k-Parent Aliquot Numbers: Counting Pre-Images Under the Sum-of-Proper-Divisors Function

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1 Model Density of k-Parent Numbers

1.1 Pollack-Pomerance Heuristics

2 Tabulation of k-Parent Numbers

2.1 Brute-Forcing Preimages

Consider the equation $s(n) = m$, where the integer m is the *image* of n under the sum-of-proper-divisors function. The problem is equivalent to the prime-factorization of n, using the method of [DeTurck, 2005]. A harder problem is to compute the *preimages* of m, all integers n such that $s(n) = m$ expressed as $s^{-1}(m)$. Restricting to even values of m this problem can be brute-forced by computing $s(n)$ for ranges of n, a conjecture similar to [Luca and Pomerance, 2015] is useful to select the values of n.

Conjecture 2.1. If $2|m$ and $s(n) = m$ then $n \in \{x|\text{ even } x, 2 \leq x \leq 2m\} \cup \{x^2|\text{ odd } x, 1 \leq x \leq m\}$

If $2|n$ then $s(n) = 1 + 2 + \frac{n}{2} + x$ so $s(n) > \frac{n}{2}$;

thus $s(n) > m$ if $n > 2m$.

If $2 \nmid n$ then $s(n) = m$ if and only if n is an odd square,

thus $s(n) = 1 + \sqrt{n} + x$ and $s(n) > \sqrt{n}$. So $s(n) > m$ if $n > m^2$.

Proposition 2.1. $s(n)$ is even if and only if

$n = (2m + 1)^2$, or

$2|n$ and $(n \neq m^2$ or $n \neq 2m^2)$.

This approach is expensive if computing $s^{-1}(m)$ for a single value of m, most values $s(n)$ will be disregarded when searching for $s(n) = m$. It is more palatable to apply this method to compute the preimages for a range of m, as determining $s^{-1}(m)$ requires $\{s^{-1}(x)|$ even x, $1 \leq x \leq m\}$ anyway. This computation is useful to count the occurrence and density of k-parent numbers, these counts can be compared to the predictions of the k-parent heuristic model to provide evidence for correctness.

2.2 Pomerance-Yang Algorithm

We can certainly do better than this brute-force approach for enumerating preimages, a technique of [Pomerance and Yang, 2014] is a useful improvement. Algorithm 1 presents the Pomerance-Yang algorithm modified to determine $\#s^{-1}(n)$ for all integers less than or equal to the bound x.

Algorithm 1 Pomerance-Yang

procedure ENUMERATE_KPARENT(x)
    $f[i] \leftarrow 1, \forall i \in [1, x]$
    for all odd $m \in [1, x]$ **do**
        if $2|\sigma(m)$ **then**
            $t \leftarrow 3\sigma(m) - 2m$
            while $t \leq x$ **do**
                $f[t] \leftarrow f[t] + 1$
                $t \leftarrow 2t + \sigma(m)$
            end while
        end if
        if $\sigma(m) = m + 1$ **then**   ▷ if m is prime
            $f[m + 1] \leftarrow f[m + 1] + 1$
        end if
    end for
    for all odd composite $m \in [1, x^{2/3})$ **do**
        if $s(m^2) \leq x$ **then**
            $f[s(m^2)] \leftarrow f[s(m^2)] + 1$
        end if
    end for
    return f
end procedure

Algorithm 1 was computed on a large scale by [Chum et al., 2018] who used the result to compute the geometric mean of $s(n)/n$ weighted by $\#s^{-1}(n)$. Extending this work to compute the density of k-parent numbers I have reimplemented the algorithm and can report a substantial performance improvement.

References

[Chum et al., 2018] Chum, K., Guy, R. K., Jacobson, J. M. J., and Mosunov, A. S. (2018). Numerical and statistical analysis of aliquot sequences. *Experimental Mathematics*, 29(4):414–425.

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