On the computation and inversion of the Normal Inverse Gaussian cumulative distribution function

Guillermo Navas-Palencia

g.navas.palencia@gmail.com

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Abstract

Abstract

1 Introduction

Applications:

- Generalized Hyperbolic distribution [14]
- Variance-Gamma [12]
- NIG in Energy markets: [4]
- NIG in volatility modelling: [1]

2 Distribution properties

Variance-mean mixture distribution

$$Z \sim \mathcal{IG}(\delta\gamma, \gamma^2), \quad X \sim \mathcal{N}(\mu + \beta Z, Z), \tag{2.1}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. The domain of the parameters is

$$0 \leq |\beta| < \alpha, \quad \mu \in \mathbb{R}, \quad \delta > 0. \tag{2.2}$$

2.1 Density function

The density function is given as

$$f(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta}{\pi} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}} e^{\delta\gamma + \beta(x-\mu)} \tag{2.3}$$

Parameterization: Standard case $\mu = 0$ and $\delta = 1$. The parameters have the following interpretation: α is the tail heaviness, β is the asymmetry or skewness, μ is the location parameter and δ the scale parameter. Where μ is the location of the density, β is the skewness parameter, α measures the heaviness of the tails.

2.2 Cumulative distribution function

The cumulative distribution function is given by

$$F(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi} \int_{-\infty}^{x} \frac{K_1\left(\alpha\sqrt{\delta^2 + (t-\mu)^2}\right)}{\sqrt{\delta^2 + (t-\mu)^2}} e^{\beta(t-\mu)}\, dt \tag{2.4}$$

$$F(x; \alpha, \beta, \mu, \delta) = \frac{\delta}{\sqrt{2\pi}} \int_{0}^{\infty} \Phi\left(\frac{x - (\mu + \beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{(\delta - \gamma t)^2}{2t}}\, dt \tag{2.5}$$

Also denote

$$\tilde{F}(x; \alpha, \beta, \mu, \delta) = 1 - F(-x; \alpha, -\beta, -\mu, \delta). \tag{2.6}$$

Follows from the reversion formula of $\Phi(x)$.

Proposition 2.1 *For $x-\mu<0$, an incomplete Laplace-type integral representation in terms of modified Bessel function $K_0(x)$ is given by*

$$F(x;\alpha,\beta,\mu,\delta)=\frac{\sqrt{2}\delta e^{\delta\gamma}}{\pi}\int_{\beta/\sqrt{2}}^{\infty}e^{\sqrt{2}(x-\mu)t}K_0\left(\sqrt{2\left((x-\mu)^2+\delta^2\right)}\sqrt{\frac{\gamma^2}{2}+t^2}\right)dt, \tag{2.7}$$

and (2.6)*, otherwise.*

Proof: Consider the integral representation of the function $\Phi\left(\frac{x-\mu}{\sqrt{t}}-\beta\sqrt{t}\right)$

$$\Phi\left(\frac{x-\mu}{\sqrt{t}}-\beta\sqrt{t}\right)=\sqrt{\frac{t}{\pi}}e^{-\frac{(x-\mu)^2}{2t}}\int_{\beta/\sqrt{2}}^{\infty}e^{-(tu^2-\sqrt{2}(x-\mu)u)}\,du\,.$$

Replacing in (2.5) and interchanging the order of integration we obtain

$$F(x;\alpha,\beta,\mu,\delta)=\frac{\delta e^{\delta\gamma}}{\sqrt{2}\pi}\int_{\beta/\sqrt{2}}^{\infty}e^{\sqrt{2}(x-\mu)u}\int_0^{\infty}t^{-1}e^{-\frac{\left((x-\mu)^2+\delta^2\right)}{2t}-\left(\frac{\gamma^2}{2}+u^2\right)t}\,dt\,du\,.$$

The observation that the inner integral can be represented in terms of the modified Bessel function $K_0(x)$

$$\int_0^{\infty}t^{-1}e^{-\frac{\left((x-\mu)^2+\delta^2\right)}{2t}-\left(\frac{\gamma^2}{2}+u^2\right)t}\,dt=2K_0\left(2\sqrt{\frac{(x-\mu)^2+\delta^2}{2}}\sqrt{\frac{\gamma^2}{2}+t^2}\right).$$

□

Proposition 2.2 *The Fourier sine transform of the cumulative distribution function in terms of elementary functions is given by*

$$F(x;\alpha,\beta,\mu,\delta)=1-\frac{e^{\delta\gamma}}{\pi}\int_0^{\infty}\frac{te^{-(x-\mu)\left(\sqrt{t^2+\alpha^2}-\beta\right)}}{\sqrt{t^2+\alpha^2}\left(\sqrt{t^2+\alpha^2}-\beta\right)}\sin(\delta t)\,dt,\quad x-\mu>0, \tag{2.8}$$

and apply (2.6) *for $x-\mu<0$.*

Proof: Consider the integral representation (2.4). We use the sine transform [3, §2.4]

$$\frac{K_1(\alpha\sqrt{t^2+\delta^2}}{\sqrt{t^2+\delta^2}}=\frac{1}{\alpha\delta}\int_0^{\infty}\frac{ze^{-t\sqrt{z^2+\alpha^2}}}{\sqrt{z^2+\alpha^2}}\sin(\delta z)\,dz,$$

valid for $t\geq 0$. Replacing in (2.4) and interchanging the order of integration, we have

$$\begin{aligned}F(x;\alpha,\beta,\mu,\delta)&=1-\frac{\alpha\delta e^{\delta\gamma}}{\pi}\frac{1}{\alpha\delta}\int_0^{\infty}\frac{z\sin(\delta z)}{\sqrt{z^2+\alpha^2}}\int_{x-\mu}^{\infty}e^{-t\sqrt{z^2+\alpha^2}+\beta t}\,dt\,dz\\&=1-\frac{e^{\delta\gamma}}{\pi}\int_0^{\infty}\frac{z\sin(\delta z)}{\sqrt{z^2+\alpha^2}}\frac{e^{-(x-\mu)\left(\sqrt{z^2+\alpha^2}-\beta\right)}}{\sqrt{z^2+\alpha^2}-\beta}\,dz\end{aligned}$$

where the inner integral converges for $t\in[x-\mu,\infty)$. □

In a similar manner, we can obtain a Fourier cosine transform in terms of the exponential integral.

Proposition 2.3 *The Fourier cosine transform integral representation in terms of the exponential integral $E_1(x)$ is given by*

$$F(x;\alpha,\beta,\mu,\delta)=1-\frac{\delta e^{\delta\gamma}}{\pi}\int_0^{\infty}E_1\left((x-\mu)(\sqrt{t^2+\alpha^2}-\beta)\right)\cos(\delta t)\,dt,\quad x-\mu>0. \tag{2.9}$$

Proof: This result can be derived from the sine transform in (2.8) using integration by parts. Alternatively, we can use the cosine transform [3, §1.4]

$$\frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}=\frac{1}{\alpha t}\int_0^{\infty}e^{-t\sqrt{z^2+\alpha^2}}\cos(\delta z)\,dz\,.$$

Thus,

$$F(x;\alpha,\beta,\mu,\delta)=1-\frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{x-\mu}^{\infty}\frac{e^{\beta t}}{\alpha t}\int_0^{\infty}e^{-t\sqrt{z^2+\alpha^2}}\cos(\delta z)\,dz\,dt,$$

where the inner integral is expressible in closed form in terms of the exponential integral $E_1(x)$

$$\int_{x-\mu}^{\infty}\frac{1}{t}e^{-(\sqrt{z^2+\alpha^2}-\beta)t}\,dt=E_1\left((x-\mu)(\sqrt{z^2+\alpha^2}-\beta)\right).$$

□

2.3 Moments and cumulants

$$\mathbb{E}[X^m] = \frac{\alpha\delta}{\pi} \int_{-\infty}^{\infty} t^m \frac{K_1\left(\alpha\sqrt{\delta^2+(t-\mu)^2}\right)}{\sqrt{\delta^2+(t-\mu)^2}} e^{\delta\gamma+\beta(t-\mu)}\, dt\,. \tag{2.10}$$

$$\mathbb{E}[X^m] = \frac{\alpha\delta}{\pi} e^{\delta\gamma-\beta\mu} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \int_{-\infty}^{\infty} (t-\mu)^{m+k} \frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}\, dt\,. \tag{2.11}$$

Use binomial theorem, and compute coefficients recursively (binomial sum of Bessel functions). Treat special case $\mu = 0$ and $\beta = 0$.

3 Methods of computation

In this Section, we describe the methods used for efficient computation of the CDF for the general case and various special cases. Subsequently, we discuss several approaches for computing the inverse of the CDF.

3.1 Expansions: case $\beta = 0$

The case $\beta = 0$ corresponds to the symmetric case. The symmetric NIG distribution has been widely used in financial modelling, for example for pricing collateralized default obligations (CDO) [9].

We emphasize that many of the techniques and numerical methods introduced in this Section shall also be used to develop expansions of other special cases and the general case in Section 3.3.

3.1.1 Expansions $|x-\mu| \to 0$

For developing a series expansion for the case $|x-\mu| \to 0$, we start from the integral representation in (2.5), after expanding the square

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\sqrt{2\pi}} \int_0^{\infty} \Phi\left(\frac{x-\mu}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\alpha^2}{2}t}\, dt\,. \tag{3.1}$$

We proceed expanding the term $\Phi\left(\frac{x-\mu}{\sqrt{t}}\right)$, by using the two well-known absolutely convergent series expansions of $\Phi(x)$ [10, §2]

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k k!(2k+1)}, \tag{3.2}$$

and

$$\Phi(x) = \frac{1}{2} + \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!!}. \tag{3.3}$$

If we choose the expansion (3.2) and interchange the order of integration and summation, the resulting integral has the form

$$F(x;\alpha,0,\mu,\delta) = \frac{\delta e^{\delta\alpha}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k (x-\mu)^{2k+1}}{2^k k!(2k+1)} \int_0^{\infty} t^{-k-2} e^{-\frac{\delta^2}{2t}-\frac{\alpha^2}{2}t}\, dt, \tag{3.4}$$

where the integral has a closed-form in terms of the modified Bessel function

$$\int_0^{\infty} t^{\lambda-1} e^{-a/t-zt}\, dt = 2\left(\frac{\alpha}{z}\right)^{\lambda/2} K_\lambda(2\sqrt{\alpha z}). \tag{3.5}$$

Inserting (3.5) in (3.4) and rearranging terms, we obtain the the alternating series

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (x-\mu)^{2k+1}}{2^k k!(2k+1)} \left(\frac{\alpha}{\delta}\right)^{k+1} K_{k+1}(\alpha\delta). \tag{3.6}$$

Moreover, to obtain a similar series with positive terms, we choose the expansion of $\Phi(x)$ in (3.3), yielding

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \sum_{k=0}^{\infty} \frac{(x-\mu)^{2k+1}}{(2k+1)!!} \left(\frac{\alpha}{\omega}\right)^{k+1} K_{k+1}(\alpha\omega), \quad \omega = \sqrt{\delta^2+(x-\mu)^2}. \tag{3.7}$$

Note that the series expansions (3.6) and (3.7) can be written as a truncated series with the corresponding remainder term. For example, truncating the series (3.7) at N, we can write

$$\sum_{k=0}^{\infty} T_k = \sum_{k=0}^{N-1} T_k + \sum_{k=N}^{\infty} T_k, \quad T_k = \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^k \frac{K_{k+1}(\alpha\omega)}{(2k+1)!!}. \tag{3.8}$$

Thus, one has the series with remainder term $R_N = \sum_{k=N}^{\infty} T_k$

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega}\left(\sum_{k=0}^{N-1} T_k + R_N\right). \tag{3.9}$$

For a rigorous evaluation of error bounds given a number of terms N, it is convenient to calculate an upper bound for R_N in (3.8). We can, for example, bound R_N by comparison with a geometric series

$$|R_N| \le \frac{|T_N|}{1-C}, \quad C = \left|\frac{T_{N+1}}{T_N}\right| \tag{3.10}$$

iff $C < 1$, where T_N is the first omitted term in the expansion. The following lemma provides an upper bound for $K_{\nu+1}(x)$, required for a posterior derivation of an upper bound for the term T_N.

Lemma 3.1 *For $x \ge 0$ and $\nu \ge -\frac{1}{2}$ we have*

$$K_{\nu+1}(x) < \frac{\Gamma(\nu+1)2^{\nu}}{x^{\nu+1}}. \tag{3.11}$$

Proof: The proof reduces to combining the uniform bound in [6]

$$xK_{\nu+1}(x)I_{\nu}(x) \le 1. \tag{3.12}$$

with the lower bound [11]

$$\left(\frac{x}{2}\right)^{\nu}\frac{1}{\Gamma(\nu+1)} < I_{\nu}(x). \tag{3.13}$$

Then, it follows that

$$K_{\nu+1}(x) \le \frac{1}{xI_{\nu}(x)} < \frac{\Gamma(\nu+1)2^{\nu}}{x^{\nu+1}}. \tag{3.14}$$

□

Theorem 3.2 *Given $\alpha > 0$, $\omega > 0$, $x-\mu \in \mathbb{R}$ and $N \in \mathbb{N}$, the remainder term in (3.9) satisfies*

$$R_N \le \frac{|T_N|}{1-C}, \tag{3.15}$$

where

$$T_N < \frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.16}$$

and

$$C < \left(\frac{x-\mu}{\omega}\right)^2 \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{2N+3}. \tag{3.17}$$

Proof: The use of Lemma 3.1 gives

$$\begin{aligned} T_N &= \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^N \frac{K_{N+1}(\alpha\omega)}{(2N+1)!!} \\ &< \left(\frac{(x-\mu)^2\alpha}{\omega}\right)^N \frac{\Gamma(N+1)2^N}{(\alpha\omega)^{N+1}(2N+1)!!} \\ &= \frac{1}{\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\frac{(N!2^N)^2}{(2N+1)!}. \end{aligned}$$

An upper bound for the ratio of factorials in the previous inequality is given by

$$\frac{(N!2^N)^2}{(2N+1)!} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(N+1)}{\Gamma(N+3/2)} \le \frac{1}{2}\sqrt{\frac{\pi}{N+1/2}},$$

where we use the fact that $\Gamma(N+3/2) = (N+1/2)\Gamma(N+1/2)$ and the upper bound of the ratio of gamma functions [19]

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \le (x+s)^{1-s}, \quad s \in (0,1). \tag{3.18}$$

Thus, the following bound for T_N holds

$$T_N < \frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.19}$$

For the ratio C, an explicit formula in terms of the ratio of modified Bessel functions is

$$C = \frac{T_{N+1}}{T_N} = \frac{(x-\mu)^2\alpha}{\omega(2N+3)}\frac{K_{N+2}(\alpha\omega)}{K_{N+1}(\alpha\omega)}. \tag{3.20}$$

The ratio can be bounded using a sharp bound for the ratio of modified Bessel functions [15], yielding

$$\frac{K_{N+2}(\alpha\omega)}{K_{N+1}(\alpha\omega)} < \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{\alpha\omega}. \tag{3.21}$$

Then, we have

$$C < \left(\frac{x-\mu}{\omega}\right)^2 \frac{N+3/2+\sqrt{(N+3/2)^2+(\alpha\omega)^2}}{2N+3}.$$

□

To study the regime of applicability for the expansion, we can estimate the required number of terms N equating the bound of T_N in (3.16) times the normalizing factor with the requested absolute error ϵ and solving for N, which gives

$$N \approx -\frac{\Re(W_{-1}(D))}{4\log(A)} + \frac{1}{2}, \quad A = \frac{x-\mu}{\omega}, \quad B = \frac{A\delta e^{\delta\alpha}}{2\omega\pi}, \quad C = \frac{\epsilon^2}{B^2\pi}, \quad D = -\frac{4\log(A)}{CA^2} \tag{3.22}$$

where $W_k(x)$ denotes the Lambert W function [5, §4.13]. The branch $k=-1$ is used to obtain the maximum real N. Note that since $A<1$ by definition, $D>0$. When CA^2 is tiny, $W_{-1}(D)$ can be approximated as $\Re(W_{-1}(D)) \sim \log(D) - \log(\log(D))$ using the first two terms of the asymptotic expansion in [5, §4.13.10].

The previous analysis performed on the expansion (3.7) is repeated for the alternating expansion (3.6). The main results are summarized below for the purpose of brevity. The expansion (3.6) rewritten including the remainder term follows

$$F(x;\alpha,0,\mu,\delta) = \frac{1}{2} + \frac{(x-\mu)\alpha e^{\delta\alpha}}{\pi}\left(\sum_{k=0}^{N-1} T_k + R_N\right), \quad T_k = \left(-\frac{(x-\mu)^2\alpha}{\delta}\right)^k \frac{K_{k+1}(\alpha\delta)}{2^k k!(2k+1)}. \tag{3.23}$$

The last omitted term T_N satisfies[1]

$$|T_N| < \frac{1}{\alpha\delta}\left(\frac{x-\mu}{\delta}\right)^{2N} \frac{1}{2N+1}, \tag{3.24}$$

and the number of terms N can be determined employing the Lambert W function for a given error ϵ

$$N \approx \frac{\Re(W_{-1}(D)) + \log(A)}{2\log(A)}, \quad A = \frac{x-\mu}{\delta}, \quad B = \frac{Ae^{\delta\alpha}}{\pi}, \quad C = \frac{\epsilon}{B}, \quad D = -\frac{\log(A)}{CA}. \tag{3.25}$$

Now we are in position to compare both series and their respective domains of applicability. A first important observation is that the alternating series (3.6) does not converge when $|x-\mu|>\delta$, and the number of terms N increases rapidly when $|x-\mu|\sim\delta$. In contrast, the series (3.7) is absolutely convergent. The convergence of the latter can be reliably assessed applying to T_k in (3.8) the asymptotic estimates of $K_{k+1}(\alpha\omega)$ for $\alpha\omega\to 0$ and $\alpha\omega\to\infty$, (B.1) and (B.2), respectively:

$$T_k \sim \frac{\sqrt{\pi}}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2k} \frac{\Gamma(k+1)}{\Gamma(k+3/2)}, \qquad \alpha\omega\to 0,$$

$$T_k \sim \sqrt{\frac{\pi}{2\alpha\omega}}\left(\frac{(x-\mu)^2\alpha}{\omega}\right)^k \frac{e^{-\alpha\omega}}{(2k+1)!!}, \qquad \alpha\omega\to\infty.$$

The asymptotic estimates of T_k show that the series expansion (3.7) is slowly convergent when

$$\left(\frac{x-\mu}{\omega}\right)^2 \to 1 \Longleftrightarrow \delta\to 0,$$

and the ratio of convergence improves when $\delta\to\infty$, then it can be viewed as an asymptotic expansion for large δ. For $k\to\infty$, T_k follows the asymptotic behaviour

$$T_k \sim \frac{1}{\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2k}\sqrt{\frac{\pi}{e}}\frac{1}{\sqrt{4k+2}}\left(\frac{2k+2}{2k+1}\right)^{k+\frac{1}{2}}, \quad k\to\infty,$$

where we use the asymptotic estimate of modified Bessel function for large order (B.3) and apply Stirling's approximation for the double factorial. It remains to analyze the accuracy of the bound in (3.15) for different parameters. Table 1 shows the effectiveness of the bound (3.15) after estimating N to achieve machine-precision absolute error using (3.22). For small values of $\alpha\omega$ the bound is sharp, but it is conservative for larger values, precisely where the rate of convergence improves. As a remark, the estimation of N for large $\alpha\omega$ can be enhanced by selecting N using the asymptotic estimate for $\alpha\omega\to\infty$ via binary search. Although, the bound might overestimate N for some parameters, the main purpose of the estimation of N using (3.22) is to decide whether the series expansion should be selected as the method of computation given a certain

[1] The corresponding upper bound for C can be computed straightforwardly following the procedure described in Theorem 3.2.

parameter region. Table 1 also shows that for small δ and $\alpha\omega$, the required number of terms makes the series expansion impractical. In the next section, we present a convergence acceleration method to obtain a rapidly convergent series for theses cases.

x	α	μ	δ	$\alpha\omega$	N (3.22)	R_N	Bound (3.15)
1	5	1/4	1	6.25	42	$8.8 \cdot 10^{-19}$	$1.1 \cdot 10^{-18}$
1/2	1/3	1/4	1/10	0.09	236	$2.9 \cdot 10^{-17}$	$2.9 \cdot 10^{-17}$
1/3	10	1/5	1/50	1.35	1,494	$2.1 \cdot 10^{-16}$	$2.2 \cdot 10^{-16}$
1	10	1/5	5	50.64	25	$1.9 \cdot 10^{-29}$	$4.0 \cdot 10^{-21}$
3	10	1/5	10	103.85	53	$2.4 \cdot 10^{-36}$	$1.4 \cdot 10^{-19}$
10	1/10	1/5	10	1	53	$3.8 \cdot 10^{-18}$	$3.9 \cdot 10^{-18}$

Table 1: The remainder of the series expansion (3.7) and bound (3.15), estimating N using (3.22) to achieve machine precision.

3.1.2 Convergence acceleration of the expansion $|x - \mu| \to 0$

Table 1 showed that for small values of δ and $\alpha\omega$ the required number of terms grows considerably, thereby, resorting to numerical integration (see Section 3.4) might be a more efficient approach. Alternatively, a common technique to reduce the number of terms of slowly convergent series is to use series acceleration methods (Shank's transformation and Levin-type transformations among others). Attempts to use Shank's transformation were unsuccessful; we did not observe a significant reduction in N while achieving only around ten correct digits systematically for all cases. In addition, a major drawback is that Shank's acceleration requires higher-precision arithmetic to compensate for cancellation effects, discarding them for any implementation in double-precision arithmetic.

In the following, we consider the use of exponentially improved asymptotic expansions. When the information about the remainder is available, this technique consists of re-expanding the remainder, obtaining an expansion exhibiting faster convergence. For further details, we refer to [13, §14].

Consider the convergent series expansion in (3.8)

$$F(x; \alpha, 0, \mu, \delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \frac{(x-\mu)\alpha}{\omega} \left(\sum_{k=0}^{N-1} T_k + \sum_{k=N}^{\infty} T_k \right), \tag{3.26}$$

where

$$T_k = \frac{K_{k+1}(\alpha\omega)}{(2k+1)!!} z^k, \quad z = \frac{(x-\mu)^2 \alpha}{\omega}. \tag{3.27}$$

and remainder $R_N = \sum_{k=N}^{\infty} T_k$. An integral representation of R_N can be obtained using Basset's integral [5, §10.32.11] representation of the modified Bessel function

$$K_{k+1}(\alpha\omega) = \frac{\Gamma(k+3/2)}{\sqrt{\pi}} \left(\frac{2\alpha}{\omega}\right)^{k+1} \int_0^{\infty} \frac{\cos(\omega t)}{(t^2+\alpha^2)^{k+3/2}} \, dt\,. \tag{3.28}$$

Basset's integral is chosen to split the problematic term $\alpha\omega$. Thus, it follows that inserting (3.28) into the remainder in (3.8), we obtain an integral representation of R_N

$$\begin{aligned} R_N &= \frac{1}{\sqrt{\pi}} \left(\frac{2\alpha}{\omega}\right) \int_0^{\infty} \frac{\cos(\omega t)}{(t^2+\alpha^2)^{3/2}} \left[\sum_{k=N}^{\infty} \left(\frac{2z\alpha}{\omega(t^2+\alpha^2)} \right)^k \frac{\Gamma(k+3/2)}{(2k+1)!!} \right] dt \\ &= C \int_0^{\infty} \frac{\cos(\omega t)}{(t^2+\alpha^2)^{N+3/2}} \frac{1}{\left(2 - \frac{m}{t^2+\alpha^2}\right)} \, dt\,. \end{aligned} \tag{3.29}$$

where, for the purpose of brevity, we use

$$m = \frac{2z\alpha}{\omega} = 2\left(\frac{(x-\mu)\alpha}{\omega}\right)^2, \quad C = \frac{2}{\sqrt{\pi}} \left(\frac{2\alpha}{\omega}\right) \frac{\Gamma(N+3/2)}{(2N+1)!!} m^N. \tag{3.30}$$

Now, we expand the term $\cos(\omega t)$, recall that $\omega = \sqrt{\delta^2 + (x-\mu)^2}$, yielding

$$R_N = C \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k}}{(2k)!} \int_0^{\infty} \frac{t^{2k}}{(t^2+\alpha^2)^{N+3/2}} \frac{1}{\left(2 - \frac{m}{t^2+\alpha^2}\right)} \, dt, \tag{3.31}$$

and note that the integrals above converge for $k \leq N$. Next, with the assistance of Mathematica [20], we obtain a closed-form expression for the previous integral in terms of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$

$$\int_0^\infty \frac{t^{2k}}{(t^2+\alpha^2)^{N+3/2}} \frac{1}{\left(2 - \frac{m}{t^2+\alpha^2}\right)} \, dt = P_k + Q_k, \tag{3.32}$$

with

$$P_k = \frac{2^{N-2}\alpha^{2(k-N-1)}\Gamma(N+1-k)\Gamma(k-1/2)}{\sqrt{\pi}(2N+1)!!} \, {}_2F_1\left(1, N+1-k; \frac{3}{2}-k, 1-\frac{m}{2\alpha^2}\right), \tag{3.33}$$

$$Q_k = \frac{2^{N-1-k}(2\alpha^2-m)^{k-1/2}\pi\sec(k\pi)}{m^{N+1/2}}. \tag{3.34}$$

The Gauss hypergeometric function is defined by the hypergeometric series

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol or rising factorial. The series is defined on the disk $|z| < 1$, and by analytic continuation with respect to z elsewhere. Substituting (3.32) in (3.31), we write R_N as the sum of two series $R_N = S_P + S_Q$ given by

$$S_P = \frac{(2m)^N}{\alpha^{2N+1}\omega\pi} \frac{\Gamma(N+3/2)}{(2N+1)!!(2N-1)!!} \sum_{k=0}^{N} \frac{(-1)^k(\alpha\omega)^{2k}}{(2k)!} \Gamma(N+1-k)\Gamma(k-1/2) \, {}_2F_1\left(1, N+1-k; \frac{3}{2}-k, 1-\frac{m}{2\alpha^2}\right) \tag{3.35}$$

and

$$\begin{aligned} S_Q &= C\frac{2^{N-1}}{m^{N+1/2}\sqrt{2\alpha^2-m}}\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\omega^2(2\alpha^2-m)}{2}\right)^k \sec(k\pi) \\ &= \frac{\Gamma(N+3/2)2^{N+1}}{(2N+1)!!} \frac{\alpha}{\omega} \sqrt{\frac{\pi}{m(2\alpha^2-m)}} \cosh\left(\sqrt{\frac{\omega^2(2\alpha^2-m)}{2}}\right). \end{aligned} \tag{3.36}$$

The sum S_P is terminating at $k = N$ due to the term $\Gamma(N+1-k)$ in the series. Thus, the resulting series to compute $F(x; \alpha, 0, \mu, \delta)$ is

$$F(x; \alpha, 0, \mu, \delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi} \frac{(x-\mu)\alpha}{\omega} \left(\sum_{k=0}^{N-1} T_k + S_P + S_Q\right). \tag{3.37}$$

Subsequently, we focus on the determination of the optimal N for a desired precision ϵ. The smallest term in S_P occurs when $k = N$

$$\frac{(-1)^N(\alpha\omega)^{2N}}{(2N)!}\Gamma(N-1/2) \, {}_2F_1\left(1, 1; \frac{3}{2}-N, 1-\frac{m}{2\alpha^2}\right). \tag{3.38}$$

Moreover, taking the argument of ${}_2F_1$ in (3.35)

$$1 - \frac{m}{2\alpha^2} = 1 - \left(\frac{x-\mu}{\omega}\right)^2 < 1, \tag{3.39}$$

we see that for $\alpha\omega \to 0$, $\delta \to 0$, the argument is close to 1. Therefore, for the parameter regime of interest, the last term can be effectively approximated taking the limit $\lim_{x\to 0} {}_2F_1\left(1, 1, \frac{3}{2}-k, x\right) = 1$. Then, it remains to solve the following equation for N

$$\frac{(\sqrt{2m}\omega)^{2N}}{\alpha\omega\pi} \frac{\Gamma(N-1/2)\Gamma(N+3/2)}{(2N)!(2N+1)!!(2N-1)!!} = \epsilon. \tag{3.40}$$

We use Stirling approximation of the ratio of gamma functions and double factorials

$$\frac{\Gamma(N-1/2)\Gamma(N+3/2)}{(2N)!(2N+1)!!(2N-1)!!} \sim \frac{\sqrt{\pi}}{4}\left(\frac{e}{N}\right)^{2N} 2^{-4N}, \quad N \to \infty. \tag{3.41}$$

obtaining a simplified equation

$$(\sqrt{2m}\omega)^{2N}\left(\frac{e}{N}\right)^{2N} 2^{-4N} = 4\sqrt{\pi}\alpha\omega\epsilon. \tag{3.42}$$

The solution of the last equation permits a closed-form in terms of principal branch of the Lambert W-function

$$N \approx -\frac{\log(A)}{W\left(-\frac{\log(A)}{2e\sqrt{2m\omega}}\right)}, \quad A = 4\sqrt{\pi}\alpha\omega\epsilon. \tag{3.43}$$

Table 2 shows the estimated number of terms of the accelerated convergent series (3.37), N_{acc}, and the actual number of terms N^* to achieve machine-precision for small values of $\alpha\omega$ and δ. The first and more notable observation is the reduction in the number of terms compared with convergent series (3.22), especially for $\alpha \leq 1$. The second observation is the accuracy of the estimate in N_{acc} in (3.22), which only seems to slightly underestimate N for large $\alpha\omega$.

x	α	μ	δ	$\alpha\omega$	N (3.22)	N_{acc} (3.43)	error	N^*	error
1	50	1/5	1/3	43.33	324	69	$2.9 \cdot 10^{-13}$	73	$1.1 \cdot 10^{-17}$
2	5	1/5	1/10	9.01	10,242	25	$4.6 \cdot 10^{-21}$	22	$1.1 \cdot 10^{-16}$
1	1/10	1/5	1/100	0.08	179,715	5	$1.3 \cdot 10^{-21}$	4	$2.4 \cdot 10^{-17}$
5	1	1/5	1/100	4.8	5,645,686	18	$1.1 \cdot 10^{-22}$	15	$1.5 \cdot 10^{-17}$
20	1/100	1/5	1/100	0.198	84,914,922	6	$1.1 \cdot 10^{-22}$	5	$4.5 \cdot 10^{-19}$

Table 2: Absolute error and bound (3.15) estimating N using (3.22) for the series expansion (3.7) with machine-precision absolute error.

Remark 3.3 *A simpler bound for R_N, compared to* (3.15)*, can be obtained from the integral representation in* (3.29) *as follows*

$$\begin{aligned} R_N &= C\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{N+3/2}} \frac{1}{\left(2-\frac{m}{t^2+\alpha^2}\right)}\,dt \\ &\leq \frac{C}{\left(2-\frac{m}{\alpha^2}\right)}\int_0^\infty \frac{\cos(\omega t)}{(t^2+\alpha^2)^{N+3/2}}\,dt \\ &= \frac{1}{1-\left(\frac{x-\mu}{\omega}\right)^2} T_N \end{aligned}$$

using the upper bound for T_N in (3.16)*, we obtain*

$$R_N < \frac{1}{1-\left(\frac{x-\mu}{\omega}\right)^2}\frac{1}{2\alpha\omega}\left(\frac{x-\mu}{\omega}\right)^{2N}\sqrt{\frac{\pi}{N+1/2}}. \tag{3.44}$$

3.1.3 Expansion $|x-\mu| \to \infty$

We use the expansion derived in (A.2) as a starting point, taking $a = x - \mu$ and $b = 0$. Similar to the derivation of the expansion in (3.6), we interchange summation and integration, and the resulting integral is expressible in closed-form as a modified Bessel function, see Equation (3.5). Thus, for $x - \mu < 0$ we have the alternating asymptotic expansion

$$\begin{aligned} F(x;\alpha,0,\mu,\delta) &= \frac{\delta e^{\delta\alpha}}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!}\frac{\Gamma(2k+1)}{2^k(x-\mu)^{2k+1}}\int_0^\infty t^{k-1}e^{-\frac{\omega^2}{2t}-\frac{\alpha^2}{2}t}\,dt \\ &= -\frac{\delta e^{\delta\alpha}}{\pi(x-\mu)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(2k+1)}{k!}\left(\frac{\omega}{2(x-\mu)^2\alpha}\right)^k K_k(\alpha\omega), \end{aligned} \tag{3.45}$$

where $\omega = \sqrt{(x-\mu)^2+\delta^2}$. Note that the remainder is bounded in magnitude by the first neglected term. Moreover, the convergence of the expansion improves for large α and $\alpha/\omega > 1$. Finally, for $x - \mu > 0$ we use the reflection formula (2.6).

3.1.4 Uniform expansion $\alpha \to \infty$, $\alpha \sim \delta$ and $|x-\mu| \gg 0$

For large α, we consider the uniform asymptotic expansion in terms of modified Bessel functions described in [17] and [18, §27]. We write the Laplace-type integral (2.5) in the standard form

$$F_\lambda(z,r) = C\int_0^\infty t^{\lambda-1}e^{-z\left(t+r^2/t\right)}f(t)\,dt,$$

where C is a normalizing constant, $\lambda = -1/2$, $z = \alpha^2/2$, $r = \delta/\alpha$ and $f(t) = \Phi((x-\mu)/\sqrt{t})$. The saddle point of $e^{-z\left(t+r^2/t\right)}$ occurs at $\pm r$, but only the positive saddle point r lies inside the interval of integration.

Thus, we expand $f(t)$ at the saddle point r

$$f(t) = \sum_{k=0}^{\infty} c_k(r)(t-r)^k,$$

after interchanging the order of summation and integration, we obtain

$$F_\lambda(z,r) \sim \frac{1}{z^\lambda} \sum_{k=0}^{\infty} \frac{c_k(r) Q_k(\zeta)}{z^k}, \quad z \to \infty, \tag{3.46}$$

where

$$Q_k(\zeta) = \zeta^{\lambda+k} \int_0^\infty t^{\lambda-1}(t-1)^k e^{-\zeta(t+1/t)}\, dt, \quad \zeta = rz.$$

For $f(t) = \Phi((x-\mu)/\sqrt{t})$ the coefficients at $t = r$ satisfy the recurrence in (A.14) setting $a = x - \mu$ and $b = 0$. In particular, the recurrence can be simplified as follows

$$c_0(r) = \Phi\left(\frac{x-\mu}{\sqrt{r}}\right), \quad c_1(r) = -\frac{(x-\mu)}{2r^{3/2}} \phi\left(\frac{x-\mu}{\sqrt{r}}\right) \tag{3.47}$$

and

$$c_k(r) = \frac{(k-1)((x-\mu)^2 - 4r(k-2) - 3r)c_{k-1}(r) - (2(k-2)^2 + k - 2)c_{k-2}(r)}{2r^2(k-1)k}, \quad k \geq 2. \tag{3.48}$$

The functions $Q_k(\zeta)$ can be expressed as a binomial sum of modified Bessel functions, and satisfy the recurrence relation [18, §27.3.28]

$$Q_{k+2}(\zeta) = \left(k + \frac{1}{2} - 2\zeta\right) Q_{k+1}(\zeta) + \zeta\left(2k + \frac{1}{2}\right) Q_k(\zeta) + k\zeta^2 Q_{k-1}(\zeta), \quad k \geq 1, \tag{3.49}$$

with initial values

$$Q_0(\zeta) = \frac{2}{\sqrt{\zeta}} K_{\frac{1}{2}}(2\zeta), \quad Q_1(\zeta) = 0, \quad Q_2(\zeta) = 2\zeta^{3/2}\left(K_{\frac{3}{2}}(2\zeta) - K_{\frac{1}{2}}(2\zeta)\right), \tag{3.50}$$

where the special case $K_{n+1/2}(z)$, $n \in \mathbb{N}$, is a terminating sum of elementary functions requiring n terms, see (B.4). In particular the cases $n = 0$ and $n = 1$ are

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{z}, \quad K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}(z+1)}{z^{3/2}}.$$

Thus, rearranging terms we have

$$F(x;\alpha,0,\mu,\delta) = \frac{\alpha\delta e^{\delta\alpha}}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k c_k\left(\frac{\delta}{\alpha}\right) Q_k\left(\frac{\alpha\delta}{2}\right)}{\alpha^{2k}}, \quad \alpha \to \infty. \tag{3.51}$$

The expansion (3.51) is a uniform expansion as $\alpha \to \infty$, uniformly with respect to $\delta/\alpha > 0$. Note that large values of δ improves the rate of convergence of the expansion, as observed taking well-know asymptotic estimates for large argument of the modified Bessel function. We remark that expansions (3.6) and (3.7) are also adequate for large values of α and δ, but unlike the present expansion, the number of terms increases significantly for $|x - \mu| \gg 0$.

3.2 Expansions: case $x = \mu$

First consider the case $x = \mu$ and $\beta = 0$. Then, the distribution is symmetric and centered at $x = \mu$, and it follows that

$$F(\mu;\alpha,0,\mu,\delta) = \frac{1}{2}. \tag{3.52}$$

For the case $\beta \neq 0$, the integral representation of this special case is given by simple substitution in (2.4)

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi} \int_{-\infty}^{0} \frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}} e^{\beta t}\, dt\,. \tag{3.53}$$

Using series expansion of the exponential function and interchanging the order of integration and summation in (3.53) gives that

$$F(\mu;\alpha,\beta,\mu,\delta) = 1 - \frac{\alpha\delta e^{\delta\gamma}}{\pi} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \int_0^\infty t^k \frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}\, dt\,.$$

The integral is expressible in closed form using [7, §6.596]

$$\int_0^\infty t^k \frac{K_1\left(\alpha\sqrt{\delta^2+t^2}\right)}{\sqrt{\delta^2+t^2}}\,dt = \frac{2^{\frac{k-1}{2}}\Gamma\left(\frac{k+1}{2}\right)}{\alpha^{\frac{k+1}{2}}\delta^{\frac{-k+1}{2}}} K_{\frac{k-1}{2}}(\alpha\delta),$$

and rearranging terms and using the connection formula $2^{k/2}\Gamma\left(\frac{k+1}{2}\right)/k! = \sqrt{\pi}2^{-k/2}/\Gamma\left(\frac{k}{2}+1\right)$ yields

$$F(\mu;\alpha,\beta,\mu,\delta) = 1 - \sqrt{\frac{\alpha\delta}{2\pi}}e^{\delta\gamma}\sum_{k=0}^\infty \frac{\beta^k}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)}\left(\frac{\delta}{\alpha}\right)^{\frac{k}{2}} K_{\frac{k-1}{2}}(\alpha\delta) \tag{3.54}$$

$$= \sqrt{\frac{\alpha\delta}{2\pi}}e^{\delta\gamma}\sum_{k=0}^\infty \frac{(-\beta)^k}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)}\left(\frac{\delta}{\alpha}\right)^{\frac{k}{2}} K_{\frac{k-1}{2}}(\alpha\delta). \tag{3.55}$$

The resulting expansions are convergent and the latter series expansion is preferred for $\beta < 0$ to avoid cancellation errors. A more rapidly convergent expansion for $\delta < \gamma$ or large values of δ and γ can be obtained using the integral (2.5) and expanding the term $\Phi(-\beta\sqrt{t})$. Replacing $\Phi(-\beta\sqrt{t})$ in the integral (2.5) with the expansion (3.2) and interchanging the order of integration and summation, we obtain

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{2\pi}\sum_{k=0}^\infty \frac{(-1)^k(-\beta)^{2k+1}}{2^k k!(2k+1)}\int_0^\infty t^{k-1}e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\,dt, \tag{3.56}$$

where we can express the integral in terms of the modified Bessel function (3.5). Plugging in (3.56), now yields the alternating series

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi}\sum_{k=0}^\infty \frac{(-1)^k(-\beta)^{2k+1}}{2^k k!(2k+1)}\left(\frac{\delta}{\gamma}\right)^k K_k(\gamma\delta). \tag{3.57}$$

Similarly, using (3.3) yields

$$F(\mu;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{\pi}\sum_{k=0}^\infty \frac{(-\beta)^{2k+1}}{(2k+1)!!}\left(\frac{\delta}{\alpha}\right)^k K_k(\alpha\delta). \tag{3.58}$$

The latter expansion being more convenient since $\alpha > \gamma$.

3.3 Expansions: general case

3.3.1 Expansions $|x-\mu| \to 0$ (option 1)

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}\int_0^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)t^{-3/2}e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\,dt,$$

Using expansion (3.3)

$$F(x;\alpha,\beta,\mu,\delta) = \frac{1}{2} + \frac{\delta e^{\delta\gamma}}{2\pi}\sum_{k=0}^\infty \frac{1}{(2k+1)!!}\int_0^\infty \left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)^{2k+1} t^{-3/2}e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}e^{-\frac{(x-(\mu+\beta t))^2}{2t}}\,dt$$

$$= \frac{1}{2} + \frac{\delta e^{\delta\gamma+(x-\mu)\beta}}{2\pi}\sum_{k=0}^\infty \frac{(-\beta)^{2k+1}}{(2k+1)!!}\int_0^\infty \left(t-\frac{x-\mu}{\beta}\right)^{2k+1} t^{-k-2}e^{-\frac{\omega^2}{2t}-\frac{\alpha^2}{2}t}\,dt \tag{3.59}$$

Denote the integral I_k as

$$I_k = \int_0^\infty (t-t_+)^{2k+1}t^{-k-2}e^{-z(t+\beta^2/t)}\,dt, \tag{3.60}$$

with $t_+ = \frac{x-\mu}{\beta}$, $z = \alpha^2/2$ and $\beta^2 = \omega^2/\alpha^2$. The integral I_k can be expressed in terms of the modified Bessel expansion after applying the binomial expansion

$$I_k = 2\sum_{j=0}^{2k+1}\binom{2k+1}{j}(-t_+)^{2k+1-j}\beta^{j-k-1}K_{k+1-j}(2z\beta). \tag{3.61}$$

Show the first two terms I_1 and I_2. Theorem with recursion.

3.3.2 Expansions $|x-\mu| \to 0$ (option 2)

The starting point is the integral representation in (2.5) after expanding the exponential

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \int_0^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\,dt,$$

and replacing $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ by the expansion in (A.9)

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{2^{k/2}(x-\mu)^k}{k!} \int_0^\infty \Gamma\left(\frac{k+1}{2}, \frac{\beta^2}{2}t\right) t^{-3/2-k/2} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\,dt,$$

where $\omega = \sqrt{\delta^2+(x-\mu)^2}$. Consider the ascending series of the incomplete gamma function given by [5, §8.7]

$$\Gamma(a,x) = \Gamma(a) - \sum_{j=0}^\infty \frac{(-1)^j x^{a+j}}{j!(a+j)}. \tag{3.62}$$

We proceed splitting the inner integral into two terms

$$T_1 = \Gamma\left(\frac{k+1}{2}\right) \int_0^\infty t^{-3/2-k/2} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\,dt \tag{3.63}$$

$$T_2 = \sum_{j=0}^\infty \frac{(-1)^j \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)} \int_0^\infty t^{j-1} e^{-\frac{\omega^2}{2t}-\frac{\gamma^2}{2}t}\,dt, \tag{3.64}$$

and observe that both integrals are expressible in terms of modified Bessel function, resulting in the sums S_1 and S_2, such that $F(x;\alpha,\beta,\mu,\delta) = C(S_1-S2)$, defined as follows

$$S_1 = \sum_{k=0}^\infty \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) 2K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}} \tag{3.65}$$

$$S_2 = \sum_{k=0}^\infty \frac{2^{k/2}(x-\mu)^k}{k!} \sum_{j=0}^\infty \frac{(-1)^j \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}+j}}{j!(\frac{k+1}{2}+j)} 2K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j. \tag{3.66}$$

Interchanging the order of summation in S_2, we observe that the sum in k is convergent an expressible in terms of the lower incomplete gamma function $\gamma(a,x)$. Assuming $\beta > 0$

$$\sum_{k=0}^\infty \frac{2^{k/2}(x-\mu)^k}{k!(\frac{k+1}{2}+j)} \left(\frac{\beta^2}{2}\right)^{\frac{k+1}{2}} = -\frac{\sqrt{2}}{(x-\mu)^{2j+1}\beta^{2j}} \gamma\left(2j+1, -(x-\mu)\beta\right). \tag{3.67}$$

Thus,

$$S_2 = -2\sqrt{2} \sum_{j=0}^\infty \frac{(-1)^j}{j!} \frac{\gamma\left(2j+1,-(x-\mu)\beta\right)}{(x-\mu)^{2j+1}\beta^{2j}} \left(\frac{\beta^2}{2}\right)^j K_j(\omega\gamma) \left(\frac{\omega}{\gamma}\right)^j$$

Rearranging terms

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{\pi\sqrt{2}} \Bigg[&\sum_{k=0}^\infty \frac{2^{k/2}(x-\mu)^k}{k!} \Gamma\left(\frac{k+1}{2}\right) K_{\frac{k+1}{2}}(\omega\gamma) \left(\frac{\gamma}{\omega}\right)^{\frac{k+1}{2}} \\ &+ \sqrt{2} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{\gamma\left(2k+1,-(x-\mu)\beta\right)}{(x-\mu)^{2k+1}} K_k(\omega\gamma) \left(\frac{\omega}{2\gamma}\right)^k \Bigg] \end{aligned} \tag{3.68}$$

If $\beta < 0$ then $F(x;\alpha,\beta,\mu,\delta) = 1 - F(-x;\alpha,-\beta,-\mu,\delta)$. The expansion is convergent for small $x-\mu$ and fixed values of the rest of parameters. Moreover, the convergence improves when $\gamma \sim \omega$, also valid for large values for these two parameters.

3.3.3 Expansions $|x-\mu| \to 0$ (option 3)

3.3.4 Expansion $\beta \to 0$

$$F(x;\alpha,\beta,\mu,\delta) = F(x;\gamma,0,\mu,\delta) + \frac{\delta e^{\delta\gamma}}{2\pi} \sum_{k=0}^\infty \frac{(-1)^k(-\beta)^{k+1}}{(k+1)!2^{k/2}} \int_0^\infty H_k\left(\frac{x-\mu}{\sqrt{2t}}\right) t^{k/2-1} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\,dt\,. \tag{3.69}$$

3.3.5 Uniform asymptotic $\gamma \to \infty, \gamma \sim \delta$ and $|x-\mu| \gg 0$

For large γ, we employ the uniform asymptotic described in Section 3.1.4. The only difference is the calculation of the coefficients $c_k(r)$ using the recurrence in (A.14) with $a = x - \mu$ and $b = -\beta$. The resulting uniform asymptotic expansion reads

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\gamma\delta e^{\delta\gamma}}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k c_k\left(\frac{\delta}{\gamma}\right) Q_k\left(\frac{\gamma\delta}{2}\right)}{\gamma^{2k}}, \quad \gamma \to \infty. \tag{3.70}$$

3.3.6 Expansion $|x-\mu| \to \infty$

The expansion of $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ at $t \to 0$ is given in terms of the Bessel polynomial $y_k(x)$. To simplify notation, we take $a = x - \mu$ and $b = -\beta$. Then,

$$\Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right) = 1 + \frac{e^{-a^2/(2t)-ab-b^2/2t}}{a\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k y_k\left(\frac{1}{ab}\right) t^{\frac{1}{2}+k} \tag{3.71}$$

Using the connection of the Bessel polynomials with the modified Bessel function of the second kind $K_n(x)$ given by

$$y_n(x) = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{n+\frac{1}{2}}\left(\frac{1}{x}\right), \tag{3.72}$$

we replace in the integral, we obtain

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \delta e^{\delta\gamma} \sqrt{\frac{b}{2a\pi^3}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k K_{k+\frac{1}{2}}(ab) \int_0^{\infty} t^{k-1} e^{-\frac{\delta^2+a^2}{2t} - \frac{\gamma^2+b^2}{2}t} \, dt \tag{3.73}$$

$$\int_0^{\infty} t^{k-1} e^{-\frac{\delta^2+a^2}{2t} - \frac{\gamma^2+b^2}{2}t} \, dt = 2K_k(\omega\alpha)\left(\frac{\omega}{\alpha}\right)^k, \tag{3.74}$$

where $\omega = \sqrt{\delta^2 + (x-\mu)^2}$. Rearranging terms

$$F(x;\alpha,\beta,\mu,\delta) = 1 + \delta e^{\delta\gamma} \sqrt{\frac{2\beta}{(\mu - x)\pi^3}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{\beta\omega}{\alpha(\mu - x)}\right)^k K_{k+\frac{1}{2}}((\mu - x)\beta) K_k(\omega\alpha) \tag{3.75}$$

3.3.7 Expansion $\delta \to \infty$

Starting from the integral representation in (2.5) and replacing the $\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)$ by the asymptotic expansion for $t \to \infty$ in (A.13) in terms of the incomplete gamma function. Taking $a = x - \mu$ and $b = -\beta$, we obtain

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\delta e^{\delta\gamma}}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{\Gamma(2k+1, -(x-\mu)\beta)}{(-\beta)^{2k+1}} \int_0^{\infty} t^{-k-2} e^{-\frac{\delta^2}{2t} - \frac{\gamma^2+\beta^2}{2}t} \, dt. \tag{3.76}$$

The integral is expressible in terms of modified Bessel function

$$\int_0^{\infty} t^{-k-2} e^{-\frac{\delta^2}{2t} - \frac{\gamma^2+\beta^2}{2}t} \, dt = 2\left(\frac{\alpha}{\delta}\right)^{k+1} K_{k+1}(\alpha\delta), \tag{3.77}$$

where we use the fact that $\gamma^2 + \beta^2 = \alpha^2$. After rearranging terms, we have for $\beta > 0$

$$\begin{aligned}
F(x;\alpha,\beta,\mu,\delta) &= \frac{\alpha e^{\delta\gamma}}{\pi\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma(2k+1, -(x-\mu)\beta) \left(\frac{\alpha}{2\beta^2\delta}\right)^k K_{k+1}(\alpha\delta) && (3.78)\\
&= \frac{\alpha e^{\delta\gamma}}{\pi\beta} \sum_{k=0}^{\infty} Q(2k+1, -(x-\mu)\beta) \frac{\Gamma(2k+1)}{\Gamma(k+1)} \left(-\frac{\alpha}{2\beta^2\delta}\right)^k K_{k+1}(\alpha\delta) && (3.79)\\
&= \frac{\alpha e^{\delta\gamma}}{\pi^{3/2}\beta} \sum_{k=0}^{\infty} Q(2k+1, -(x-\mu)\beta) \Gamma(k+1/2) \left(-\frac{2\alpha}{\beta^2\delta}\right)^k K_{k+1}(\alpha\delta), && (3.80)
\end{aligned}$$

and $1 + F(x;\alpha,\beta,\mu,\delta)$ for $\beta < 0$. We rewrite the asymptotic series in terms of the regularized incomplete gamma function $Q(a,x)$, see appendix C. The evaluation of the asymptotic series using the regularized incomplete gamma is more numerically stable, as described in Section 4.3.

3.4 Numerical integration

For cases do not covered by the described expansions, we need to resort to numerical integration. The Laplace-type integral (2.5), whose integrand includes the complementary error function, should be faster to evaluate than the Bessel integral in (2.4).

To use numerical integration methods requiring a finite interval, we truncate the integral (2.5) at some point N, such that

$$I = \int_0^N \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt + \int_N^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt,$$

and $F(x;\alpha,\beta,\mu,\delta) = CI$, where $C = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}$. The truncation error can be bounded by

$$\int_N^\infty \Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right) t^{-3/2} e^{-\frac{\delta^2}{2t}-\frac{\gamma^2}{2}t}\, dt \leq \frac{e^{-\frac{\delta^2}{2N}}}{N^{3/2}} \int_N^\infty e^{-\frac{\gamma^2}{2}t}\, dt \leq \frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2}.$$

We can select N for a desired absolute tolerance ϵ via a bisection procedure or by solving using root-finding methods the equation

$$\frac{2e^{-\frac{\delta^2}{2N}-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C}. \tag{3.81}$$

Moreover, a slightly lesser sharper bound allows a closed-form solution of the above equation in terms of the principal branch of the Lambert W function [5, §4.13]

$$\frac{2e^{-\frac{\gamma^2}{2}N}}{N^{3/2}\gamma^2} = \frac{\epsilon}{C} \longrightarrow N = \frac{3}{\gamma^2} W_0\left(\frac{\gamma^2}{3u}\right), \quad u = \left(\frac{\gamma^2\epsilon}{2C}\right)^{2/3}. \tag{3.82}$$

To accurately estimate N to achieve a relative tolerance, we need an estimate of the order of magnitude of I. First, we rewrite the integrand as $e^{g(t)}$, where

$$g(t) = -\frac{\delta^2}{2t} - \frac{\gamma^2}{2}t - \frac{3}{2}\log(t) + \log\left(\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)\right),$$

and

$$g'(t) = \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} - \varphi(x;\beta,\mu), \quad \varphi(x;\beta,\mu) = \frac{1}{2}\left(\frac{x-\mu}{t^{3/2}} + \frac{\beta}{\sqrt{t}}\right) \frac{\phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}{\Phi\left(\frac{x-(\mu+\beta t)}{\sqrt{t}}\right)}$$

The saddle point t_0 and maximum contribution $e^{g(t_0)}$ of the integrand is obtained as the solution of the equation $g'(t) = 0$. Thus, N for relative tolerance can be estimated after replacing ϵ with $\epsilon e^{g(t_0)}$ in (3.82). For the case where γ and δ are both large and β and $x-\mu$ are fixed, the last term in $g'(t)$ can be neglected, obtaining the quadratic equation

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + (\gamma\delta)^2}}{\gamma^2}, \tag{3.83}$$

taking the positive internal saddle point t_0. The case where γ and δ are small requires further analysis. If $x-\mu > 0$, as $x-\mu$ increases the contribution of $\varphi(x;\beta,\mu)$ vanishes and (3.83) is valid. Contrarily, if $x-\mu < 0$ and $\beta \to 0$ (since $|\beta| < \gamma < \alpha$), $\varphi(x;\beta,\mu)$ can be approximated as follows

$$\frac{\phi\left(\frac{x-\mu}{\sqrt{t}}\right)}{\Phi\left(\frac{x-\mu}{\sqrt{t}}\right)} \approx -\frac{x-\mu}{\sqrt{t}}, \quad \varphi(x;\beta,\mu) \approx -\frac{(x-\mu)^2}{2t^2},$$

then, we have another quadratic equation

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} + \frac{(x-\mu)^2}{2t^2}, \quad t_0 = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} + \gamma^2\left((x-\mu)^2 + \delta^2\right)}}{\gamma^2}.$$

If $\beta < 0$, a better approximation is

$$g'(t) \approx \frac{\delta^2}{2t^2} - \frac{\gamma^2}{2} - \frac{3}{2t} + \frac{(x-\mu)^2}{2t^2} + \frac{\beta(x-\mu)}{2t}, \quad t_0 = \frac{h + \sqrt{h^2 + \gamma^2\left((x-\mu)^2 + \delta^2\right)}}{\gamma^2}, \quad h = \frac{\beta(x-\mu) - 3}{2}.$$

The saddle point estimates t_0 can also be used as a starting point for root-finding, however, for the purpose of approximating the order of magnitude of I, the approximations are sufficient.

3.5 Double-exponential quadrature

Upper bound $W_0(x)$, for $x > 1$ given in [8].

$$W_0(x) \leq \log(x)^{\frac{\log(x)}{1+\log(x)}}. \tag{3.84}$$

Approximation $W_{-1}(z)$ with maximum relative error of $3.5e{-}3$ presented in [2]

$$W_{-1}(z) \approx \log(-z) - \frac{2}{a}\left(1 - \left(1 + a\sqrt{-\frac{1+\log(-z)}{2}}\right)^{-1}\right), \quad a = 0.3205. \tag{3.85}$$

TODO: describe algorithm and reference. Use of approximation for Lambert W function and references.

A double-exponential integration arises as follows. Because $|\beta| < \alpha$, we can write $\beta = \alpha\tanh(\theta)$. Substituting in (2.4) $x - \mu = \delta\sinh(\theta + u)$ we obtain

$$F(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{-\infty}^{\tau} K_1\left(\alpha\delta\cosh(\theta + u)\right) e^{\beta\delta\sinh(\theta+u)}\, du, \tag{3.86}$$

where

$$\tau = \operatorname{arcsinh}\left(\frac{x-\mu}{\delta}\right) - \theta. \tag{3.87}$$

3.5.1 Gaussian quadrature and double-exponential quadrature

- Gauss-Legendre
- Double-exponential tanh-sinh numerical integration

Mention the numerical integration method implemented in other libraries where the NIG distribution is available.

3.6 Inversion methods

Ideas:

- Central region
  1. The moment generating function is simple. The computation of its central moments is easy.
  2. Use multiple central moments to estimate the quantile using a Cornish-Fisher expansion.
- Tails (asymptotic methods) [18, §42]
  1. Direct application using the standard form integral representation (2.4).
- Root-finding: Halley's or Schwarzian-Newton method.

4 Algorithmic details and implementation

4.1 Handling large parameters

Exponent overflow issues in $\exp(\alpha\delta)$. Logarithmic transformation. Use scaled Bessel function.

4.2 Evaluation of Bessel-type expansions

Recurrence relation in terms of ration of modified Bessel functions. More numerically stable.

4.2.1 Partial sums recurrence

As an example, series (3.7). It is worth noticing that performing a naive computation of the terms T_k in double-precision arithmetic for large N poses underflow and overflow problems since the numerator (denominator) rapidly goes to infinity (zero) as N increases. In Section 4.2, we discuss alternative summation methods to avoid cancellation issues and precision loss for large N.

$$F(x; \alpha, 0, \mu, \delta) = \frac{1}{2} + \frac{\delta e^{\delta\alpha}}{\pi}\frac{(x-\mu)\alpha}{\omega} S_K, \tag{4.1}$$

where S_K is the k-th partial sum. The first partial sums are

$$S_0 = 0, \quad S_1 = K_1(\alpha\omega), \quad S_2 = S_1 + \frac{K_2(\alpha\omega)z}{3}, \tag{4.2}$$

and for $k \geq 0$, the partial sums satisfy the recursion relation

$$S_{k+3} = \frac{-\alpha\omega z^2 S_k + z\left(-2(2+k)(3+2k)+\alpha\omega\right)S_{k+1} + (3+2k)\left((5+2k)\alpha\omega + 2(2+k)z\right)S_{k+2}}{(3+2k)(5+2k)\alpha\omega}, \tag{4.3}$$

where

$$z = \frac{(x-\mu)^2\alpha}{\omega}. \tag{4.4}$$

Stopping criterion is

$$\left|1 - \frac{S_K}{S_{K-1}}\right| < \epsilon. \tag{4.5}$$

- Computation of 2F1, use partial sums recursion with Mathematica.

4.3 Evaluation of asymptotic expansions

4.4 Implementation

4.4.1 Case $\beta = 0$

4.4.2 Case $x = \mu$

4.4.3 General case

5 Numerical experiments

6 Conclusions

A The function $\Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$

In this section, we present some results to be used throughout this work.

The function $F(t;a,b) = \Phi\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right)$ is part of the integrand of the integral representation in (2.5). Given its relevance throughout this work, we introduce here some results that shall be used subsequently. $F(t;a,b)$ has the following integral representation [5, §7.7.6]

$$F(t;a,b) = \frac{1}{2}\operatorname{erfc}\left(-\frac{\frac{a}{\sqrt{t}} + b\sqrt{t}}{\sqrt{2}}\right) = \sqrt{\frac{t}{\pi}}e^{-\frac{a^2}{2t}}\int_{-b/\sqrt{2}}^{\infty} e^{-(tu^2-\sqrt{2}au)}\,du \tag{A.1}$$

A.0.1 Expansion $t \to 0$

Let us consider the case $a < 0$, since we can use the mirror property $\Phi(z) = 1 - \Phi(-z)$ otherwise. To obtain an expansion for $t \to 0$, we expand e^{-tu^2} and interchange summation and integration obtaining

$$F(t;a,b) = \sqrt{\frac{t}{\pi}}e^{-\frac{a^2}{2t}}\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au}u^{2k}\,du\,.$$

For $a < 0$ the integral can be expressed in closed form in terms of the incomplete gamma function, $\Gamma(a,x)$

$$\int_{-b/\sqrt{2}}^{\infty} e^{\sqrt{2}au}u^{2k}\,du = \frac{\Gamma(2k+1,-ab)}{(\sqrt{2}a)^{2k+1}},$$

and for the special case $b = 0$, it reduces to

$$\int_{0}^{\infty} e^{\sqrt{2}au}u^{2k}\,du = \frac{\Gamma(2k+1)}{(\sqrt{2}a)^{2k+1}}.$$

Then, we obtain the series expansion valid for $t \to 0$, $a \to -\infty$ and fixed b

$$F(t;a,b) = \sqrt{\frac{t}{\pi}}e^{-\frac{a^2}{2t}}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}t^k}{k!}\frac{\Gamma(2k+1,ab)}{(\sqrt{2}a)^{2k+1}}. \tag{A.2}$$

NOTE: same expansion as $\operatorname{erfc}(x), x \to \infty$.

Moreover, another expansion valid for large values of $a > 0$ and b can be obtained after expanding $F(t;a,b)$ at $t = 0$. The first coefficients are

$$c_0 = \frac{1}{a}, \quad c_1 = \frac{ab+1}{a^3}, \quad c_2 = \frac{a^2b+3ab+3}{a^5}, \quad c_3 = \frac{a^3b^3+6a^2b^3+15ab+15}{a^7} \tag{A.3}$$

and the expansion reads

$$F(t;a,b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}}+b\sqrt{t}\right)^2}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} c_k t^{k+\frac{1}{2}}. \tag{A.4}$$

The coefficients are expressible in terms of Bessel polynomials $y_k(x)$ [16, §A001498], and it follows that

$$F(t;a,b) = 1 + \frac{e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}}+b\sqrt{t}\right)^2}}{a\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k y_k\left(\frac{1}{ab}\right) t^{k+\frac{1}{2}}, \tag{A.5}$$

where $y_k(x)$ has an explicit formula

$$y_k(x) = \sum_{m=0}^{k} \binom{k}{m} (k+1)_m \left(\frac{x}{2}\right)^m. \tag{A.6}$$

Using the connection of the Bessel polynomials with the modified Bessel function of the second kind $K_k(x)$ given by [18, §33.1.3]

$$y_k(x) = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{k+\frac{1}{2}}\left(\frac{1}{x}\right), \tag{A.7}$$

the resulting expansion is represented as a Bessel-type expansion

$$F(t;a,b) = 1 + \frac{e^{-\frac{a^2}{2t}-\frac{b^2}{2}t}}{\pi} \sqrt{\frac{b}{a}} \sum_{k=0}^{\infty} (-1)^{k+1} \left(\frac{b}{a}\right)^k K_{k+\frac{1}{2}}(ab) t^{k+\frac{1}{2}}. \tag{A.8}$$

The expansion is convergent for $t < 1$. The convergence follows from the asymptotic estimate of $(b/a)^k K_k(ab) \sim (b/a)^k \sqrt{\frac{\pi}{2ab}} e^{-ab}$ as $|ab| \to \infty$. The expansion can be seen as an asymptotic expansion for large a, or as a uniform asymptotic expansion for $a \sim b$. The coefficients can be computed by using a recurrence relation for the modified Bessel function.

A.0.2 Expansion $t \to \infty$

Let us focus on the case $t \to \infty$. We can develop an asymptotic expansion after expanding the term $e^{\sqrt{2}au}$ in (A.1), which yields

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k \, du\,.$$

Considering the case $b < 0$ (again, we can use the mirror property), the integral has a closed-form

$$\int_{-b/\sqrt{2}}^{\infty} e^{-tu^2} u^k \, du = \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{2t^{\frac{k+1}{2}}}.$$

Thus,

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} \frac{e^{-\frac{a^2}{2t}}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{2}a)^k}{k!} \frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}}. \tag{A.9}$$

The asymptotic behaviour of the terms in the series is

$$\frac{\Gamma\left(\frac{k+1}{2}, \frac{b^2}{2}t\right)}{t^{\frac{k+1}{2}}} \sim \left(\frac{b^2}{2}\right)^{\frac{k+1}{2}} e^{-\frac{b^2}{2}t}, \quad t \to \infty.$$

In fact this series is convergent, as can be observed taking the asymptotic estimate of $\Gamma(k,x)$ as $k \to \infty$. A simpler convergent expansion can be obtained transforming the integral in (A.1)

$$\sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}} \int_{-b/\sqrt{2}}^{\infty} e^{-(tu^2-\sqrt{2}au)} \, du = \sqrt{\frac{t}{\pi}} e^{-\frac{a^2}{2t}-ab-\frac{b^2}{2}t} \int_0^{\infty} e^{\sqrt{2}(a+bt)u} e^{-tu^2} \, dt,$$

and expanding $e^{\sqrt{2}(a+bt)u}$ obtaining

$$F(t;a,b) = \sqrt{\frac{t}{\pi}} \frac{e^{-\frac{a^2}{2t}-ab-\frac{b^2}{2}t}}{2} \sum_{k=0}^{\infty} \frac{(\sqrt{2}(a+bt))^k}{k!} \frac{\Gamma\left(\frac{k+1}{2}\right)}{t^{\frac{k+1}{2}}}. \tag{A.10}$$

Similarly to the expansion at $t \to 0$, we can obtain an asymptotic expansion expanding $F(t;a,b)$ at $t \to \infty$. For $b > 0$, the first terms of the expansion are

$$c_0 = 1, \quad c_1 = 2 + 2ab + a^2b^2, \quad c_2 = 24 + 24ab + 12a^2b^2 + 4a^3b^3 + a^4b^4, \tag{A.11}$$

$$F(t;a,b) = 1 + \frac{e^{-ab-\frac{b^2}{2}t}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{c_k}{b^{2k+1}} \left(\frac{1}{t}\right)^{k+\frac{1}{2}}. \tag{A.12}$$

The coefficients c_k are expressible in terms of the incomplete gamma function, since

$$c_k = \sum_{j=0}^{2k} \frac{(2k)!}{j!}(ab)^j = e^{ab}\Gamma(2k+1, ab).$$

Rearranging terms, we get

$$F(t;a,b) = 1 + \frac{e^{-\frac{b^2}{2}t}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^k k!} \frac{\Gamma(2k+1,ab)}{b^{2k+1}} \left(\frac{1}{t}\right)^{k+\frac{1}{2}}. \tag{A.13}$$

A.0.3 Expansion $t \to u$

Lastly, we study the expansion of $F(t;a,b)$ at $t = u$. This expansion shall be crucial when developing various Bessel-type asymptotic expansions later on. The first coefficients of the Taylor series are

$$c_0 = \Phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_0, \quad c_1 = \phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_1, \quad c_2 = -\phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_2, \quad c_3 = \phi\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) d_3,$$

where

$$\begin{aligned}
d_0 &= 1 \\
d_1 &= \frac{-a+bu}{2u^{3/2}} \\
d_2 &= \frac{a^3 - 3au - a^2bu + bu^2 - ab^2u^2 + b^3u^3}{8u^{7/2}} \\
d_3 &= \frac{-a^5 + 10a^3u + a^4bu - 15au^2 - 6a^2bu^2 + 2a^3b^2u^2 + 3bu^3 - 6ab^2u^3 - 2a^2b^3u^3 + 2b^3u^4 - ab^4u^4 + b^5u^5}{48u^{11/2}},
\end{aligned}$$

and $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ is the probability density function of the standard normal distribution. Thus, we have

$$F(t;a,b) = \sum_{k=0}^{\infty} c_k (t-u)^k. \tag{A.14}$$

Additional terms satisfy the following recurrence

$$c_{k+4} = \frac{f_0(k)c_k + f_1(k)c_{k+1} + f_2(k)c_{k+2} + f_3(k)c_{k+3}}{f_4(k)}, \quad k \geq 0 \tag{A.15}$$

where

$$\begin{aligned}
f_0(k) &= -kb^3 \\
f_1(k) &= -(1+k)b(1+2k-ab+3b^3u) \\
f_2(k) &= (2+k)(5a+2ka+a^2b-8bu-6kbu+2ab^2u-3b^3u^2) \\
f_3(k) &= -(3+k)(a^3-11au-4kau-a^2bu+13bu^2+6kbu^2-ab^2u^2+b^3u^3) \\
f_4(k) &= 2(3+k)(4+k)u^2(-a+bu)
\end{aligned}$$

B The modified Bessel function of the second kind

Asymptotic behaviour with respect to the argument

$$K_\nu(x) \sim \frac{2^{|\nu|-1}\Gamma(|\nu|)}{x^{|\nu|}}, \quad x \to 0, \quad \nu \neq 0. \tag{B.1}$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}, \quad x \to \infty, \quad \nu \in \mathbb{R}. \tag{B.2}$$

Asymptotic behaviour with respect to the order

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}, \quad \nu \to \infty, \quad x \neq 0. \tag{B.3}$$

$$K_{n+1/2,z} = \sqrt{\frac{\pi}{2z}} \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!}(2z)^{-j}e^{-z}, \quad z \in \mathcal{C}. \tag{B.4}$$

C The incomplete gamma functions

$$Q(a,x) = \frac{\Gamma(a,x)}{\Gamma(a)} \tag{C.1}$$

$$P(a,x) = \frac{\gamma(a,x)}{\Gamma(a)} \tag{C.2}$$

$$\Gamma(a,x) + \gamma(a,x) = \Gamma(a) \tag{C.3}$$

$$P(a,x) + Q(a,x) = 1 \tag{C.4}$$

- Series for integer a
- Recursions

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