

# Densities of $k$ Parent Aliquot Numbers

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# Aliquot Sequences

## Sum of Divisors

For any natural number  $n$  the sum of divisors of  $n$  is defined  
 $\sigma(n) = \sum_{d|n} d$

And its close relative the sum of proper divisor function

## Sum of Proper Divisors

$$s(n) = \sigma(n) - n$$

If  $s(m) = n$  then  $m$  is known as the parent of  $n$  and  $n$  is referred to as *aliquot*. It is possible that  $n$  could have multiple parents.

A number not in the image of  $s(\cdot)$  is known as *non-aliquot* or more colourfully as an *aliquot orphan*

## Aliquot Sequences, cont

An aliquot sequence is the iteration of  $s(n)$

$$s(10) = 1 + 2 + 5$$

$$s(8) = 1 + 2 + 4$$

$$s(7) = 1$$

$$s(1) = 0$$

Above is an example of a sequence terminating after hitting a prime. Sequences can also enter a cycle on a perfect number or on a set of sociable numbers:

$$s(1184) \rightarrow s(1210) \rightarrow s(1184) \rightarrow \dots$$

It is an open question whether all aliquot sequences converge or terminate

### Guy-Selfridge Conjecture

An infinite amount of aliquot sequences are unbounded

# Natural Density

For some sequence  $S$  of positive integers define a counting function  
 $A(x) = |\{n \in S : 1 \leq n \leq x\}|$

## Natural Density of Sequence $S$

$$d(S) = \lim_{x \rightarrow \infty} \frac{A(x)}{x}$$

Natural density is a tool suited for measuring what proportion of all integers belong to the sequence being investigated

# Motivation from Dr.Guy

Think of a number!! Say 36%, which is nice and divisible. It appears that about 36% of the even numbers are "orphans".

Divide by 1. For about 36% of the (even) values of  $n$  there is just one positive integer  $m$  such that  $s(m) = n$ . These values of  $n$  have just one "parent".

Divide by 2. About 18% of the even values of  $n$  have exactly two parents.

Divide by 3. About 6% of the even values of  $n$  have three parents.

Divide by 4. About 1 1/2 % of the even values of  $n$  have just 4 parents.

This suggests that  $1/(p! e)$  of the even numbers have  $p$  parents.

Experiments suggests that these values are a bit large for small values of  $p$  and a bit small for larger values of  $p$ . Can anything be proved?

## Questions

There already exists an excellent body of work on the density of aliquot orphans, however there seems to minimal work on the densities of  $k$  parent aliquot numbers.

Let  $\Delta_k$  be the natural density of  $k$  parent aliquot numbers.

Dr. Guy's observations raise a couple of questions about  $\Delta_k$ :

- What is the relationship between the calculation of the density of non-aliquots and  $\Delta_k$
- How do the counts of  $k$  parent aliquot numbers compare to density estimates
- What has been proved about the similar densities and how can we extend that to  $\Delta_k$

# Existing Work

There are two general approaches to the density of non-aliquots:

## Heuristic:

Pollack and Pomerance (2016) propose a statistical model for the density of non-aliquot numbers.

$$\Delta = \lim_{y \rightarrow \infty} \frac{1}{\log y} \sum_{\substack{a \leq y \\ 2|a}} \frac{1}{a} e^{-a/s(a)}$$

This approach appears to be highly accurate with  $\Delta \approx 0.171822$  and counts of non-aliquot numbers up to  $10^{10}$  producing a density of 0.1682

## Existing Work, cont.

### **Provable:**

Several authors have produced provable lower bounds for the density of non-aliquot numbers. The most accurate of these would be the work of Chen and Zhao (2011) in which they prove that the set of non-aliquot numbers must make up a set with density greater than 0.06.

With the measured density  $\approx 0.17$  there seems to be a marked difference between what has been proven and the real density of non-aliquot numbers.

# My Project

## Goals for this project:

- Extend the Pollack and Pomerance probabilistic model for the density of non-aliquot numbers to model  $\Delta_k$ 
  - Tabulate counts of  $k$  parent aliquot numbers from existing data and compare results to model
- Scale up computation to improve estimates for  $\Delta_k$
- Extend the provable densities of non-aliquot numbers to  $k$  parent aliquot numbers
  - The plan of attack will be to first extend the lower bound proved by Banks and Luca (2005). The Chen and Zhao paper is not accessible directly through the library.

# Results so Far

## Conjectural Density of Aliquot Numbers with $k$ Parents

$$\Delta_k = \lim_{y \rightarrow \infty} \frac{1}{\log y} \sum_{a \leq y} \frac{a^{k-1}}{k! \cdot s(a)^k} \cdot e^{-a/s(a)}$$

However there is still work needed to completely justify this result.

# Tentative Numerical Results

$y$	$\Delta_0$	$\Delta_1$	$\Delta_2$	$\Delta_3$	$\Delta_4$
50000	0.16367	0.16577	0.09741	0.04376	0.01651
100000	0.16458	0.16592	0.09714	0.04351	0.01637
150000	0.16506	0.16601	0.09699	0.04337	0.01630
200000	0.16538	0.16606	0.09689	0.04328	0.01625
250000	0.16561	0.16611	0.09682	0.04321	0.01622
300000	0.16580	0.16614	0.09676	0.04316	0.01619
$1/(p! \cdot e)$	$1/(0! \cdot e)$	$1/(1! \cdot e)$	$1/(2! \cdot e)$	$1/(3! \cdot e)$	$1/(4! \cdot e)$
-	0.36787	0.36787	0.18393	0.06131	0.01532
$1/2(p! \cdot e)$	$1/2(0! \cdot e)$	$1/2(1! \cdot e)$	$1/2(2! \cdot e)$	$1/2(3! \cdot e)$	$1/2(4! \cdot e)$
-	0.18393	0.18393	0.09196	0.03065	0.00766

Table: Approximation of  $\Delta_k$  (restricted to even  $a$ ) compared to Dr. Guy's estimates

Where  $\frac{1}{p! \cdot e}$  is the estimated density of  $k$  parents aliquot numbers over the evens.

Where  $\frac{1}{2(p! \cdot e)}$  is the estimated density of **even**  $k$  parent aliquot numbers over all integers.

# Bounty Problem

Pollack and Pomerance observe that:

$$|T_a(x)| \sim \frac{\phi(A_y) \cdot x}{A_y \cdot a}$$

Where:

- $\phi(x)$  is Euler's totient function
- $A_y = \text{lcm}[1, 2, \dots, y]$
- $T_a = \{n : \gcd(n, A_y) = a\}$
- $T_a(x) = T_a \cap [1, x]$

If anyone knows why this holds I'll send you a picture of one the rabbits that live in the park near me



Figure: Example Rabbit