Chapter 10: The Linear Search Theorem.

In which we introduce the simplest searching algorithm.

Suppose we are given an array f[0..N) of integer, where $\{1 \le N\}$, which is guaranteed to contain at least one occurrence of the value X, and we are asked to find the location of the leftmost X in f, i.e. the smallest index n where f.n = X.

We begin as usual with a problem specification.

$$\{\langle \exists\, j : 0 \le j < N : f.j = X \rangle\}$$

$$S$$

$$\{\langle \forall\, j : 0 \le j < n : f.j \ne X\rangle \wedge f.n = X\}$$

As usual, we begin by developing a model of the problem domain.

* (0) $C.n \quad \equiv \quad \langle \forall\, j : 0 \le j < n : f.j \ne X\rangle \qquad , 0 \le n \le N$

Appealing to the empty range and associativity we get the following theorems

Consider.

$$\begin{array}{ll} & C.0 \\ = & \qquad \{(0) \text{ in model }\} \\ & \langle \forall\, j : 0 \le j < 0: f.j \ne X\rangle \\ = & \qquad \{\text{ empty range }\} \\ & true \end{array}$$

\- (1) $C.0 \quad \equiv \quad true$

Consider

$$\begin{array}{ll} & C.(n+1) \\ = & \qquad \{(0) \text{ in model }\} \\ & \langle \forall\, j : 0 \le j < n+1: f.j \ne X\rangle \\ = & \qquad \{\text{ split off } j = n \text{ term}\} \\ & \langle \forall\, j : 0 \le j < n : f.j \ne X\rangle \wedge f.n \ne X \\ = & \qquad \{(0) \text{ in model}\} \\ & C.n \wedge f.n \ne X \end{array}$$

\- (2) $C.(n+1) \quad \equiv \quad C.n \wedge f.n \ne X \qquad , 0 \le n < N$

Rewrite postcondition using the model.

$$\text{Post} : C.n \wedge f.n = X$$

Choose Invariants.

We choose as our invariants

$$P0: C.n$$
$$P1: 0 \leq n \leq N$$

Establish Invariants.

Theorem (1) in our model shows us that we can establish P0 by the assignment

$$n := 0$$

This also establishes P1.

Termination.

We note that

$$P0 \wedge P1 \wedge f.n = X \Rightarrow \text{Post}$$

Guard

We choose our loop guard to be

$$B : f.n \neq X$$

Variant.

As our variant function we choose

$$K - n$$

Where $0 \leq K < N$ and K is the (as yet unknown) index of the leftmost occurrence of X.

Calculate Loop body.

Decreasing the variant by the assignment n := n+1 is a standard step and maintains P1. Let us see what effect it has on P0

$$
\begin{array}{ll}
 & (n := n+1).\ P0 \\
= & \qquad \{\text{textual substitution}\} \\
 & C.(n+1) \\
= & \qquad \{(2)\ \text{above}\} \\
 & C.n \wedge f.n \neq X \\
= & \qquad \{\ f.n \neq X \text{ at start of loop body}\} \\
 & C.n \\
= & \qquad \{\ P0 \text{ is invariant }\} \\
 & \text{true}
\end{array}
$$

Finished program.

So our finished program is

```
  n := 0
; do f.n ≠ X →

            n := n+1

  od
{C.n ∧ f.n = X}
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