Globally Optimal Hierarchical Reinforcement Learning for Linearly-Solvable Markov Decision Processes

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Overview

Introduction

Introduction

- Hierachical Reinforcement Learning aims to make learning more efficient by exploiting of large problems (Wen et al., 2020).
- Novel approach to HRL using Linearly-solvable Markov Decision Processes (LMDPs) by combining the value function for subtasks induced by partitions on the state space.
- LMDPs are a computationally efficient way to model sequential decision problems (Todorov, 2006).
- Our method presented here retrieves the globally optimal value function (Dietterich, 2000).

Background

Background - LMDPs (i)

An LMDP (Kappen et al., 2012; Todorov, 2006) is as a tuple $\mathcal{L} = \langle \mathcal{S}, \mathcal{T}, \mathcal{P}, \mathcal{R}, \mathcal{J} \rangle$, where:

- $\mathcal{S}$ is a discrete set of non-terminal states.
- $\mathcal{T}$ is a set of terminal states.
- We use $\mathcal{S}^+ = \mathcal{S} \cup \mathcal{T}$ to denote the full set of states.
- $\mathcal{P} : \mathcal{S} \to \Delta(\mathcal{S}^+)$ is an uncontrolled transition function.
- $\mathcal{R} : \mathcal{S} \to \mathbb{R}$ is a reward function for non-terminal states, assumed to be non-negative.
- $\mathcal{J} : \mathcal{T} \to \mathbb{R}$ is a reward function for terminal states, assumed to be non-negative.

The learning agent follows a policy $\pi : \mathcal{S} \to \Delta(\mathcal{S}^+)$, which is a conditional probability distribution over next states $\pi(\cdot|s_t)$.

Background - LMDPs (ii)

- At time-step t, the agent observes s_t and receives a reward

$$\mathcal{R}(s_t, \pi) = \mathcal{R}(s_t) - \lambda \cdot \mathrm{KL}(\pi(\cdot|s_t) \| \mathcal{P}(\cdot|s_t)).$$

- The aim of the agent is to maximize

$$v^\pi(s) = \mathbb{E}\left[\sum_{t=1}^{T-1} \mathcal{R}(S_t, \pi) + \mathcal{J}(S_T) \,\middle|\, S_1 = s\right].$$

- We obtain the following Bellman optimality equation

$$\begin{aligned}\frac{1}{\lambda} v(s) &= \frac{1}{\lambda} \max_\pi \left[\mathcal{R}(s, \pi) + \mathbb{E}_{s' \sim \pi(\cdot|s)} v(s')\right] \\ &= \frac{1}{\lambda} \mathcal{R}(s) + \max_\pi \mathbb{E}_{s' \sim \pi(\cdot|s)} \left[\frac{1}{\lambda} v(s') - \log \frac{\pi(s'|s)}{\mathcal{P}(s'|s)}\right] \quad (\forall s).\end{aligned}$$

Background - LMDPs (iii) - Eigenvector setting

- Taking the exponential transformation $z(s) = e^{v(s)/\lambda}$ for each $s \in \mathcal{S}^+$ leads to Bellman equations that are linear in z

$$z(s) = e^{\mathcal{R}(s)/\lambda} \sum_{s'} \mathcal{P}(s'|s) z(s').$$

- If $\mathcal{P}$ and $\mathcal{R}$ are known, this problem can be formulated as an eigenvector problem

$$\mathbf{z} = RP\mathbf{z}^+ \text{ where } R = \mathrm{diag}(e^{\mathcal{R}(\cdot)/\lambda})$$

where initially $\mathbf{z} = \mathbf{1}$.

Background - LMDPs (iii) - Online setting

- Alternatively, there exists an online (corrected) update rule

$$\hat{z}(s_t) \leftarrow (1-\alpha_t)\hat{z}(s_t) + \alpha_t e^{r_t/\lambda}\hat{z}(s_{t+1})\frac{\mathcal{P}(s_{t+1}|s_t)}{\hat{\pi}(s_{t+1}|s_t)}.$$

- Given a z, policies are derived using

$$\pi(s'|s) = \frac{\mathcal{P}(s'|s)z(s')}{\sum_{s''}\mathcal{P}(s''|s)z(s'')}.$$

- This process is equivalent to a probabilistic inference task (Kappen et al., 2012).

Background - LMDPs (iv) - Compositionality

- Let $\{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$ be a collection of LMDPs.
- Each LMDP $\mathcal{L}_i$ only differs in the reward structure of each terminal state $\mathsf{t} \in \mathcal{T}$ (i.e. $\mathcal{J}_i(\mathsf{t})$).
- Let us consider a new LMDP $\mathcal{L}$ whose (exponential) reward structure at terminal states can be expressed as follows (Todorov, 2009)

$$e^{\mathcal{J}(\mathsf{t})/\lambda} = z(\mathsf{t}) = \sum_{k=1}^{n} w_k e^{\mathcal{J}_k(\mathsf{t})/\lambda}.$$

- Since the Bellman equation is linear in z, then the optimal value function of any $s \in \mathcal{S}$ satisfies the same equation above

$$z(s) = \sum_{k=1}^{n} w_k z_k(s)$$

Hierarchical LMDPs

Hierarchical LMDPs (i)

- Hierarchical decomposition inspired by the work of Wen et al. (Wen et al., 2020).
- Given an LMDP $\mathcal{L}$, its state space $\mathcal{S}$ is partitioned into L subsets $\{\mathcal{S}_i\}_{i=1}^{L}$.
- Each such subset $\mathcal{S}_i$ induces a subtask, represented by an LMDP $\mathcal{L}_i = \langle \mathcal{S}_i, \mathcal{T}_i, \mathcal{P}_i, \mathcal{R}_i, \mathcal{J}_i \rangle$:
  1. The set of non-terminal states is $\mathcal{S}_i$.
  2. The set of terminal states $\mathcal{T}_i$ includes all states in $\mathcal{S}^+ \setminus \mathcal{S}_i$ that are reachable in one step from a state in $\mathcal{S}_i$.
  3. $\mathcal{P}_i : \mathcal{S}_i \to \mathcal{S}_i^+$ and $\mathcal{R}_i : \mathcal{S}_i \to \mathbb{R}$ are restrictions of $\mathcal{P}$ and $\mathcal{R}$ to $\mathcal{S}_i$.
  4. Reward at $\tau \in \mathcal{T}_i$ equals $\mathcal{J}_i(\tau) = \mathcal{J}(\tau)$ if $\tau \in \mathcal{T}$, and $\mathcal{J}_i(\tau) = \hat{v}(\tau)$ otherwise, where $\hat{v}(\tau)$ is the estimated value in $\mathcal{L}$ of the non-terminal state $\tau \in \mathcal{S} \setminus \mathcal{S}_i$.

Hierarchical LMDPs (ii)

Definition
Two subtasks $\mathcal{L}_i$ and $\mathcal{L}_j$ are equivalent if there exists a bijection $f : \mathcal{S}_i \to \mathcal{S}_j$ such that the transition probabilities and rewards of non-terminal states are equivalent through f.

- We define a set of *exit states* $\mathcal{E} = \cup_{i=1}^{L} \mathcal{T}_i$
- We also use $\mathcal{E}_i = \mathcal{E} \cap \mathcal{S}_i$ to denote the set of (non-terminal) exit states in the subtask $\mathcal{L}_i$.
- A set of equivalence classes $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_C\}$, $C \leq L$, i.e. a partition of the set of subtasks $\{\mathcal{L}_1, \ldots, \mathcal{L}_L\}$ such that all subtasks in a given partition are equivalent.
- We represent a single subtask $\mathcal{L}_j = \langle \mathcal{S}_j, \mathcal{T}_j, \mathcal{P}_j, \mathcal{R}_j, \mathcal{J}_j \rangle$ per equivalence class $\mathcal{C}_j \in \mathcal{C}$.

Hierarchical LMDPs (iii) - Illustration

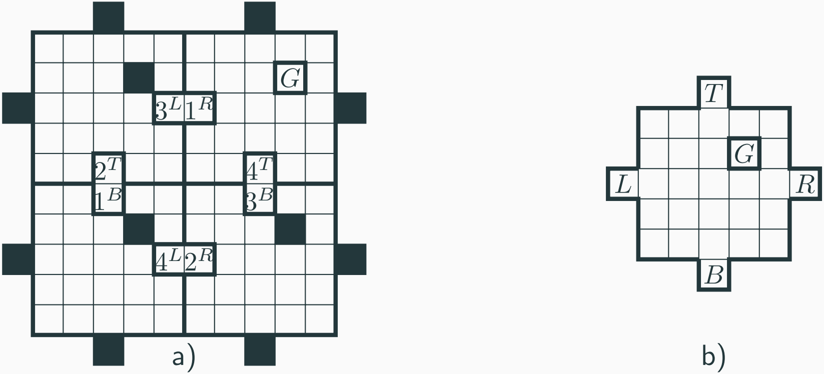


Figure 1: a) A 4-room LMDP, with all exit states highlighted; b) a single subtask with 5 terminal states G, L, R, T, B that is equivalent to all 4 room subtasks.

Hierarchical LMDPs (iv) - Subtask compositionality (i)

- For a subtask $\mathcal{L}_j = \langle \mathcal{S}_j, \mathcal{T}_j, \mathcal{P}_j, \mathcal{R}_j, \mathcal{J}_j \rangle$ as defined previously, consider its terminal set $\mathcal{T}_j = \{\tau_1, \ldots, \tau_n\}$.
- We define n base LMDPs $\mathcal{L}_j^1, \ldots, \mathcal{L}_j^n$, where each base LMDP $\mathcal{L}_j^k$ only differ in the reward of terminal states $\mathcal{J}_j^k$.
- Concretely, $z_j^k(\tau) = 1$ if $\tau = \tau_k$, and $z_j^k(\tau) = 0$ otherwise. This corresponds to an actual reward of $\mathcal{J}_j^k(\tau) = 0$ for $\tau = \tau_k$, and $\mathcal{J}_j^k(\tau) = -\infty$ otherwise.
- Thus, we can solve these base LMDPs to obtain $z_j^1, \ldots, z_j^n$.
- Having an estimate $\hat{z}(s)$ for each $t \in \mathcal{T}_j$, then thanks to compositionality, the estimated value of each non-terminal state $s \in \mathcal{S}_i$ is

$$\hat{z}(s) = \hat{z}(\tau_1) z_j^1(s) + \cdots + \hat{z}(\tau_n) z_j^n(s) \;\; \forall s \in \mathcal{S}_i, \forall \mathcal{L}_i \in \mathcal{C}_j.$$

Hierarchical LMDPs (v) - Subtask compositionality (ii)

- For all subtaks, terminal states $\tau_1 \dots \tau_n$ are by definition in $\mathcal{E}$.
- Having access to a value estimate $\hat{z}_{\mathcal{E}} : \mathcal{E} \to \mathbb{R}$ and the base LMDPs value functions $z_j^1, \dots, z_j^n$ is enough to express the value estimate of every other state without learning.
- No need to store an explicit estimate $\hat{z}(s)$.
- Once we have solved the base LMDPs, there is no need to solve again each individual subtask. That is why we represent a single subtask $\mathcal{L}_j$ for each equivalence class $\mathcal{C}_j$.

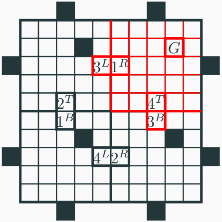
Hierarchical LMDPs (vi) - Subtask compositionality (iii)

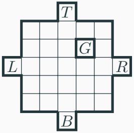
Remark

If $\hat{z}(s)$ is optimal for $s \in \mathcal{E}$, then $\hat{z}(s)$ for $s \in \mathcal{S}_i$ will also be optimal. Thanks to compositionality we have

$$\hat{z}(s) = \hat{z}(3^L) * z_L(s) + \hat{z}(3^B) * z_B(s) + \hat{z}(G) * z_G(s)$$

For any state s in $\mathcal{S}_i$ represented in red. Thus, if $\hat{z} = z^*$ for terminal states, then it will be optimal for the interior states as well.





Algorithms

Eigenvector algorithm (i)

- We can reformulate the system of equations yielded by

$$\hat{z}(s) = \hat{z}(\tau_1) z_j^1(s) + \cdots + \hat{z}(\tau_n) z_j^n(s) \;\; \forall s \in \mathcal{S}_i, \forall \mathcal{L}_i \in \mathcal{C}_j.$$

  to be defined only in the states $s \in \mathcal{E}$.
- Thus, we define

$$\mathbf{z}_{\mathcal{E}} = G \mathbf{z}_{\mathcal{E}}.$$

- G contains the value of the base LMDPs, while $\mathbf{z}_{\mathcal{E}}$ is initialized with value 1 for all $s \in \mathcal{E}$.
- No need to keep an estimate of the interior states in the higher level, the values for states in $\mathcal{E}$ are sufficient.

Eigenvector algorithm (ii) - Convergence proof

Lemma (1)
If the reward of each terminal state $t \in \mathcal{T}_i$ equals its optimal value in $\mathcal{L}$, i.e. $z_i(t) = z(t)$, the optimal value of each non-terminal state $s \in \mathcal{S}_i$ equals its optimal value in $\mathcal{L}$, i.e. $z_i(s) = z(s)$.

Lemma (2)
The solution to $\mathbf{z}_{\mathcal{E}} = G\mathbf{z}_{\mathcal{E}}$ is unique.

Lemma (3)
For each subtask $\mathcal{L}_i$ and state $s \in \mathcal{S}_i^+$, it holds that

$$z_i^1(s) + \cdots + z_i^n(s) \leq 1$$

Online algorithm (i)

- We keep an estimate of the base value functions $\hat{z}_j^1, \ldots, \hat{z}_j^n$ for each $\mathcal{L}_j$.
- A single transition is enough to update the value functions of all base LMDPs associated with $\mathcal{L}_j$ by using intra-task learning (Kaelbling, 1993).
- The update rule for the states in the exit set becomes

$$\hat{z}_{\mathcal{E}}(s) \leftarrow (1-\alpha_\ell)\hat{z}_{\mathcal{E}}(s) + \alpha_\ell[\hat{z}_{\mathcal{E}}(t_1)\hat{z}_j^1(s) + \cdots + \hat{z}_{\mathcal{E}}(t_n)\hat{z}_j^n(s)].$$

- Estimates at any level are learned in an episodic fashion.
- When to update states in $\mathcal{E}$ is still a question to answer (next slide).

Online algorithm (ii)

We propose the following alternatives:

V_1: Update the value of an exit state $s \in \mathcal{E}_i$ each time we take a transition from s.

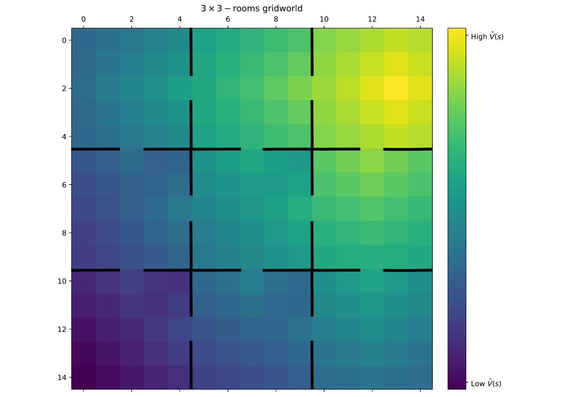
V_2: When we reach a terminal state of the subtask $\mathcal{L}_i$, update the values of all exit states in $\mathcal{E}_i$.

V_3: When we reach a terminal state of the subtask $\mathcal{L}_i$, update the values of all exit states in $\mathcal{E}_i$ and all exit states of subtasks in the equivalence class $\mathcal{C}_j$ of $\mathcal{L}_i$.

Experiments and results

Experiments - Rooms domain

- We varied the size of the rooms as well as the number of rooms.



Results - Rooms domain

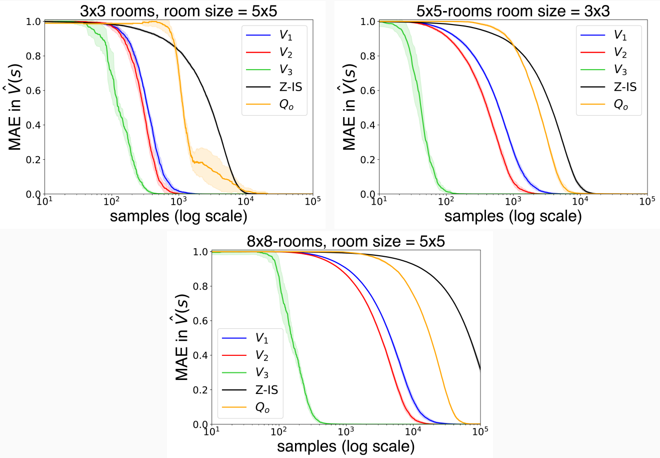
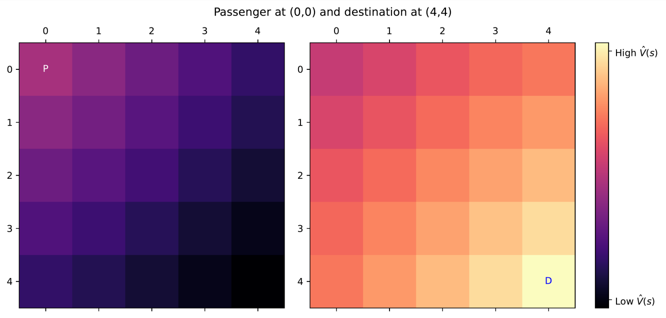


Figure 2: MAE over time for 3×3 (left), 5×5 (middle) and 8×8 (right) room instances.

Experiments - Taxi domain

- A passenger is located at one of the four corners and he must be carried to a certain corner (excluding the pickup location).
- Base LMDPs here are going to each of the corners.
- Sometimes there exist natural equivalence classes in facotres MDPs



Results - Taxi domain

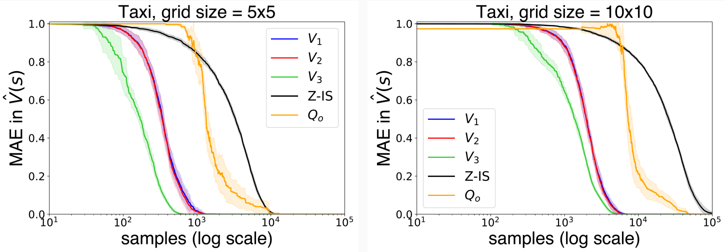


Figure 3: MAE over time for 5×5 (left) and 10×10 (right) grids of Taxi domain.

Contributions and Conclusion

Contributions

- We define a novel scheme based on compositionality for solving subtasks.
- The subtasks decomposition is at the level of the value function, thus our approach does not suffer from non-stationarity in the online setting.
- Under mild assumptions, our method converges to the optimal value function.
- Unlike the options setting, we can retrieve the optimal value function for the full state space using the value estimates for the exit states.

Conclusion

- Unlike typical HRL approaches, we no longer need a high-level policy that selects among subgoals (e.g. non-stationarity in SMDPs).
- Instead, we are able to retrieve the optimal value function for each state thanks to the decomposition of the value function in terms of the values of base LMDPs.

References

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