第 1.6 节：数列的极限与第二个重要极限（Limits of Sequences and the second Important Limit）

一、内容提要（contents）

① 数列的有界性：如果对于数列$\{a_n\}$，存在一个实数M使得$a_n \le M$对于每个n都成立，则M称为数列$\{a_n\}$的一个上界，数列$\{a_n\}$叫做是有上界的（bounded above）；如果对于数列$\{a_n\}$，存在一个实数m使得$m \le a_n$对于每个n都成立，则m称为数列$\{a_n\}$的一个下界，数列$\{a_n\}$叫做是有下界的(bounded below)。

②数列极限的夹逼定理（Squeeze Theorem/ Sandwich Theorem）:如果数列$\{a_n\},\{b_n\},\{c_n\}$满足当$n \ge N$时，有$a_n \le b_n \le c_n$ ，且$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$，则$\lim\limits_{n\to\infty} b_n = L$。

③单调数列极限存在准则（Monotonic Sequence Theorem）:单调有界数列极限一定存在。（即：单调递增的有上界的数列，其极限一定存在；单调递降的有下界的数列，其极限一定存在。）

④第二个重要极限：

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e, \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e, \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e, \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

以及第二个重要极限的其他表现形式：

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e, \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1, \lim_{x\to 0}\frac{e^x-1}{x} = 1, \lim_{x\to 0}\frac{a^x-1}{x} = \ln a \quad (a>0)$$

⑤定理：如果$\lim f(x) = A, A>0, \lim g(x) = B$,则$\lim f(x)^{g(x)} = A^B$。

二、习题解答 （answers）

Exercise 1.6

1.(1) A sequence a_n with $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2+a_n}$,

Proof. The sequence a_n is increasing and bounded above.

$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2+\sqrt{2}}, \quad a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$, It is evident that $\{a_n\}$ is increasing.

$a_1 = \sqrt{2} < 2$.

Suppose that $a_n < 2$, then $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$.

According to mathematical induction, we know that $\{a_n\}$ is bounded above by 2,

I mean that bounded above by 3. So by the monotonic sequence Theorem, we know that

$\lim\limits_{n\to\infty} a_n = A$ exists.

$a_{n+1} = \sqrt{2+a_n}$, squaring both sides, we have

${a_{n+1}}^2 = 2 + a_n$, taking limit on both sides, we get

$A^2 = 2 + A, A^2 - A - 2 = 0$, then $A = 2, \quad A = -1$ (omited)

The reason it that the order-preserving property, that is,

$If \quad a_n > 0 \quad ,then \quad \lim\limits_{n\to\infty} a_n \geq 0$

(Note . Alternatively, if you want to prove the sequence $\{a_n\}$ is increasing, you can by means of mathematical induction.

Firstly, it is evident that $a_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = a_2$;

Secondly, suppose that $a_n < a_{n+1}$, then we can prove that $\sqrt{2+a_n} < \sqrt{2+a_{n+1}}$, that is,

$a_{n+1} < a_{n+2}$.

Then according to induction, we know that the sequence $\{a_n\}$ is increasing.)

2. Show that the sequence defined by

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}$$

Is increasing and $a_n < 3$ for all n .Deduce that $\{a_n\}$ is convergent and find its limit.

Proof. Let us compute the difference of two consecutive terms of $\{a_n\}$,

$$a_{n+2} - a_{n+1} = \left(3 - \frac{1}{a_{n+1}}\right) - \left(3 - \frac{1}{a_n}\right) = \frac{1}{a_n} - \frac{1}{a_{n+1}} = \frac{(a_{n+1} - a_n)}{a_n a_{n+1}},$$

Then $\dfrac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} = \dfrac{1}{a_n a_{n+1}}$, we know that $a_1 = 1, \quad a_2 = 2$, then

$\dfrac{a_3 - a_2}{a_2 - a_1} = \dfrac{1}{a_1 a_2} > 0$, so the denominator and the numerator are both positive , so

$a_3 > a_2 > a_1$, inductively, if $a_{n+1} > a_n$, and , $\dfrac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} = \dfrac{1}{a_n a_{n+1}}$ is positive, so

$a_{n+2} > a_{n+1}$.

Therefore, $\{a_n\}$ is increasing and bounded above by 3. According to the monotonic Sequence Theorem, the $\lim_{n\to\infty} a_n$ exist. Let $\lim_{n\to\infty} a_n = A$, by $a_{n+1} = 3 - \frac{1}{a_n}$, take limits on both sides, we have

$\lim_{n\to\infty} a_{n+1} = 3 - \frac{1}{\lim_{n\to\infty} a_n}$, $A = 3 - \frac{1}{A}$, solve the equation, we have $A = \frac{3+\sqrt{5}}{2}$,

$A = \frac{3-\sqrt{5}}{2}$ does not hold. Because $\{a_n\}$ is increasing and $1 \le a_n$ for all n. By order-preserving property, we know that $1 \le \lim_{n\to\infty} a_n = A$.

3. Find the following limits.

(1)

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{3x} = \lim_{x\to\infty}\left[1+\left(-\frac{2}{x}\right)\right]^{\frac{1}{-\frac{2}{x}}\cdot(-6)} = \lim_{x\to\infty}\left\{\left[1+\left(-\frac{2}{x}\right)\right]^{\frac{1}{-\frac{2}{x}}}\right\}^{-6} = e^{-6}$$

(2)

$$\lim_{x\to 0}(1-2x)^{\frac{1}{x}} = \lim_{x\to 0}\left[(1-2x)^{\frac{1}{-2x}}\right]^{-2} = e^{-2}$$

(3)

$$\lim_{n\to\infty} 3^n \sin\frac{\pi}{3^n} = \lim_{n\to\infty}\frac{\sin\frac{\pi}{3^n}}{\frac{\pi}{3^n}}\cdot\pi = \pi$$, by using the first important limit.

(4)

$$\lim_{n\to\infty}\left(1+\frac{2}{5^n}\right)^{5^n} = \lim_{n\to\infty}\left[\left(1+\frac{2}{5^n}\right)^{\frac{5^n}{2}}\right]^2 = e^2$$

(5)

$$\lim_{x\to 0^+}(\cos x)^{\frac{1}{x}} = \lim_{x\to 0^+}\left[(1-2\sin^2\frac{x}{2})^{\frac{1}{-2\sin^2\frac{x}{2}}}\right]^{\frac{-2\sin^2\frac{x}{2}}{x}} = e^0 = 1$$

By using the Theorem

If $\lim f(x) = A > 0, \lim g(x) = B$, then $\lim f(x)^{g(x)} = A^B$.

(6).

$$\lim_{x\to 1}(2-x)^{\sec\frac{\pi x}{2}} = \lim_{x\to 1}\left\{\left[1+(1-x)\right]^{\frac{1}{1-x}}\right\}^{(1-x)\sec\frac{\pi x}{2}}$$

Let $g(x) = (1-x)\sec\frac{\pi x}{2}$, find the limit

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}(1-x)\sec\frac{\pi x}{2} = \lim_{x\to 1}\frac{(1-x)}{\cos\frac{\pi x}{2}} = \lim_{t\to 0}\frac{t}{\cos\frac{\pi}{2}(1-t)} = \lim_{t\to 0}\frac{t}{\sin\frac{\pi}{2}t} = \lim_{t\to 0}\frac{\frac{\pi}{2}t}{\sin\frac{\pi}{2}t}\cdot\frac{2}{\pi} = \frac{2}{\pi}$$

So,

$$\lim_{x\to 1}(2-x)^{\sec\frac{\pi x}{2}} = \lim_{x\to 1}\left\{\left[1+(1-x)\right]^{\frac{1}{1-x}}\right\}^{(1-x)\sec\frac{\pi x}{2}} = e^{\frac{2}{\pi}}$$

Prove that

$$\lim_{n\to\infty}\sqrt[n]{n} = 1$$

Proof. We need to know that any number greater than 1 , its positive power is also greater than 1.

Which means that

And We have known that the arithmetic average of several positive numbers is no less than its geometric average, which can be expressed in the following formula,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \quad (where \quad x_i > 0)$$

So,

$$< 1\sqrt[n]{n} = \left(\sqrt{n} \cdot \sqrt{n} \cdot 1 \cdots 1\right)^{\frac{1}{n}} \leq \frac{\sqrt{n} + \sqrt{n} + 1 + 1 + \cdots 1}{n} = \frac{2\sqrt{n}}{n} + \frac{n-2}{n} = \frac{2}{\sqrt{n}} + 1 - \frac{2}{n},$$

Then, by using Sandwich Theorem, the limit $\lim_{n \to \infty} \sqrt[n]{n} = 1$