Hand In:

Your answers should be compiled into a report in pdf format, containing the solutions, the plots with proper axes labels, unit of measurements and legend and explanations of the exercises. Your MATLAB programs shall be inserted adjacent to your answers. Students can use Python if the prefer it to MATLAB.

Assessment Criteria:

We will assess your submissions using the following criteria:

- Clarity of solution and the supporting explanations, demonstrating your understanding of orbital motion of satellites.
- Choice and clarity of plots and diagrams to support your solutions
- Operational and concise MATLAB/Python code, all of which is included within your report.
- Overall layout, style and grammatical content of your report.

Mission Analysis and Orbital Mechanics Assignment

The aim of this assignment is to demonstrate understanding of two-body equations of motion. The use of two methods, solving Kepler's Equation and numerical integration of the equations of motion, will provide an understanding of satellite orbits and orbital elements. The challenges of orbit phasing and rendezvous are illustrated in the third part of the assignment.

The two-body problem:

References: Vallado D., *Fundamentals of Astrodynamics and Applications* [Chapter 1], Microcosm
Curtis H., *Orbital Mechanics for Engineering Students* [Chapter 2], Elsevier

We begin with the definition of the equations for the two-body problem between the Earth and a spacecraft:

$$\ddot{\mathbf{r}} + \frac{\mu_E}{r^3}\mathbf{r} = 0 \tag{1}$$

In these equations, we assume:

- The mass of the smaller body is negligible compared to the central body,
- The coordinate system is inertial,
- The two bodies, Earth and satellite, are spherically symmetric with uniform density, enabling us to treat the bodies as point masses, and
- No other forces act on the system, apart from the gravitational forces along the line joining the centres of mass.

The trajectory equation gives the radius (r) and velocity (v) as a function of the true anomaly (θ), where p is the semilatus rectum, a the semi-major axis, e the eccentricity, h the magnitude of the specific angular momentum, r_p,r_a the periapsis and apoapsis radius and T the orbital period:

$$r = \frac{p}{1+e\cos\theta} \tag{2}$$

$$\frac{h^2}{\mu} = a\left(1-e^2\right) = p \tag{3}$$

$$v = \sqrt{\mu\left(\frac{2}{r}-\frac{1}{a}\right)} \tag{4}$$

$$r\left(\theta=0\right) = r_p = \frac{p}{1+e} \;;\; r\left(\theta=\pi\right) = r_a = \frac{p}{1-e} \tag{5}$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{6}$$

1. Solving Kepler's Equation:

Your lectures have introduced Kepler's equation which links the mean anomaly (M) with the eccentric anomaly (E) and eccentricity (*e*)

$$E - e\sin E = M = n\left(t - t_0\right) \tag{7}$$

$$n = \sqrt{\frac{\mu}{a^3}} = \frac{2\pi}{\tau} \tag{8}$$

where n is the mean motion, t the time, t_0 the time of periapsis passage and μ the gravitational parameter of the central body.

The true anomaly θ defines the position of the satellite in the orbital plane as the angular distance from periapsis along the direction of motion and is related to the eccentric anomaly by:

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} \tag{9}$$

Kepler's equation is transcendental and requires numerical methods for its solution.

The Newton-Raphson method is an iterative method for solving equations of the form $f(x) = 0$ by successive approximations. By setting a chosen tolerance on the difference between the current solution and the next, it is possible to determine the convergence to a solution within the required accuracy.

More specifically, making an initial guess x_1 of the solution we want to find, we generate a sequence x_1, x_2, x_3, x_4 ... using the formula:

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}$$

until we approach the solution with the given tolerance.

You will need to use the following constants in arriving at your answers:

R_E=6378 km (Radius of the Earth)
μ_E= 3.986x10^5 km^3/s^2 (The gravitational parameter of the Earth)

<table>
<tr>
<td>1 a)</td>
<td>Create a MATLAB function “[E] = Kepler(M,e,tol)” - to solve Kepler’s equation numerically using the Newton-Raphson method returning E, the Eccentric Anomaly (in radians), from the following inputs:

mean anomaly (in radians) “M”, eccentricity “e”, tolerance “tol”.

Note: use M as first guess for the Newton-Raphson method</td>
<td>10 marks</td>
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<tr>
<td>b)</td>
<td>Execute the MATLAB function “[E] = Kepler(M,e,tol)” created in 1a, with M=21 degrees and e=0.25 as inputs, to calculate the following:
• The value of the Eccentric Anomaly after one iteration.
• Number of iterations to reduce the error – “tol” – below 10^{-6}.

Repeat the above, i.e. recalculate eccentric anomaly and number of iterations with tol changed to 10^{-12}
• Using the result for eccentric anomaly, calculate the true anomaly (θ) and the radial distance r for semi-major axis = 24000 km.
• What happens if you repeat calculations for M = 180? What does r represent in this case?</td>
<td>5 marks</td>
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<tr>
<td>c)</td>
<td>Using the function “[E] = Kepler(M,e,tol)” compute a MEO satellite orbit, computed every 15 seconds, with the following parameters:
• At t_0 the spacecraft is at perigee
• Semi-major axis a = 24000 km
• Eccentricity e = 0.72
Plot the satellite’s altitude as a function of time
• for one orbit [x axis: Time (% of orbital period) – y axis: altitude (km)]
• for one day [x axis: Time (hours) – y axis: altitude (km)] What
would you have to change if the satellite were in apogee at t_0?</td>
<td>5 marks</td>
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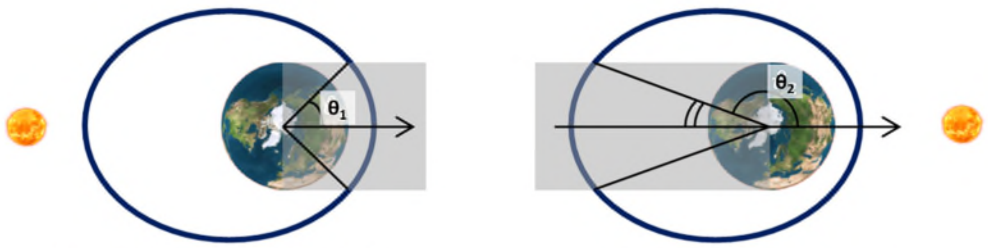
<tr>
<td>d)</td>
<td>Analysis of eclipse times is a key aspect of mission design.
Using the orbit of question 1c, consider the two cases when the Sun lies in the orbital plane and the Earth eclipses either the periapsis (1) or the apoapsis (2).



Case 1: Eclipse at periapsis Case 2: Eclipse at apoapsis

• Using data from 1c, plot the 2D orbit and the Earth (represented as a circle of radius equal to R_E).
• Compute the two solutions θ_1, θ_2 of $\alpha\cos^2(\theta)+\beta\cos(\theta)+\gamma=0$ where $\alpha= R_E^2 e^2+ p^2$, $\beta= 2R_E^2 e$, $\gamma = R_E^2- p^2$ and add to the previous plot the point along the orbit correspondent to $\pm\theta_1$, $\pm\theta_2$ highlighted by ‘*’ and ‘o’ markers respectively.
• Using Equations (7,8,9) find the Eccentric Anomalies and via the Mean Anomalies compute the eclipse time (in minutes) spent by the satellite over one orbit for each of the two cases. Compare these times and discuss the result.</td>
<td>10 marks</td>
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(10)

2. Numerical Integration of the Equations of Motion Using MATLAB Differential Equation Solvers

A system of ordinary differential equations of first order can be solved numerically using different algorithms. In this part of the assignment, we will use two of MATLAB's built-in functions called ***ode45***. In order to do so, a function with the equations of motion has to be implemented. We will call this function ***f***=*twobody*(*t*,***X***).

Given a time *t* and a state vector $\mathbf{X}$ this function returns the derivatives of the state $\dot{\mathbf{X}} = f\left(t, \mathbf{X}\right)$. In MATLAB we solve the equations by calling ode45 in the following way:

[*tout*,*yout*]=ode45(@*twobody*, [t_0:t_s:t_f], $\boldsymbol{X_o}$, *options*)

This integrates the system of differential equations in *twobody.m* from time t_0 to t_f and outputs the solution at every t_s. The integration is done with initial state vector $\boldsymbol{X_0}$ which defines the initial position and velocity. $\boldsymbol{X_0}$ needs to be a column vector. In *options*, we define the error tolerances accepted for the numerical method – you are given data for *options* as required.

The function ***f***=*twobody*(*t*,***X***), for a scalar *t* and a column vector $\mathbf{X}$, must return a column vector ***f*** corresponding to *f*(*t*,***X***). Each row in the solution array *yout* corresponds to a time returned in column vector *tout*. MATLAB has a substantial help page and other useful resources can be found online that explain the use of the ODE solvers.

You will need to use the following constants in arriving at your answers:
R_E=6378 km (Radius of the Earth)
μ_E= 3.986x10^5 km^3/s^2 (The gravitational parameter of the Earth)

2a)	Write down the two body equations of motion in 3D, as a system of differential equations in the form $\dot{\mathbf{X}} = f\left(t, \mathbf{X}\right)$, where: $$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} \text{ and } \dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} X_4 \\ X_5 \\ X_6 \\ f_1\left(X_1, X_2, X_3\right) \\ f_2\left(X_1, X_2, X_3\right) \\ f_3\left(X_1, X_2, X_3\right) \end{bmatrix}$$	5 marks

b)	Write a MATLAB function with the following call ***f***=*twobody*(*t*,***X***) that describes the equations of exercise 2a. $\mathbf{X}$ represents the 6-dimensional vector $$\mathbf{X}=\begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix}$$	5 marks
c)	By calling *twobody*(*t*,***X***) from a suitable MATLAB script, use *ode45* to integrate the equations of motion of the two-body problem for one day every 10 seconds. The initial position and velocity state vector $\mathbf{X}_0=(\mathbf{r},\mathbf{v})^{\mathrm{T}}$ is: **r** = [7115.804; 3391.696; 3492.221] km **v** = [-3.762; 4.063; 4.184] km/s options = odeset('RelTol',1e-12,'AbsTol', 1e-12); • Plot the magnitude of the position vector versus time [x axis: Time (hours) – y axis: radius (km)] • Plot the magnitude of the velocity vector versus time [x axis: Time (hours) – y axis: velocity (km/s)] • Plot the orbit in 3D including a suitably dimensioned sphere for Earth.	10 marks
d)	For the orbit in 2c write a MATLAB script that generates the following plots: ·Specific Energies (Kinetic, Potential and Total) in the same plot as a function of time [x axis: Time (hours) – y axis: Energy (km^2/s^2)] ·Specific Angular Momentum [x axis: Time (hours) – y axis: Angular Momentum (km^2/s)] Comment on the variation over the orbit. **Note: Please choose your axes scaling carefully.**	5 marks
e)	A satellite has the initial position and velocity state vector $\mathbf{X}_0=(\mathbf{r},\mathbf{v})^{\mathrm{T}}$: **r** = [0; 0; 8550] km **v** = [0; -7.0; 0] km/s Using ODE45 and your *twobody*(*t*,***X***) subroutine with *options* = odeset('RelTol',1e-12,'AbsTol', 1e-12), determine the type of orbit that this represents. You should support your conclusions with plots of radial distance and velocity over at least one day and determine orbit period, semi-major axis and inclination. Hints: Recall that the Orbital Specific Energy ε= −μ/(2*a*) where *a* is the semimajor axis. The inclination of a satellite's orbit is the angle between the direction of its angular momentum vector and the z-axis.	10 marks

3. Orbit Phasing and Rendezvous

The international space station (ISS) orbits the Earth. To provide food and water to the ISS crew a Progress capsule is send to the ISS three to four times a year. After the supplies are delivered the capsule is filled with waste and de-orbited. Your mission is to determine the orbit parameters for the "chaser" spacecraft (i.e. Progress), catch up and dock with the ISS, and apply the correct change in velocity to de-orbit it and cause it to burn-up upon re-entry to the Earth's atmosphere.

To simplify the problem, we assume that the ISS is in a circular, equatorial orbit at an altitude of 404 km above the Earth. The chaser satellite is placed in an equatorial orbit with an eccentricity such that it will intercept the ISS at apogee, i.e. the apogee of the chaser orbit coincides with the orbital radius of the ISS. Initially the angular separation of between the chaser and the ISS is $\Delta\Theta$, with the ISS $\Delta\Theta$ radians ahead of the chaser as illustrated in Figure 1.

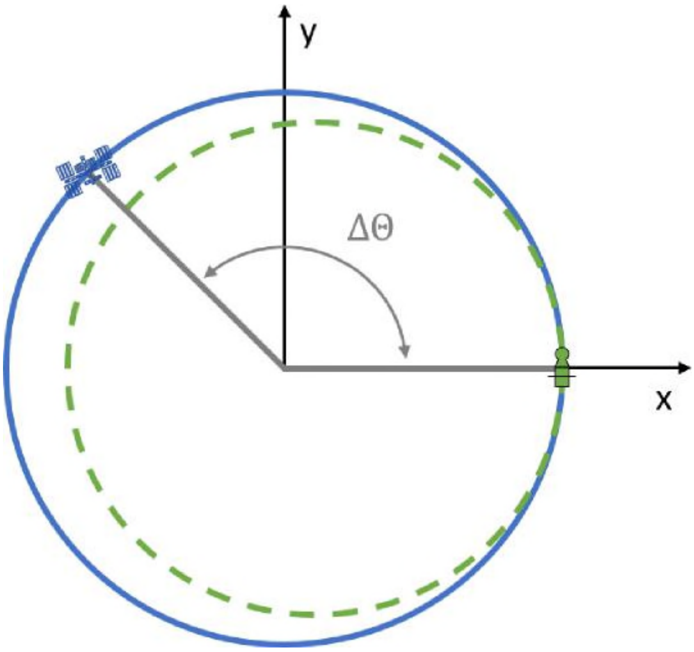

Figure 1. Chaser and ISS initial orbital configuration.

Due to the different semi-major axes the two objects have different orbital periods and in particular the orbital period of the chaser is shorter than that of the ISS. The semi-major axis of the chaser satellite is selected such that it catches up with the ISS in N_{rev} orbits. To enforce this condition, the following equation can be written

$$N_{rev} T_{chaser} \left(n_{chaser} - n_{ISS} \right) = \Delta\Theta \tag{11}$$

where T_{chaser} is the orbital period of the chaser and n_{chaser} and n_{ISS} are the mean motion of the chaser and ISS, respectively. Eq. (11) basically states that the relative angular velocity (n_{chaser}-n_{ISS}) must be chosen such that the angular separation between $\Delta\Theta$ goes to zero in an interval of time $N_{rev}T_{chaser}$. By substituting the expressions for the mean motion and orbital period, it can be found that:

$$a_{chaser} = \left(1 - \frac{\Delta\Theta}{2\pi N_{rev}} \right)^{2/3} \cdot a_{ISS} \tag{12}$$

Finally, noting that the apogee of the chaser is equal to the orbital radius of the ISS, the value of the eccentricity of the chaser orbit can be calculate:

$$e_{chaser} = \frac{a_{ISS}}{a_{chaser}} - 1 \tag{13}$$

3a)	Calculate the initial state vector $\mathbf{X}_0 = (\mathbf{r}, \mathbf{v})^{\mathrm{T}}$ of the ISS using the coordinate system as in Figure 1 and with $\Delta\Theta$ =100 deg. Plot the orbit using *ode45*. What is the orbit period?	5 marks
b)	Using expressions (12) and (13) above, calculate the semimajor axis and eccentricity required to intercept the ISS after 12 orbits. Calculate the initial state vector $\mathbf{X}_0 = (\mathbf{r}, \mathbf{v})^{\mathrm{T}}$ for the chaser. Propagate the orbit to plot the distance between the ISS and chaser as the latter catches up.	10 marks
c)	The chaser will also need a ΔV to match its apogee speed to the ISS before it can dock. Reconsider point (3b) and instead of N_{rev}=12, now assume N_{rev}=[2:30]. Then, compute the ΔV required to match the apogee of the chaser to the speed of the ISS before it docks for each Nrev in the interval. Comment upon the variation, and the reason, in the ΔV required.	10 marks
d)	Once Progress has docked the ISS, and thus is in a circular orbit, it will need to undock and re-enter. Calculate and plot as a graph, the ΔV required to lower the perigee of the Progress capsule to the range of values 60 km to 210 km while keeping the apogee unchanged.	10 marks