Chapter 13 : Fibonacci.

In which we look at a different shape of Invariant and look at a different way of motivating decreasing the variant vf.

Sometimes, when we are presented with a problem we are given the domain model to start so we do not have to go and develop one ourselves. Here is such an example. Suppose we are given the fibonacci function defined as follows.

* (0) $fib.0 \quad = \quad 0$

* (1) $fib.1 \quad = \quad 1$

* (2) $fib.(n+2) \quad = \quad fib.n + fib.(n+1) \qquad , 0 \le n$

We are given a natural number N and asked to write a program to compute fib.N

Qualities of a good loop invariant.

* It must be easy to establish at the start (easy to make it true)

* It must eventually lead you to a solution of the problem. It must help you make the Postcondition true.

* It must be "relatively easy" to keep it true as you decrease the variant (vf) inside the loop body.

Choosing invariants.

We propose as invariants

P0 : $\alpha * fib.n + \beta * fib.(n+1) = fib.N$
P1 : $0 \le n \le N$

Establishing the Invariants.

The following assignment establishes the invariants

$n, \alpha, \beta := N, 1, 0$

Termination.

We note the following.

$$\begin{array}{ll}
 & P0 \wedge P1 \wedge n=N \\
= & \quad \{\text{definitions of P0, P1}\} \\
 & \alpha * fib.n + \beta * fib.(n+1) = fib.N \;\wedge\; 0 \le n \le N \;\wedge n=0 \\
\Rightarrow & \quad \{\text{algebra}\} \\
 & \alpha * fib.0 + \beta * fib.1 = fib.N \;\wedge\; true \\
= & \quad \{\text{predicate calculus}\} \\
 & \alpha * fib.0 + \beta * fib.1 = fib.N \\
= & \quad \{\text{defn. fib}\} \\
 & \alpha * 0 + \beta * 1 = fib.N \\
= & \quad \{\text{arithmetic}\} \\
 & \beta = fib.N
\end{array}$$

Calculate the loop body.

$$\begin{array}{ll}
 & \alpha * fib.n + \beta * fib.(n+1) = fib.N \\
= & \quad \{P1 \wedge n \neq 0, (2)\} \\
 & \alpha * fib.n + \beta * (fib.(n-1) + fib.n) = fib.N \\
= & \quad \{*/+\} \\
 & \alpha * fib.n + \beta * fib.(n-1) + \beta * fib.n = fib.N \\
= & \quad \{\text{algebra, grouping terms}\} \\
 & \beta * fib.(n-1) + (\alpha + \beta)* fib.n = fib.N \\
= & \quad \{\text{WP calculus}\} \\
 & (n, \alpha, \beta := n-1, \beta, \alpha + \beta).P0
\end{array}$$

Finished program.

And so the entire program is as follows.

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$n, \alpha, \beta := N, 1, 0$ $\{P0 \wedge P1\}$
;do $n \neq 0$ $\rightarrow$   $\{P0 \wedge P1 \wedge n \neq 0\}$

                  $n, \alpha, \beta := n-1, \beta, \alpha + \beta$

                  $\{P0 \wedge P1\}$
od
$\{\beta = fib.N\}$
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