Chapter 5
Rotation of a Rigid Body About a Fixed Axis

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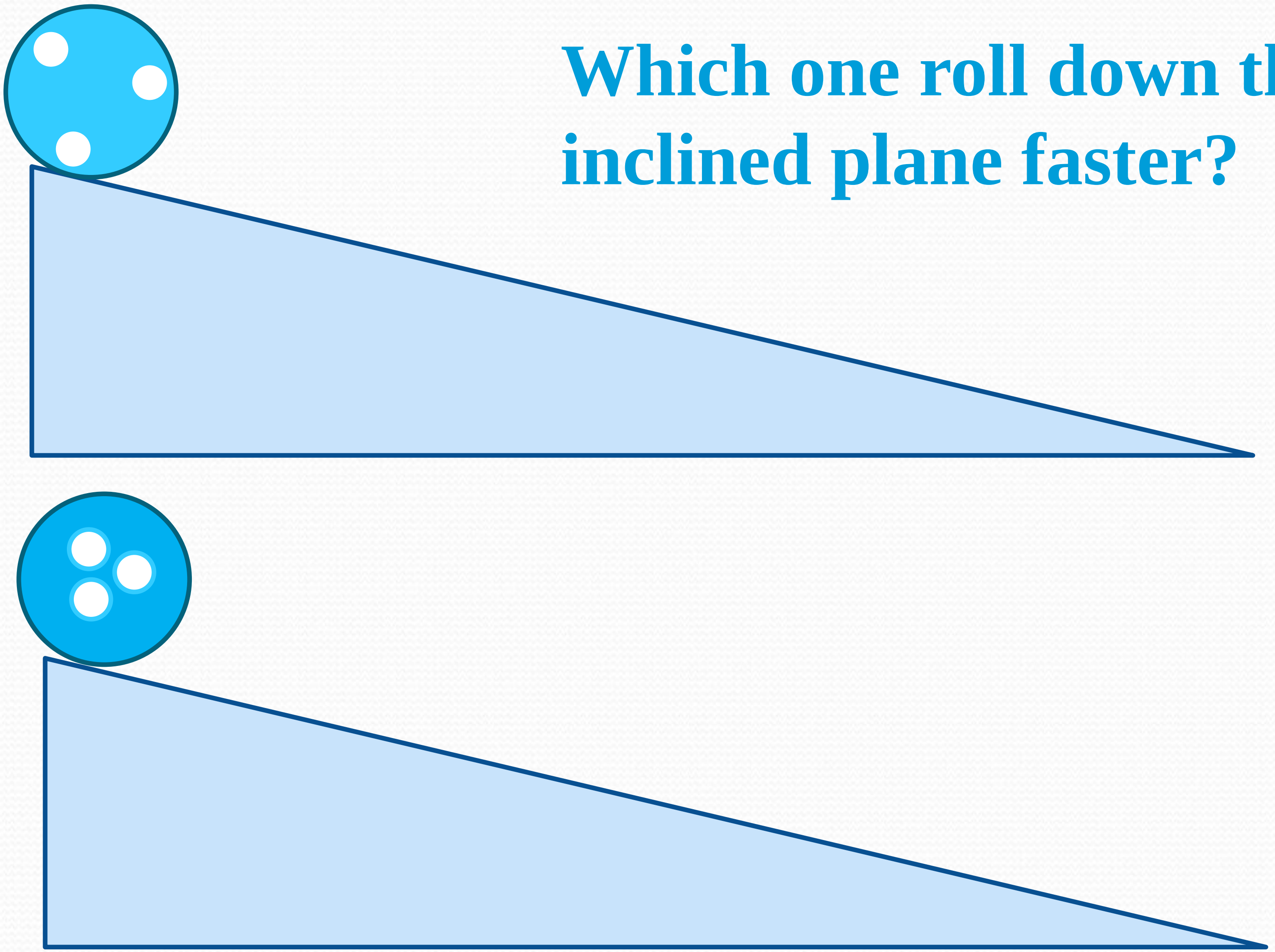
Which one roll down the inclined plane faster?

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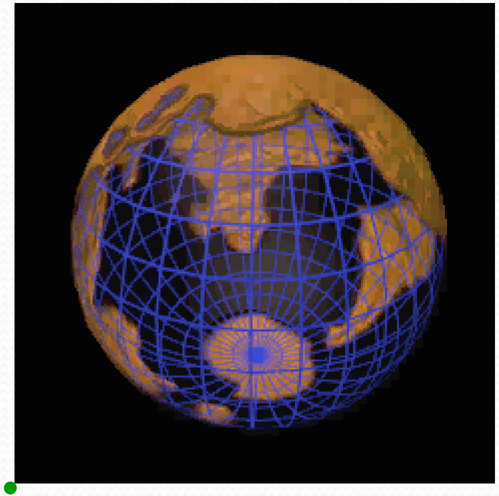
§ 5.1 The Rotational kinematics of a Rigid Body about a Fixed Axis

Rigid bodies are objects of fixed form that do not distort or deform (change shape) as they move.



As the body rotates

Each particle moves

Relative positions don’t change

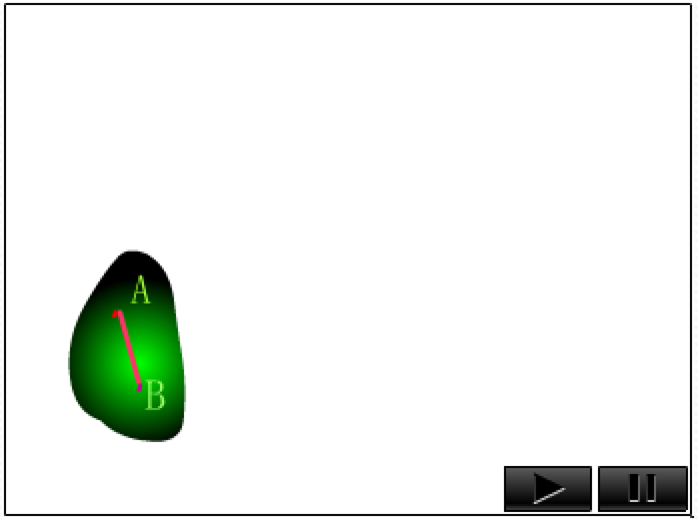
Cities on the earth are always moving

But they don’t get closer together.

◆Translational motion of a rigid body

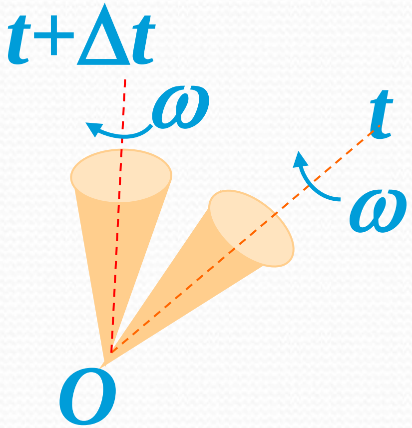
During translational motion, all points of the body describe identical trajectories, that is, trajectories coincident when superposed, and have at every instant velocities and accelerations that are the same in magnitude and direction.

The translational motion of a body is therefore treated in much the same way as the kinematics of a particle.



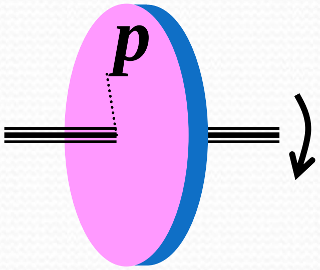
◆ Rotates in three dimensions of a rigid body

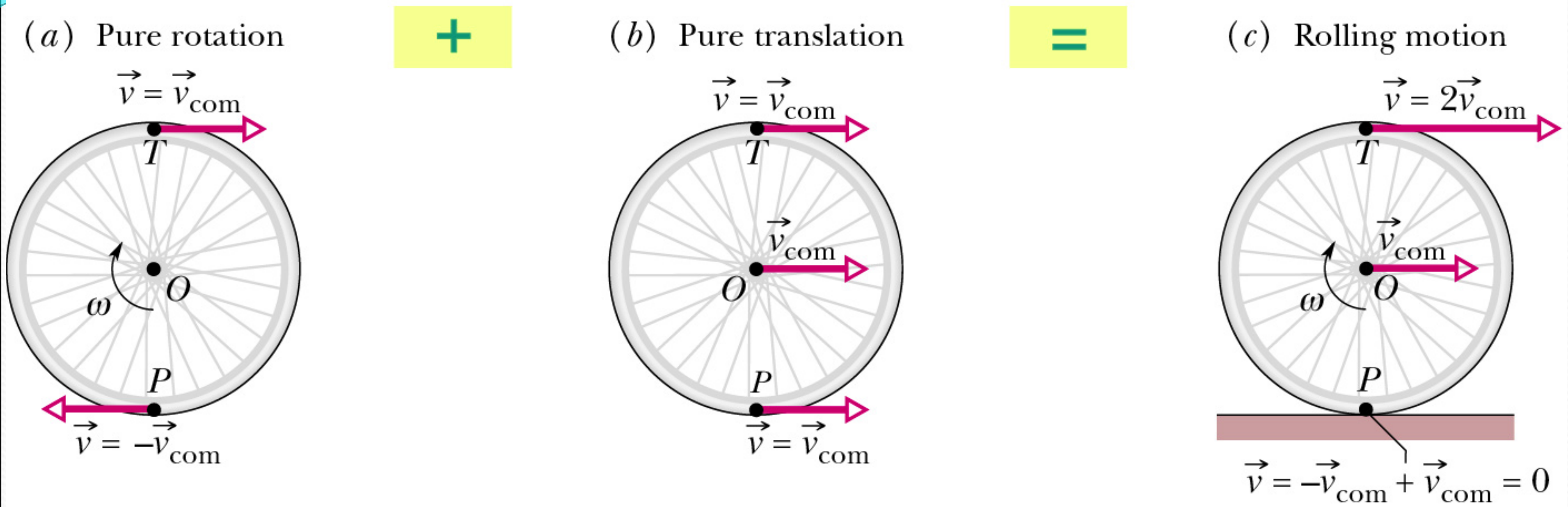
There exists a fixed point, from which any other point *P* maintains a constant distance. This means that the point *P* can at most undergo a rotation relative to the fixed point.



◆ Rotates in Two dimensions of a rigid body

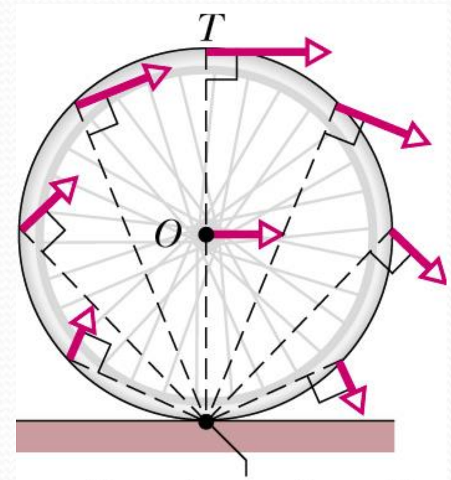
There exists a fixed axis, from which any other point *P* maintains a constant distance. At any instant, every part of a rotating rigid body has the same angular velocity.





Motion of the wheel can be viewed as rotation about the contact point.

In this reference frame the point of contact is at rest.

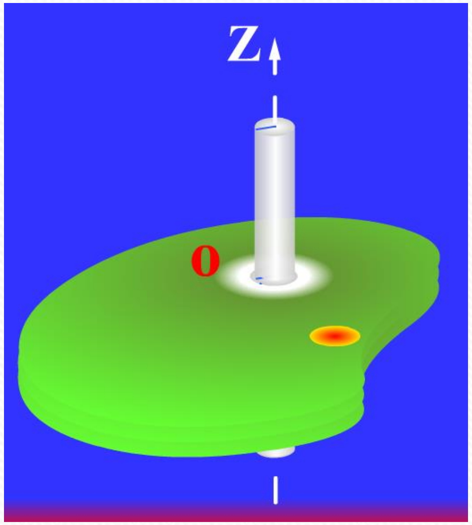


◆ The rotation of rigid body about a fixed axis

Fixed axis: axis which is at rest in some inertial frame of reference and does not change direction relative to that frame.

First, recall what we learned about Uniform Circular Motion:

$$\begin{cases} \omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t} \\ \alpha = \dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \end{cases} \qquad \begin{cases} a_n = R\omega^2 \\ a_t = R\alpha \end{cases}$$



Summary
(with comparison to 1-D kinematics)

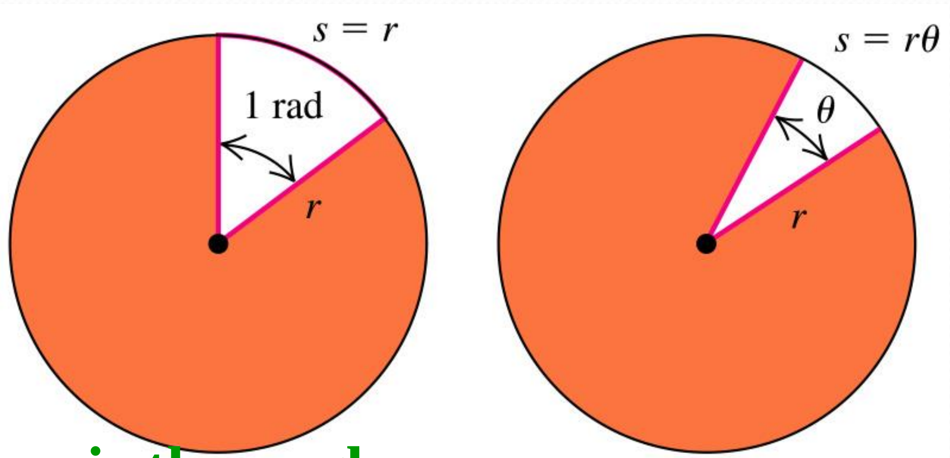
Angular	Linear
$\alpha = \text{constant}$	$a = \text{constant}$
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_0 t + \frac{1}{2}at^2$

And for a point at a distance *R* from the rotation axis:

$$S = R\theta \qquad v = R\,\omega \qquad a_t = R\alpha$$

- **Both angular position and angular displacement will most commonly be expressed in *radians*. To convert between radians, revolutions, and degrees use the conversion:**

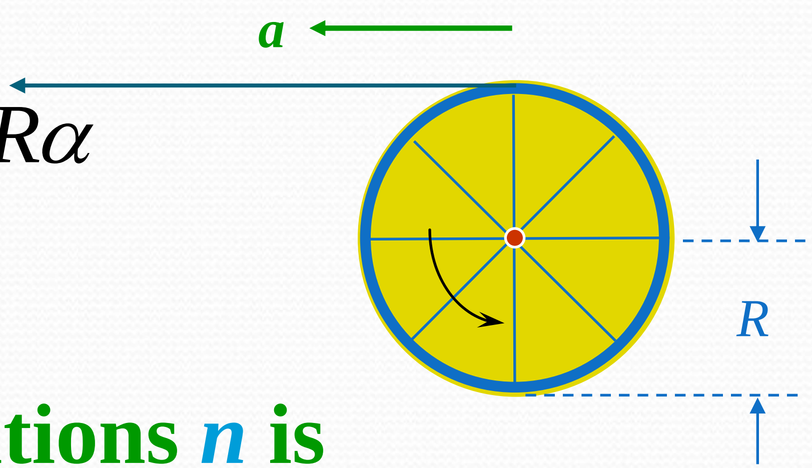
1 revolution = 2π radians = 360 degrees



One radian is the angle subtended at the center of a circle by an arc with a length equal to the radius of this circle.

Angle θ is subtended by an arc with a length S on a circle of the radius r

Example5.1 A wheel with radius R=0.4m rotates freely about a fixed axle. There is a rope wound around the wheel. Starting from rest at t =0, the rope is pulled such that it has a constant acceleration a=4m/s^2. How many revolutions has the wheel made after 10s?



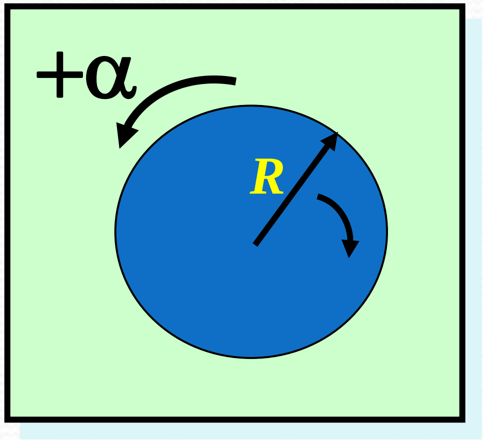
Solution According to $a_t = R\alpha$

We have $\alpha = \dfrac{a_t}{R} = \dfrac{a}{R} = 10/\mathrm{s}^2$

Then the number of revolutions n is

$$n = \frac{\theta}{2\pi} = \frac{1}{2\pi} \cdot \frac{1}{2}\alpha t^2 = \frac{500}{2\pi} \approx 80$$

Example 5.2: A drum is rotating clockwise initially at 100 rpm and undergoes a constant counterclockwise acceleration of 3 rad/s^2 for 2s. What is the angular displacement?



Solution Given: $\omega_o = -100$ rpm;
$t=2s$ $\alpha = +3\ \text{rad/s}^2$

$$100\,\frac{\text{rev}}{\text{min}}\left(\frac{1\ \text{min}}{60\ \text{s}}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right) = 10.5\ \text{rad/s}$$

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2 = (-10.5)(2) + \frac{1}{2}(3)(2)^2$$

θ = -21rad + 6 rad = -15 rad

Net angular displacement is clockwise (-)

Example 5.3 **The equation of motion of a point on the rim of a flywheel with radius R=2m is S=0.1t^3 (m). What are the tangential and normal accelerations when speed is v=30 m/s?**

Solution $v = \mathrm{d}S/\mathrm{d}t = 0.3t^2$ m/s

Let v=30 m/s, we get t=10s

So $a_t = \mathrm{d}v/\mathrm{d}t = 0.6t$ m/s^2

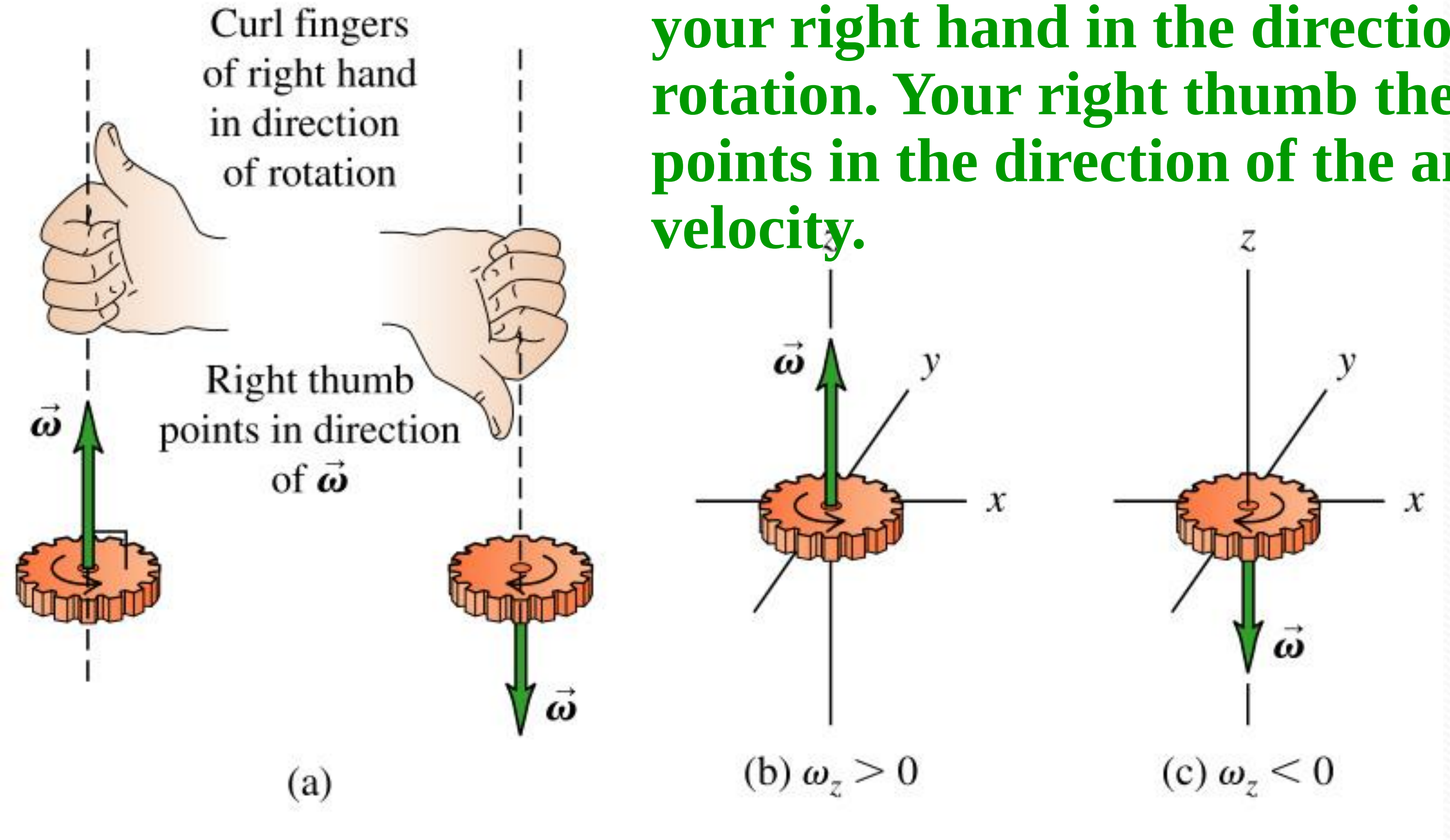
Substituting t=10s, **we obtain**

$a_t = 0.6 \times 10 = 6$ m/s^2

And $a_n = v^2/R = (30)^2/2 = 450$ m/s^2

◆Angular velocity is a vector quantity

- The direction of angular velocity is defined using the ***right-hand rule***.
- To use this rule, curl the fingers of your right hand in the direction of rotation. Your right thumb then points in the direction of the angular velocity.

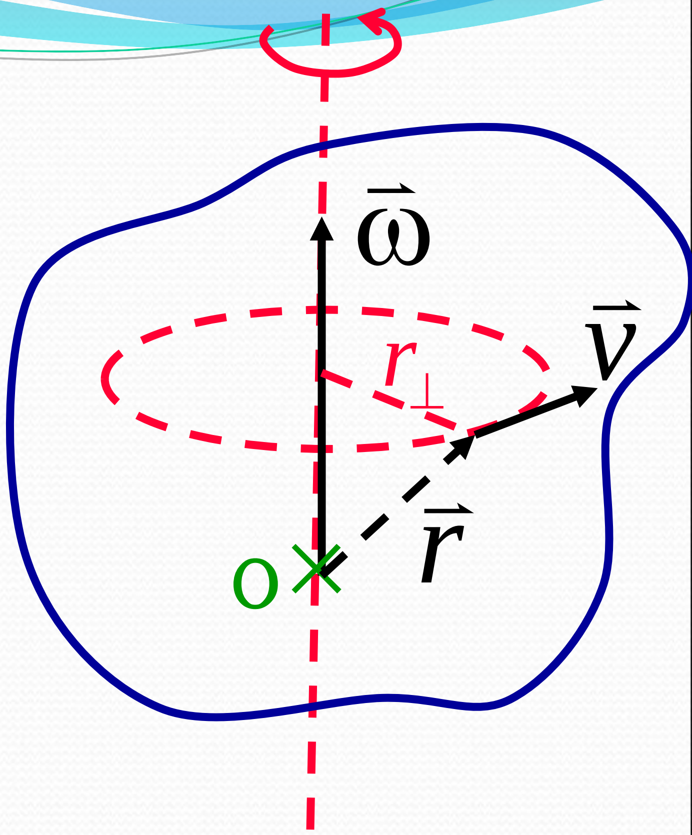


$$\vec{v} = \vec{\omega} \times \vec{r}$$

Example5.4 $\vec{\omega} = 60\vec{k}$ rev/min

$$\vec{r} = (3\vec{i} + 4\vec{j} + 5\vec{k}) \times 10^{-2} \text{ m}$$

Find the linear velocity $\vec{v}$ **.**



Solution

$$\vec{\omega} = (60 \times 2\pi / 60)\vec{k}$$

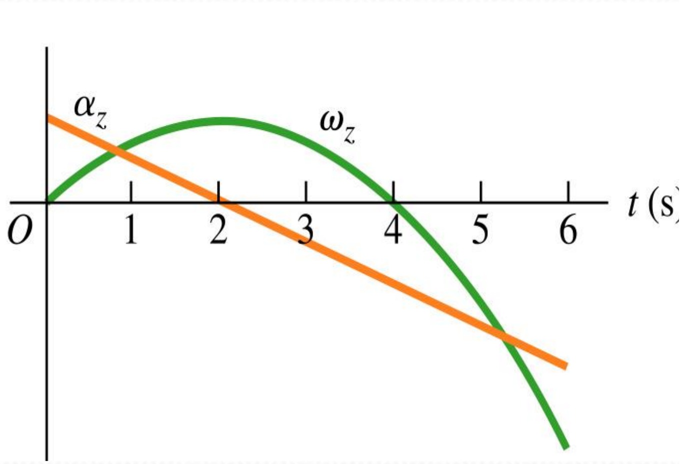
$$= 2\pi\vec{k} \text{ (rad/s)}$$

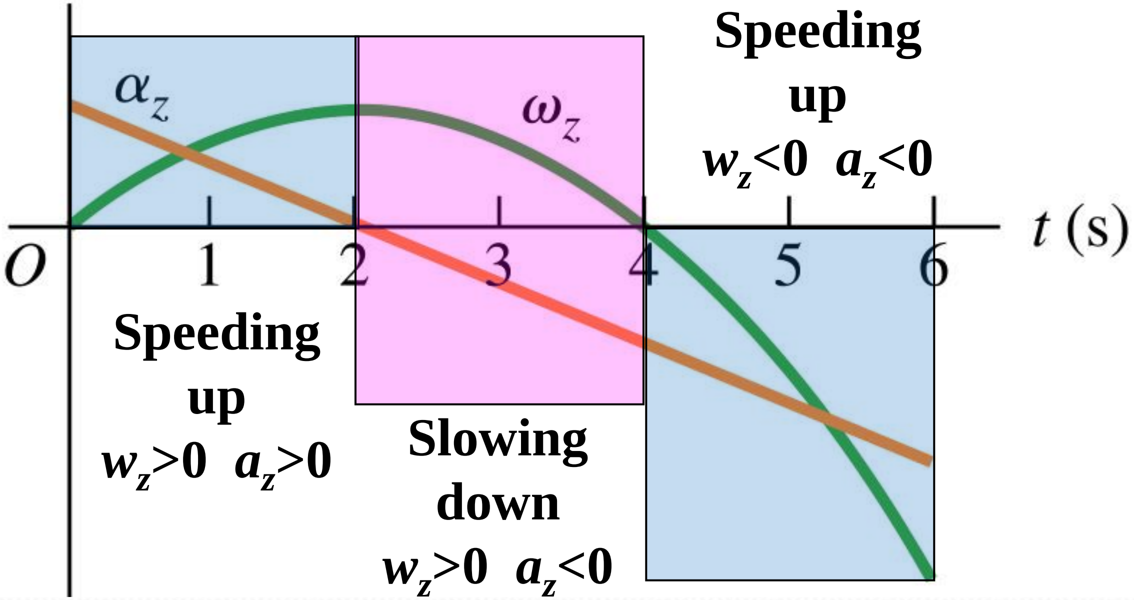
$$\vec{v} = \vec{\omega} \times \vec{r} = 2\pi\,\vec{k} \times (3\vec{i} + 4\vec{j} + 5\vec{k}) \times 10^{-2}$$

$$= (6\pi\vec{j} - 8\pi\vec{i}) \times 10^{-2}$$

$$= -0.251\vec{i} + 0.188\vec{j} \text{ (m/s)}$$

Example5.5 The graph of w_z and α_z versus time for a rotating object. During which time intervals is the rotation "speeding up"? During which time intervals is the rotation "slowing down"?

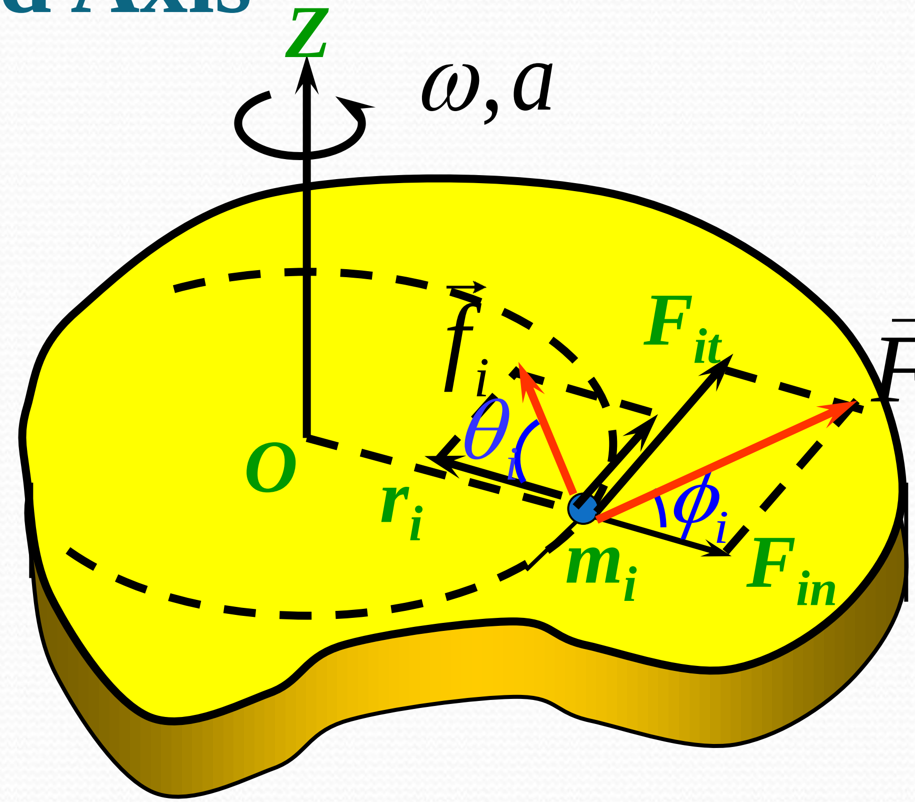




§ 5.2 The Law of Rotation of the Rigid Body about a Fixed Axis

Newton's second law

$$\vec{F}_i + \vec{f}_i = m_i \vec{a}_i$$



Tangential motion equation

$$F_i \sin\varphi_i + f_i \sin\theta_i = m_i a_{it} = m_i r_i \alpha$$

Multiplying the both sides of this equation by r_i

$$F_i r_i \sin\varphi_i + f_i r_i \sin\theta_i = m_i r_i^2 \alpha$$

Summing over all mass points, we have

$$\underbrace{\sum_i F_i r_i \sin\varphi_i}_{M_Z} + \underbrace{\sum_i f_i r_i \sin\theta_i}_{0} = \underbrace{(\sum_i m_i r_i^2)\alpha}_{J\alpha}$$

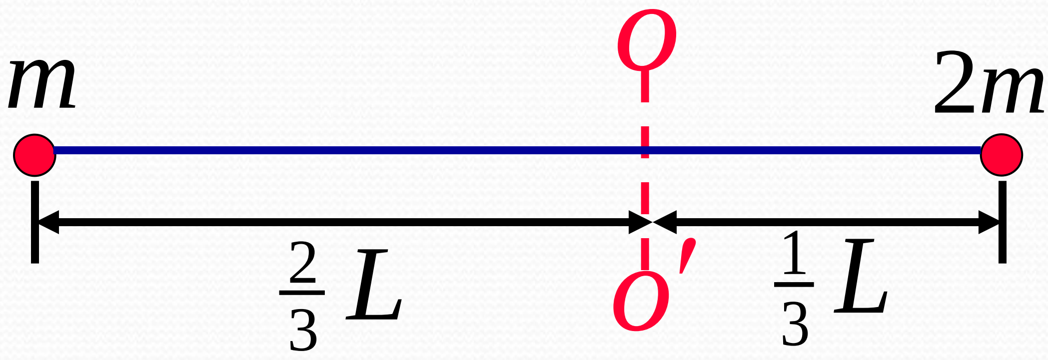
The moment of inertia $J = \sum_i m_i r_i^2 = \int r_i^2 dm$

Net torque

$$\boxed{M_Z = J\alpha = J\frac{d\omega}{dt}}$$

angular acceleration

Example5.6 Find the rotational inertia of the two mass points about the *OO′* axis.



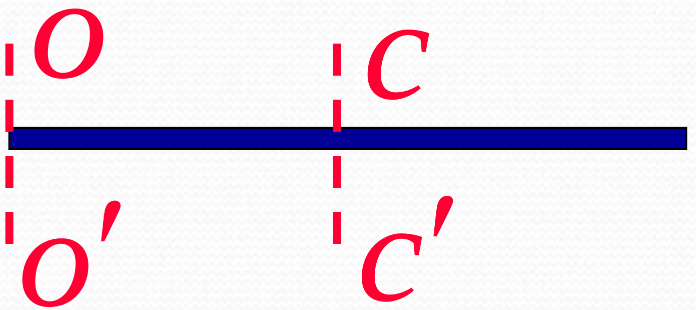
Solution

$$J = \sum_i m_i r_i^2$$

$$= m(\tfrac{2}{3}L)^2 + 2m(\tfrac{1}{3}L)^2$$

$$= \tfrac{2}{3}mL^2$$

Example5.7 Given: a slender rod with mass *m* and length *L*. Find the rotational inertia about *OO′* and *CC′* axes.



Solution

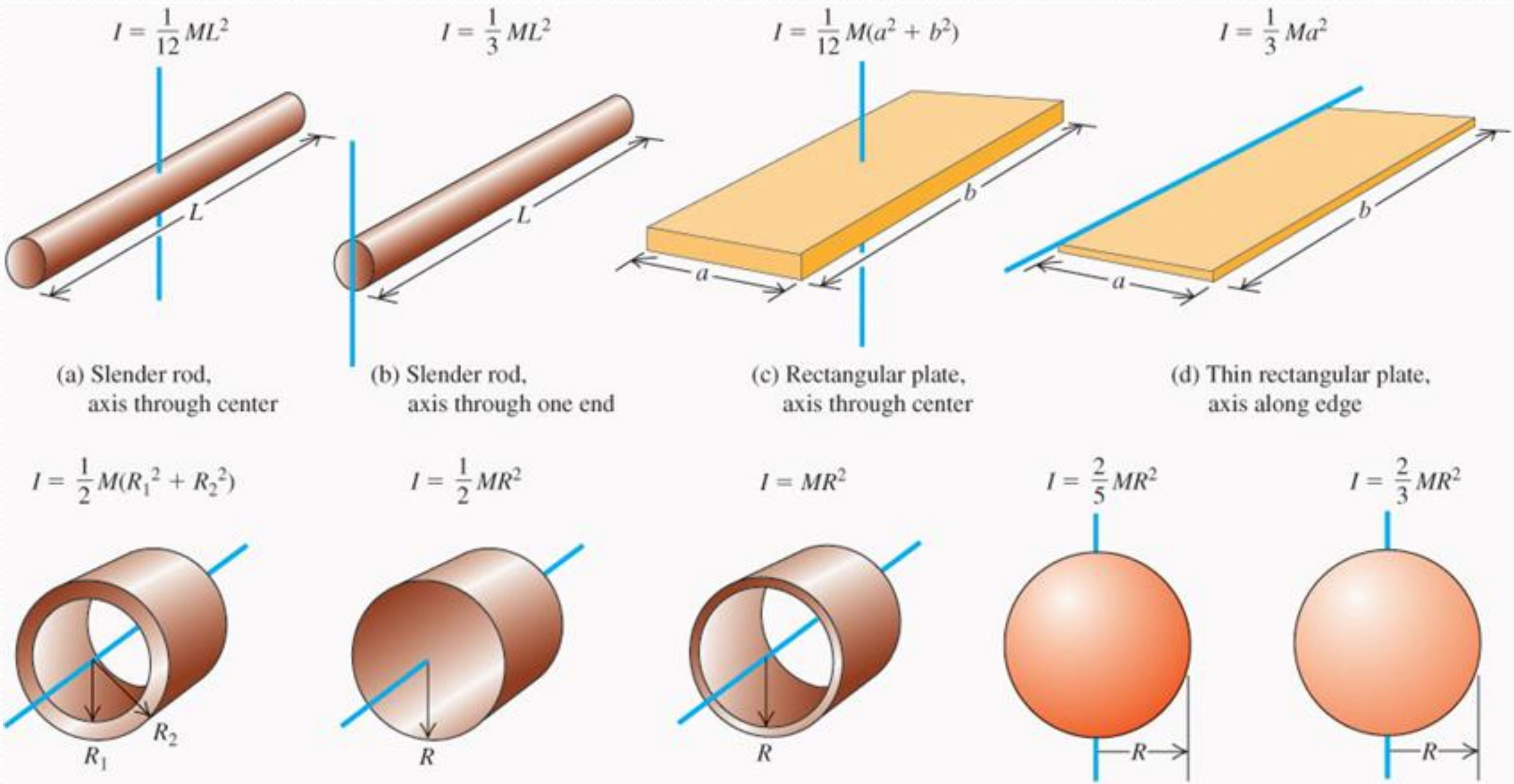
$$J_{OO'} = \sum_i m_i r_i^2 = \int r^2 dm$$

$$= \int_0^L r^2 \frac{m}{L} \mathrm{d}r = \frac{mL^2}{3}$$

For *CC′* axis $J_{CC'} = \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \frac{m}{L} \mathrm{d}r = \frac{mL^2}{12}$

- **The *moment of inertia* is also called *rotational inertia*;**
- **Rotational inertia depends on *the shape, mass distribution and the position of the rotational axis* of the rigid body.**
- **The *rotational inertia* of an object is a measure of the resistance of the object to changes in its rotational motion;**
- ***SI unit* of moment of inertia is the $kg \cdot m^2$**
- **For a solid object the rotational inertia is found by evaluating an *integral*.**

Moment of inertia of various bodies



(a) Slender rod, axis through center

(b) Slender rod, axis through one end

(c) Rectangular plate, axis through center

(d) Thin rectangular plate, axis along edge

(e) Hollow cylinder

(f) Solid cylinder

(g) Thin-walled hollow cylinder

(h) Solid sphere

(i) Thin-walled hollow sphere

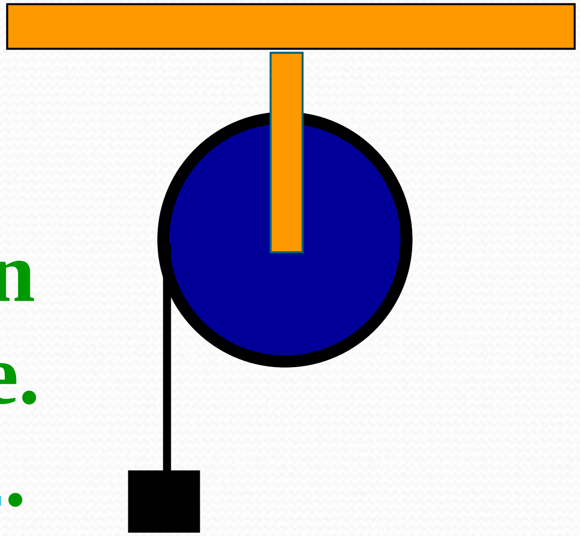
Example5.8 **Two thin circular ring A and B have the same radius and mass. The mass distribution of A is uniform but that of B is not. Compare the moment of inertia J_A and J_B about the axes through the circle center and perpendicular to the circular plane?**

(A) $J_A > J_B$ **(B)** $J_A < J_B$ **(C)** $J_A = J_B$

(D) Be uncertain which one is larger

Solution (C)

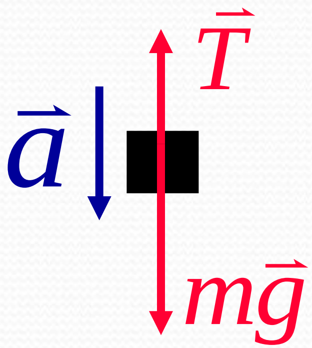
Example5.9 **A pulley can be considered as a uniform disk of mass *M* and radius *R*, mounted on a frictionless fixed horizontal axle. The rotational inertia is $J=MR^2/2$.**

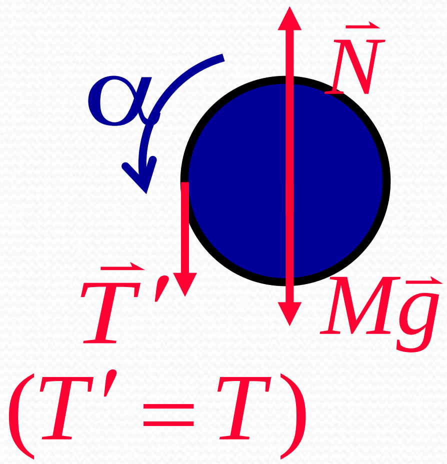


A block of mass *m* hangs from a light cord that is wrapped around the rim of the pulley (Suppose the cord does not slip or stretch). Calculate the acceleration of the falling block, the tension in the cord, and the angular acceleration of the pulley.

Solution

Force diagram





For the block: $mg - T = ma$ ①

For the pulley: $TR = \frac{1}{2}MR^2 \cdot \alpha = \frac{1}{2}MRa$ ②

①②→ $$a = g\frac{2m}{M + 2m}$$

$$\alpha = \frac{a}{R} = \frac{2m}{R(M + 2m)}g$$

And the tension in the cord

$$T = mg\frac{M}{M+2m}$$

We can also get the speed of the block

$$v = at = \frac{2mgt}{2m+M}$$

? If we replace the block by a force F=2mg, the angular acceleration of the pulley is β now. What is the relation of the angular accelerations α and β?

$$\beta > 2\alpha$$

Example5.10 **Several forces acting on a rigid body with a fixed axis simultaneously. If $\sum_i \vec{F}_i = 0$ the rigid body [ ]**

(A) will not rotate.

(B) the angular speed will not change.

(C) the angular speed will change.

(D) the angular speed will change or not.

Solution (D)

The net force is zero but the net torque can be nonzero.

§ 5.3 Work and Energy in Rotational Motion

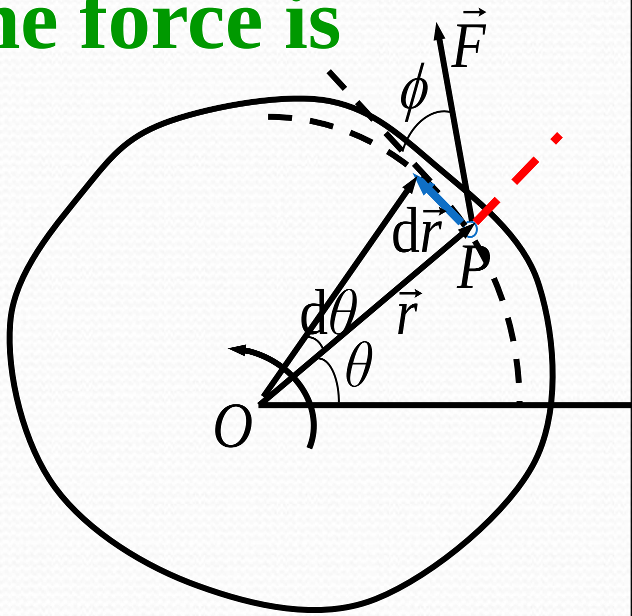
§ 5.3.1 Work done by a torque

Assume a rigid body rotates with an angular displacement $\mathrm{d}\theta$ under the force $\vec{F}$, the work done in this displacement by the force is

$$\mathrm{d}W = \vec{F}\cdot\mathrm{d}\vec{r}$$

$$= F\cos\phi\left|\mathrm{d}\vec{r}\right|$$

$$= F\cos\phi r\mathrm{d}\theta$$



where the torque of the force about the rotational axis is

$$M = F\sin(90^\circ - \phi)r = F\cos\phi r$$

then

$$\mathrm{d}W = M\mathrm{d}\theta$$

For a rotation from angle θ_1 to θ_2

$$W = \int_{\theta_1}^{\theta_2} M\mathrm{d}\theta$$

§ 5.3.2 Rotational kinetic energy

Assume the rigid body consists of n mass points. Take mass point i on the rigid body with mass Δm_i, and it rotates at speed v_i with radius r_i about the Oz axis

$$E_{\mathrm{k}i} = \frac{1}{2}\Delta m_i v_i^2 = \frac{1}{2}\Delta m_i r_i^2 \omega^2$$

The total kinetic energy of the rigid body′s rotation is the sum of all the kinetic energies associated with all of its mass points

$$E_k = \tfrac{1}{2} J\omega^2$$

§ 5.3.2 The theorem of kinetic energy of a rigid body rotating about a fixed axis

$$W = \frac{1}{2}J\omega_2^2 - \frac{1}{2}J\omega_1^2$$

work done by the combined external torque

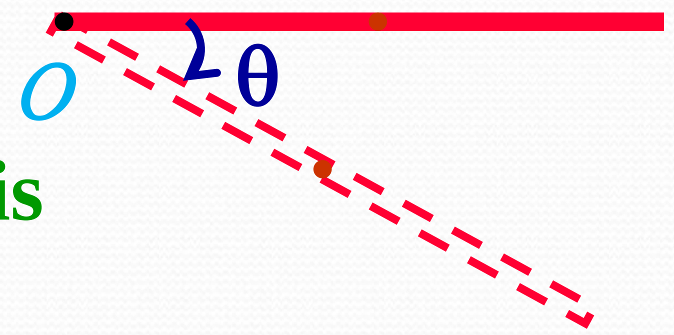
increment of the rotational kinetic energy

The above equation shows that the work done on a rigid body rotating about a fixed axis by the combined external torque is equal to the increment of the rotational kinetic energy of the rigid body.

Note

The gravity potential energy of a rigid body equals to the gravity potential energy of a point with total mass of the body at the center of mass of the body.

Example5.11 A uniform rod of length L and mass m can pivot freely about a horizontal axis through the fixed end O in the vertical plane. Initially, it rests horizontally. Find the Angular speed ω when the angle with the horizontal axis is θ. The rotational inertia of the rod about O is $J=mL^2/3$.



Solution 1 The mechanical energy of the rod-Earth system is conservative:

$$E_p+E_k= \text{const.}$$

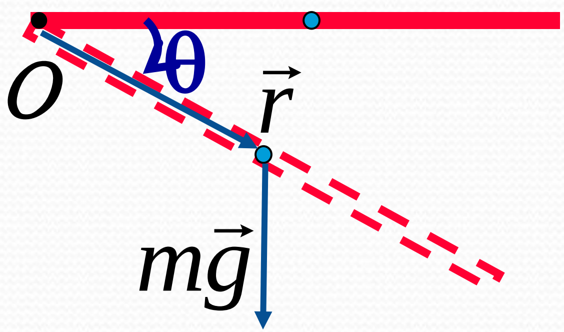
Let the gravity potential energy at the horizontal position zero，we have

$$-mg\cdot\frac{L}{2}\sin\theta+\frac{1}{2}\cdot\frac{1}{3}mL^2\cdot\omega^2=0$$

Then we get $$\omega=\sqrt{\frac{3g\sin\theta}{L}}$$

Solution 2 The magnitude of the torque acting on the rod due to the gravity is

$$M=\frac{1}{2}mgL\cos\theta$$



According to the law of rotation of the rigid body about a fixed axis

$$M=\frac{1}{2}mgL\cos\theta=J\alpha=\frac{1}{3}mL^2\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

$$\frac{1}{2}mgL\cos\theta=\frac{1}{3}mL^2\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{3}mL^2\omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$$

Take the integral of both sides of the above equation

$$\int_0^\theta g\cos\theta\mathrm{d}\theta=\int_0^\omega\frac{2}{3}L\omega\mathrm{d}\omega$$

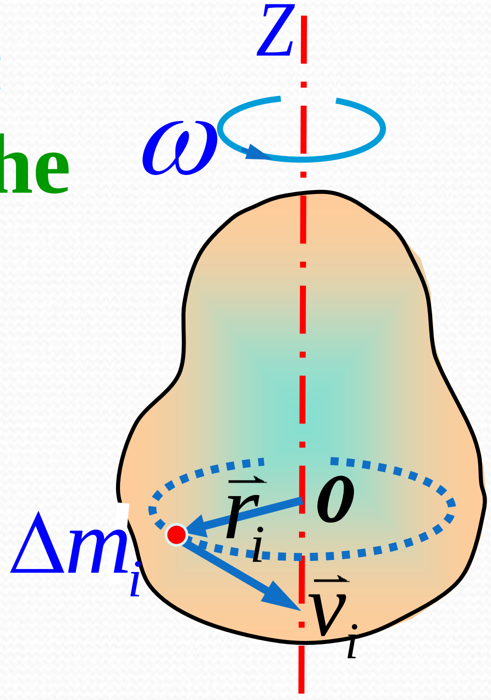
we get

$$\omega=\sqrt{\frac{3g\sin\theta}{L}}$$

§ 5.4 The Law of Conservation of Angular Momentum about a fixed Axis

◆ The Angular Momentum of a Rigid Body Rotating about a Fixed Axis

Assume the rigid body consists of n mass points. Take mass point i on the rigid body with mass Δm_i, and it rotates at speed v_i with radius r_i about the z axis. The angular momentum of Δm_i is



$$L_{iZ} = \Delta m_i v_i r_i = \Delta m_i r_i^2 \omega$$

The angular momentum of the rigid body with respect to the *Oz* axis is

$$L_Z = \sum L_{iZ} = \left(\sum \Delta m_i r_i^2\right)\omega = J_z\omega$$

For the mass of the rigid body is distributed continuously

$$L_Z = \left(\int r^2 dm\right)\omega = J_z\omega$$

Then

$$\frac{dL_Z}{dt} = \frac{d(J_Z\omega)}{dt} = J_Z\frac{d\omega}{dt} = J_Z\alpha = M_Z$$

the time rate of change of the object′s angular momentum

net torque

◆ The Theorem of Angular Momentum of a Rigid Body Rotating about a Fixed Axis

$$\frac{dL_Z}{dt} = M_Z$$

Take the integral of both sides of the above equation

$$\int_{t_1}^{t_2} M_Z dt = \int_{L_1}^{L_2} dL_Z = L_{Z2} - L_{Z1} = J_Z \omega_2 - J_Z \omega_1$$

impulsive moment

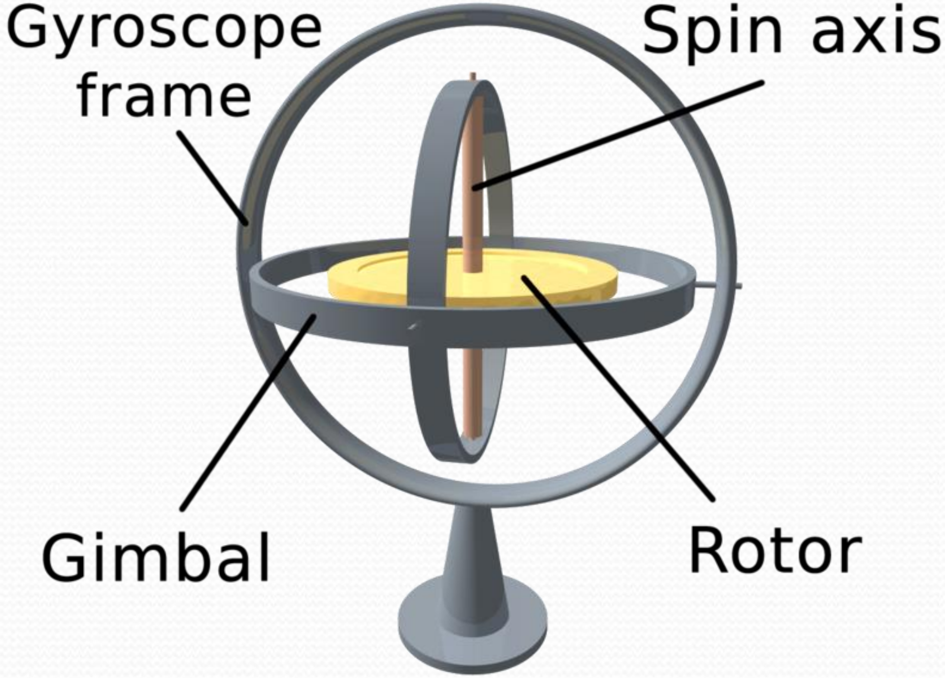
increment of the angular momentum

◆ The Law of Angular Momentum Conservation of a Rigid Body Rotating about a Fixed Axis

$$M_z = 0 \Longrightarrow \frac{dL_Z}{dt} = 0 \Longrightarrow \mathrm{L}_Z = J_Z\omega = const$$

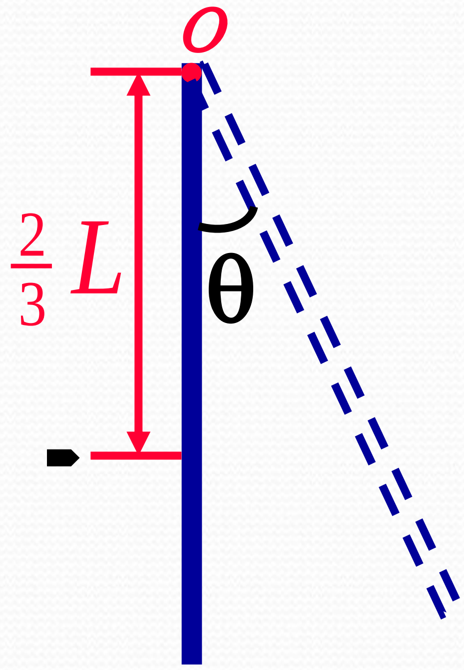
Example5.7

gyroscope



A gyroscope is a device for measuring or maintaining orientation, based on the principles of angular momentum. Mechanically, a gyroscope is a spinning wheel or disk in which the axle is free to assume any orientation. Although this orientation does not remain fixed, it changes in response to an external torque much less and in a different direction than it would without the large angular momentum associated with the disk's high rate of spin and moment of inertia. The device's orientation remains nearly fixed, regardless of the mounting platform's motion, because mounting the device in a gimbal minimizes external torque.

Example5.12 a thin and uniform rod of length L and mass M can rotate without friction about horizontal axis O which passes through the top of the rod. A bullet with mass m is shot to the point with distance $2L/3$ away from the pivot. The bullet is shot in the rod with velocity $\vec{v}_0$. The rotational inertia of the rod about O is $J=mL^2/3$ and $M=4m$. Find the maximum deflection angle of the rod and the bullet with respect to the vertical axis.



Solution **For the bullet-rod system, the net external torque is zero, angular momentum is conserved about a fixed axis.**

$$mv_0 \cdot \tfrac{2}{3}L = [m(\tfrac{2}{3}L)^2 + \tfrac{1}{3}ML^2]\omega$$

Then we get $$\omega = \frac{3v_0}{8L}$$

The mechanical energy of the rod-Earth system is conservative in the process of swing, i.e.

$$\boldsymbol{E_p + E_k = \text{const.}}$$

Let the gravity potential energy at the initial position be zero， we have

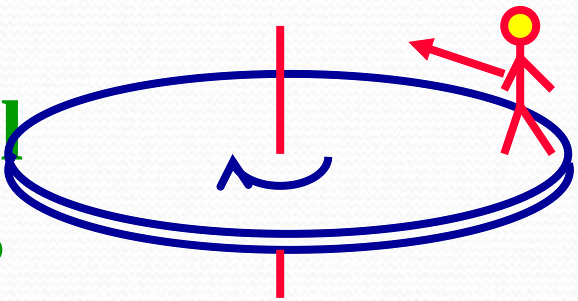
$$\frac{1}{2}[m(\frac{2}{3}L)^2+\frac{1}{3}ML^2]\omega^2 =$$

$$mg\cdot\frac{2}{3}L(1-\cos\theta)+Mg\cdot\frac{1}{2}L(1-\cos\theta)$$

So the maximum deflection angle is

$$\theta=\arccos(1-\frac{3v_0^2}{64gL})$$

Example5.13 **A woman with mass *m* stands at the rim of a horizontal turntable with mass *M* and radius *R* having a moment of inertia of $J=MR^2/2$. The turntable is initially at rest and is free to rotate about a frictionless vertical axle through its center. The woman then starts walking around the rim. Find the angle the table turned relative to the ground when the woman takes a round along the rim.**



Solution **In the process of walking, the angular momentum of the woman-table system is conservative. Assume the angular speed of the woman relative to the table is ω' and the angular speed of the table relative to the ground is ω at instant t, then we have**

$$mR^2(\omega' - \omega) - \tfrac{1}{2}MR^2\omega = 0$$

Then we get $$\omega = \frac{2m}{2m+M}\omega'$$

Multiplied by d*t* on both sides of the above equation

$$\omega \mathrm{d}t = \frac{2m}{2m+M}\omega' \mathrm{d}t$$

And integral the both sides

$$\int_0^{\phi} d\phi = \frac{2m}{2m+M}\int_0^{2\pi} d\phi'$$

The angle the table turned relative to the ground is

$$\phi = \frac{4\pi m}{2m+M}$$

Translational and rotational motions

Translational motion	Rotational motion
m	J
$\vec{v}$	$\vec{\omega}$
$\vec{F}$	$\vec{M}$
$\vec{F} = m\vec{a}$	$\vec{M} = J\alpha$
$\vec{p} = m\vec{v}$	$\vec{L} = J\vec{\omega}$
$E_k = \frac{1}{2}mv^2$	$E_k = \frac{1}{2}J\omega^2$
$A = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$	$A = \frac{1}{2}J\omega_b^2 - \frac{1}{2}J\omega_a^2$
$\int_a^b \vec{F}dt = \vec{P}_b - \vec{P}_a$	$\int_a^b \vec{M}dt = \vec{L}_b - \vec{L}_a$

See you next time!