第 1.7 节：两个无穷小量的比较（Comparing with two Infnitesimals）

一、内容提要（contents）

①无穷小量：以零为极限的变量。

②高阶无穷小量：若在某个变化过程中$\alpha \to 0, \beta \to 0$且$\lim \frac{\beta}{\alpha} = 0$，则称$\beta$为$\alpha$的高阶无穷小量，记做$\beta = o(\alpha)$。

③同阶无穷小量：若在某个变化过程中$\alpha \to 0, \beta \to 0$且$\lim \frac{\beta}{\alpha} = c \neq 0$，则称$\beta$为$\alpha$的同阶无穷小量。

④等价无穷小量：若在某个变化过程中$\alpha \to 0, \beta \to 0$且$\lim \frac{\beta}{\alpha} = 1$，则称$\beta$为$\alpha$的等价无穷小量,记做$\beta \sim \alpha$。

⑤k阶无穷小量：若在某个变化过程中$\alpha \to 0, \beta \to 0$且$\lim \frac{\beta}{\alpha^k} = c \neq 0$，则称$\beta$为$\alpha$的$k$阶无穷小量（$k$为正整数）。

⑥等价无穷小代换定理（Equivalent Infinitesimal Substitution Theorem）:若在某个变化过程中$\alpha \to 0, \beta \to 0$，$\alpha' \to 0, \beta' \to 0$，且$\alpha \sim \alpha'$，$\beta \sim \beta'$，$\lim \frac{\beta'}{\alpha'}$存在，则$\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$。

⑦当$x \to 0$时，有$x \sim \sin x \sim \tan x \sim \arctan x \sim \arcsin x \sim \ln(1+x) \sim e^x - 1$。

⑧当$x \to 0$时，$(1+x)^\alpha - 1 \sim \alpha x$，其中$\alpha$为任意实数。

二、习题解答

Exercise 1.7

1. Prove that as $x \to 0$, the following statements are true.

(1) $\arctan x \sim x$

Because $\lim_{x \to 0} \frac{\arctan x}{x} = 1$

(2) (2) $\sec x - 1 \sim \frac{x^2}{2}$

Because

$$\lim_{x \to 0} \frac{\sec x - 1}{\frac{x^2}{2}} = \lim_{x \to 0} \frac{\frac{1}{\cos x} - 1}{\frac{x^2}{2}} = \lim_{x \to 0} \frac{1 - \cos x}{\frac{x^2}{2}} \cdot \frac{1}{\cos x} = \lim_{x \to 0} \frac{1 - \cos^2 x}{\frac{x^2}{2}} \cdot \frac{1}{(1 + \cos x)\cos x} = 1$$

2. Finding the following limits by using the Equivalent Infinitesimal Substitution Theorem.

(1).

$$\lim_{x\to 0}\frac{\tan 3x}{2x}=\lim_{x\to 0}\frac{3x}{2x}=\frac{3}{2}$$

(2).

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\tan x(1-\cos x)}{\sin^3 x}=\lim_{x\to 0}\frac{x(1-\cos x)}{x^3}=\lim_{x\to 0}\frac{(1-\cos x)}{x^2}=\lim_{x\to 0}\frac{\frac{x^2}{2}}{x^2}=\frac{1}{2}$$

(3)

$$\lim_{x\to 0}\frac{\sin x-\tan x}{(\sqrt[3]{1+x^2}-1)(\sqrt{1+\sin x}-1)}=\lim_{x\to 0}\frac{\tan x(\cos x-1)}{(\sqrt[3]{1+x^2}-1)(\sqrt{1+\sin x}-1)}$$

$$=\lim_{x\to 0}\frac{x\cdot(-\frac{x^2}{2})}{(\frac{1}{3}x^2)\cdot(\frac{1}{2}\sin x)}$$

$$=\lim_{x\to 0}\frac{x\cdot(-\frac{x^2}{2})}{(\frac{1}{3}x^2)\cdot(\frac{1}{2}x)}$$

$$=-3$$

Note:

$(1+x)^{\alpha}-1\sim\ \alpha x$ as $x\to 0$. So $\sqrt[3]{1+x^2}-1\sim\ \frac{1}{3}x^2$, and so on.

(4)

By using Equivalent Infinitesimal Substitution Theorem, $\lim_{x\to 0}\frac{e^x-1}{\sqrt[4]{1+x}-1}=\lim_{x\to 0}\frac{x}{\frac{1}{4}x}=4$