\omega)^{N+1}(2N+1)!!} \\
&= \frac{1}{\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\frac{(N!2^N)^2}{(2N+1)!}.
\end{aligned}$$

An upper bound for the ratio of factorials in the previous inequality is given by

$$\frac{(N!2^N)^2}{(2N+1)!} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(N+1)}{\Gamma(N+3/2)} \leq \frac{1}{2}\sqrt{\frac{\pi}{N+1/2}},$$

where we use the fact that $\Gamma(N+3/2) = (N+1/2)\Gamma(N+1/2)$ and the upper bound of the ratio of gamma functions [43]

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+s)^{1-s}, \quad s \in (0,1). \tag{3.18}$$

Thus, the following bound for T_N holds

$$T_N < \frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.19}$$

For the ratio C, an explicit formula in terms of the ratio of modified Bessel functions is

$$C = \frac{T_{N+1}}{T_N} = \frac{(x-\mu)^2\alpha}{\omega(2N+3)}\frac{K_{N+2}(\alpha\omega)}{K_{N+1}(\alpha\omega)}. \tag{3.20}$$

The ratio can be bounded using a sharp bound for the ratio of modified Bessel functions [35], yielding

$$\frac{K_{N+2}(\alpha\omega)}{K_{N+1}(\alpha\omega)} < \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{\alpha\omega}. \tag{3.21}$$

Then, we have

$$C < \left(\frac{x-\mu}{\omega}\right)^2 \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{2N+3}.$$

□

To study the regime of applicability for the expansion, we can estimate the required number of terms N equating the bound of T_N in (3.16) times the normalizing factor with the requested absolute error ϵ and solving for N, which gives

$$N \approx -\frac{\Re(W_{-1}(D))}{4\log(A)} + \frac{1}{2}, \quad A = \frac{x-\mu}{\omega}, \quad B = \frac{A\delta e^{\delta\alpha}}{2\omega\pi}, \quad C = \frac{\epsilon^2}{B^2\pi}, \quad D = -\frac{4\log(A)}{CA^2} \tag{3.22}$$

where $W_k(x)$ denotes the Lambert W function [11, §4.13]. The branch $k = -1$ is used to obtain the maximum real N. Note that since $A < 1$ by definition, $D > 0$. When CA^2 is tiny, $W_{-1}(D)$ can be approximated as $\Re(W_{-1}(D)) \sim \log(D) - \log(\log(D))$ using the first two terms of the asymptotic expansion in [11, §4.13.10].

The previous analysis performed on the expansion (3.7) is repeated for the alternating expansion (3.6). The main results are summarized below for the purpose of brevity. The expansion (3.6) rewritten including the remainder term follows

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{(x-\mu)\alpha e^{\delta\alpha}}{\pi}\left(\sum_{k=0}^{N-1} T_k + R_N\right), \quad T_k = \left(-\frac{(x-\mu)^2\alpha}{\delta}\right)^k \frac{K_{k+1}(\alpha\delta)}{2^k k!(2k+1)}. \tag{3.23}$$

The last omitted term T_N satisfies[1]

$$|T_N| < \frac{1}{\alpha\delta}\left(\frac{x-\mu}{\delta}\right)^{2N}\frac{1}{2N+1}, \tag{3.24}$$

and the number of terms N can be determined employing the Lambert W function for a given error ϵ

$$N \approx \frac{\Re(W_{-1}(D)) + \log(A)}{2\log(A)}, \quad A = \frac{x-\mu}{\delta}, \quad B = \frac{Ae^{\delta\alpha}}{\pi}, \quad C = \frac{\epsilon}{B}, \quad D = -\frac{\log(A)}{CA}. \tag{3.25}$$

Now we are in position to compare both series and their respective domains of applicability. A first important observation is that the alternating series (3.6) does not converge when $|x-\mu| > \delta$, and the number of terms N increases rapidly when $|x-\mu| \sim \delta$. In contrast, the series (3.7) is absolutely convergent. The convergence of the latter can be reliably assessed applying to T_k in (3.8) the asymptotic estimates of $K_{k+1}(\alpha\omega)$ for $\alpha\omega \to 0$ and $\alpha\omega \to \infty$, (C.1) and (C.2), respectively:

$$\begin{aligned} T_k &\sim \frac{\sqrt{\pi}}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2k}\frac{\Gamma(k+1)}{\Gamma(k+3/2)}, && \alpha\omega \to 0,\\ T_k &\sim \sqrt{\frac{\pi}{2\alpha\omega}}\left(\frac{(x-\mu)^2\alpha}{\omega}\right)^k \frac{e^{-\alpha\omega}}{(2k+1)!!}, && \alpha\omega \to \infty. \end{aligned}$$

The asymptotic estimates of T_k show that the series expansion (3.7) is slowly convergent when

$$\left(\frac{x-\mu}{\omega}\right)^2 \to 1 \Longleftrightarrow \delta \to 0,$$

and the ratio of convergence improves when $\delta \to \infty$, then it can be viewed as an asymptotic expansion for large δ. For $k \to \infty$, T_k follows the asymptotic behaviour

$$T_k \sim \frac{1}{\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2k}\sqrt{\frac{\pi}{e}}\frac{1}{\sqrt{4k+2}}\left(\frac{2k+2}{2k+1}\right)^{k+\frac{1}{2}}, \quad k \to \infty,$$

where we use the asymptotic estimate of modified Bessel function for large order (C.3) and apply Stirling's approximation for the double factorial. It remains to analyze the accuracy of the bound in (3.15) for different parameters. Table 1 shows the effectiveness of the bound (3.15) after estimating N to achieve machine-precision using (3.22). For small values of $\alpha\omega$ the bound is sharp, but it is conservative for larger values, precisely where the rate of convergence improves. As a remark, the estimation of N for large $\alpha\omega$ can be enhanced by selecting N using the asymptotic estimate for $\alpha\omega \to \infty$ via binary search. Although, the bound might overestimate N for some parameters, the main purpose of the estimation of N using (3.22) is to decide whether the series expansion should be selected as the method of computation given a certain parameter region (see Section 4.3). Table 1 also shows that for small δ and $\alpha\omega$, the required number of terms makes the series expansion impractical. In the next section, we present a convergence acceleration method to obtain a rapidly convergent series for these cases.

x	α	μ	δ	$\alpha\omega$	N (3.22)	R_N	Bound (3.15)
1	5	1/4	1	6.25	42	$8.8 \cdot 10^{-19}$	$1.1 \cdot 10^{-18}$
1/2	1/3	1/4	1/10	0.09	236	$2.9 \cdot 10^{-17}$	$2.9 \cdot 10^{-17}$
1/3	10	1/5	1/50	1.35	1,494	$2.1 \cdot 10^{-16}$	$2.2 \cdot 10^{-16}$
1	10	1/5	5	50.64	25	$1.9 \cdot 10^{-29}$	$4.0 \cdot 10^{-21}$
3	10	1/5	10	103.85	53	$2.4 \cdot 10^{-36}$	$1.4 \cdot 10^{-19}$
10	1/10	1/5	10	1	53	$3.8 \cdot 10^{-18}$	$3.9 \cdot 10^{-18}$

Table 1: The remainder of the series expansion (3.7) and bound (3.15), estimating N using (3.22) to achieve machine precision.

[1] The corresponding upper bound for C can be computed straightforwardly following the procedure described in Theorem 3.2.

3.1.2 Convergence acceleration of the expansion $|x-\mu| \to 0$

Table 1 showed that for small values of δ and $\alpha\omega$ the required number of terms grows considerably, thereby, resorting to numerical integration (see Section 3.4) might be a more efficient approach. Alternatively, a common technique to reduce the number of terms of slowly convergent series is to use series acceleration methods (Shank's transformation and Levin-type transformations among others [15]). Attempts to use Shank's transformation were unsuccessful; we did not observe a significant reduction in N while achieving only around ten correct digits systematically for all cases. In addition, a major drawback is that Shank's acceleration requires higher-precision arithmetic to compensate for cancellation effects, which makes us discard this option for an implementation in double-precision arithmetic.

In the following, we investigate the use of exponentially improved asymptotic expansions. When the information about the remainder is available, this technique consists of re-expanding the remainder, obtaining an expansion exhibiting faster convergence. For further details, we refer to [30, §14]. Consider the convergent series expansion in (3.8)

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega}\left(\sum_{k=0}^{N-1} T_k + \sum_{k=N}^{\infty} T_k\right), \tag{3.26}$$

where

$$T_k = \frac{K_{k+1}(\alpha\omega)}{(2k+1)!!} z^k, \quad z = \frac{(x-\mu)^2\alpha}{\omega}. \tag{3.27}$$

and remainder $R_N = \sum_{k=N}^{\infty} T_k$. An integral representation of R_N can be obtained using Basset's integral [11, §10.32.11] representation of the modified Bessel function

$$K_{k+1}(\alpha\omega) = \frac{\Gamma(k+3/2)}{\sqrt{\pi}}\left(\frac{2\alpha}{\omega}\right)^{k+1}\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{k+3/2}}\, dt\,. \tag{3.28}$$

Basset's integral is chosen to transform the term $\alpha\omega$ into α/ω. Thus, it follows that inserting (3.28) into the remainder in (3.8), we have an integral representation of R_N

$$\begin{aligned} R_N &= \frac{1}{\sqrt{\pi}}\left(\frac{2\alpha}{\omega}\right)\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{3/2}}\left[\sum_{k=N}^{\infty}\left(\frac{2z\alpha}{\omega(t^2+\alpha^2)}\right)^k \frac{\Gamma(k+3/2)}{(2k+1)!!}\right] dt \\ &= C\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{N+3/2}}\frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\, dt\,. \end{aligned} \tag{3.29}$$

where, for the purpose of brevity, we use

$$m = \frac{2z\alpha}{\omega} = 2\left(\frac{(x-\mu)\alpha}{\omega}\right)^2, \quad C = \frac{2}{\sqrt{\pi}}\left(\frac{2\alpha}{\omega}\right)\frac{\Gamma(N+3/2)}{(2N+1)!!} m^N. \tag{3.30}$$

Now, we expand the term $\cos(\omega t)$, recall that $\omega = \sqrt{\delta^2 + (x-\mu)^2}$, yielding

$$R_N = C\sum_{k=0}^{\infty}\frac{(-1)^k\omega^{2k}}{(2k)!}\int_0^\infty \frac{t^{2k}}{(t^2+\alpha^2)^{N+3/2}}\frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\, dt, \tag{3.31}$$

and note that the integrals above converge for $k \le N$. Next, after various iterations with the assistance of Mathematica [44], we arrive to a closed-form expression for the previous integral in terms of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$

$$\int_0^\infty \frac{t^{2k}}{(t^2+\alpha^2)^{N+3/2}}\frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\, dt = P_k + Q_k, \tag{3.32}$$

with

$$P_k = \frac{2^{N-2}\alpha^{2(k-N-1)}\Gamma(N+1-k)\Gamma(k-1/2)}{\sqrt{\pi}(2N+1)!!}\,{}_2F_1\left(1, N+1-k; \frac{3}{2}-k, 1-\frac{m}{2\alpha^2}\right), \tag{3.33}$$

$$Q_k = \frac{2^{N-1-k}(2\alpha^2-m)^{k-1/2}\pi\sec(k\pi)}{m^{N+1/2}}. \tag{3.34}$$

The Gauss hypergeometric function is defined by the hypergeometric series

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol or rising factorial. The series is defined on the disk $|z| < 1$, and by analytic continuation with respect to z elsewhere. Subsequently, substituting (3.32) in (3.31), we write R_N as the sum of two series $R_N = S_P + S_Q$ given by

$$S_P = \frac{(2m)^N}{\alpha^{2N+1}\omega\pi} \frac{\Gamma(N+3/2)}{(2N+1)!!(2N-1)!!} \sum_{k=0}^{N} \frac{(-1)^k(\alpha\omega)^{2k}}{(2k)!} \Gamma(N+1-k)\Gamma(k-1/2)\, {}_2F_1\left(1, N+1-k; \frac{3}{2}-k, 1-\frac{m}{2\alpha^2}\right) \tag{3.35}$$

and

$$\begin{aligned} S_Q &= C\frac{2^{N-1}}{m^{N+1/2}\sqrt{2\alpha^2-m}}\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{\omega^2(2\alpha^2-m)}{2}\right)^k \sec(k\pi) \\ &= \frac{\Gamma(N+3/2)2^{N+1}}{(2N+1)!!}\frac{\alpha}{\omega}\sqrt{\frac{\pi}{m(2\alpha^2-m)}}\cosh\left(\sqrt{\frac{\omega^2(2\alpha^2-m)}{2}}\right). \end{aligned} \tag{3.36}$$

The sum S_P is terminating at $k = N$ due to the term $\Gamma(N+1-k)$ in the series. Thus, the resulting series to compute $F(x; \alpha, 0, \mu, \delta)$ is

$$F(x; \alpha, 0, \mu, \delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega}\left(\sum_{k=0}^{N-1} T_k + S_P + S_Q\right). \tag{3.37}$$

Next, we focus on the determination of the optimal N for a desired precision ϵ. The smallest term in S_P occurs when $k = N$, corresponding to

$$S_P = \frac{(-1)^N(\alpha\omega)^{2N}}{(2N)!}\Gamma(N-1/2)\, {}_2F_1\left(1, 1; \frac{3}{2}-N, 1-\frac{m}{2\alpha^2}\right). \tag{3.38}$$

Moreover, inspecting the argument of ${}_2F_1$ in (3.35),

$$1 - \frac{m}{2\alpha^2} = 1 - \left(\frac{x-\mu}{\omega}\right)^2 < 1, \tag{3.39}$$

we see that for $\alpha\omega \to 0$, $\delta \to 0$, the argument is close to 1. Therefore, for the parameter regime of interest, the last term can be effectively approximated taking the limit $\lim_{x\to 0} {}_2F_1\left(1, 1, \frac{3}{2}-k, x\right) = 1$. Then, it remains to solve the following equation for N

$$\frac{(\sqrt{2m}\omega)^{2N}}{\alpha\omega\pi}\frac{\Gamma(N-1/2)\Gamma(N+3/2)}{(2N)!(2N+1)!!(2N-1)!!} = \epsilon. \tag{3.40}$$

An asymptotic approximation follows by using Stirling approximation of the ratio of gamma functions and double factorials

$$\frac{\Gamma(N-1/2)\Gamma(N+3/2)}{(2N)!(2N+1)!!(2N-1)!!} \sim \frac{\sqrt{\pi}}{4}\left(\frac{e}{N}\right)^{2N} 2^{-4N}, \quad N \to \infty. \tag{3.41}$$

obtaining a simplified equation

$$(\sqrt{2m}\omega)^{2N}\left(\frac{e}{N}\right)^{2N} 2^{-4N} = 4\sqrt{\pi}\alpha\omega\epsilon. \tag{3.42}$$

The solution of the latter permits a closed-form in terms of principal branch of the Lambert W function

$$N \approx -\frac{\log(A)}{W_0\left(-\frac{\log(A)}{2e\sqrt{2m}\omega}\right)}, \quad A = 4\sqrt{\pi}\alpha\omega\epsilon. \tag{3.43}$$

Table 2 shows the estimated number of terms of the accelerated convergent series (3.37), N_{acc}, and the actual number of terms N^* to achieve machine-precision for small values of $\alpha\omega$ and δ. The first and more notable observation is the reduction in the number of terms compared with convergent series (3.22), especially for $\alpha \leq 1$. The second observation is the accuracy of the estimate in N_{acc} in (3.22), which only seems to slightly underestimate N for large $\alpha\omega$.

x	α	μ	δ	$\alpha\omega$	N (3.22)	N_{acc} (3.43)	error	N^*	error
1	50	1/5	1/3	43.33	324	69	$2.9 \cdot 10^{-13}$	73	$1.1 \cdot 10^{-17}$
2	5	1/5	1/10	9.01	10,242	25	$4.6 \cdot 10^{-21}$	22	$1.1 \cdot 10^{-16}$
1	1/10	1/5	1/100	0.08	179,715	5	$1.3 \cdot 10^{-21}$	4	$2.4 \cdot 10^{-17}$
5	1	1/5	1/100	4.8	5,645,686	18	$1.1 \cdot 10^{-22}$	15	$1.5 \cdot 10^{-17}$
20	1/100	1/5	1/100	0.198	84,914,922	6	$1.1 \cdot 10^{-22}$	5	$4.5 \cdot 10^{-19}$

Table 2: Absolute error and bound (3.15) estimating N using (3.22) for the series expansion (3.7) with machine-precision absolute error.

Remark 3.3 *A simpler bound for R_N, compared to* (3.15)*, can be derived from the integral representation in* (3.29) *as follows*

$$\begin{aligned} R_N &= C\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{N+3/2}}\frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\,dt \\ &\le \frac{C}{\left(2-\frac{m}{\alpha^2}\right)}\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{N+3/2}}\,dt \\ &= \frac{1}{1-\left(\frac{x-\mu}{\omega}\right)^2}T_N \end{aligned}$$

using the upper bound for T_N in (3.16)*, we obtain*

$$R_N < \frac{1}{1-\left(\frac{x-\mu}{\omega}\right)^2}\frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.44}$$

3.1.3 Expansion $|x-\mu| \to \infty$

To derive an asymptotic expansion for large $|x-\mu|$, we use the expansion derived in (B.2) as a starting point, taking $a = x-\mu$ and $b = 0$. Similar to the derivation of the expansion in (3.6), we interchange summation and integration, and the resulting integral is expressible in closed-form as a modified Bessel function, see Equation (3.5). Thus, for $x-\mu < 0$ we have the alternating asymptotic expansion

$$\begin{aligned} F(x;\alpha,0,\mu,\delta) &= \frac{\delta e^{\delta\alpha}}{2\pi}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{k!}\frac{\Gamma(2k+1)}{2^k(x-\mu)^{2k+1}}\int_0^\infty t^{k-1}e^{-\frac{\omega^2}{2t}-\frac{\alpha^2}{2}t}\,dt \\ &= -\frac{\delta e^{\delta\alpha}}{\pi(x-\mu)}\sum_{k=0}^\infty \frac{(-1)^k\Gamma(2k+1)}{k!}\left(\frac{\omega}{2(x-\mu)^2\alpha}\right)^k K_k(\alpha\omega). \end{aligned} \tag{3.45}$$

Note that the remainder is bounded in magnitude by the first neglected term. Moreover, the convergence of the expansion improves for large α and $\alpha/\omega > 1$. Finally, for the case $x-\mu > 0$ we use the reflection formula (2.6).

3.1.4 Uniform expansion $\alpha \to \infty$, $\alpha \sim \delta$ and $|x-\mu| \gg 0$

For large α, we consider the uniform asymptotic expansion in terms of modified Bessel functions described in [40] and [42, §27]. We write the Laplace-type integral (2.5) in the standard form

$$F_\lambda(z,r) = C\int_0^\infty t^{\lambda-1}e^{-z\left(t+r^2/t\right)}f(t)\,dt,$$

where C is a normalizing constant, $\lambda = -1/2$, $z = \alpha^2/2$, $r = \delta/\alpha$ and $f(t) = \Phi((x-\mu)/\sqrt{t})$. The saddle point of $e^{-z\left(t+r^2/t\right)}$ occurs at $\pm r$, but only the positive saddle point r lies inside the interval of integration. Thus, we expand $f(t)$ at the saddle point r

$$f(t) = \sum_{k=0}^\infty c_k(r)(t-r)^k,$$

after interchanging the order of summation and integration, we obtain

$$F_\lambda(z,r) \sim \frac{1}{z^\lambda}\sum_{k=0}^\infty \frac{c_k(r)Q_k(\zeta)}{z^k}, \quad z\to\infty, \tag{3.46}$$

where

$$Q_k(\zeta) = \zeta^{\lambda+k}\int_0^\infty t^{\lambda-1}(t-1)^k e^{-\zeta(t+1/t)}\,dt, \quad \zeta = rz.$$

For $f(t) = \Phi((x-\mu)/\sqrt{t})$ the coefficients at $t = r$ satisfy the recurrence in (B.14) setting $a = x-\mu$ and $b = 0$. In particular, the recurrence can be simplified as follows

$$c_0(r) = \Phi\left(\frac{x-\mu}{\sqrt{r}}\right), \quad c_1(r) = -\frac{(x-\mu)}{2r^{3/2}}\phi\left(\frac{x-\mu}{\sqrt{r}}\right) \tag{3.47}$$

and

$$c_k(r) = \frac{(k-1)((x-\mu)^2-4r(k-2)-3r)c_{k-1}(r)-(2(k-2)^2+k-2)c_{k-2}(r)}{2r^2(k-1)k}, \quad k\ge 2. \tag{3.48}$$

The functions $Q_k(\zeta)$ can be expressed as a binomial sum of modified Bessel functions, and satisfy the recurrence relation [42, §27.3.28]

$$Q_{k+2}(\zeta) = \left(k + \frac{1}{2} - 2\zeta\right) Q_{k+1}(\zeta) + \zeta \left(2k + \frac{1}{2}\right) Q_k(\zeta) + k\zeta^2 Q_{k-1}(\zeta), \quad k \geq 1, \tag{3.49}$$

with initial values

$$Q_0(\zeta) = \frac{2}{\sqrt{\zeta}} K_{\frac{1}{2}}(2\zeta), \quad Q_1(\zeta) = 0, \quad Q_2(\zeta) = 2\zeta^{3/2} \left(K_{\frac{3}{2}}(2\zeta) - K_{\frac{1}{2}}(2\zeta)\right), \tag{3.50}$$

where the special case $K_{n+1/2}(z)$, $n \in \mathbb{N}$, is a terminating sum of elementary functions requiring n terms, see (C.4). In particular the cases $n = 0$ and $n = 1$ are

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}(z+1)}{z},$$

and subsequent terms can be computed via recursion.

Thus, rearranging terms we have

$$F(x; \alpha, 0, \mu, \delta) = \frac{\alpha\delta e^{\delta\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k c_k\left(\frac{\delta}{\alpha}\right) Q_k\left(\frac{\alpha\delta}{2}\right)}{\alpha^{2k}}, \quad \alpha \to \infty. \tag{3.51}$$

The expansion (3.51) is a uniform expansion as $\alpha \to \infty$, uniformly with respect to $\delta/\alpha > 0$. Note that large values of δ improves the rate of convergence of the expansion, as observed taking well-know asymptotic estimates for large argument of the modified Bessel function. We remark that expansions (3.6) and (3.7) are also adequate for large values of α and δ, but unlike the present expansion, the number of terms increases significantly when $|x - \mu| \gg 0$.

3.2 Special case $x = \mu$

The second special case of interest, $x = \mu$, is arguably less common and has fewer applications, but it is nonetheless required to obtain a robust implementation. In particular, the attentive reader might have noticed that none of the series expansions already introduced are suitable for this specific case.

First consider the case $x = \mu$ and $\beta = 0$. Then, the distribution is symmetric and centered at $x = \mu$, and it follows that

$$F(\mu; \alpha, 0, \mu, \delta) = \frac{1}{2}. \tag{3.52}$$

Secondly, for the case $\beta \neq 0$, the integral representation of this special case is given by simple substitution in (2.4)

$$F(\mu; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi} \int_{-\infty}^{0} \frac{K_1\left(\alpha\sqrt{\delta^2 + t^2}\right)}{\sqrt{\delta^2 + t^2}} e^{\beta t}\, dt\,. \tag{3.53}$$

Using the series expansion of the exponential function and interchanging the order of integration and summation in (3.53) gives that

$$F(\mu; \alpha, \beta, \mu, \delta) = 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \int_0^{\infty} t^k \frac{K_1\left(\alpha\sqrt{\delta^2 + t^2}\right)}{\sqrt{\delta^2 + t^2}}\, dt\,.$$

The integral is expressible in closed form using [16, §6.596]

$$\int_0^{\infty} t^k \frac{K_1\left(\alpha\sqrt{\delta^2 + t^2}\right)}{\sqrt{\delta^2 + t^2}}\, dt = \frac{2^{\frac{k-1}{2}} \Gamma\left(\frac{k+1}{2}\right)}{\alpha^{\frac{k+1}{2}} \delta^{\frac{-k+1}{2}}} K_{\frac{k-1}{2}}(\alpha\delta),$$

and rearranging terms and using the connection formula $2^{k/2}\Gamma\left(\frac{k+1}{2}\right)/k! = \sqrt{\pi}2^{-k/2}/\Gamma\left(\frac{k}{2}+1\right)$ yields

$$F(\mu; \alpha, \beta, \mu, \delta) = 1 - \sqrt{\frac{\alpha\delta}{2\pi}} e^{\delta\gamma} \sum_{k=0}^{\infty} \frac{\beta^k}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)} \left(\frac{\delta}{\alpha}\right)^{\frac{k}{2}} K_{\frac{k-1}{2}}(\alpha\delta) \tag{3.54}$$

$$= \sqrt{\frac{\alpha\delta}{2\pi}} e^{\delta\gamma} \sum_{k=0}^{\infty} \frac{(-\beta)^k}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)} \left(\frac{\delta}{\alpha}\right)^{\frac{k}{2}} K_{\frac{k-1}{2}}(\alpha\delta). \tag{3.55}$$

The resulting expansions are convergent and the latter series expansion is preferred for $\beta < 0$ to avoid cancellation errors. A more rapidly convergent expansion for $\delta < \gamma$ or large values of δ and γ can be obtained using the integral (2.5) and expanding the term $\Phi(-\beta\sqrt{t})$. Replacing $\Phi(-\beta\sqrt{t})$ in the integral (2.5) with the expansion (3.2) and interchanging the order of integration and summation, we obtain

$$F(\mu; \alpha, \beta, \mu, \delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k(-\beta)^{2k+1}}{2^k k!(2k+1)} \int_0^{\infty} t^{k-1} e^{-\frac{\delta^2}{2t} - \frac{\gamma^2}{2}t}\, dt, \tag{3.56}$$

where we can express the integral in terms of the modified Bessel function (3.5). Plugging in (3.56), now yields the alternating series

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(-\beta)^{2k+1}}{2^k k!(2k+1)}\left(\frac{\delta}{\gamma}\right)^k K_k(\gamma\delta). \tag{3.57}$$

Similarly, using (3.3) yields

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi}\sum_{k=0}^{\infty}\frac{(-\beta)^{2k+1}}{(2k+1)!!}\left(\frac{\delta}{\alpha}\right)^k K_k(\alpha\delta). \tag{3.58}$$

The latter expansion being more convenient since $\alpha \geq \gamma$.

Finally, to complement the series expansion, a special case of the asymptotic expansion defined in Equation (3.98), detailed in the next Section, can be employed for large values of δ and large α, and its convergence improves as $\beta \sim \alpha$.

3.3 General case

In the general case, we continue using similar techniques to devise a set of series expansions and asymptotic expansions. Two of the series expansions involving Hermite polynomials make use of the two special cases studied in detail in the preceding sections.

3.3.1 Expansions $|x-\mu| \to 0$ in terms of modified Bessel functions

We start deriving a series expansion for small $|x-\mu|$ following similar steps to those proposed for the special case $\beta = 0$. In the same manner, we use the expansion of $\Phi(x)$ in (3.3) to obtain

$$F(x;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{2\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!!}I_k, \tag{3.59}$$

$$I_k = \int_0^\infty \left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)^{2k+1} t^{-3/2}e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}e^{-\frac{(x-(\mu+\beta t))^2}{2t}}\,dt\,. \tag{3.60}$$

The series expansion written in a more standard form is given by

$$F(x;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma+(x-\mu)\beta}}{2\pi}\sum_{k=0}^{\infty}\frac{(-\beta)^{2k+1}}{(2k+1)!!}\int_0^\infty\left(t-\frac{x-\mu}{\beta}\right)^{2k+1}t^{-k-2}e^{-\frac{\omega^2}{2t}-\frac{\alpha^2}{2}t}\,dt, \tag{3.61}$$

where we denote J_k the integral just introduced

$$J_k = \int_0^\infty\left(t-\frac{x-\mu}{\beta}\right)^{2k+1}t^{-k-2}e^{-\frac{\omega^2}{2t}-\frac{\alpha^2}{2}t}\,dt\,. \tag{3.62}$$

The integral J_k is expressible in terms of a finite series of modified Bessel functions after applying the binomial expansion

$$J_k = 2\left(\frac{-(x-\mu)}{\beta}\right)^{2k+1}\left(\frac{\alpha}{\omega}\right)^{k+1}\sum_{j=0}^{2k+1}(-1)^j\binom{2k+1}{j}\left(\frac{\omega\beta}{\alpha(x-\mu)}\right)^j K_{k+1-j}(\alpha\omega). \tag{3.63}$$

Then, we have the series expansion

$$F(x;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{(x-\mu)\alpha\delta e^{\delta\gamma+(x-\mu)\beta}}{\pi\omega}\sum_{k=0}^{\infty}\frac{A_k}{(2k+1)!!}\left(\frac{(x-\mu)^2\alpha}{\omega}\right)^k \tag{3.64}$$

$$A_k = \sum_{j=0}^{2k+1}(-1)^j\binom{2k+1}{j}\left(\frac{\omega\beta}{\alpha(x-\mu)}\right)^j K_{k+1-j}(\alpha\omega) \tag{3.65}$$

Observe that the convergence of the series improves for large values of α and δ, hence it can be seen as a uniform asymptotic expansion with respect to these parameters. The first three coefficients A_k are

$$\begin{aligned}
A_0 &= -uK_0(z) + K_1(z)\\
A_1 &= 3u^2K_0(z) - u(3+u^2)K_1(z) + K_2(z)\\
A_2 &= -10u^3K_0(z) + 5u^2(2+u^2)K_1(z) - u(5+u^4)K_2(z) + K_3(z)
\end{aligned}$$

where, to compact notation, we use u and z defined as

$$u = \frac{\omega\beta}{\alpha(x-\mu)}, \quad z = \alpha\omega. \tag{3.66}$$

Furthermore, the series expansion can be written as a truncated series with remainder. Using the representation in (3.61) we have a series in the form

$$\sum_{k=0}^{\infty} T_k = \sum_{k=0}^{N-1} T_k + \sum_{k=N}^{\infty} T_k, \quad T_k = \frac{(-\beta)^{2k+1}}{(2k+1)!!} I_k. \tag{3.67}$$

The remainder, $R_N = \sum_{k=N}^{\infty} T_k$, integral representation is obtained as

$$R_N = \int_0^\infty e^{-\frac{\omega^2}{2t} - \frac{\alpha^2}{2}t} \left[\sum_{k=N}^{\infty} \frac{(-\beta)^{2k+1}}{(2k+1)!!} \left(t - \frac{x-\mu}{\beta} \right)^{2k+1} t^{-k-2} \right] dt \tag{3.68}$$

$$= \frac{(-1)^{2N+1} 2^{N-1/2}(2N+1)}{(2N+1)!!} e^{-\beta(x-\mu)} \int_0^\infty t^{-3/2} e^{-\frac{\delta^2}{2t} - \frac{\gamma^2}{2}t} \gamma\left(N+1/2, \frac{(\beta t - (x-\mu))^2}{2t} \right) dt, \tag{3.69}$$

where $\gamma(a,z)$ is the lower incomplete gamma function. It remains to bound the remainder R_N. A simple but weak bound, since the dependence on N disappears, is obtained by using the regularized incomplete gamma function $P(a,z)\Gamma(a) = \gamma(a,z)$ and the bound $P(a,z) \leq 1$ for $z \geq 0$ (recall the asymptotic estimate $P(a,z) \to 1$ as $z \to \infty$). A sharper bound for $\gamma(a,z)$ derived from the asymptotic expansion for $a > z$ in [42, §7.3], $\gamma(a,z) \sim \frac{z^a e^{-z}}{a}$ as $a \to \infty$, is given by

$$\gamma(a,z) \leq \frac{z^a e^{-z}}{a-z-1}, \quad a > z+1, \quad z > 0. \tag{3.70}$$

Since the use of this bound introduces a term involving t in the denominator, complicating subsequent calculations, we use the asymptotic estimate instead. Thus, $|R_N|$ is approximated as

$$|R_N| \approx \frac{e^{-\beta(x-\mu)}}{(2N+1)!!} I_N \tag{3.71}$$

$$\approx \frac{2\beta^{2N+1}}{(2N+1)!!} \left(\frac{\omega}{\alpha}\right)^N K_N(\alpha\omega) \quad (|x-\mu| \to 0), \tag{3.72}$$

and the latter approximation of $|R_N|$ can be simplified further by considering the asymptotic approximation in (C.3), which gives us

$$|R_N| \approx \sqrt{\frac{2\pi}{N}} \left(\frac{e\alpha^2}{2N} \right)^{-N} \frac{\beta^{2N+1}}{(2N+1)!!}. \tag{3.73}$$

Observe that the approximation is only adequate for the case $\delta \to 0$ since its contribution is removed as a result of applying the specified asymptotic estimate. Table 3 compares the absolute value of the remainder in expression (3.68), $|R_N|$, with the upper bound using (3.70) and the approximations in (3.71) and (3.72) for different sets of parameters with small $|x-\mu|$. We can see that the bound is sharp for all cases considered, but requires the evaluation of an integral in terms of non-elementary functions. On the other hand, both approximations are close, only an order of magnitude off in most of the cases, and the approximation in terms of the modified Bessel function (3.72) might be seen as a suitable and fast approximation at least for small δ.

x	α	β	μ	δ	N	R_N	Bound (3.70)	(3.71)	(3.72)
1/2	2	1	1/4	3	10	$4.7 \cdot 10^{-9}$	$5.6 \cdot 10^{-9}$	$3.6 \cdot 10^{-9}$	$9.9 \cdot 10^{-9}$
1/3	5	-1	1/4	1	50	$2.5 \cdot 10^{-72}$	$2.6 \cdot 10^{-72}$	$2.4 \cdot 10^{-73}$	$2.8 \cdot 10^{-73}$
4	15	-6	7/2	10	20	$2.7 \cdot 10^{-60}$	$5.6 \cdot 10^{-60}$	$6.5 \cdot 10^{-61}$	$8.6 \cdot 10^{-63}$
2/10	1	1/2	1/10	1/ 2	50	$1.1 \cdot 10^{-33}$	$1.1 \cdot 10^{-33}$	$7.9 \cdot 10^{-34}$	$9.8 \cdot 10^{-34}$

Table 3: Comparison of various approximations ((3.71) and (3.72)) and the upper bound (3.70) for the estimation of the remainder (3.68).

3.3.2 Expansion $|x-\mu| \to 0$ in terms of modified Bessel and incomplete gamma functions

In the following, we present an alternative derivation of a series expansion for small $|x-\mu|$ where the series coefficients are easier to compute via recurrence relations. The starting point is the integral representation in (2.5) after expanding the exponential and replacing the term $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ by the expansion in (B.9) in terms of upper incomplete gamma function $\Gamma(a,z)$. The resulting series is

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \int_0^{\infty} \Gamma\left(\frac{k+1}{2}, \frac{\beta^2}{2}t\right) t^{-3/2-k/2} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\, dt\,. \tag{3.74}$$

Next, consider the ascending series of the incomplete gamma function given by [11, §8.7]

$$\Gamma(a,z) = \Gamma(a) - \gamma(a,z) = \Gamma(a) - \sum_{j=0}^{\infty} \frac{(-1)^j z^{a+j}}{j!(a+j)}, \tag{3.75}$$

and insert it into (3.74), splitting the inner integral into two terms

$$T_1 = \Gamma\left(\frac{k+1}{2}\right) \int_0^{\infty} t^{-3/2-k/2} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\, dt, \tag{3.76}$$

$$T_2 = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)} \int_0^{\infty} t^{j-1} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\, dt\,. \tag{3.77}$$

Both integrals are expressible in terms of modified Bessel functions, resulting in the sums S_1 and S_2, respectively, such that $F(x;\alpha,\beta,\mu,\delta) = C(S_1 - S2)$. These series are defined as follows

$$S_1 = \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) 2K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}}, \tag{3.78}$$

$$S_2 = \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)} 2K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j. \tag{3.79}$$

Interchanging the order of summation in S_2, we observe that the sum in k is convergent an expressible in terms of the lower incomplete gamma function $\gamma(a,x)$. Assuming $\beta > 0$ (otherwise, use (2.6)), we have

$$\sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!(\frac{k+1}{2}+j)} \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}} = -\frac{\sqrt{2}}{(x-\mu)^{2j+1}\beta^{2j}} \gamma\left(2j+1, -(x-\mu)\beta\right). \tag{3.80}$$

Thus,

$$S_2 = -2\sqrt{2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\gamma\left(2j+1, -(x-\mu)\beta\right)}{(x-\mu)^{2j+1}\beta^{2j}} \left(\frac{\beta^2}{2}\right)^j K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j. \tag{3.81}$$

Rearranging terms, we obtain

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\pi\sqrt{2}} \Bigg[& \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}} \\ & + \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\gamma\left(2k+1, -(x-\mu)\beta\right)}{(x-\mu)^{2k+1}} K_k(\omega\gamma) \left(\frac{\omega}{2\gamma}\right)^k \Bigg] \end{aligned} \tag{3.82}$$

The expansion converges rapidly for small $|x-\mu|$ and fixed values of the rest of parameters. Moreover, the convergence improves when $\gamma \sim \omega$, also valid for large values of these two parameters. In principle, the computation of the lower incomplete gamma functions in the second series S_2 can be performed using the recurrence relation

$$\gamma(a+1,z) = a\gamma(a,z) - z^a e^{-z}. \tag{3.83}$$

However, one must take into account that forward recurrence for positive parameters a and z is unstable; we refer to [41, §13] for details on how to deal with unstable recurrence relations. Compared to the previous expansion, where modified Bessel functions are cached to avoid recalculations when evaluating coefficients A_k in (3.64), this series expansion is faster to evaluate since all the terms can be computed recursively.

3.3.3 Expansion $|x-\mu| \to 0$ in terms of Hermite polynomials

Furthermore, we obtain an additional series expansion in terms of Hermite polynomials and the special case $x = \mu$, $F(\mu;\alpha,\beta,\mu,\delta)$. As in previous derivations, our starting point is the substitution of the term $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ in (2.5) by the Hermite-type expansion derived in (B.17). After inserting the expansion we have

$$F(x;\alpha,\beta,\mu,\delta) = F(\mu;\alpha,\beta,\mu,\delta) + \frac{\delta e^{\delta\gamma}}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (x-\mu)^{k+1}}{(k+1)! 2^{k/2}} \int_0^{\infty} t^{-k/2-2} H_k\left(k, -\beta\sqrt{\frac{t}{2}}\right) e^{-\frac{\delta^2}{2t}-\frac{\alpha^2}{2}t}\, dt\,. \tag{3.84}$$

Denote I_k the integral in the series coefficients. The finite power series of $H_k(x)$ [11, §18.5.13] allows the representation of I_k as a finite series of modified Bessel functions. In this way, we obtain

$$I_k = \int_0^\infty t^{-k/2-2} H_k\left(k, -\beta\sqrt{\frac{t}{2}}\right) e^{-\frac{\delta^2}{2t} - \frac{\alpha^2}{2}t}\, dt \tag{3.85}$$

$$= 2k! \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{j!(k-2j)!} (-\beta\sqrt{2})^{k-2j} \left(\frac{\alpha}{\delta}\right)^{j+1} K_{j+1}(\alpha\delta). \tag{3.86}$$

After rearranging terms, the series expansion reads as

$$F(x;\alpha,\beta,\mu,\delta) = F(\mu;\alpha,\beta,\mu,\delta) + \frac{e^{\delta\gamma}(x-\mu)\alpha}{\pi} \sum_{k=0}^\infty \frac{(\beta(x-\mu))^k}{k+1} A_k, \tag{3.87}$$

$$A_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{j!(k-2j)!} \left(\frac{\alpha}{2\delta\beta^2}\right)^j K_{j+1}(\alpha\delta). \tag{3.88}$$

Table 4 compares the number of terms of the three presented series expansion for the case $|x-\mu| \to 0$ for various sets of parameters to achieve a machine-precision target accuracy. As a first observation, we note that the Hermite-type expansion (3.87) appears to be the most efficient when $|x-\mu| \ll 1$. Also, in this regime, both series expansions (3.82) and (3.64) require a similar number of terms N. On the other hand, in the region $|x-\mu| \geq 1$, the expansion (3.64) becomes the most efficient in the cases considered.

x	α	β	μ	δ	(3.64)	(3.82)	(3.87)
1/2	2	1	1/4	3	27	32	15
1/3	5	-1	1/4	1	12	17	14
4	15	-6	7/2	10	63	67	17
2/10	1	1/2	1/10	1/2	22	25	19
2	3	2	1	2	42	152	52
3	5	1/2	1	4	29	63	66

Table 4: Comparison on the number of terms to achieve machine precision with the three series expansions for $|x-\mu| \to 0$.

Note, however, that a complexity analysis gives a different picture. Given a number of terms N:

- Series expansion (3.64): $(N+1)(N+2) \sim \mathcal{O}(N^2)$
- Series expansion (3.82): $N_1 + N_2 \sim \mathcal{O}(N)$
- Series expansion (3.87): $\frac{1}{2}\left(\lfloor \frac{N-1}{2} \rfloor \left(\lfloor \frac{N-1}{2} \rfloor + 1\right) + \lfloor \frac{N}{2} \rfloor \left(\lfloor \frac{N}{2} \rfloor + 1\right) + 2(N+1)\right) \sim \mathcal{O}(N^2)$

We turn our attention to the series expansions (3.64) and (3.87) because the fact that both require the storage of modified Bessel functions make them directly comparable. We remark that, although the series (3.64) might require fewer terms in some cases, asymptotically, it requires 4 times more terms than (3.87); indeed taking the limit we have

$$\lim_{N\to\infty} \frac{(N+1)(N+2)}{\frac{1}{2}\left(\lfloor \frac{N-1}{2} \rfloor \left(\lfloor \frac{N-1}{2} \rfloor + 1\right) + \lfloor \frac{N}{2} \rfloor \left(\lfloor \frac{N}{2} \rfloor + 1\right) + 2(N+1)\right)} = 4.$$

This observation shall be taken into consideration when designing the algorithm for the general case in Section 4.3.

3.3.4 Expansion $\beta \to 0$

For the case $\beta \to 0$, we follow closely the derivation of the Hermite-type expansion for $|x-\mu| \to 0$ in (3.87), hence some steps are omitted. In this case, the expansion involves the special case $\beta = 0$ studied in detail in Section 3.1. Replacing the Hermite-type expansion defined in (B.18) in integral (2.5), we obtain

$$F(x;\alpha,\beta,\mu,\delta) = F(x;\gamma,0,\mu,\delta) + \frac{\delta e^{\delta\gamma}}{2\pi} \sum_{k=0}^\infty \frac{(-1)^k(-\beta)^{k+1}}{(k+1)!2^{k/2}} \int_0^\infty H_k\left(\frac{x-\mu}{\sqrt{2t}}\right) t^{k/2-1} e^{-\frac{\omega^2}{2t} - \frac{\gamma^2}{2}t}\, dt\,. \tag{3.89}$$

We may apply some algebraic manipulations to obtain an expansion in the same form as (3.87), obtaining

$$F(x;\alpha,\beta,\mu,\delta) = F(x;\gamma,0,\mu,\delta) - \frac{\beta\delta e^{\delta\gamma}}{\pi} \sum_{k=0}^\infty \frac{(\beta(x-\mu))^k}{k+1} B_k, \tag{3.90}$$

$$B_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{j!(k-2j)!} \left(\frac{\omega}{2\gamma(x-\mu)^2}\right)^j K_j(\gamma\omega), \tag{3.91}$$

where we can observe the similarity of the terms in both Hermite-type expansions.

3.3.5 Expansion $\delta \to \infty$

Start from the integral representation in (2.5) replacing the term $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ by the asymptotic expansion for $t \to \infty$ in (B.13) in terms of the incomplete gamma function. Taking $a = x - \mu$ and $b = -\beta$, we obtain

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{\Gamma(2k+1, -(x-\mu)\beta)}{(-\beta)^{2k+1}} \int_0^{\infty} t^{-k-2} e^{-\frac{\delta^2}{2t} - \frac{\gamma^2+\beta^2}{2} t}\, dt\,. \tag{3.92}$$

The integral is expressible in terms of modified Bessel function

$$\int_0^{\infty} t^{-k-2} e^{-\frac{\delta^2}{2t} - \frac{\gamma^2+\beta^2}{2} t}\, dt = 2\left(\frac{\alpha}{\delta}\right)^{k+1} K_{k+1}(\alpha\delta), \tag{3.93}$$

where we use the fact that $\gamma^2 + \beta^2 = \alpha^2$. After rearranging terms, we have for $\beta > 0$

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) &= \frac{\alpha e^{\delta\gamma}}{\pi\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma(2k+1, -(x-\mu)\beta) \left(\frac{\alpha}{2\beta^2\delta}\right)^k K_{k+1}(\alpha\delta) & (3.94)\\ &= \frac{\alpha e^{\delta\gamma}}{\pi\beta} \sum_{k=0}^{\infty} Q(2k+1, -(x-\mu)\beta) \frac{\Gamma(2k+1)}{\Gamma(k+1)} \left(-\frac{\alpha}{2\beta^2\delta}\right)^k K_{k+1}(\alpha\delta) & (3.95)\\ &= \frac{\alpha e^{\delta\gamma}}{\pi^{3/2}\beta} \sum_{k=0}^{\infty} Q(2k+1, -(x-\mu)\beta) \Gamma(k+1/2) \left(-\frac{2\alpha}{\beta^2\delta}\right)^k K_{k+1}(\alpha\delta), & (3.96) \end{aligned}$$

and $1+F(x;\alpha,\beta,\mu,\delta)$ for $\beta < 0$. We can rewrite the asymptotic series in terms of the regularized incomplete gamma function $Q(a,x) = \Gamma(a,x)/\Gamma(a)$, to avoid ratios with large numerator and denominator. In addition, the evaluation of the upper incomplete gamma function , $\Gamma(a,z)$, and its regularized version via recurrence is numerically stable for positive parameters a and z,

$$\Gamma(a+1, z) = a\Gamma(a,z) + z^a e^{-z}. \tag{3.97}$$

Furthermore, the special case $x = \mu$ simplifies the asymptotic expansion considerably since $\Gamma(a, 0) = 1$ for $a > 0$, thus obtaining for $\beta > 0$

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{\alpha e^{\delta\gamma}}{\pi^{3/2}\beta} \sum_{k=0}^{\infty} \Gamma(k+1/2) \left(-\frac{2\alpha}{\beta^2\delta}\right)^k K_{k+1}(\alpha\delta). \tag{3.98}$$

3.3.6 Expansion $|x-\mu| \to \infty$

In a similar manner, using instead the expansion in (B.2), we obtain for $x - \mu < 0$ the asymptotic expansion

$$F(x;\alpha,\beta,\mu,\delta) = -\frac{\delta e^{\delta\gamma}}{\pi(x-\mu)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma(2k+1, -(x-\mu)\beta) \left(\frac{\omega}{2\gamma(x-\mu)^2}\right) K_k(\gamma\omega), \tag{3.99}$$

and use $1 + F(x;\alpha,\beta,\mu,\delta)$ for $x - \mu > 0$. The same comments about the computation of the series using the recurrence of the upper incomplete gamma and modified Bessel function apply.

3.3.7 Uniform asymptotic $\gamma \to \infty, \gamma \sim \delta$ and $|x-\mu| \gg 0$

For large γ, we can employ the uniform asymptotic described in Section 3.1.4. The only difference is the calculation of the coefficients $c_k(r)$ using the recurrence in (B.14) with $a = x - \mu$ and $b = -\beta$. The resulting uniform asymptotic expansion reads

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\gamma\delta e^{\delta\gamma}}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k c_k\left(\frac{\delta}{\gamma}\right) Q_k\left(\frac{\gamma\delta}{2}\right)}{\gamma^{2k}}, \quad \gamma \to \infty. \tag{3.100}$$

Similar to the comment in Section 3.1.4, the series expansion (3.90) for $\beta \to 0$ can also be interpreted as an asymptotic for $\gamma \to \infty$, but only admits small values of $|x-\mu|$.

3.4 Numerical integration

For cases not covered by the described expansions, we have no other alternative than resorting to numerical integration. However, although numerical integration is an excellent backup method when series expansions are not appropriate, there is an important efficiency aspect to be considered since it might be orders of magnitude slower than series expansions.

3.4.1 Integrand in terms of the error function

First, we focus on the Laplace-type integral (2.5), whose integrand involves the complementary error function. As a preliminary step, we truncate the improper integral (2.5) at some point $N > 0$, such that

$$I_N = \int_0^N \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt + \int_N^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt, \tag{3.101}$$

and $F(x;\alpha,\beta,\mu,\delta) = CI_N$, where $C = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}$. The truncation error can be bounded by

$$\int_N^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt \leq \frac{e^{-\frac{\delta^2}{2N}}}{N^{3/2}} \int_N^\infty e^{-\frac{\gamma^2}{2}t}\, dt \leq \frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2}, \tag{3.102}$$

and we can select N for a desired absolute tolerance ϵ solving numerically the equation

$$\frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C}. \tag{3.103}$$

Moreover, a slightly lesser sharper bound allows a closed-form solution of the above equation in terms of the principal branch of the Lambert W function, $W_0(z)$

$$\frac{2e^{-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C} \longrightarrow N = \frac{3}{\gamma^2} W_0\left(\frac{\gamma^2}{3u}\right), \quad u = \left(\frac{\gamma^2\epsilon}{2C}\right)^{2/3}. \tag{3.104}$$

To reduce the cost of evaluating N, we can accurately approximate $W_0(z)$ for $z > 1$ using the upper bound derived in [20]

$$W_0(z) \leq \log(z)^{\frac{\log(z)}{1+\log(z)}}, \tag{3.105}$$

implying that we can safely use this approximation when the following inequality is satisfied (which should occur for any reasonable set of parameters)

$$\delta\gamma e^{\delta\gamma} > \sqrt{\frac{27\pi}{2}}\epsilon. \tag{3.106}$$

Note that, from an implementation perspective, the use of the upper bound (3.105) to estimate N comes in handy to avoid overflow/underflow issues for large parameters by direct use of logarithmic properties. To accurately estimate N to achieve a relative tolerance, we need an estimate of the order of magnitude of I. The integrand can be written as $e^{g(t)}$, where

$$g(t) = -\frac{\delta^2}{2t} - \frac{\gamma^2}{2}t - \frac{3}{2}\log(t) + \log\left(\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)\right), \tag{3.107}$$

and its derivative with respect to t

$$g'(t) = \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} - \frac{1}{2}\left(\frac{x-\mu}{t^{3/2}} + \frac{\beta}{\sqrt{t}}\right) \frac{\phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}{\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}. \tag{3.108}$$

The saddle point t_0 and maximum contribution $e^{g(t_0)}$ of the integrand is obtained as the solution of the equation $g'(t) = 0$. Thus, N for relative tolerance can be estimated after replacing ϵ with $\epsilon e^{g(t_0)}$ in (3.104). For the case where γ and δ are both large and β and $x-\mu$ are fixed, the last term in $g'(t)$ can be neglected, obtaining the quadratic equation

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + (\gamma\delta)^2}}{\gamma^2}, \tag{3.109}$$

taking the positive internal saddle point t_0. The case where γ and δ are small requires further analysis. If $x-\mu > 0$, t_0 in (3.109) is still valid. Contrarily, if $x-\mu < 0$, the ratio of the cumulative distribution function and the density function can be approximated via

$$\frac{\phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}{\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)} \approx -\frac{x-(\mu+\beta t)}{\sqrt{t}},$$

then, replacing in (3.108), we have another quadratic equation

$$g'(t) \approx \frac{\delta^2 + (x-\mu)^2}{2t^2} - \frac{\alpha^2}{2} - \frac{3}{2t}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + \alpha^2\left((x-\mu)^2 + \delta^2\right)}}{\alpha^2}. \tag{3.110}$$

The saddle point estimate t_0 computed by (3.109) or (3.110) is used as a starting point for the root-finding method. In particular, we find that the Newton's method setting an absolute error 10^{-4} only requires a few iterations (typically less than 5) to refine the initial estimate. After estimating N using the calculated magnitude with the saddle point t_0, $\epsilon e^{g(t_0)}$, the truncated integral (3.101) can be computed using standard numerical integration methods such as double exponential (tanh-sinh) quadrature or Gaussian quadrature (Gauss-Legendre). Technical details are discussed in Section 4.2.

3.4.2 Integrand in terms of elementary functions

Most of the integral representations presented in Section 2.2 involve special functions in their integrands (e.g., $K_1(x)$ or $\mathrm{erfc}(x)$), which necessarily increases the computation cost of their evaluation. Therefore, for computational efficiency, it is generally preferred to evaluate integrals representable in terms of elementary functions. A suitable integral representation satisfying this requirement was given in Equation (2.8). Following the steps previously described, the integral (2.8) can be approximated by the truncated integral

$$I_N = \int_0^N \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)} \sin(\delta t)\, dt + \int_N^\infty \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)} \sin(\delta t)\, dt, \tag{3.111}$$

and $F(x;\alpha,\beta,\mu,\delta) = 1 - CI_N$, where $C = \frac{e^{\delta\gamma}}{\pi}$. Consider the tail given by

$$T_N = \int_N^\infty \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)} \sin(\delta t)\, dt\,. \tag{3.112}$$

An upper bound for the tail holds:

$$\begin{aligned}
|T_N| &\leq \int_N^\infty \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)}\, dt \\
&\leq \frac{e^{(x-\mu)\beta}}{\sqrt{N^2+\alpha^2}-\beta} \int_N^\infty e^{-(x-\mu)\sqrt{t^2+\alpha^2}}\, dt \\
&\leq \frac{e^{(x-\mu)\beta}}{\sqrt{N^2+\alpha^2}-\beta} \sqrt{N^2+\alpha^2} K_1((x-\mu)\sqrt{N^2+\alpha^2}),
\end{aligned}$$

where we use the fact that $|\sin(\delta t)| \leq 1$, and a bound for the latter integral expressible in closed form using [16, §3.461]

$$\int_N^\infty e^{-(x-\mu)\sqrt{t^2+\alpha^2}}\, dt \leq \int_0^\infty e^{-(x-\mu)\sqrt{t^2+N^2+\alpha^2}}\, dt = \sqrt{N^2+\alpha^2} K_1((x-\mu)\sqrt{N^2+\alpha^2}).$$

Remark 3.4 *The previous bound involving the modified Bessel function $K_1(z)$ can be simplified at the expense of obtaining a slightly less sharper bound. Using the monotonicity properties of $K_\nu(z)$ for $z > 0$, we have that $K_1(z) < K_{3/2}(z)$, with $K_{3/2}(z)$ being expressible via elementary functions*

$$K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}(z+1)}{z^{3/2}}.$$

Hence, one can obtain

$$|T_N| \leq \sqrt{\frac{\pi}{2}} \frac{\sqrt{N^2+\alpha^2}}{\sqrt{N^2+\alpha^2}-\beta} \frac{(x-\mu)\sqrt{N^2+\alpha^2}+1}{\left((x-\mu)\sqrt{N^2+\alpha^2}\right)^{3/2}} e^{-(x-\mu)\left(\sqrt{N^2+\alpha^2}-\beta\right)}. \tag{3.113}$$

Although the derived bounds are sharp, the inversion to obtain N is far from trivial. In contrast, a practical approximation, improving as N increases ($N \gg \alpha > |\beta|$) is given by

$$|T_N| \approx \sqrt{\frac{\pi}{2}} \frac{e^{-(x-\mu)N}}{(x-\mu)N}. \tag{3.114}$$

To select N, we take the approximation and equate to a prescribed accuracy ϵ. The solution admits a closed-form in terms of $W_0(z)$

$$\frac{e^{-(x-\mu)N}}{(x-\mu)N} = \frac{\epsilon}{C} \longrightarrow N = \frac{1}{x-\mu} W_0\left(\frac{C\sqrt{\frac{\pi}{2}}}{\epsilon}\right). \tag{3.115}$$

It is crucial to mention that the integrand in (3.111) becomes highly oscillatory as δ increases. Therefore, the direct computation in the presence of high oscillations leads to catastrophic cancellation, requiring a prohibited number of quadrature points for convergence. In the literature, there is a myriad of numerical quadrature methods for highly oscillatory integrals, for example [31], and we encourage the interested reader to explore them. Nevertheless, in this work, we decided solely to investigate the application of the steepest descent method, which we found adequate for this case. We transform the integral with sine kernel into the complex form

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{e^{\delta\gamma}}{\pi}\Im\left(\int_0^\infty \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)+\delta t i}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)}\,dt\right), \quad x-\mu>0, \tag{3.116}$$

and apply the change of variable $t = iq/\delta$ which yields

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \frac{e^{\delta\gamma+(x-\mu)\beta}}{\pi\delta^2}\Im\left(\int_0^\infty \frac{qe^{-(x-\mu)\sqrt{\alpha^2-q^2/\delta^2}-q}}{\sqrt{\alpha^2-q^2/\delta^2}\left(\sqrt{\alpha^2-q^2/\delta^2}-\beta\right)}\,dq\right). \tag{3.117}$$

The integrand has two poles located at $q = \delta\alpha$ and $q = \delta\sqrt{\alpha^2-\beta^2} = \delta\gamma$, but observe that the imaginary part of the integral only differs from zero when $q > \delta\alpha$. Consequently, the interval of integration can be shifted to $(\delta\alpha, \infty)$ removing the pole ($\delta\alpha \geq \delta\gamma$) which gives

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \frac{e^{\delta\gamma+(x-\mu)\beta}}{\pi\delta^2}\Im\left(\int_{\delta\alpha}^\infty \frac{qe^{-(x-\mu)i\sqrt{q^2/\delta^2-\alpha^2}-q}}{i\sqrt{q^2/\delta^2-\alpha^2}\left(i\sqrt{q^2/\delta^2-\alpha^2}-\beta\right)}\,dq\right), \tag{3.118}$$

where the resulting integrand prevents the evaluation of complex square roots. The previously mentioned numerical integration methods are applicable to the non-oscillatory integral (3.118), but also Gauss-Laguerre quadrature with weight function $q^r e^{-q}$, $r \in \{0, 1\}$.

4 Algorithmic details and implementation

4.1 Evaluation of Bessel-type expansions

4.1.1 Handling large parameters

As observed, all the expansions introduced in Section 3 involve the modified Bessel function. A common method to compute the modified Bessel function is based on the recurrence identity for consecutive neighbors [1]

$$K_{\nu+1}(z) = K_{\nu-1}(z) + \frac{2\nu}{z}K_\nu(z), \tag{4.1}$$

given the initial values of the immediately previous two terms. However, for large order ν and large argument z, the recurrence is quickly affected by overflow/underflow issues. A common approach to circumvent this problem is the use of the exponentially scaled function (e.g., implemented in [10]) defined as

$$K_\nu^*(z) = e^z K_\nu(z). \tag{4.2}$$

Another method to obtain a stable recursion entails the ratio of modified Bessel functions defined as

$$r_\nu(z) = \frac{K_{\nu+1}(z)}{K_\nu(z)}, \tag{4.3}$$

which satisfy the recurrence identity [46]

$$r_\nu(z) = \frac{1}{r_{\nu-1}(z)} + 2\frac{\nu}{z}. \tag{4.4}$$

Furthermore, another option to handle large parameters is to apply logarithmic transformations [46], although we found that the forward recursion of the ratios combined with the use of the exponential scaling suffice to obtain a numerically stable implementation of the recursion.

Finally, the evaluation of a series expansion using partial sums recurrence can also alleviate instability issues. As an example, we consider the series (3.7)

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega}S_K, \tag{4.5}$$

where S_K is the k-th partial sum. The first partial sums are

$$S_0 = 0, \quad S_1 = K_1(\alpha\omega), \quad S_2 = S_1 + \frac{K_2(\alpha\omega)z}{3}, \tag{4.6}$$

and for $k \geq 0$, the partial sums satisfy the recursion relation

$$S_{k+3} = \frac{-\alpha\omega z^2 S_k + z\left(-2(2+k)(3+2k)+\alpha\omega\right)S_{k+1} + (3+2k)\left((5+2k)\alpha\omega + 2(2+k)z\right)S_{k+2}}{(3+2k)(5+2k)\alpha\omega}, \tag{4.7}$$

where

$$z = \frac{(x-\mu)^2\alpha}{\omega}. \tag{4.8}$$

4.1.2 Performance comparison modified Bessel functions $K_0(x)$ and $K_1(x)$

The computation of the two special cases, $K_0(x)$ and $K_1(x)$, of the modified Bessel function is required in several series expansions to apply the forward recurrence relation described in the previous section, so it is essential to guarantee a good performance. Since the C++17 standard, the function `std::cyl_bessel_k` was made available, but some recent performance experiments[2] for the function `std::erfc` encouraged us to perform a benchmarking exercise. In particular, we compare `std::cyl_bessel_k` with our own C++ implementation of the minimax rational approximation in SPECFUN [10], functions `CALCK0` and `CALCK1`, respectively.

The experiments have been performed on an AMD Ryzen 7 5825U running at 2.0 GHz using compilers `clang` 18.1.3 and `gcc` 13.3.0. with compiler optimization flag `-O3` on Linux. In this experiment, we evaluate $K_0(x)$ on 10^5 evenly spaced points in the interval $x \in (0, 700)$. The timings in seconds show that the minimax rational approximation, evaluated on the described hardware and compilers, is approximately two orders of magnitude faster:

- `std::cyl_bessel_k`: $5.9 \cdot 10^{-3}$.
- SPECFUN: $8.1 \cdot 10^{-5}$.

The same differences are obtained when evaluating $K_1(x)$.

4.2 Numerical integration

For the computation of the truncated half-line integral in (3.101), we choose the double-exponential quadrature method, also known as tanh-sinh quadrature, originally developed in [39]. The double-exponential quadrature has risen in popularity in scientific computing, and compared to the classical Gaussian quadratures, computing the nodes and weights for integration is fast while effectively handling endpoint singularities. Our C++ implementation is inspired by the Python library `tanh-sinh` `https://github.com/sigma-py/tanh-sinh` which carefully implements many of the optimizations described in [2]. In particular, it includes an analysis of the determination of the optimal step size, and the number of quadrature points by level, which demands multiple evaluations of the Lambert W function $W_{-1}(z)$. For that purpose, as an optimization, we replace its exact computation by the approximation in [5]

$$W_{-1}(z) \approx \log(-z) - \frac{2}{a}\left(1 - \left(1 + a\sqrt{-\frac{1+\log(-z)}{2}}\right)^{-1}\right), \quad a = 0.3205, \tag{4.9}$$

with maximum relative error of $3.5 \cdot 10^{-3}$.

4.3 Implementation

A reliable implementation of the cumulative distribution function for the normal inverse Gaussian distribution in double-precision arithmetic presents several difficulties. Consequently, it becomes necessary to find the right combination of methods to perform efficient and accurate computations on a wide range of the parameters' domain. To devise this combination (i.e., decide the method of choice in each parameter regime) one could formulate an optimization problem with two possible objective functions: minimization of terms (complexity) and maximization of robustness, or a blended approach (multi-objective optimization).

As discussed throughout this work, the achievable accuracy of each method can be effectively estimated by developing rigorous error bounds or, in the absence of error bounds, via linear search with a rough initial estimate. However, it is not always practical to obtain the optimal solution for each set of parameters, and a heuristic might suffice for the vast majority of cases. One can treat the problem of automatically choosing the most suitable method for each region as a classification problem where Machine Learning algorithms are suitable [36, 29]. More specifically, we can consider a multi-class problem, where each class corresponds to a method, or a multi-label classification problem since multiple methods might be selected for a given set of parameters. Our approach consists of generating a dataset covering small and large regions of the parameters' domain, computing the CDF via `mpmath` [27] using numerical quadrature, and training a decision tree for each method to decide the regions with absolute relative error below a prescribed tolerance of $5 \cdot 10^{-13}$. As a refinement, we combine and simplify the model using another decision tree and rounding the boundaries. The result of the process are the algorithms 1-3. Some remarks:

- Numerical integration is used as a backup method whenever series expansions and asymptotic expansions are not suitable or, they fail to converge.
- Expansion $|x-\mu| \to 0$ in terms of the lower incomplete gamma function in (3.74) was discarded because numerical instability prevents its implementation in double-precision arithmetic.
- For asymptotic expansions, we slightly relaxed tolerance requirements to reduce the use of numerical integration to improve performance.

[2] `https://nag.com/insights/quant-essentials-normdist/`

Algorithm 1 Algorithm for special case $\beta = 0$, $F(x; \alpha, 0, \mu, \delta)$

Input: $x \in \mathbb{R}$, $\alpha > 0$, $\mu \in \mathbb{R}$, $\delta > 0$
Output: $F(x; \alpha, 0, \mu, \delta)$
  $C_1 = |x - \mu| \leq 5$ and $\alpha/\omega \leq 0.25$ and $\delta/2 \geq |x - \mu|$
  $C_2 = (x - \mu)^2 \leq 1.25$ and $\alpha/\omega \leq 1$
  if (C_1 or C_2) and $\delta \geq 1$ **then**
    series expansion (3.7)
  else if $(x - \mu)^2 \leq 2.5$ and $\alpha \geq 5$ and $\delta \geq 10$ and $\delta\alpha \geq 200$ **then**
    uniform asymptotic expansion (3.51)
  else if $(x - \mu)^2 \geq 70$ and $\alpha/\omega \geq 1$ **then**
    asymptotic expansion (3.45)
  else
    numerical integration (3.101)
  end if

Algorithm 2 Algorithm for special case $x = \mu$, $F(\mu; \alpha, \beta, \mu, \delta)$

Input: $x \in \mathbb{R}$, $\alpha > 0$, $0 \leq |\beta| < \alpha$, $\delta > 0$
Output: $F(\mu; \alpha, \beta, \mu, \delta)$
  if $\alpha \leq 10$ and $\delta \leq 10$ and $|\beta| \leq 1.5$ and $|\beta|/\alpha \leq 0.9$ **then**
    series expansion (3.58)
  else if $|\beta|/\alpha \geq 0.75$ and $\delta\alpha \geq 300$ and $\delta \geq 15$ **then**
    uniform asymptotic expansion (3.98)
  else
    numerical integration (3.101)
  end if

Algorithm 3 Algorithm for $F(x; \alpha, \beta, \mu, \delta)$

Input: $x \in \mathbb{R}$, $\alpha > 0$, $0 \leq |\beta| < \alpha$, $\mu \in \mathbb{R}$, $\delta > 0$
Output: $F(x; \alpha, \beta, \mu, \delta)$
  if ($|\beta| \leq 1$ and $\gamma \geq 1.5$) or ($|\beta| \leq 0.5$ and $\gamma \geq 0.75$) **then**
    series expansion (3.90)
  else if $(x - \mu)^2 \leq 2.25$ and $\delta \geq 2.5$ **then**
    series expansion (3.87)
  else if $(x - \mu)^2 \leq 3$ and $\delta \geq 1$ and $|\beta| \leq 1.5$ and $\gamma \geq 0.75$ **then**
    series expansion (3.64)
  else if $(x - \mu)^2 \leq 20$ and $\alpha \geq 5$ and $|\beta|/\alpha \geq 0.5$ and $\delta \geq 15$ **then**
    asymptotic expansion (3.94)
  else if $(x - \mu)^2 \geq 100$ and $\alpha/\omega \geq 0.25$ and $\gamma \geq 10$ and $\delta \leq 10$ and $\alpha/|\beta| \geq 5$ **then**
    asymptotic expansion (3.99)
  else
    numerical integration (3.101)
  end if

Finally, the resulting algorithm implemented in C++17 is freely available on GitHub: `https://github.com/guillermo-navas-palencia/normal-inverse-gaussian`.

5 Numerical experiments

The number of available open-source and commercial implementations of the normal inverse Gaussian distribution is limited. In the open-source space, this statistical distribution is not included in widely used numerical libraries such as Boost [8], GNU Scientific Library (GSL) [13] or CERN ROOT [9] to mention a few. To the author's knowledge, the two exceptions are SciPy `stats.norminvgauss` [21] and the R packages `GeneralizedHyperbolic`[3] and `ghyp`[4]. The implementation of the cumulative distribution function in these libraries solely relies on the use of numerical integration, more specifically performing adaptive Gaussian quadrature using the Fortran library QUADPACK [33]. SciPy performs direct computation via adaptive Gaussian quadrature, whereas `GeneralizedHyperbolic` partitions the real line into multiple regions, and the numerical integration routine is used in each to integrate the density function.

[3] `https://rdrr.io/rforge/GeneralizedHyperbolic/`
[4] https://rdrr.io/cran/ghyp/

We compare our implementation to the SciPy `stats.norminvgauss` since it seems the fastest and most robust of the aforementioned options. To test accuracy, we generate two sample sets, small and large, with 10^4 instances for each special case and the general case. The small set, which mainly comprises small parameters, was generated to test the accuracy of the series expansions. On the other hand, the large set assesses the accuracy of the asymptotic expansions. For comparing accuracy, we take as a reference the value obtained by computing the truncated integral (3.101) using `mpmath` with 100 digits of precision. A run is considered successful if the absolute relative error is below the threshold $5 \cdot 10^{-13}$ ($\sim 200\epsilon$). Furthermore, we define as hard instances cases where `mpmath` does not converge and an increment of working precision to 300 digits is required. In terms of execution speed, we run the C++ compiled library from Python using `ctypes` to guarantee a fair comparison. Tests were run on an AMD Ryzen 7 5825U running at 2.0 GHz using the `clang` compiler with optimization flag `-O3`.

Following, we outline the main observations for each of the cases:

Case $\beta = 0$

- Tables 5 and 6 show the characteristics of the sample sets and the accuracy and performance summary. In terms of accuracy, the SciPy implementation has significantly higher mean absolute error for all the regions, and, therefore, a higher number of failures. In contrast, our implementation shows superior robustness, maintaining a mean error below 10^{-13}. Regarding performance, our implementation is consistently around $10 - 20$ times faster than SciPy.
- Tables 7 and 8 show that numerical integration is used in above 60% of the cases. However, the results in Table 9 show that for large hard instances, generally corresponding to the case where $|x - \mu|$ is large, the asymptotic expansion (3.45) is the preferred choice.

Case $x = \mu$

- Tables 10 and 11 show the characteristics of the sample sets and the accuracy and performance summary. The SciPy shows more reliability in the small region, but its accuracy worsens in the large region, especially for hard instances, where just 1.77% of these are computed successfully.
- In the small region, contrasting to the case $\beta = 0$, Table 17 shows that the series expansion (3.58) can be safely chosen in more than 80% of the instances hence the speedup compared to SciPy increases, being ~ 60 times faster.
- Tables 13 and 14 shares the same conclusions than the case $\beta = 0$, numerical integration dominates in the large region, and for hard instances (large parameters α and/or δ), the asymptotic expansion (3.98) is effective.

General case

- Tables 15 and 16 show the characteristics of the sample sets and the accuracy and performance summary. The previous observations largely apply, except for the fact, that the accuracy for large (hard) instances is remarkably lower. The symbol $*$ indicates that struggles to converge with 300 digits after various attempts varying the integral upper bound N, returning systematically a value of zero. Nevertheless, the proposed algorithms seems less reliable for large values of $|x - \mu|$ for the general case.
- Table 17 and 18 exhibit that although the use of numerical integration is also dominant, the various expansions achieve similar accuracy. Nonetheless, table 19 illustrates that for some large hard instances, some expansions suffer to obtain more than 11 correct digits, forcing a switch to numerical integration to improve reliability in these cases.
- Table 20 compares the SciPy adaptive quadrature implementation with our double-exponential quadrature implementation, the latter demonstrating superior accuracy and performance, being $5 - 20$ times faster. Nevertheless, we emphasize that, although an implementation uniquely based on numerical integration is possible, the use of various expansions can reliably complement with benefits in terms of CPU times (as previously stated, numerical integration tends to be up to 100 times slower than an efficient implementation of expansions exploiting recurrence relations).

6 Conclusions

In the work presented, we developed and implemented an algorithm for fast and robust computation of the normal inverse Gaussian cumulative distribution function. With a thorough treatment, we showed the effectiveness of combining multiple series and asymptotic expansions with numerical integration to obtain significant acceleration compared to current state-of-the-art open-source implementations while keeping accuracy to near machine precision.

Several potential improvements remain to be explored in future work. First, the parameter regime of the convergent series expansions could be widened if computed increasing the working precision to long double (80-bit) or quadruple precision (128 bit), taking into account that quadruple precision is not natively supported in hardware, and necessarily it must be simulated[5], which in turn produces a code at least 10x slower. An alternative is using double-double precision (approximately 30 digits), an approach implemented in libraries such as QD library [19], and it was successfully implemented for the computation of special functions [28].

Second, it would also be interesting to evaluate the use of fixed-order Gauss-Legendre quadrature instead of the double-exponential quadrature, studying the appropriate number of quadrature points for different regions of the parameters' domain. In addition, it might be worth investigating the implementation of the integral expressible in terms of elementary function for small δ values.

Lastly, the techniques introduced throughout this work for developing many of the expansions can, of course, be seamlessly extendible to develop efficient algorithms for the computation of the generalized hyperbolic distribution and other special cases, such as the variance gamma distribution.

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A Benchmark

A.1 Case $\beta = 0$

Region	x	α	μ	δ
small	(-5, 5)	(0.001, 5)	(-5, 5)	(0.001, 5)
large	(-10, 10)	(0.001, 50)	(-10, 10)	(0.001, 50)

Table 5: Case $\beta = 0$: Parameter range for small and large region.

Library	Region	Success	Median	Mean	Time (s)	Time (μs)
SciPy	Small	3952 / 5000 (79.04%)	$9.44 \cdot 10^{-16}$	$9.90 \cdot 10^{-4}$	2.45	490
Paper	Small	4988 / 5000 (99.76%)	$2.66 \cdot 10^{-15}$	$1.53 \cdot 10^{-14}$	0.11	21
SciPy	Large	3549 / 4868 (72.90%)	$1.49 \cdot 10^{-14}$	$5.63 \cdot 10^{-3}$	2.60	535
Paper	Large	4868 / 4868 (100%)	$1.11 \cdot 10^{-15}$	$1.27 \cdot 10^{-14}$	0.22	45
SciPy	Large (hard)	25 / 132 (18.94%)	$1.44 \cdot 10^{-4}$	$6.45 \cdot 10^{-3}$	0.028	214
Paper	Large (hard)	127 / 132 (96.21%)	$2.46 \cdot 10^{-14}$	$8.84 \cdot 10^{-14}$	0.002	12

Table 6: Summary of the accuracy and performance comparison for the case $\beta = 0$.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	4121 (82.42%)	0	$1.22 \cdot 10^{-15}$	$4.15 \cdot 10^{-15}$	$1.44 \cdot 10^{-14}$	$8.40 \cdot 10^{-13}$
Series (3.7)	879 (17.58%)	0	0	$2.22 \cdot 10^{-16}$	$1.11 \cdot 10^{-15}$	$6.58 \cdot 10^{-14}$

Table 7: Precision metrics of the numerical methods used for computing in the small region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	3137 (64.44%)	0	$3.33 \cdot 10^{-16}$	$8.88 \cdot 10^{-16}$	$4.11 \cdot 10^{-15}$	$4.84 \cdot 10^{-13}$
Series (3.7)	810 (16.64%)	0	$3.33 \cdot 10^{-16}$	$1.33 \cdot 10^{-15}$	$8.41 \cdot 10^{-15}$	$4.28 \cdot 10^{-13}$
Asymptotic (3.45)	620 (12.74%)	0	0	0	$2.54 \cdot 10^{-14}$	$2.31 \cdot 10^{-13}$
Asymptotic (3.51)	301 (6.18%)	$2.22 \cdot 10^{-16}$	$4.44 \cdot 10^{-15}$	$1.24 \cdot 10^{-14}$	$3.46 \cdot 10^{-14}$	$2.26 \cdot 10^{-13}$

Table 8: Precision metrics of the numerical methods used for computing in the large region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

[5]For example, using libquadmath, the GCC Quad-Precision Math Library.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	22 (16.67%)	0	$6.94 \cdot 10^{-16}$	$3.66 \cdot 10^{-15}$	$1.76 \cdot 10^{-15}$	$5.98 \cdot 10^{-14}$
Series (3.7)	7 (5.30%)	$4.44 \cdot 10^{-16}$	$6.11 \cdot 10^{-16}$	$2.00 \cdot 10^{-15}$	$5.39 \cdot 10^{-13}$	$3.22 \cdot 10^{-12}$
Asymptotic (3.45)	100 (75.76%)	$8.88 \cdot 10^{-16}$	$1.35 \cdot 10^{-14}$	$3.00 \cdot 10^{-14}$	$5.57 \cdot 10^{-14}$	$1.30 \cdot 10^{-13}$
Asymptotic (3.51)	3 (2.27%)	$5.76 \cdot 10^{-13}$	$8.84 \cdot 10^{-13}$	$1.19 \cdot 10^{-12}$	$1.42 \cdot 10^{-12}$	$1.64 \cdot 10^{-12}$

Table 9: Precision metrics of the numerical methods used for computing in the large (hard) region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

A.2 Case $x = \mu$

Region	α	β	δ
small	(0.001, 5)	(-5, 5)	(0.001, 5)
large	(0.001, 50)	(-50, 50)	(0.001, 50)

Table 10: Case $x = \mu$: Parameter range for small and large region.

Library	Region	Success	Median	Mean	Time (s)	Time (μs)
SciPy	Small	4838 / 5000 (96.76%)	$8.88 \cdot 10^{-16}$	$8.40 \cdot 10^{-11}$	2.02	405
Paper	Small	5000 / 5000 (100%)	$2.22 \cdot 10^{-16}$	$9.60 \cdot 10^{-16}$	0.03	7
SciPy	Large	3084 / 4548 (67.81%)	$1.96 \cdot 10^{-15}$	$1.34 \cdot 10^{-12}$	2.58	516
Paper	Large	4548 / 4548 (100%)	$6.66 \cdot 10^{-16}$	$8.25 \cdot 10^{-15}$	0.27	53
SciPy	Large (hard)	8 / 452 (1.77%)	$1.26 \cdot 10^{-1}$	$1.65 \cdot 10^{-1}$	0.104	229
Paper	Large (hard)	425 / 452 (94.03%)	$3.86 \cdot 10^{-14}$	$2.47 \cdot 10^{-12}$	0.024	53

Table 11: Summary of the accuracy and performance comparison for the case $x = \mu$.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	949 (18.48%)	0	0	$2.22 \cdot 10^{-16}$	$4.44 \cdot 10^{-16}$	$6.00 \cdot 10^{-15}$
Series (3.58)	4051 (81.02%)	0	0	$2.22 \cdot 10^{-16}$	$5.55 \cdot 10^{-16}$	$2.48 \cdot 10^{-13}$

Table 12: Precision metrics of the numerical methods used for computing in the small region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	4289 (94.31%)	0	$2.22 \cdot 10^{-16}$	$5.55 \cdot 10^{-16}$	$3.77 \cdot 10^{-15}$	$4.81 \cdot 10^{-13}$
Series (3.58)	126 (2.77%)	0	$2.22 \cdot 10^{-16}$	$4.44 \cdot 10^{-16}$	$1.67 \cdot 10^{-15}$	$2.60 \cdot 10^{-13}$
Asymptotic (3.98)	133 (2.92%)	$5.55 \cdot 10^{-16}$	$1.70 \cdot 10^{-14}$	$2.70 \cdot 10^{-14}$	$4.30 \cdot 10^{-14}$	$1.15 \cdot 10^{-13}$

Table 13: Precision metrics of the numerical methods used for computing in the large region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	204 (45.13%)	$2.22 \cdot 10^{-16}$	$1.73 \cdot 10^{-14}$	$3.22 \cdot 10^{-14}$	$4.73 \cdot 10^{-14}$	$1.19 \cdot 10^{-13}$
Series (3.58)	1 (0.22%)	$7.76 \cdot 10^{-13}$	$7.76 \cdot 10^{-13}$	$7.76 \cdot 10^{-13}$	$7.76 \cdot 10^{-13}$	$7.76 \cdot 10^{-13}$
Asymptotic (3.98)	247 (54.65%)	0	$3.09 \cdot 10^{-14}$	$5.43 \cdot 10^{-14}$	$9.60 \cdot 10^{-14}$	–

Table 14: Precision metrics of the numerical methods used for computing in the large (hard) region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

A.3 General case

Region	x	α	β	μ	δ
small	(-5, 5)	(0.001, 5)	(-5, 5)	(-5, 5)	(0.001, 5)
large	(-10, 10)	(0.001, 50)	(-50, 50)	(-10, 10)	(0.001, 50)

Table 15: General case: Parameter range for small and large region.

Library	Region	Success	Median	Mean	Time (s)	Time (μs)
SciPy	Small	3983 / 5000 (79.66%)	$1.78 \cdot 10^{-15}$	$1.89 \cdot 10^{-5}$	2.53	506
Paper	Small	4939 / 5000 (98.78%)	$3.89 \cdot 10^{-15}$	$1.01 \cdot 10^{-13}$	0.12	24
SciPy	Large	2949 / 4296 (68.65%)	$4.49 \cdot 10^{-14}$	$7.91 \cdot 10^{-3}$	2.79	559
Paper	Large	4296 / 4296 (100%)	$6.66 \cdot 10^{-16}$	$1.09 \cdot 10^{-14}$	0.44	88
SciPy	Large (hard)	70 / 704 (9.94%)	$0.16 \cdot 10^{-1}$	$*$	0.16	223
Paper	Large (hard)	409 / 704 (58.10%)	$8.07 \cdot 10^{-14}$	$*$	0.07	95

Table 16: Summary of the accuracy and performance comparison for the general case. Symbol $*$ indicates numerical issues were encountered.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	3585 (71.70%)	0	$6.66 \cdot 10^{-16}$	$5.00 \cdot 10^{-15}$	$1.78 \cdot 10^{-14}$	$7.56 \cdot 10^{-11}$
Series (3.64)	212 (4.24%)	0	$4.44 \cdot 10^{-16}$	$1.61 \cdot 10^{-15}$	$1.14 \cdot 10^{-14}$	$2.68 \cdot 10^{-12}$
Series (3.87)	606 (12.12%)	0	$3.33 \cdot 10^{-16}$	$8.88 \cdot 10^{-16}$	$5.30 \cdot 10^{-15}$	$3.08 \cdot 10^{-11}$
Series (3.90)	597 (11.94%)	0	$8.88 \cdot 10^{-16}$	$4.22 \cdot 10^{-15}$	$1.47 \cdot 10^{-14}$	$1.70 \cdot 10^{-12}$

Table 17: Precision metrics of the numerical methods used for computing in the small region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	3344 (77.84%)	0	$2.22 \cdot 10^{-16}$	$7.77 \cdot 10^{-16}$	$5.19 \cdot 10^{-15}$	$4.54 \cdot 10^{-13}$
Series (3.64)	9 (0.21%)	$2.22 \cdot 10^{-16}$	$1.11 \cdot 10^{-15}$	$2.00 \cdot 10^{-15}$	$1.09 \cdot 10^{-14}$	$1.46 \cdot 10^{-13}$
Series (3.87)	516 (12.01%)	0	$2.22 \cdot 10^{-16}$	$6.66 \cdot 10^{-16}$	$8.83 \cdot 10^{-15}$	$4.90 \cdot 10^{-13}$
Series (3.90)	156 (3.63%)	0	$4.44 \cdot 10^{-16}$	$2.33 \cdot 10^{-15}$	$1.87 \cdot 10^{-14}$	$4.71 \cdot 10^{-13}$
Asymptotic (3.99)	13 (0.30%)	0	0	0	0	$2.11 \cdot 10^{-14}$
Asymptotic (3.94)	258 (6.01%)	0	0	0	0	$3.62 \cdot 10^{-13}$

Table 18: Precision metrics of the numerical methods used for computing in the small region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

Method	Count	min	25%	Median	75%	max
Integration (3.101)	400 (56.82%)	$2.22 \cdot 10^{-15}$	$2.75 \cdot 10^{-14}$	$4.76 \cdot 10^{-14}$	$7.70 \cdot 10^{-14}$	–
Series (3.64)	2 (0.29%)	$9.32 \cdot 10^{-13}$	$1.67 \cdot 10^{-11}$	$3.24 \cdot 10^{-11}$	$4.81 \cdot 10^{-11}$	$6.81 \cdot 10^{-11}$
Series (3.87)	155 (22.02%)	$9.99 \cdot 10^{-16}$	$1.11 \cdot 10^{-12}$	$3.75 \cdot 10^{-10}$	–	–
Series (3.90)	13 (1.85%)	$2.55 \cdot 10^{-13}$	$7.09 \cdot 10^{-13}$	$1.62 \cdot 10^{-12}$	$7.45 \cdot 10^{-12}$	$7.96 \cdot 10^{-11}$
Asymptotic (3.99)	14 (1.99%)	$2.66 \cdot 10^{-15}$	$5.22 \cdot 10^{-14}$	$3.65 \cdot 10^{-12}$	$2.08 \cdot 10^{-10}$	–
Asymptotic (3.94)	120 (17.05%)	$2.11 \cdot 10^{-15}$	$3.87 \cdot 10^{-13}$	$1.02 \cdot 10^{-9}$	–	–

Table 19: Precision metrics of the numerical methods used for computing in the small region. The errors are the absolute relative errors compared to the reference solutions obtained using mpmath. Percentiles: 25, 50 (median), 75.

Library	Region	Success	Median	Mean	Time (s)	Time (μs)
SciPy	Small	3983 / 5000 (79.66%)	$1.78 \cdot 10^{-15}$	$1.89 \cdot 10^{-5}$	2.53	506
Paper	Small	4980 / 5000 (99.60%)	$3.99 \cdot 10^{-15}$	$1.64 \cdot 10^{-14}$	0.12	27
SciPy	Large	3139 / 5000 (62.78%)	$0.16 \cdot 10^{-1}$	$*$	2.74	550
Paper	Large	4964 / 5000 (99.28%)	$8.07 \cdot 10^{-14}$	$*$	0.51	102

Table 20: Summary of the accuracy and performance comparison for the general case only using numerical integration. Symbol $*$ indicates numerical issues were encountered.

B The function $\Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$

The function $F(t; a, b) = \Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$ is part of the integrand of the integral representation in (2.5). Given its relevance throughout this work, we introduce here some results that shall be used subsequently. $F(t; a, b)$ has the following integral representation [11, §7.7.6]

$$F(t; a, b) = \frac{1}{2} \operatorname{erfc}\left(-\frac{\frac{a}{\sqrt{t}} + b\sqrt{t}}{\sqrt{2}}\right) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \int_{-b/\sqrt{2}}^{\infty} e^{-(tu^2 - \sqrt{2}au)}\, du \tag{B.1}$$

B.1 Expansions t

B.1.1 Expansion $t \to 0$

Let us consider the case $a < 0$, since we can use the mirror property $\Phi(z) = 1 - \Phi(-z)$ otherwise. To obtain an expansion for $t \to 0$, we expand e^{-tu^2} and interchange summation and integration obtaining

$$F(t; a, b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au} u^{2k}\, du\,.$$

For $a < 0$ the integral can be expressed in closed form in terms of the incomplete gamma function, $\Gamma(a, x)$

$$\int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au} u^{2k}\, du = \frac{\Gamma(2k+1, -ab)}{(\sqrt{2}a)^{2k+1}},$$

and for the special case $b = 0$, it reduces to

$$\int_{0}^{\infty} e^{\sqrt{2}au} u^{2k}\, du = \frac{\Gamma(2k+1)}{(\sqrt{2}a)^{2k+1}}.$$

Then, we obtain the series expansion valid for $t \to 0$, $a \to -\infty$ and fixed b

$$F(t; a, b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} t^k}{k!} \frac{\Gamma(2k+1, ab)}{(\sqrt{2}a)^{2k+1}}. \tag{B.2}$$

NOTE: same expansion as $\mathrm{erfc}(x), x \to \infty$.

Moreover, another expansion valid for large values of $a > 0$ and $b > 0$ can be obtained after expanding $F(t; a, b)$ at $t = 0$. The first coefficients are

$$c_0 = \frac{1}{a}, \quad c_1 = \frac{ab+1}{a^3}, \quad c_2 = \frac{a^2b + 3ab + 3}{a^5}, \quad c_3 = \frac{a^3b^3 + 6a^2b^3 + 15ab + 15}{a^7} \tag{B.3}$$

and the expansion reads

$$F(t; a, b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)^2}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} c_k t^{k+\frac{1}{2}}. \tag{B.4}$$

The coefficients are expressible in terms of Bessel polynomials $y_k(x)$ [37, §A001498], and it follows that

$$F(t; a, b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)^2}}{a\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k y_k\left(\frac{1}{ab}\right) t^{k+\frac{1}{2}}, \tag{B.5}$$

where $y_k(x)$ has an explicit formula

$$y_k(x) = \sum_{m=0}^{k} \binom{k}{m} (k+1)_m \left(\frac{x}{2}\right)^m. \tag{B.6}$$

Using the connection of the Bessel polynomials with the modified Bessel function of the second kind $K_k(x)$ given by [42, §33.1.3]

$$y_k(x) = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{k+\frac{1}{2}}\left(\frac{1}{x}\right), \tag{B.7}$$

the resulting expansion is represented as a Bessel-type expansion

$$F(t; a, b) = 1 + \frac{e^{-\frac{a^2}{2t} - \frac{b^2}{2}t}}{\pi} \sqrt{\frac{b}{a}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k K_{k+\frac{1}{2}}(ab) t^{k+\frac{1}{2}}. \tag{B.8}$$

The expansion is convergent for $t < 1$. The convergence follows from the asymptotic estimate of $(b/a)^k K_k(ab) \sim (b/a)^k \sqrt{\frac{\pi}{2ab}} e^{-ab}$ as $|ab| \to \infty$. The expansion can be seen as an asymptotic expansion for large a, or as a uniform asymptotic expansion for $a \sim b$. The coefficients can be computed by using a recurrence relation for the modified Bessel function.

B.1.2 Expansion $t \to \infty$

Let us focus on the case $t \to \infty$. We can develop an asymptotic expansion after expanding the term $e^{\sqrt{2}au}$ in (B.1), which yields

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k \, du \, .$$

Considering the case $b < 0$ (again, we can use the mirror property), the integral has a closed-form

$$\int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k \, du = \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{2t^{\frac{k+1}{2}}}.$$

Thus,

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} \frac{e^{-\frac{a^2}{2t}}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}}. \tag{B.9}$$

The asymptotic behaviour of the terms in the series is

$$\frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}} \sim \left(\frac{b^2}{2}\right)^{\frac{k+1}{2}} e^{-\frac{b^2}{2}t}, \quad t \to \infty.$$

In fact this series is convergent, as can be observed taking the asymptotic estimate of $\Gamma(k,x)$ as $k \to \infty$. A simpler convergent expansion can be obtained transforming the integral in (B.1)

$$\sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \int_{-b/\sqrt{2}}^{\infty} e^{-(tu^2 - \sqrt{2}au)} \, du = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t} - ab - \frac{b^2}{2}t} \int_{0}^{\infty} e^{\sqrt{2}(a+bt)u} e^{-tu^2} \, dt,$$

and expanding $e^{\sqrt{2}(a+bt)u}$ obtaining

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} \frac{e^{-\frac{a^2}{2t} - ab - \frac{b^2}{2}t}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{2}(a+bt))^k}{k!} \frac{\Gamma\left(\frac{k+1}{2}\right)}{t^{\frac{k+1}{2}}}. \tag{B.10}$$

Similarly to the expansion at $t \to 0$, we can obtain an asymptotic expansion expanding $F(t;a,b)$ at $t \to \infty$. For $b > 0$, the first terms of the expansion are

$$c_0 = 1, \quad c_1 = 2 + 2ab + a^2b^2, \quad c_2 = 24 + 24ab + 12a^2b^2 + 4a^3b^3 + a^4b^4, \tag{B.11}$$

$$F(t;a,b) = 1 + \frac{e^{-ab-\frac{b^2}{2}t}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{c_k}{b^{2k+1}} \left(\frac{1}{t}\right)^{k+\frac{1}{2}}. \tag{B.12}$$

The coefficients c_k are expressible in terms of the incomplete gamma function, since

$$c_k = \sum_{j=0}^{2k} \frac{(2k)!}{j!} (ab)^j = e^{ab} \Gamma(2k+1, ab).$$

Rearranging terms, we get

$$F(t;a,b) = 1 + \frac{e^{-\frac{b^2}{2}t}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{\Gamma(2k+1, ab)}{b^{2k+1}} \left(\frac{1}{t}\right)^{k+\frac{1}{2}}. \tag{B.13}$$

B.1.3 Expansion $t \to u$

Lastly, we study the expansion of $F(t;a,b)$ at $t = u$. This expansion shall be crucial when developing various Bessel-type asymptotic expansions later on. The first coefficients of the Taylor series are

$$c_0 = \Phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_0, \quad c_1 = \phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_1, \quad c_2 = -\phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_2, \quad c_3 = \phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_3,$$

where

$$\begin{aligned}
d_0 &= 1 \\
d_1 &= \frac{-a + bu}{2u^{3/2}} \\
d_2 &= \frac{a^3 - 3au - a^2bu + bu^2 - ab^2u^2 + b^3u^3}{8u^{7/2}} \\
d_3 &= \frac{-a^5 + 10a^3u + a^4bu - 15au^2 - 6a^2bu^2 + 2a^3b^2u^2 + 3bu^3 - 6ab^2u^3 - 2a^2b^3u^3 + 2b^3u^4 - ab^4u^4 + b^5u^5}{48u^{11/2}},
\end{aligned}$$

and $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ is the probability density function of the standard normal distribution. Thus, we have

$$F(t; a, b) = \sum_{k=0}^{\infty} c_k (t-u)^k. \tag{B.14}$$

Additional terms satisfy the following recurrence

$$c_{k+4} = \frac{f_0(k)c_k + f_1(k)c_{k+1} + f_2(k)c_{k+2} + f_3(k)c_{k+3}}{f_4(k)}, \quad k \geq 0 \tag{B.15}$$

where

$$\begin{aligned}
f_0(k) &= -kb^3 \\
f_1(k) &= -(1+k)b(1+2k-ab+3b^3u) \\
f_2(k) &= (2+k)(5a+2ka+a^2b-8bu-6kbu+2ab^2u-3b^3u^2) \\
f_3(k) &= -(3+k)(a^3-11au-4kau-a^2bu+13bu^2+6kbu^2-ab^2u^2+b^3u^3) \\
f_4(k) &= 2(3+k)(4+k)u^2(-a+bu)
\end{aligned}$$

B.2 Expansion $a \to 0$

We also consider expansions for small values of the parameters. If we expand $F(t; a, b)$ at $a = 0$, we obtain the series

$$F(t; a, b) = \Phi\left(b\sqrt{t}\right) + \frac{e^{-\frac{b^2 t}{2}}}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a^k P_{k-1}(t; b)}{k!}. \tag{B.16}$$

The first coefficients $P_k(t; b)$ are

$$\begin{aligned}
P_0(t; b) &= 1 \\
P_1(t; b) &= b \\
P_2(t; b) &= \frac{b^2 t - 1}{t} \\
P_3(t; b) &= \frac{b^3 t - 3b}{t} \\
P_4(t; b) &= \frac{b^4 t^2 - 6b^2 t + 3}{t^2} \\
P_5(t; b) &= \frac{b^5 t^2 - 10b^3 t + 15b}{t^2}
\end{aligned}$$

Looking at the first coefficients, it is not hard to observe that coefficients $P_k(t; b)$ are expressible in terms of probabilist Hermite polynomials. Thus, after using the connection formula between probabilist and classical Hermite polynomials, we obtain the following convergent series expansion

$$F(t; a, b) = \Phi\left(b\sqrt{t}\right) + \frac{e^{-\frac{b^2 t}{2}}}{\sqrt{2\pi t}} \sum_{k=0}^{\infty} \frac{(-1)^k a^{k+1}}{(k+1)!} \frac{1}{(2t)^{k/2}} H_k\left(k, b\sqrt{\frac{t}{2}}\right) \tag{B.17}$$

B.3 Expansion $b \to 0$

The Hermite-type expansion for the case $b \to 0$ is obtained analogously after expanding $F(t; a, b)$ at $b = 0$. Thus,

$$F(t; a, b) = \Phi\left(\frac{a}{\sqrt{t}}\right) + e^{-\frac{a^2 t}{2}} \sqrt{\frac{t}{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k b^{k+1}}{(k+1)!} \left(\frac{t}{2}\right)^{k/2} H_k\left(\frac{a}{\sqrt{2t}}\right). \tag{B.18}$$

C The modified Bessel function of the second kind

For small values of x we have

$$K_\nu(x) \sim \frac{2^{|\nu|-1}\Gamma(|\nu|)}{x^{|\nu|}}, \quad x \to 0, \quad \nu \neq 0. \tag{C.1}$$

Asymptotic behaviour with respect to the argument

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \to \infty, \quad \nu \in \mathbb{R}. \tag{C.2}$$

Asymptotic behaviour with respect to the order

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu}, \quad \nu \to \infty, \quad x \neq 0. \tag{C.3}$$

The modified Bessel function can be stated explicitly for $\nu = n + 1/2$ with $n \in \mathbb{Z}$,

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}} \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!} (2z)^{-j} e^{-z}, \quad z \in \mathcal{C}. \tag{C.4}$$

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