On the computation and inversion of the Normal Inverse Gaussian cumulative distribution function

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May 18, 2024

Abstract

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Contents

1 Introduction

2 Distribution properties

Variance-mean mixture distribution

$$Z \sim \mathcal{IG}(\delta\gamma, \gamma^2), \quad X \sim \mathcal{N}(\mu + \beta Z, Z), \tag{2.1}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. The domain of the parameters is

$$0 \leq |\beta| < \alpha, \quad \mu \in \mathbb{R}, \quad \delta > 0. \tag{2.2}$$

2.1 Density function

The density function is given as

$$f(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta}{\pi} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}} e^{\delta\gamma + \beta(x-\mu)} \tag{2.3}$$

Parameterization: Standard case $\mu = 0$ and $\delta = 1$. The parameters have the following interpretation: α is the tail heaviness, β is the asymmetry or skewness, μ is the location parameter and δ the scale parameter. Where μ is the location of the density, β is the skewness parameter, α measures the heaviness of the tails.

2.2 Cumulative distribution function

The cumulative distribution function is given by

$$F(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi} \int_{-\infty}^{x} \frac{K_1\left(\alpha\sqrt{\delta^2 + (t-\mu)^2}\right)}{\sqrt{\delta^2 + (t-\mu)^2}} e^{\beta(t-\mu)}\, dt \tag{2.4}$$

$$F(x; \alpha, \beta, \mu, \delta) = \frac{\delta}{\sqrt{2\pi}} \int_{0}^{\infty} \Phi\left(\frac{x - (\mu + \beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{(\delta - \gamma t)^2}{2t}}\, dt \tag{2.5}$$

Also denote

$$\tilde{F}(x; \alpha, \beta, \mu, \delta) = 1 - F(-x; \alpha, -\beta, -\mu, \delta). \tag{2.6}$$

Follows from the reversion formula of $\Phi(x)$.

Proposition 2.1 *For $x - \mu < 0$, an incomplete Laplace-type integral representation in terms of modified Bessel function $K_0(x)$ is given by*

$$F(x; \alpha, \beta, \mu, \delta) = \frac{\sqrt{2}\delta e^{\delta\gamma}}{\pi} \int_{\beta/\sqrt{2}}^{\infty} e^{\sqrt{2}(x-\mu)t} K_0\left(\sqrt{2\left((x-\mu)^2 + \delta^2\right)}\sqrt{\frac{\gamma^2}{2} + t^2}\right) dt, \tag{2.7}$$

and (2.6)*, otherwise.*

Proof: Consider the integral representation of the function $\Phi\left(\frac{x-\mu}{\sqrt{t}} - \beta\sqrt{t}\right)$

$$\Phi\left(\frac{x-\mu}{\sqrt{t}} - \beta\sqrt{t}\right) = \sqrt{\frac{t}{\pi}} e^{-\frac{(x-\mu)^2}{2t}} \int_{\beta/\sqrt{2}}^{\infty} e^{-(tu^2 - \sqrt{2}(x-\mu)u)}\, du\,.$$

Replacing in (2.5) and interchanging the order of integration we obtain

$$F(x; \alpha, \beta, \mu, \delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \int_{\beta/\sqrt{2}}^{\infty} e^{\sqrt{2}(x-\mu)u} \int_{0}^{\infty} t^{-1} e^{-\frac{\left((x-\mu)^2 + \delta^2\right)}{2t} - \left(\frac{\gamma^2}{2} + u^2\right)t}\, dt\, du\,.$$

The observation that the inner integral can be represented in terms of the modified Bessel function $K_0(x)$

$$\int_{0}^{\infty} t^{-1} e^{-\frac{\left((x-\mu)^2 + \delta^2\right)}{2t} - \left(\frac{\gamma^2}{2} + u^2\right)t}\, dt = 2K_0\left(2\sqrt{\frac{(x-\mu)^2 + \delta^2}{2}}\sqrt{\frac{\gamma^2}{2} + t^2}\right).$$

$\square$

Proposition 2.2 *The Fourier sine transform of the cumulative distribution function in terms of elementary functions is given by*

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{e^{\delta\gamma}}{\pi}\int_0^\infty \frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)}\sin(\delta t)\,dt, \quad x-\mu>0, \tag{2.8}$$

and apply (2.6) *for* $x-\mu<0$.

Proof: Consider the integral representation (2.4). We use the sine transform [1, §2.4]

$$\frac{K_1(\alpha\sqrt{t^2+\delta^2}}{\sqrt{t^2+\delta^2}} = \frac{1}{\alpha\delta}\int_0^\infty \frac{ze^{-t\sqrt{z^2+\alpha^2}}}{\sqrt{z^2+\alpha^2}}\sin(\delta z)\,dz,$$

valid for $t\geq 0$. Replacing in (2.4) and interchanging the order of integration, we have

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) &= 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi}\frac{1}{\alpha\delta}\int_0^\infty \frac{z\sin(\delta z)}{\sqrt{z^2+\alpha^2}}\int_{x-\mu}^\infty e^{-t\sqrt{z^2+\alpha^2}+\beta t}\,dt\,dz \\ &= 1 - \frac{e^{\delta\gamma}}{\pi}\int_0^\infty \frac{z\sin(\delta z)}{\sqrt{z^2+\alpha^2}}\frac{e^{-(x-\mu)\left(\sqrt{z^2+\alpha^2}-\beta\right)}}{\sqrt{z^2+\alpha^2}-\beta}\,dz \end{aligned}$$

where the inner integral converges for $t\in[x-\mu,\infty)$. □

In a similar manner, we can obtain a Fourier cosine transform in terms of the exponential integral.

Proposition 2.3 *The Fourier cosine transform integral representation in terms of the exponential integral* $E_1(x)$ *is given by*

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{\delta e^{\delta\gamma}}{\pi}\int_0^\infty E_1\left((x-\mu)(\sqrt{t^2+\alpha^2}-\beta)\right)\cos(\delta t)\,dt, \quad x-\mu>0. \tag{2.9}$$

Proof: This result can be derived from the sine transform in (2.8) using integration by parts. Alternatively, we can use the cosine transform [1, §1.4]

$$\frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}} = \frac{1}{\alpha t}\int_0^\infty e^{-t\sqrt{z^2+\alpha^2}}\cos(\delta z)\,dz\,.$$

Thus,

$$F(x;\alpha,\beta,\mu,\delta) = 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{x-\mu}^\infty \frac{e^{\beta t}}{\alpha t}\int_0^\infty e^{-t\sqrt{z^2+\alpha^2}}\cos(\delta z)\,dz\,dt,$$

where the inner integral is expressible in closed form in terms of the exponential integral $E_1(x)$

$$\int_{x-\mu}^\infty \frac{1}{t}e^{-(\sqrt{z^2+\alpha^2}-\beta)t}\,dt = E_1\left((x-\mu)(\sqrt{z^2+\alpha^2}-\beta)\right).$$

□

2.3 Moments and cumulants

$$\mathbb{E}[X^m] = \frac{\alpha\delta}{\pi}\int_{-\infty}^\infty t^m\frac{K_1\left(\alpha\sqrt{\delta^2+(t-\mu)^2}\right)}{\sqrt{\delta^2+(t-\mu)^2}}e^{\delta\gamma+\beta(t-\mu)}\,dt\,. \tag{2.10}$$

$$\mathbb{E}[X^m] = \frac{\alpha\delta}{\pi}e^{\delta\gamma-\beta\mu}\sum_{k=0}^\infty \frac{\beta^k}{k!}\int_{-\infty}^\infty (t-\mu)^{m+k}\frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}\,dt\,. \tag{2.11}$$

Use binomial theorem, and compute coefficients recursively (binomial sum of Bessel functions). Treat special case $\mu=0$ and $\beta=0$.

3 Methods of computation

In this Section, we describe the methods used for efficient computation of the CDF for the general case and various special cases. Subsequently, we discuss several approaches for computing the inverse of the CDF.

3.1 Expansions: case $\beta = 0$

3.1.1 Expansions $|x-\mu| \to 0$

Consider the integral representation

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\sqrt{2\pi}} \int_0^\infty \Phi\left(\frac{x-\mu}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\alpha^2}{2}t}\, dt\,. \tag{3.1}$$

To obtain a series expansion for $|x-\mu| \to 0$, we can proceed expanding $\Phi\left(\frac{x-\mu}{\sqrt{t}}\right)$, using the two well-known absolutely convergent series expansions of $\Phi(x)$ [5, §2]

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k k!(2k+1)}, \tag{3.2}$$

and

$$\Phi(x) = \frac{1}{2} + \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!!}. \tag{3.3}$$

If we choose the expansion (3.2) and interchange the order of integration and summation, the resulting integral has the form

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\sqrt{2\pi}} \int_0^\infty \sum_{k=0}^{\infty} \frac{(-1)^k (x-\mu)^{2k+1}}{2^k k!(2k+1)} \int_0^\infty t^{-k-2} e^{-\frac{\delta^2}{2t}-\frac{\alpha^2}{2}t}\, dt, \tag{3.4}$$

which has a closed-form in terms of the modified Bessel function

$$\int_0^\infty t^{\lambda-1} e^{-a/t-zt}\, dt = 2\left(\frac{\alpha}{z}\right)^{\lambda/2} K_\lambda(2\sqrt{\alpha z}). \tag{3.5}$$

Inserting (3.5) in (3.4) and rearranging terms, we obtain the the alternating series

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (x-\mu)^{2k+1}}{2^k k!(2k+1)} \left(\frac{\alpha}{\delta}\right)^{k+1} K_{k+1}(\alpha\delta). \tag{3.6}$$

To obtain a series with positive terms for $x-\mu > 0$, we choose the alternative expansion of $\Phi(x)$ in (3.3), obtaining

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \sum_{k=0}^{\infty} \frac{(x-\mu)^{2k+1}}{(2k+1)!!} \left(\frac{\alpha}{\omega}\right)^{k+1} K_{k+1}(\alpha\omega), \quad \omega = \sqrt{\delta^2+(x-\mu)^2}. \tag{3.7}$$

The series expansions (3.6) and (3.7) can be written as a truncated series with the corresponding remainder term. For example, truncating the series (3.7) at N, we can write

$$\sum_{k=0}^{\infty} T_k = \sum_{k=0}^{N-1} T_k + \sum_{k=N}^{\infty} T_k, \quad T_k = \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^k \frac{K_{k+1}(\alpha\omega)}{(2k+1)!!}, \tag{3.8}$$

and

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \frac{(x-\mu)\alpha}{\omega} \sum_{k=0}^{N-1} T_k + R_N.$$

The error term in (3.8) can be bounded by comparison with a geometric series

$$\left|\sum_{k=N}^{\infty} T_k\right| \le \frac{|T_N|}{1-C}, \quad C = \left|\frac{T_{N+1}}{T_N}\right| \tag{3.9}$$

iff $C < 1$, where T_N is the first omitted term in the expansion.

The following lemma provides an upper bound for $K_{\nu+1}(x)$, required for deriving an upper bound for the term T_N.

Lemma 3.1 *For $x \ge 0$ and $\nu \ge -\frac{1}{2}$ we have*

$$K_{\nu+1}(x) < \frac{\Gamma(\nu+1)2^\nu}{x^{\nu+1}}. \tag{3.10}$$

Proof: The proof reduces to combining the uniform bound in [3]

$$xK_{\nu+1}(x)I_\nu(x) \leq 1. \tag{3.11}$$

with the lower bound [6]

$$\left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} < I_\nu(x). \tag{3.12}$$

Hence,

$$K_{\nu+1}(x) \leq \frac{1}{xI_\nu(x)} < \frac{\Gamma(\nu+1)2^\nu}{x^{\nu+1}}. \tag{3.13}$$

□

Theorem 3.2 *Given* $\alpha > 0$ *and* $\omega > 0$, $x - \mu \in \mathbb{R}$ *and* $N \in \mathbb{N}$, *the error term in* (3.8) *satisfies*

$$\left|\sum_{k=N}^{\infty} T_k\right| \leq \frac{|T_N|}{1-C}, \tag{3.14}$$

where

$$T_N < \frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.15}$$

and

$$C < \left(\frac{x-\mu}{\omega}\right)^2 \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{2N+3}. \tag{3.16}$$

Proof: The use of Lemma 3.1 gives

$$\begin{aligned} T_N &= \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^N \frac{K_{N+1}(\alpha\omega)}{(2N+1)!!} \\ &< \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^N \frac{\Gamma(N+1)2^N}{(\alpha\omega)^{N+1}(2N+1)!!} \\ &= \frac{1}{\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N} \frac{(N!2^N)^2}{(2N+1)!}. \end{aligned}$$

An upper bound for the ratio of factorials in the previous inequality is given by

$$\frac{(N!2^N)^2}{(2N+1)!} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(N+1)}{\Gamma(N+3/2)} \leq \frac{1}{2}\sqrt{\frac{\pi}{N+1/2}},$$

where we use the fact that $\Gamma(N+3/2) = (N+1/2)\Gamma(N+1/2)$ and the upper bound of the ratio of gamma functions [11]

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+s)^{1-s}, \quad s \in (0,1). \tag{3.17}$$

Thus, the following bound for T_N holds

$$T_N < \frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.18}$$

An explicit formula for C in terms of the ratio of modified Bessel functions is

$$C = \frac{T_{N+1}}{T_N} = \frac{(x-\mu)^2\alpha}{\omega(2N+3)}\frac{K_{N+2}(\alpha\omega)}{K_{N+1}(\alpha\omega)}. \tag{3.19}$$

To bound C, we use the sharp bound for the ratio of modified Bessel functions [7]

$$\frac{K_{N+2}(\alpha\omega)}{K_{N+1}(\alpha\omega)} < \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{\alpha\omega}, \tag{3.20}$$

and we obtain

$$C < \left(\frac{x-\mu}{\omega}\right)^2 \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{2N+3}.$$

□

To study the regime of applicability for the expansion, we can estimate the required number of terms N equating the bound of T_N in (3.15) times the normalizing factor with the requested absolute error ϵ and solving for N, which gives

$$N \approx -\frac{\Re(W_{-1}(D))}{4\log(A)} + \frac{1}{2}, \quad A = \frac{x-\mu}{\omega}, \quad B = \frac{A\delta e^{\delta\alpha}}{2\omega\pi}, \quad C = \frac{\epsilon^2}{B^2\pi}, \quad D = -\frac{4\log(A)}{CA^2} \tag{3.21}$$

where $W_k(x)$ denotes the Lambert W function [2, §4.13]. The branch $k = -1$ is used to obtain the maximum real N. Note that since $A < 1$ by definition, $D > 0$. When CA^2 is tiny, $W_{-1}(D)$ can be approximated as $\Re(W_{-1}(D)) \sim \log(D) - \log(\log(D))$ using the first two terms of the asymptotic expansion in [2, §4.13.10].

The previous analysis performed for the expansion (3.7) is repeated for the alternating expansion (3.6). The main results are summarized below for the purpose of brevity. The expansion (3.6) rewritten including the remainder term follows

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{(x-\mu)\alpha e^{\delta\alpha}}{\pi}\left(\sum_{k=0}^{N-1} T_k + \sum_{k=N}^{\infty} T_k\right), \quad T_k = \left(-\frac{(x-\mu)^2\alpha}{\delta}\right)^k \frac{K_{k+1}(\alpha\delta)}{2^k k!(2k+1)}. \tag{3.22}$$

The last omitted term T_N satisfies

$$|T_N| < \frac{1}{\alpha\delta}\left(\frac{x-\mu}{\delta}\right)^{2N}\frac{1}{2N+1}, \tag{3.23}$$

and the number of terms N can be determined employing the Lambert W function for a given error ϵ

$$N \approx \frac{\Re(W_{-1}(D)) + \log(A)}{2\log(A)}, \quad A = \frac{x-\mu}{\delta}, \quad B = \frac{Ae^{\delta\alpha}}{\pi}, \quad C = \frac{\epsilon}{B}, \quad D = -\frac{\log(A)}{CA}. \tag{3.24}$$

Now we are in position to compare both series and their corresponding domains of applicability. A first important observation is that the alternating series (3.6) does not converge when $|x-\mu| > \delta$, and the number of terms N increases rapidly when $|x-\mu| \sim \delta$. In contrast, the series (3.7) is absolutely convergent, although N also grows considerably when $\alpha\omega \to 0$. The convergence of the latter can be reliably assessed applying to T_k in (3.8) the asymptotic estimates of $K_{k+1}(\alpha\omega)$ for $\alpha\omega \to 0$ and $\alpha\omega \to \infty$, (B.1) and (B.2), respectively:

$$T_k \sim \frac{\sqrt{\pi}}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2k}\frac{\Gamma(k+1)}{\Gamma(k+3/2)}, \qquad \alpha\omega \to 0,$$
$$T_k \sim \sqrt{\frac{\pi}{2\alpha\omega}}\left(\frac{(x-\mu)^2\alpha}{\omega}\right)^k \frac{e^{-\alpha\omega}}{(2k+1)!!}, \qquad \alpha\omega \to \infty.$$

These asymptotic estimates of T_k show that the series expansion (3.7) is slowly convergent for $\alpha\omega \to 0$, and the ratio of convergence improves when $\alpha\omega \to \infty$, requiring only a few terms when performing computations using machine precision. For $k \to \infty$, occurring for large values of $|x-\mu|$ or when $\alpha\omega$ is small, T_k follows the asymptotic behaviour

$$T_k \sim \frac{1}{\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2k}\sqrt{\frac{\pi}{e}}\frac{1}{\sqrt{4k+2}}\left(\frac{2k+2}{2k+1}\right)^{k+\frac{1}{2}}, \quad k \to \infty,$$

where we use the asymptotic estimate of modified Bessel function for large order (B.3) and apply Stirling's approximation for the double factorial. It is worth noticing that performing a naive computation of the terms T_k in double-precision arithmetic for large N poses underflow and overflow problems since the numerator (denominator) rapidly goes to infinity (zero) as N increases. In Section 4.2, we discuss alternative summation methods to avoid cancellation issues and precision loss, and the usage of convergence acceleration methods for large N.

It remains to analyze the accuracy of the bound in (3.14) for different parameters. Table 1 shows the effectiveness of the bound (3.14) for small values $\alpha\omega$, but it is conservative for larger values, where the rate of convergence improves. TODO: As a remark, the estimation of N for large $\alpha\omega$ can be enhanced... use

x	α	μ	δ	$\alpha\omega$	N (3.21)	R_N	Bound (3.14)
1	5	1/4	1	6.25	42	$8.8 \cdot 10^{-19}$	$1.1 \cdot 10^{-18}$
1/2	1/3	1/4	1/10	0.09	236	$2.9 \cdot 10^{-17}$	$2.9 \cdot 10^{-17}$
1/3	10	1/5	1/50	1.35	1494	$2.1 \cdot 10^{-16}$	$2.2 \cdot 10^{-16}$
1	10	1/5	5	50.64	25	$1.9 \cdot 10^{-29}$	$4.0 \cdot 10^{-21}$
3	10	1/5	10	103.85	53	$2.4 \cdot 10^{-36}$	$1.4 \cdot 10^{-19}$
10	1/10	1/5	10	1	53	$3.8 \cdot 10^{-18}$	$3.9 \cdot 10^{-18}$

Table 1: Absolute error and bound (3.14) estimating N using (3.21) for the series expansion (3.7) with machine-precision absolute error.

TODO: Nevertheless, in practice, the main purpose of the estimation of N using (3.21) is to decide whether the series expansion should selected as the method of computation given a certain parameter region. For a detailed discussion, we refer to Section 4.4.

3.1.2 Convergence acceleration of the expansion $|x-\mu| \to 0$

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega}\left(\sum_{k=0}^{N-1} T_k + \sum_{k=N}^{\infty} T_k\right), \tag{3.25}$$

where

$$T_k = \frac{K_{k+1}(\alpha\omega)}{(2k+1)!!} z^k, \quad z = \frac{(x-\mu)^2\alpha}{\omega}. \tag{3.26}$$

Remainder $R_N = \sum_{k=N}^{\infty} T_k$.

Basset's integral [2, §10.32.11]

$$K_{k+1}(\alpha\omega) = \frac{\Gamma(k+3/2)}{\sqrt{\pi}}\left(\frac{2\alpha}{\omega}\right)^{k+1}\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{k+3/2}}\,dt \tag{3.27}$$

$$\begin{aligned} R_N &= \frac{1}{\sqrt{\pi}}\left(\frac{2\alpha}{\omega}\right)\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{3/2}}\left[\sum_{k=N}^{\infty}\left(\frac{2z\alpha}{\omega(t^2+\alpha^2)}\right)^k \frac{\Gamma(k+3/2)}{(2k+1)!!}\right] dt & (3.28)\\ &= C\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{N+3/2}}\frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\,dt\,. & (3.29)\end{aligned}$$

To simplify, we used

$$m = \frac{2z\alpha}{\omega} = 2\left(\frac{(x-\mu)\alpha}{\omega}\right)^2, \quad C = \frac{2}{\sqrt{\pi}}\left(\frac{2\alpha}{\omega}\right)\frac{\Gamma(N+3/2)}{(2N+1)!!} m^N \tag{3.30}$$

Expand $\cos(\omega t)$

$$R_N = C\sum_{k=0}^{\infty}\frac{(-1)^k\omega^{2k}}{(2k)!}\int_0^\infty \frac{t^{2k}}{(t^2+\alpha^2)^{N+3/2}}\frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\,dt\,. \tag{3.31}$$

Comment about convergence of the integral ($k \le N$).

$$\int_0^\infty \frac{t^{2k}}{(t^2+\alpha^2)^{N+3/2}}\frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\,dt = P_k + Q_k, \tag{3.32}$$

$$\begin{aligned} P_k &= \frac{2^{N-2}\alpha^{2(k-N-1)}\Gamma(N+1-k)\Gamma(k-1/2)}{\sqrt{\pi}(2N+1)!!}\,{}_2F_1\left(1, N+1-k; \frac{3}{2}-k, 1-\frac{m}{2\alpha^2}\right) & (3.33)\\ Q_k &= \frac{2^{N-1-k}(2\alpha^2-m)^{k-1/2}\pi\sec(k\pi)}{m^{N+1/2}} & (3.34)\end{aligned}$$

$$S_P = \frac{(2m)^N}{\alpha^{2n+1}\omega\pi}\frac{\Gamma(N+3/2)}{(2N+1)!!(2N-1)!!}\sum_{k=0}^{N}\frac{(-1)^k(\alpha\omega)^{2k}}{(2k)!}\Gamma(N+1-k)\Gamma(k-1/2)\,{}_2F_1\left(1, N+1-k; \frac{3}{2}-k, 1-\frac{m}{2\alpha^2}\right) \tag{3.35}$$

$$\begin{aligned} S_Q &= C\frac{2^{N-1}}{m^{N+1/2}\sqrt{2\alpha^2-m}}\pi\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\omega^2(2\alpha^2-m)}{2}\right)^k\sec(k\pi)\\ &= \frac{\Gamma(N+3/2)2^{N+1}}{(2N+1)!!}\frac{\alpha}{\omega}\sqrt{\frac{\pi}{m(2\alpha^2-m)}}\cosh\left(\sqrt{\frac{\omega^2(2\alpha^2-m)}{2}}\right) & (3.36)\end{aligned}$$

$$R_N = S_P + S_Q. \tag{3.37}$$

3.1.3 Expansion $|x-\mu| \to \infty$

For $x-\mu < 0$

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\Gamma(2k+1)}{2^k k!(x-\mu)^{2k+1}}\left(\frac{\omega}{\alpha}\right)^k K_k(\alpha\omega), \quad \omega = \sqrt{(x-\mu)^2+\delta^2}. \tag{3.38}$$

3.1.4 Uniform expansion $\alpha \to \infty$, $\alpha \sim \delta$ and $|x-\mu| \gg 0$

For large α, we consider the uniform asymptotic expansion in terms of modified Bessel functions described in [9] and [10, §27]. We write the Laplace-type integral (2.5) in the standard form

$$F_\lambda(z,r) = C\int_0^\infty t^{\lambda-1} e^{-z\left(t+r^2/t\right)} f(t)\,dt,$$

where C is a normalizing constant, $\lambda = -1/2$, $z = \alpha^2/2$, $r = \delta/\alpha$ and $f(t) = \Phi((x-\mu)/\sqrt{t})$. The saddle point of $e^{-z\left(t+r^2/t\right)}$ occurs at $\pm r$, but only the positive saddle point r lies inside the interval of integration. Thus, we expand $f(t)$ at the saddle point r

$$f(t) = \sum_{k=0}^{\infty} c_k(r)(t-r)^k,$$

after interchanging the order of summation and integration, we obtain

$$F_\lambda(z,r) \sim \frac{1}{z^\lambda}\sum_{k=0}^{\infty}\frac{c_k(r)Q_k(\zeta)}{z^k}, \quad z\to\infty, \tag{3.39}$$

where

$$Q_k(\zeta) = \zeta^{\lambda+k}\int_0^\infty t^{\lambda-1}(t-1)^k e^{-\zeta(t+1/t)}\,dt, \quad \zeta = rz.$$

For $f(t) = \Phi((x-\mu)/\sqrt{t})$ the coefficients at $t=r$ satisfy the recurrence in (A.14) setting $a = x-\mu$ and $b=0$. In particular, the recurrence can be simplified as follows

$$c_0(r) = \Phi\left(\frac{x-\mu}{\sqrt{r}}\right), \quad c_1(r) = -\frac{(x-\mu)}{2r^{3/2}}\phi\left(\frac{x-\mu}{\sqrt{r}}\right) \tag{3.40}$$

and

$$c_k(r) = \frac{(k-1)((x-\mu)^2 - 4r(k-2) - 3r)c_{k-1}(r) - (2(k-2)^2 + k - 2)c_{k-2}(r)}{2r^2(k-1)k}, \quad k\geq 2. \tag{3.41}$$

The functions $Q_k(\zeta)$ can be expressed as a binomial sum of modified Bessel functions, and satisfy the recurrence relation [10, §27.3.28]

$$Q_{k+2}(\zeta) = \left(k+\frac{1}{2}-2\zeta\right)Q_{k+1}(\zeta) + \zeta\left(2k+\frac{1}{2}\right)Q_k(\zeta) + k\zeta^2 Q_{k-1}(\zeta), \quad k\geq 1, \tag{3.42}$$

with initial values

$$Q_0(\zeta) = \frac{2}{\sqrt{\zeta}}K_{\frac{1}{2}}(2\zeta), \quad Q_1(\zeta) = 0, \quad Q_2(\zeta) = 2\zeta^{3/2}\left(K_{\frac{3}{2}}(2\zeta) - K_{\frac{1}{2}}(2\zeta)\right), \tag{3.43}$$

where the special case $K_{m+1/2}(x)$, $m\in\mathbb{N}$, is a terminating sum of elementary functions requiring m terms, see [Ref to Appendix]. Thus, rearranging terms we have

$$F(x;\alpha,0,\mu,\delta) = \frac{\alpha\delta e^{\delta\alpha}}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{2^k c_k\left(\frac{\delta}{\alpha}\right)Q_k\left(\frac{\alpha\delta}{2}\right)}{\alpha^{2k}}, \quad \alpha\to\infty. \tag{3.44}$$

The expansion (3.43) is a uniform expansion as $\alpha\to\infty$, uniformly with respect to $\delta/\alpha > 0$. Note that large values of δ improves the rate of convergence of the expansion, as observed taking well-know asymptotic estimates for large argument of the modified Bessel function. We remark that expansions (3.6) and (3.7) are also adequate for large values of α and δ, but unlike the present expansion, the number of terms increases significantly for $|x-\mu|\gg 0$.

3.2 Expansions: case $x = \mu$

First consider the case $x=\mu$ and $\beta = 0$. Then, the distribution is symmetric and centered at $x=\mu$, and it follows that

$$F(\mu;\alpha,0,\mu,\delta) = \frac{1}{2}. \tag{3.45}$$

For the case $\beta\neq 0$, the integral representation of this special case is given by simple substitution in (2.4)

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{-\infty}^{0}\frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}e^{\beta t}\,dt\,. \tag{3.46}$$

Using series expansion of the exponential function and interchanging the order of integration and summation in (3.45) gives that

$$F(\mu;\alpha,\beta,\mu,\delta) = 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi}\sum_{k=0}^{\infty}\frac{\beta^k}{k!}\int_0^{\infty} t^k \frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}\, dt\,.$$

The integral is expressible in closed form using [4, §6.596]

$$\int_0^{\infty} t^k \frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}\, dt = \frac{2^{\frac{k-1}{2}}\Gamma\left(\frac{k+1}{2}\right)}{\alpha^{\frac{k+1}{2}}\delta^{\frac{-k+1}{2}}} K_{\frac{k-1}{2}}(\alpha\delta),$$

and rearranging terms and using the connection formula $2^{k/2}\Gamma\left(\frac{k+1}{2}\right)/k! = \sqrt{\pi}2^{-k/2}/\Gamma\left(\frac{k}{2}+1\right)$ yields

$$F(\mu;\alpha,\beta,\mu,\delta) = 1 - \sqrt{\frac{\alpha\delta}{2\pi}} e^{\delta\gamma}\sum_{k=0}^{\infty}\frac{\beta^k}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)}\left(\frac{\delta}{\alpha}\right)^{\frac{k}{2}} K_{\frac{k-1}{2}}(\alpha\delta) \tag{3.47}$$

$$= \sqrt{\frac{\alpha\delta}{2\pi}} e^{\delta\gamma}\sum_{k=0}^{\infty}\frac{(-\beta)^k}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)}\left(\frac{\delta}{\alpha}\right)^{\frac{k}{2}} K_{\frac{k-1}{2}}(\alpha\delta). \tag{3.48}$$

The resulting expansions are convergent and the latter series expansion is preferred for $\beta < 0$ to avoid cancellation errors. A more rapidly convergent expansion for $\delta < \gamma$ or large values of δ and γ can be obtained using the integral (2.5) and expanding the term $\Phi(-\beta\sqrt{t})$. Replacing $\Phi(-\beta\sqrt{t})$ in the integral (2.5) with the expansion (3.2) and interchanging the order of integration and summation, we obtain

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(-\beta)^{2k+1}}{2^k k!(2k+1)}\int_0^{\infty} t^{k-1} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt, \tag{3.49}$$

where we can express the integral in terms of the modified Bessel function (3.5). Plugging in (3.48), now yields the alternating series

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(-\beta)^{2k+1}}{2^k k!(2k+1)}\left(\frac{\delta}{\gamma}\right)^k K_k(\gamma\delta). \tag{3.50}$$

Similarly, using (3.3) yields

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi}\sum_{k=0}^{\infty}\frac{(-\beta)^{2k+1}}{(2k+1)!!}\left(\frac{\delta}{\alpha}\right)^k K_k(\alpha\delta). \tag{3.51}$$

The latter expansion being more convenient when $\gamma \to 0$, i.e., $\alpha \sim \beta$.

3.3 Expansions: general case

3.3.1 Expansions $|x-\mu| \to 0$ (option 1)

3.3.2 Expansions $|x-\mu| \to 0$ (option 2)

The starting point is the integral representation in (2.5) after expanding the exponential

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}\int_0^{\infty}\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt,$$

and replacing $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ by the expansion in (A.9)

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{2^{k/2}(x-\mu)^k}{k!}\int_0^{\infty}\Gamma\left(\frac{k+1}{2},\frac{\beta^2}{2}t\right) t^{-3/2-k/2} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\, dt,$$

where $\omega = \sqrt{\delta^2+(x-\mu)^2}$. Consider the ascending series of the incomplete gamma function given by [2, §8.7]

$$\Gamma(a,x) = \Gamma(a) - \sum_{j=0}^{\infty}\frac{(-1)^j x^{a+j}}{j!(a+j)}. \tag{3.52}$$

We proceed splitting the inner integral into two terms

$$T_1 = \Gamma\left(\frac{k+1}{2}\right)\int_0^{\infty} t^{-3/2-k/2} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\, dt \tag{3.53}$$

$$T_2 = \sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)}\int_0^{\infty} t^{j-1} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\, dt, \tag{3.54}$$

and observe that both integrals are expressible in terms of modified Bessel function, resulting in the sums S_1 and S_2, such that $F(x;\alpha,\beta,\mu,\delta) = C(S_1 - S2)$, defined as follows

$$S_1 = \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) 2K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}} \tag{3.55}$$

$$S_2 = \sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)} 2K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j . \tag{3.56}$$

Interchanging the order of summation in S_2, we observe that the sum in k is convergent an expressible in terms of the lower incomplete gamma function $\gamma(a,x)$. Assuming $\beta > 0$

$$\sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!(\frac{k+1}{2}+j)} \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}} = -\frac{\sqrt{2}}{(x-\mu)^{2j+1}\beta^{2j}} \gamma\left(2j+1, -(x-\mu)\beta\right). \tag{3.57}$$

Thus,

$$S_2 = -2\sqrt{2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\gamma\left(2j+1, -(x-\mu)\beta\right)}{(x-\mu)^{2j+1}\beta^{2j}} \left(\frac{\beta^2}{2}\right)^j K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j$$

Rearranging terms

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\pi\sqrt{2}} \Bigg[&\sum_{k=0}^{\infty} \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}} \\ &+ \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\gamma\left(2k+1, -(x-\mu)\beta\right)}{(x-\mu)^{2k+1}} K_k(\omega\gamma) \left(\frac{\omega}{2\gamma}\right)^k \Bigg] \end{aligned} \tag{3.58}$$

If $\beta < 0$ then $F(x;\alpha,\beta,\mu,\delta) = 1 - F(-x;\alpha,-\beta,-\mu,\delta)$. The expansion is convergent for small $x-\mu$ and fixed values of the rest of parameters. Moreover, the convergence improves when $\gamma \sim \omega$, also valid for large values for these two parameters.

3.3.3 Expansion $\alpha \to \infty, \delta \to \infty$

3.3.4 Expansion $|x-\mu| \to \infty$

The expansion of $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ at $t \to 0$ is given in terms of the Bessel polynomial $y_k(x)$. To simplify notation, we take $a = x-\mu$ and $b = -\beta$. Then,

$$\Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right) = 1 + \frac{e^{-a^2/(2t)-ab-b^2/2t}}{a\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k y_k\left(\frac{1}{ab}\right) t^{\frac{1}{2}+k} \tag{3.59}$$

Using the connection of the Bessel polynomials with the modified Bessel function of the second kind $K_n(x)$ given by

$$y_n(x) = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{n+\frac{1}{2}}\left(\frac{1}{x}\right), \tag{3.60}$$

we replace in the integral, we obtain

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \delta e^{\delta\gamma} \sqrt{\frac{b}{2a\pi^3}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k K_{k+\frac{1}{2}}(ab) \int_0^{\infty} t^{k-1} e^{-\frac{\delta^2+a^2}{2t} - \frac{\gamma^2+b^2}{2} t} \, dt \tag{3.61}$$

$$\int_0^{\infty} t^{k-1} e^{-\frac{\delta^2+a^2}{2t} - \frac{\gamma^2+b^2}{2} t} \, dt = 2K_k(\omega\alpha) \left(\frac{\omega}{\alpha}\right)^k, \tag{3.62}$$

where $\omega = \sqrt{\delta^2 + (x-\mu)^2}$. Rearranging terms

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \delta e^{\delta\gamma} \sqrt{\frac{2\beta}{(\mu-x)\pi^3}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{\beta\omega}{\alpha(\mu-x)}\right)^k K_{k+\frac{1}{2}}((\mu-x)\beta) K_k(\omega\alpha) \tag{3.63}$$

3.4 Numerical integration

For cases do not covered by the described expansions, we need to resort to numerical integration. The Laplace-type integral (2.5), whose integrand includes the complementary error function, should be faster to evaluate than the Bessel integral in (2.4).

To use numerical integration methods requiring a finite interval, we truncate the integral (2.5) at some point N, such that

$$I = \int_0^N \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt + \int_N^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt,$$

and $F(x;\alpha,\beta,\mu,\delta) = CI$, where $C = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}$. The truncation error can be bounded by

$$\int_N^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt \leq \frac{e^{-\frac{\delta^2}{2N}}}{N^{3/2}} \int_N^\infty e^{-\frac{\gamma^2}{2}t}\, dt \leq \frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2}.$$

We can select N for a desired absolute tolerance ϵ via a bisection procedure or by solving using root-finding methods the equation

$$\frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C}. \tag{3.64}$$

Moreover, a slightly lesser sharper bound allows a closed-form solution of the above equation in terms of the principal branch of the Lambert W function [2, §4.13]

$$\frac{2e^{-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C} \longrightarrow N = \frac{3}{\gamma^2} W_0\left(\frac{\gamma^2}{3u}\right), \quad u = \left(\frac{\gamma^2\epsilon}{2C}\right)^{2/3}. \tag{3.65}$$

To accurately estimate N to achieve a relative tolerance, we need an estimate of the order of magnitude of I. First, we rewrite the integrand as $e^{g(t)}$, where

$$g(t) = -\frac{\delta^2}{2t} - \frac{\gamma^2}{2}t - \frac{3}{2}\log(t) + \log\left(\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)\right),$$

and

$$g'(t) = \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} - \varphi(x;\beta,\mu), \quad \varphi(x;\beta,\mu) = \frac{1}{2}\left(\frac{x-\mu}{t^{3/2}} + \frac{\beta}{\sqrt{t}}\right) \frac{\phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}{\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}$$

The saddle point t_0 and maximum contribution $e^{g(t_0)}$ of the integrand is obtained as the solution of the equation $g'(t) = 0$. Thus, N for relative tolerance can be estimated after replacing ϵ with $\epsilon e^{g(t_0)}$ in (3.64). For the case where γ and δ are both large and β and $x-\mu$ are fixed, the last term in $g'(t)$ can be neglected, obtaining the quadratic equation

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + (\gamma\delta)^2}}{\gamma^2}, \tag{3.66}$$

taking the positive internal saddle point t_0. The case where γ and δ are small requires further analysis. If $x-\mu > 0$, as $x-\mu$ increases the contribution of $\varphi(x;\beta,\mu)$ vanishes and (3.65) is valid. Contrarily, if $x-\mu < 0$ and $\beta \to 0$ (since $|\beta| < \gamma < \alpha$), $\varphi(x;\beta,\mu)$ can be approximated as follows

$$\frac{\phi\left(\frac{x-\mu}{\sqrt{t}}\right)}{\Phi\left(\frac{x-\mu}{\sqrt{t}}\right)} \approx -\frac{x-\mu}{\sqrt{t}}, \quad \varphi(x;\beta,\mu) \approx -\frac{(x-\mu)^2}{2t^2},$$

then, we have another quadratic equation

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} + \frac{(x-\mu)^2}{2t^2}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + \gamma^2\left((x-\mu)^2 + \delta^2\right)}}{\gamma^2}.$$

If $\beta < 0$, a better approximation is

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} + \frac{(x-\mu)^2}{2t^2} + \frac{\beta(x-\mu)}{2t}, \quad t_0 = \frac{h + \sqrt{h^2 + \gamma^2\left((x-\mu)^2 + \delta^2\right)}}{\gamma^2}, \quad h = \frac{\beta(x-\mu) - 3}{2}.$$

The saddle point estimates t_0 can also be used as a starting point for root-finding, however, for the purpose of approximating the order of magnitude of I, the approximations are sufficient.

- Gauss-Legendre
- Double-exponential tanh-sinh numerical integration

A double-exponential integration arises as follows. Because $|\beta| < \alpha$, we can write $\beta = \alpha\tanh(\theta)$. Substituting in (2.4) $x - \mu = \delta\sinh(\theta + u)$ we obtain

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi} \int_{-\infty}^{\tau} K_1\left(\alpha\delta\cosh(\theta+u)\right) e^{\beta\delta\sinh(\theta+u)}\, du, \tag{3.67}$$

where

$$\tau = \operatorname{arcsinh}\left(\frac{x-\mu}{\delta}\right) - \theta. \tag{3.68}$$

3.5 Inversion methods

Ideas:

- Central region
  1. The moment generating function is simple. The computation of its central moments is easy.
  2. Use multiple central moments to estimate the quantile using a Cornish-Fisher expansion.
- Tails (asymptotic methods) [10, §42]
  1. Direct application using the standard form integral representation (2.4).
- Root-finding: Halley's or Schwarzian-Newton method.

4 Algorithmic details and implementation

4.1 Handling large parameters

Exponent overflow issues in $\exp(\alpha\delta)$. Logarithmic transformation. Use scaled Bessel function.

4.2 Evaluation of Bessel-type expansions

4.2.1 Partial sums recurrence

As an example, series (3.7).

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega} S_K, \tag{4.1}$$

where S_K is the k-th partial sum. The first partial sums are

$$S_0 = 0, \quad S_1 = K_1(\alpha\omega), \quad S_2 = S_1 + \frac{K_2(\alpha\omega)z}{3}, \tag{4.2}$$

and for $k \geq 0$, the partial sums satisfy the recursion relation

$$S_{k+3} = \frac{-\alpha\omega z^2 S_k + z\left(-2(2+k)(3+2k)+\alpha\omega\right)S_{k+1} + (3+2k)\left((5+2k)\alpha\omega + 2(2+k)z\right)S_{k+2}}{(3+2k)(5+2k)\alpha\omega}, \tag{4.3}$$

where

$$z = \frac{(x-\mu)^2\alpha}{\omega}. \tag{4.4}$$

Stopping criterion is

$$\left|1 - \frac{S_K}{S_{K-1}}\right| < \epsilon. \tag{4.5}$$

4.3 Evaluation of asymptotic expansions

4.4 Implementation

5 Numerical experiments

6 Conclusions

A The function $\Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$

In this section, we present some results to be used throughout this work.

The function $F(t;a,b) = \Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$ is part of the integrand of the integral representation in (2.5). Given its relevance throughout this work, we introduce here some results that shall be used subsequently. $F(t;a,b)$ has the following integral representation [2, §7.7.6]

$$F(t;a,b) = \frac{1}{2}\operatorname{erfc}\left(-\frac{\frac{a}{\sqrt{t}} + b\sqrt{t}}{\sqrt{2}}\right) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \int_{-b/\sqrt{2}}^{\infty} e^{-(tu^2 - \sqrt{2}au)}\, du \tag{A.1}$$

A.0.1 Expansion $t \to 0$

Let us consider the case $a < 0$, since we can use the mirror property $\Phi(z) = 1 - \Phi(-z)$ otherwise. To obtain an expansion for $t \to 0$, we expand e^{-tu^2} and interchange summation and integration obtaining

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au} u^{2k}\, du\,.$$

For $a < 0$ the integral can be expressed in closed form in terms of the incomplete gamma function, $\Gamma(a,x)$

$$\int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au} u^{2k}\, du = \frac{\Gamma(2k+1,-ab)}{(\sqrt{2}a)^{2k+1}},$$

and for the special case $b = 0$, it reduces to

$$\int_{0}^{\infty} e^{\sqrt{2}au} u^{2k}\, du = \frac{\Gamma(2k+1)}{(\sqrt{2}a)^{2k+1}}.$$

Then, we obtain the series expansion valid for $t \to 0$, $a \to -\infty$ and fixed b

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} t^k}{k!} \frac{\Gamma(2k+1,ab)}{(\sqrt{2}a)^{2k+1}}. \tag{A.2}$$

Moreover, another expansion valid for large values of $a > 0$ and b can be obtained after expanding $F(t;a,b)$ at $t = 0$. The first coefficients are

$$c_0 = \frac{1}{a}, \quad c_1 = \frac{ab+1}{a^3}, \quad c_2 = \frac{a^2 b + 3ab + 3}{a^5}, \quad c_3 = \frac{a^3 b^3 + 6a^2 b^3 + 15ab + 15}{a^7} \tag{A.3}$$

and the expansion reads

$$F(t;a,b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)^2}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} c_k t^{k+\frac{1}{2}}. \tag{A.4}$$

The coefficients are expressible in terms of Bessel polynomials $y_k(x)$ [8, §A001498], and it follows that

$$F(t;a,b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)^2}}{a\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k y_k\left(\frac{1}{ab}\right) t^{k+\frac{1}{2}}, \tag{A.5}$$

where $y_k(x)$ has an explicit formula

$$y_k(x) = \sum_{m=0}^{k} \binom{k}{m} (k+1)_m \left(\frac{x}{2}\right)^m. \tag{A.6}$$

Using the connection of the Bessel polynomials with the modified Bessel function of the second kind $K_k(x)$ given by [10, §33.1.3]

$$y_k(x) = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{k+\frac{1}{2}}\left(\frac{1}{x}\right), \tag{A.7}$$

the resulting expansion is represented as a Bessel-type expansion

$$F(t;a,b) = 1 + \frac{e^{-\frac{a^2}{2t} - \frac{b^2}{2}t}}{\pi} \sqrt{\frac{b}{a}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k K_{k+\frac{1}{2}}(ab) t^{k+\frac{1}{2}}. \tag{A.8}$$

The expansion is convergent for $t < 1$. The convergence follows from the asymptotic estimate of $(b/a)^k K_k(ab) \sim (b/a)^k \sqrt{\frac{\pi}{2ab}} e^{-ab}$ as $|ab| \to \infty$. The expansion can be seen as an asymptotic expansion for large a, or as a uniform asymptotic expansion for $a \sim b$. The coefficients can be computed by using a recurrence relation for the modified Bessel function.

A.0.2 Expansion $t \to \infty$

Let us focus on the case $t \to \infty$. We can develop an asymptotic expansion after expanding the term $e^{\sqrt{2}au}$ in (A.1), which yields

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k \, du \, .$$

Considering the case $b < 0$ (again, we can use the mirror property), the integral has a closed-form

$$\int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k \, du = \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{2t^{\frac{k+1}{2}}}.$$

Thus,

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} \frac{e^{-\frac{a^2}{2t}}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}}. \tag{A.9}$$

The asymptotic behaviour of the terms in the series is

$$\frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}} \sim \left(\frac{b^2}{2}\right)^{\frac{k+1}{2}} e^{-\frac{b^2}{2}t}, \quad t \to \infty.$$

In fact this series is convergent, as can be observed taking the asymptotic estimate of $\Gamma(k,x)$ as $k \to \infty$. A simpler convergent expansion can be obtained transforming the integral in (A.1)

$$\sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \int_{-b/\sqrt{2}}^{\infty} e^{-(tu^2-\sqrt{2}au)} \, du = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}-ab-\frac{b^2}{2}t} \int_{0}^{\infty} e^{\sqrt{2}(a+bt)u} e^{-tu^2} \, dt,$$

and expanding $e^{\sqrt{2}(a+bt)u}$ obtaining

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} \frac{e^{-\frac{a^2}{2t}-ab-\frac{b^2}{2}t}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{2}(a+bt))^k}{k!} \frac{\Gamma\left(\frac{k+1}{2}\right)}{t^{\frac{k+1}{2}}}. \tag{A.10}$$

Similarly to the expansion at $t \to 0$, we can obtain an asymptotic expansion expanding $F(t;a,b)$ at $t \to \infty$. For $b > 0$, the first terms of the expansion are

$$c_0 = 1, \quad c_1 = 2 + 2ab + a^2b^2, \quad c_2 = 24 + 24ab + 12a^2b^2 + 4a^3b^3 + a^4b^4, \tag{A.11}$$

$$F(t;a,b) = 1 + \frac{e^{-ab-\frac{b^2}{2}t}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{c_k}{b^{2k+1}} \left(\frac{1}{t}\right)^{k+\frac{1}{2}}. \tag{A.12}$$

The coefficients c_k are expressible in terms of the incomplete gamma function, since

$$c_k = \sum_{j=0}^{2k} \frac{(2k)!}{j!} (ab)^j = e^{ab} \Gamma(2k+1, ab).$$

Rearranging terms, we get

$$F(t;a,b) = 1 + \frac{e^{-\frac{b^2}{2}t}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{\Gamma(2k+1, ab)}{b^{2k+1}} \left(\frac{1}{t}\right)^{k+\frac{1}{2}}. \tag{A.13}$$

A.0.3 Expansion $t \to u$

Lastly, we study the expansion of $F(t;a,b)$ at $t = u$. This expansion shall be crucial when developing various Bessel-type asymptotic expansions later on. The first coefficients of the Taylor series are

$$c_0 = \Phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_0, \quad c_1 = \phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_1, \quad c_2 = -\phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_2, \quad c_3 = \phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_3,$$

where

$$\begin{aligned}
d_0 &= 1 \\
d_1 &= \frac{-a+bu}{2u^{3/2}} \\
d_2 &= \frac{a^3 - 3au - a^2bu + bu^2 - ab^2u^2 + b^3u^3}{8u^{7/2}} \\
d_3 &= \frac{-a^5 + 10a^3u + a^4bu - 15au^2 - 6a^2bu^2 + 2a^3b^2u^2 + 3bu^3 - 6ab^2u^3 - 2a^2b^3u^3 + 2b^3u^4 - ab^4u^4 + b^5u^5}{48u^{11/2}},
\end{aligned}$$

and $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ is the probability density function of the standard normal distribution. Thus, we have

$$F(t; a, b) = \sum_{k=0}^{\infty} c_k (t-u)^k. \tag{A.14}$$

Additional terms satisfy the following recurrence

$$c_{k+4} = \frac{f_0(k)c_k + f_1(k)c_{k+1} + f_2(k)c_{k+2} + f_3(k)c_{k+3}}{f_4(k)}, \quad k \geq 0 \tag{A.15}$$

where

$$\begin{aligned}
f_0(k) &= -kb^3 \\
f_1(k) &= -(1+k)b(1+2k-ab+3b^3u) \\
f_2(k) &= (2+k)(5a+2ka+a^2b-8bu-6kbu+2ab^2u-3b^3u^2) \\
f_3(k) &= -(3+k)(a^3-11au-4kau-a^2bu+13bu^2+6kbu^2-ab^2u^2+b^3u^3) \\
f_4(k) &= 2(3+k)(4+k)u^2(-a+bu)
\end{aligned}$$

B The modified Bessel function of the second kind

Asymptotic behaviour with respect to the argument

$$K_\nu(x) \sim \frac{2^{|\nu|-1}\Gamma(|\nu|)}{x^{|\nu|}}, \quad x \to 0, \quad \nu \neq 0. \tag{B.1}$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \to \infty, \quad \nu \in \mathbb{R}. \tag{B.2}$$

Asymptotic behaviour with respect to the order

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu}, \quad \nu \to \infty, \quad x \neq 0. \tag{B.3}$$

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