Chapter 25 : Slope search.

In which we make use of a nice mathematical property called monotonicity.

Given f[0..M,0..N] of int where $\{0 \le M \wedge 0 \le N\}$. We are told that f is ascending in both of its arguments.

The problem specification is as follows

$\{\langle \exists\ i,j : 0 \le i \le M \wedge 0 \le j \le N : f.i.j = X \rangle\}$

S

$\{ 0 \le a \le M \wedge 0 \le b \le N \wedge f.a.b = X\}$

Domain modelling.

We make a model of our problem domain.

* (0) C.m.n $=$ $\langle \exists\ i,j : m \le i \le M \wedge 0 \le j \le n : f.i.j = X \rangle$

From this we can derive the following theorems

\- (1) C.0.N $\equiv$ true[1]

We observe

C.m.n

$=$

$\langle \exists\ i,j : m \le i\ \ M \wedge 0 \le j \le n : f.i.j = X \rangle$

$=$ {Split off i = m term }

$\langle \exists\ i,j : m+1 \le i \le M \wedge 0 \le j \le n : f.i.j = X \rangle \vee \langle \exists\ j : 0 \le j \le n : f.m.j = X \rangle$

$=$ { (0), (3) }

C.(m+1).n $\vee$ D.n

So we have

\- (2) C.m.n $\equiv$ C.(m+1).n $\vee$ D.n ,$0 \le m < M$

* (3) D.n $\equiv$ $\langle \exists\ j : 0 \le j \le n : f.m.j = X \rangle$

[1] This is simply what is given in the Precondition.

Similarly, we observe

$$
\begin{array}{ll}
 & C.m.n \\
= & \\
 & \langle \exists\, i,j : m \le i \le M \wedge 0 \le j \le n : f.i.j = X \rangle \\
= & \quad \{\text{Split off } j = n \text{ term}\} \\
 & \langle \exists\, i,j : m \le i \le M \wedge 0 \le j \le n\text{-}1 : f.i.j = X \rangle \vee \langle \exists\, i : m \le i \le M : f.i.n = X \rangle \\
= & \quad \{(0), (5)\} \\
 & C.m.(n\text{-}1) \vee E.m
\end{array}
$$

- (4) C.m.n $\equiv$ C.m.(n-1) $\vee$ E.m ,$0 < n \le N$

* (5) E.m $\equiv$ $\langle \exists\, i : m \le i \le M : f.i.n = X \rangle$

Now let us consider D.n and E.m in turn.

- (6) D.n $\equiv$ false $\Leftarrow$ f.m.n $<$ X

- (7) D.n $\equiv$ true $\Leftarrow$ f.m.n $=$ X

- (8) D.n $\equiv$? $\Leftarrow$ f.m.n $>$ X

- (9) E.m $\equiv$? $\Leftarrow$ f.m.n $<$ X

- (10) E.m $\equiv$ true $\Leftarrow$ f.m.n $=$ X

- (11) E.m $\equiv$ false $\Leftarrow$ f.m.n $>$ X

Choose invariants.
For our invariants we choose the following

P0 : C.a.b
P1 : $0 \le a \le M \wedge 0 \le b \le N$

Guard.
We choose as our guard

f.a.b $\neq$ X

Establish Invariant.

$$a, b := 0, N$$

Termination.
Upon termination of the loop we note

$$P0 \wedge P1 \wedge f.a.b = X \Rightarrow Post$$

Loop Body.
We now calculate the loop body.

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition of } P0\} \\
 & C.a.b \\
= & \quad \{\ (2)\} \\
 & C.(a+1).b \vee D.b \\
= & \quad \{\text{case analysis } f.a.b < X\ (6)\} \\
 & C.(a+1).b \vee false \\
= & \quad \{\ ID \vee\} \\
 & C.(a+1).b \\
= & \quad \{WP\ ) \\
 & (a := a+1).P0
\end{array}
$$

Which gives us the program fragment

$$\text{if } f.a.b < X \rightarrow a := a+1$$

Because of the guard we do not consider the case f.a.b = X.

Now we seek to exploit E. We observe

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition of } P0\} \\
 & C.a.b \\
= & \quad \{\ (4)\ \} \\
 & C.a.(b-1) \vee E.a \\
= & \quad \{\text{case analysis } f.a.b > X\ (11)\} \\
 & C.a.(b-1) \vee false \\
= & \quad \{\ ID \vee\} \\
 & C.a.(b-1) \\
= & \quad \{WP\} \\
 & (b := b-1).P0
\end{array}
$$

This gives us the program fragment

$$\text{if } f.a.b > X \rightarrow b := b-1$$

Because of the guard we do not need to consider the case f.a.b = X.

Finished Program.

Putting this together we arrive at our finished program

```
a, b := 0, N {P0 ∧ P1}
;do f.a.b ≠ X →          {P0 ∧ P1 ∧ f.a.b ≠ X}

                          if f.a.b < X → a := a+1
                          [] f.a.b > X → b := b-1
                          fi

                          {P0 ∧ P1}
od
{Post}
```

This has temporal complexity O(M+N).