Densities of k Parent Aliquot Numbers

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Overview

Aliquot Sequences

Sum of Divisors

For any natural number n the sum of divisors of n is defined $\sigma(n) = \sum_{d|n} d$

And its close relative the sum of proper divisor function

Sum of Proper Divisors

$s(n) = \sigma(n) - n$

If $s(m) = n$ then m is known as the parent of n and n is referred to as *aliquot*. It is possible that n could have multiple parents.

A number not in the image of $s(\cdot)$ is known as *non-aliquot* or more colourfully as an *aliquot orphan*

Aliquot Sequences, cont.

An aliquot sequence is the iteration of $s(n)$

$$\begin{aligned} s(10) &= 1+2+5 \\ s(8) &= 1+2+4 \\ s(7) &= 1 \\ s(1) &= 0 \end{aligned}$$

Above is an example of a sequence terminating after hitting a prime.

Sequences can also enter a cycle on a perfect number:

$$s(8128) \to s(8128) \to \dots$$

Or enter a cycle of *sociable numbers*:

$s(14316) \to s(19116) \to s(31704) \to s(47616) \to s(83328) \to s(177792) \to s(295488) \to s(629072) \to s(589786) \to s(294896) \to s(358336) \to s(418904) \to s(366556) \to s(274924) \to s(275444) \to s(243760) \to s(376736) \to s(381028) \to s(285778) \to s(152990) \to s(122410) \to s(97946) \to s(48976) \to s(45946) \to s(22976) \to s(22744) \to s(19916) \to s(17716) \to s(14316)$

Aliquot Sequences, cont.

It is an open question whether all aliquot sequences converge

Catalan-Dickson Conjecture

Every aliquot sequence ends in a prime, a perfect number, or a set of sociable numbers.

However many seem to diverge, such as the sequence starting with $s(276)$

Guy-Selfridge Counter Conjecture

An infinite amount of aliquot sequences are unbounded

Aliquot Sequences, cont.

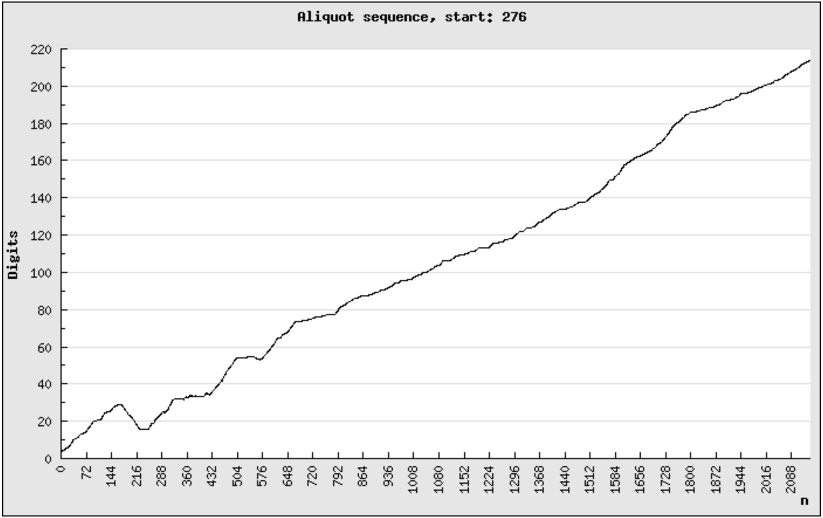

Figure: Growth of the sequence starting with $s(276)$, from [3]

Natural Density

For some sequence S of positive integers define a counting function $A(x) = |\{n \in S : 1 \leq n \leq x\}|$

Natural Density of Sequence S

$$d(S) = \lim_{x \to \infty} \frac{A(x)}{x}$$

The additive property of natural density will come in handy

Additive Property of Natural Density

Let S_1, S_2 be disjoint sequences with natural density, then:

$$d(S_1 \cup S_2) = d(S_1) + d(S_2)$$

Motivation from Dr.Guy

Think of a number!! Say 36%, which is nice and divisible. It appears that about 36% of the even numbers are "orphans".

Divide by 1. For about 36% of the (even) values of n there is just one positive integer m such that $s(m) = n$. These values of n have just one "parent".

Divide by 2. About 18% of the even values of n have exactly two parents.

Divide by 3. About 6% of the even values of n have three parents.

Divide by 4. About 1 1/2 % of the even values of n have just 4 parents.

This suggests that $1/(p!\ e)$ of the even numbers have p parents.

Experiments suggests that these values are a bit large for small values of p and a bit small for larger values of p. Can anything be proved?

Density of Aliquot Orphans

Let Δ_k be the natural density of k parent aliquot numbers.

Dr. Guy's observations suggest the densities of aliquot orphans are a specific instance of the more general Δ_k

These slides first present the heuristic model for the density of aliquot orphans proposed by Pollack and Pomerance [1] then continues to present my work in generalizing that model

Density of Non-Aliquot Numbers (3.4)

$$\Delta = \lim_{y \to \infty} \frac{1}{\log y} \sum_{\substack{a \leq y \\ 2 | a}} \frac{1}{a} e^{-a/s(a)}$$

Density of Aliquot Orphans, cont.

To understand this model it is instructive to look at an intermediate expression for Δ

Density of Non-Aliquot Numbers (3.1)

$$\Delta = \lim_{y \to \infty} \frac{\phi(A_y)}{A_y} \sum_{\substack{a \leq y \\ 2 | a}} \frac{1}{a} e^{-a/s(a)}$$

- $\phi(x)$ is Euler's totient function
- $s(x)$ is the sum-of-proper-divisors function
- $A_y = \text{lcm}[1, 2, ..., y]$

Density of Aliquot Orphans, cont.

From here I will be presenting my understanding of how the authors constructed an expression for Δ (3.1).

To find the density of non-aliquot numbers the authors construct a series of disjoint subsets, let:

$$A_y = \text{lcm}[1, 2, ..., y]$$

Then for a positive integer $a | A_y$, let:

$$T_a = \{n : \gcd(n, A_y) = a\}$$

We also bound the set:

$$T_a(x) = T_a \cap [1, x]$$

Density of Aliquot Orphans, cont.

A useful property of how these sets T_a are constructed is that the parity of a determines the parity of everything in the set; its worth quickly proving this for later use.

Claim: If a is even then n must also be even.
Proof: Assume a is even, then:

$$\gcd(n, A_y) = a = 2b$$

So $2b|n$ which proves the claim.

Aliquot Orphans, cont

Claim: If a is odd then n is also odd

Proof: Assume a is odd, also noting that for $y \geq 2$ that $2|A_y$, so we have:

$$\gcd(n, A_y) = a$$
$$\gcd(n, 2c) = 2b + 1$$

For contradiction assume that n is even, then:

$$\gcd(2d, 2c) = 2b + 1$$

But $\gcd(kr, km) = k \cdot \gcd(r, m)$ which gives: [4]

$$2 \cdot \gcd(d, c) = 2b + 1$$

This contradiction implies that n must be odd, establishing the claim

Aliquot Orphans, cont.

We have a series of disjoint subsets T_a, next we determine what proportion of each set are orphans. We need to setup some assumptions to use statistical tools to handle this problem:

- Assume that $s(\cdot)$ maps T_a to T_a for each $a|A_y$ (Asymptotically true if $n > e^{e^y}$ and $y \to \infty$)
- Assume that for $n \in T_a$ that we have $\sigma(n)/n \approx \sigma(a)/a$ (Asymptotically true up to sets of vanishing density as $y \to \infty$)
- Assume that $s(\cdot)$ is a random map

Aliquot Orphans, cont.

The classic "balls and bins" statistics problem [2] comes in handy here. Say that we have n bins and decide to toss m balls independently and randomly into those bins.

Probability of a Empty Bin

$$\mathbb{P}[\text{bin } i \text{ empty}] = (1 - \frac{1}{n})^m$$

Number of a Empty Bins

$$\#[\text{empty bins}] = n(1 - \frac{1}{n})^m$$

Aliquot Orphans, cont.

Using that model to estimate the proportion of non-aliquot numbers in a specific set T_a, naturally:

$$\# \text{ bins} = |T_a(x)|$$

Next we need an analog for the number of balls, a set B such that $m \in B$ if and only if $s(m) \in T_a(x)$

Previously we assumed that $s(\cdot)$ maps T_a back into T_a so we know that $B \subset T_a$

Aliquot Orphans, cont.

Its worth noting that $\frac{a}{s(a)}$ provides a measure of how much smaller or larger a is compared to $s(a)$. By assumption every element in T_a also shares this same approximate ratio as:

$$\frac{a}{s(a)} = \frac{1}{(\sigma(a)/a) - 1}$$

We can use this fact to find the range of values of that will map into $T_a(x)$ by:

$$x\left(\frac{a}{s(a)}\right) = \frac{x}{(\sigma(a)/a) - 1}$$

For instance let $a = 2$ and $x = 100$ then:

$$100\left(\frac{2}{s(2)}\right) = 200$$

So in this case $s(m)$ will map into $T_2(100)$ if and only if $m \in T_2(200)$

Aliquot Orphans, cont.

So we have a number of 'bins':

$$n = |T_a(x)|$$

And the number of 'balls':

$$m = \left| T_a\left(\frac{x \cdot a}{s(a)}\right) \right|$$

Plugging those values in:

$$\left(1 - \frac{1}{n}\right)^m = \left(1 - \frac{1}{|T_a(x)|}\right)^{|T_a(x \cdot a/s(a))|}$$

Which is the probability of any specific element in $T_a(x)$ being non-aliquot

Aliquot Orphans, cont.

The authors observe that:

$$|T_a(x)| \sim \frac{\phi(A_y)\cdot x}{A_y \cdot a}$$

So we can substitute in:

$$\left(1 - \frac{1}{|T_a(x)|}\right)^{|T_a(x\cdot a/s(a))|} = \left(1 - \frac{1}{\left(\frac{\phi(A_y)\cdot x}{A_y\cdot a}\right)}\right)^{(\phi(A_y)\cdot x)/(A_y\cdot s(a))}$$

Where $f(x) \sim g(x) \iff \lim_{x\to\infty} f(x)/g(x) = 1$

Asymp. Equiv.

The authors observe that:

$$|T_a(x)| \sim \frac{\phi(A_y) \cdot x}{A_y \cdot a}$$

Breaking this into terms we can make some observations about why this relation holds:

$$\phi(A_y) = |\{n \in \mathbb{Z} \mid \gcd(A_y, n) = 1 \text{ and } 1 \leq n \leq A_y\}|$$

Euler's totient function counts the amount of numbers co-prime to its input up-to that input

$$\frac{\phi(A_y)}{A_y}$$

This is the ratio of numbers that are co-prime with A_y upto A_y

Asymp. Equiv.

The core of the argument comes from the fact that:

$$\gcd(n, A_y) = 1 \implies \gcd(an, A_y) = a$$

Proof:
Assume that $\gcd(n, A_y) = 1$, remember $a|A_y$

$$\begin{aligned} \gcd(an, A_y) &= \gcd(an, am) && [m > 0 \text{ and } m|A_y] \\ &= a \cdot \gcd(n, m) && [\gcd(ax, ay) = a \cdot \gcd(x, y)] \end{aligned}$$

So to get the result all we need to show is $\gcd(n, m) = 1$ we can use another gcd property to establish this:

$$\gcd(x, yz) = 1 \iff \gcd(x, y) = 1 \text{ and } \gcd(x, z) = 1$$

Asymp. Equiv.

Plugging values into the bi-conditional:

$$\gcd(n, A_y) = \gcd(n, am) = 1 \iff \gcd(n, a) = 1 \text{ and } \gcd(n, m) = 1$$

By assumption we know $\gcd(n, am) = 1$ it follows that:

$$\gcd(n, a) = 1 \text{ and } \gcd(n, m) = 1$$

So $a \cdot \gcd(n, m) = a$ which is sufficient to establish the implication:

$$\gcd(n, A_y) = 1 \implies \gcd(an, A_y) = a$$

Asymp Equiv

While we have that:

$$\gcd(n, A_y) = 1 \implies \gcd(an, A_y) = a$$

It is not true that:

$$\gcd(an, A_y) = a \implies \gcd(n, A_y) = 1$$

As a consequence of this we know that every n such that $n \leq A_y$ and $gcd(n, A_y) = 1$ must correspond to a member of the set $T_a(A_y \cdot a)$.

However there exists elements in T_a that do not correspond to the totatives of A_y. The table on the next slide outlines some examples of this case in blue.

Asymp. Equiv.

n	T_3	Type	n	T_3	Type
1	3	$3 \cdot 1$	31	93	$3 \cdot 31$
NA	9	$3 \cdot 3^1$	NA	99	$3 \cdot (3 \cdot 11)$
7	21	$3 \cdot 7$	37	111	$3 \cdot 37$
NA	27	$3 \cdot 3^2$	NA	117	$3 \cdot (3 \cdot 13)$
11	33	$3 \cdot 11$	41	123	$3 \cdot 41$
13	39	$3 \cdot 13$	43	129	$3 \cdot 43$
17	51	$3 \cdot 17$	47	141	$3 \cdot 47$
19	57	$3 \cdot 19$	49	147	$3 \cdot 49$
NA	63	$3 \cdot (3 \cdot 7)$	NA	153	$3 \cdot (3 \cdot 17)$
23	69	$3 \cdot 23$	53	159	$3 \cdot 53$
NA	81	$3 \cdot 3^3$	NA	171	$3 \cdot (3 \cdot 19)$
29	87	$3 \cdot 31$	59	177	$3 \cdot 59$

- $A_y = \text{lcm}(1, 2, 3, 4, 5) = 60$
- n s.t. $\gcd(n, A_y) = 1$
- $T_3 = \{n \mid \gcd(n, A_y) = 3\}$

Asymp. Equiv.

So where does this leave us in proving:

$$|T_a(x)| \sim \frac{\phi(A_y) \cdot x}{A_y \cdot a}$$

I am not precisely sure, we are to show that:

$$\lim_{x \to \infty} \frac{|T_a(x)|}{\left(\frac{\phi(A_y) \cdot x}{A_y \cdot a}\right)} = 1$$

And we know that:

$$\frac{\phi(A_y) \cdot A_y \cdot a}{A_y \cdot a} \neq |T_a(A_y \cdot a)|$$

As:

$$\frac{\phi(60) \cdot 180}{60 \cdot 3} = 16 \neq 24 = |T_3(180)|$$

Asymp. Equiv.

My next step when I continue looking into this will be to prove:

$$\gcd(x, A_y) = a$$

if and only if

$$x = an \text{ and } \gcd(a, n) = 1$$

or

$$x = a^i,\ i \geq 1$$

or

$$x = a^2 n$$

Where $n \perp A_y$ and $n \leq A_y$

I know this statement is flawed as it doesn't account for $x > A_y \cdot a$, I will fix that later. The idea is to separate into classes.

Rabbit

Figure: Bounty Reward Rabbit

Asymp. Equiv.

So next we would like to find what proportion of numbers n such that:

$$\gcd(an, A_y) = a$$

Well on top of the ratio we will have:

$$\phi(A_y)$$

As this number counts how many solutions we have to $\gcd(an, A_y) = a$. Then what goes in the denominator? We are looking for the ratio of $\#an$ that satisfies the equation over all possible values of $\#an$. Since earlier we defined $n \leq A_y$ so naturally $an \leq aA_y$ we can use aA_y as the denominator of the ratio.

Asymp. Equiv.

So we have:

$$\frac{\phi(A_y)}{aA_y}$$

as the proption of solutions

Aliquot Orphans, cont.

We are looking for information about the asymptotic behaviour of this probability:

$$\lim_{x\to\infty}\left(1-\frac{1}{\left(\frac{\phi(A_y)\cdot x}{A_y\cdot a}\right)}\right)^{(\phi(A_y)\cdot x)/(A_y\cdot s(a))}$$
$$= e^{-a/s(a)}$$

Which is the probability of any element in T_a being non-aliquot

Note: $\lim_{x\to\infty}(1-\frac{1}{x})^x = e$

Aliquot Orphans, cont.

Working heuristically, since:

$$|T_a(x)| \sim \frac{\phi(A_y) \cdot x}{A_y \cdot a}$$

So we treat $\phi(A_y)/(a \cdot A_y)$ as the probability that any number is in T_a (see [5] for a similar treatment)

So the probability that some number is both in T_a and non-aliquot:

$$\frac{\phi(A_y)}{A_y \cdot a} \cdot e^{-a/s(a)}$$

Image of $s(\cdot)$

Let p and q be distinct primes. By the fundamental theorem of arithmetic the only divisors of $p \cdot q$ are in $\{1, p, q, pq\}$, so $s(pq) = 1 + p + q$.

Goldbach Conjecture (Strong Variant)

Every even number greater than 8 can be expressed as the sum of 2 distinct primes

Assuming the stronger Goldbach Conjecture $\forall m \geq 4$ there exists unique primes p and q such that $2m = p + q$.

$$s(pq) = 1 + p + q = 2m + 1$$

So every odd number greater than 8 is in the image of $s(\cdot)$, **almost all odd numbers cannot be aliquot orphans**

Aliquot Orphans, cont.

Since each set T_a is disjoint we can apply the additive property of natural density to find the density of non-aliquot numbers over the positive integers:

$$\Delta = \lim_{y\to\infty} \sum_{\substack{a|A_y \\ 2|a}} \frac{\phi(A_y)}{a \cdot A_y} \cdot e^{-a/s(a)}$$

$$= \lim_{y\to\infty} \frac{\phi(A_y)}{A_y} \sum_{\substack{a|A_y \\ 2|a}} \frac{1}{a} e^{-a/s(a)}$$

We restrict a to even values as its parity determines the parity of everything in T_a and we know that the density of odd aliquot orphans quickly vanishes

k Parent Aliquot Numbers

So how do we extend this reasoning to count aliquot numbers with k parents?

Probability of a Bin Containing k Balls

$$\mathbb{P}[k \text{ balls in bin}] = \binom{m}{k} \frac{1}{n^k} (1 - \frac{1}{n})^{m-k}$$

- k is the number of balls in a bin
- n is the number of bins
- m is the number of balls

k Parent Aliquot Numbers

Using the same values of:

$$n = \frac{\phi(A_y) \cdot x}{A_y \cdot a} \qquad m = \frac{\phi(A_y) \cdot x}{A_y \cdot s(a)}$$

Plugging those in we get:

$$\mathbb{P}[k \text{ parents}] = \lim_{x \to \infty} \binom{\frac{\phi(A_y)\cdot x}{A_y \cdot s(a)}}{k} \frac{1}{\left(\frac{\phi(A_y)\cdot x}{A_y\cdot a}\right)^k} \left(1 - \frac{1}{\left(\frac{\phi(A_y)\cdot x}{A_y \cdot a}\right)}\right)^{\frac{\phi(A_y)\cdot x}{A_y\cdot s(a)} - k}$$

Which is the probability of a specific number in T_a having k parents

k Parent Aliquot Numbers

This probability cleans up quite nicely (proof excluded for brevity)

Probability of an $n \in T_a$ having k parents

$$\mathbb{P}[k \text{ parents}] = \frac{a^k}{k! \cdot s(a)^k} \cdot \mathrm{e}^{-a/s(a)}$$

Density of k Parent Aliquot Numbers

$$\Delta_k = \lim_{y \to \infty} \frac{\phi(A_y)}{A_y} \sum_{a | A_y} \frac{a^{k-1}}{k! \cdot s(a)^k} \cdot \mathrm{e}^{-a/s(a)}$$

Note that this expression is not restricted to even values of a

k Parent Aliquot Numbers

The authors continue to establish an easier to compute expression for Δ:

$$\Delta = \lim_{y\to\infty} \frac{\phi(A_y)}{A_y} \sum_{\substack{a|A_y \\ 2|a}} \frac{1}{a} e^{-a/s(a)} \tag{3.1}$$

$$= \lim_{y\to\infty} \frac{1}{\log y} \sum_{\substack{a\leq y \\ 2|a}} \frac{1}{a} e^{-a/s(a)} \tag{3.4}$$

Continuing the trend of unjustified claims this suggests:

Density of numbers with k parents

$$\Delta_k = \lim_{y\to\infty} \frac{1}{\log y} \sum_{a\leq y} \frac{a^{k-1}}{k! \cdot s(a)^k} \cdot e^{-a/s(a)}$$

Tentative Numerical results

y	Δ_0	Δ_1	Δ_2	Δ_3
50000	0.163674852	0.211872977	0.150917967	0.091332510
100000	0.164576576	0.212205479	0.150682529	0.091010417
150000	0.165055668	0.212382167	0.150556904	0.090837746
200000	0.165376121	0.212500620	0.150473266	0.090722516
250000	0.165614500	0.212588736	0.150411231	0.090637101
300000	0.165803078	0.212658477	0.150362014	0.090569168

Table: Approximate values of Δ_k

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Think of a number!! Say 36%, which is nice and divisible. It appears that about 36% of the even numbers are "orphans".

Divide by 1. For about 36% of the (even) values of n there is just one positive integer m such that $s(m) = n$. These values of n have just one "parent".

Divide by 2. About 18% of the even values of n have exactly two parents.

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This suggests that $1/(p!\ e)$ of the even numbers have p parents.

Experiments suggests that these values are a bit large for small values of p and a bit small for larger values of p. Can anything be proved?

Tentative Numerical results

y	Δ_0	Δ_1	Δ_2	Δ_3	Δ_4
50000	0.16367	0.16577	0.09741	0.04376	0.01651
100000	0.16458	0.16592	0.09714	0.04351	0.01637
150000	0.16506	0.16601	0.09699	0.04337	0.01630
200000	0.16538	0.16606	0.09689	0.04328	0.01625
250000	0.16561	0.16611	0.09682	0.04321	0.01622
300000	0.16580	0.16614	0.09676	0.04316	0.01619
$1/(p! \cdot \mathrm{e})$	$1/(0! \cdot \mathrm{e})$	$1/(1! \cdot \mathrm{e})$	$1/(2! \cdot \mathrm{e})$	$1/(3! \cdot \mathrm{e})$	$1/(4! \cdot \mathrm{e})$
-	0.36787	0.36787	0.18393	0.06131	0.01532
$1/2(p! \cdot \mathrm{e})$	$1/2(0! \cdot \mathrm{e})$	$1/2(1! \cdot \mathrm{e})$	$1/2(2! \cdot \mathrm{e})$	$1/2(3! \cdot \mathrm{e})$	$1/2(4! \cdot \mathrm{e})$
-	0.18393	0.18393	0.09196	0.03065	0.00766

Table: Approximation of Δ_k (restricted to even a) compared to Dr. Guy's estimates

Where $\frac{1}{p! \cdot e}$ is the estimated density of k parents aliquot numbers over the evens.
Where $\frac{1}{2(p! \cdot e)}$ is the estimated density of **even** k parent aliquot numbers over all integers.

Further work

1) Compare estimates for Δ_k against numerical data on aliquot parents
2) Improve the computational accuracy of my estimates for Δ_k
3) Prove:

$$\Delta_k = \lim_{y\to\infty} \frac{\phi(A_y)}{A_y} \sum_{a|A_y} \frac{a^{k-1}}{k! \cdot s(a)^k} \cdot e^{-a/s(a)}$$

$$= \lim_{y\to\infty} \frac{1}{\log y} \sum_{a\le y} \frac{a^{k-1}}{k! \cdot s(a)^k} \cdot e^{-a/s(a)}$$

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