Chapter 4: Eindhoven Quantified Notation.

In which we learn a notation to express quantification.

In traditional mathematics you will have encountered a number of different quantified operators such as Σ, $\prod$, $\forall$, $\exists$. Each of these has their own way of being written and has their own manipulation rules. Often the notation is ambiguous or different authors will use it in different ways. This places heavy burden on us and requires us to learn a lot of rules.

We opt instead to use what is called the Eindhoven Quantified Notation[1]. This is a notation for quantified expressions, which brings together into one simple format a range of quantified notation used elsewhere in mathematics. It provides us with a very simple set of rules for manipulating expressions.

Suppose that $\oplus$ is a binary operator with the following properties

It is associative. So

$$(x \oplus y) \oplus z \quad = \quad x \oplus (y \oplus z)$$

It is symmetric. So

$$x \oplus y \;=\; y \oplus x$$

It has an identity element, written $ID_{\oplus}$. So

$$x \oplus ID_{\oplus} \quad = \quad x$$

then the following expression

$$f.0 \;\oplus\; f.1 \;\oplus\; f.2 \;\oplus \ldots\ldots\ldots \oplus\; f.(N\text{-}1)$$

can be written as

$$\langle \oplus j : 0 \leq j < N : f.j \rangle$$

in this form it is said to be in Eindhoven quantified form.

You should learn to move between the quantified form and the long version of such expressions. It is a skill that will serve you well.

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[1] Named after the Dutch city of Eindhoven where Dijkstra and his colleagues worked during the 1970's and 1980's.

For traditional reasons when the operator is $\wedge$ we use $\forall$ as the quantified form and when the operator is $\vee$ we use $\exists$ as the quantified form. Remember that we choose this notation, not just because we want a common shape for quantified expressions, but also because we can formulate standard rules for manipulating such expressions.

Rules for manipulating quantified expressions.

The empty range law (allowed because $\oplus$ has an identity element)

$\langle \oplus j : 0 \le j < 0 : f.j \rangle$

= {empty range}

$ID_{\oplus}$

The one-point rule

$\langle \oplus j : j = i : f.j \rangle$

= {one-point}

$f.i$

The split off a term law (allowed because $\oplus$ is associative)

$\langle \oplus j : 0 \le j < i+1 : f.j \rangle$ where $0 \le i < N$

= {split off j = i term}

$\langle \oplus j : 0 \le j < i : f.j \rangle \oplus f.i$

The split off a term law where $\oplus$ is idempotent[2]

$$\langle \oplus j : 0 \le j < i{+}1 : f.j \rangle \quad \text{where } 0 \le i < N$$

$=$ {split off j = i term}

$$\langle \oplus j : 0 \le j < i{+}1 : f.j \rangle \oplus f.i$$

The Strengthening law.

$$\langle \oplus j : 0 \le j < i : f.j \rangle \wedge i = N$$

$\Rightarrow$ {Leibniz[3]}

$$\langle \oplus j : 0 \le j < N : f.j \rangle$$

The split off a term law when quantifying over 2 variables

$$\langle \oplus j,k : 0 \le j \le k \le n{+}1 : f.j.k \rangle \quad \text{where } 0 \le n < N$$

$=$ {split off k = n+1 term}

$$\langle \oplus j,k : 0 \le j \le k \le n : f.j.k \rangle \oplus \langle \oplus j : 0 \le j \le n{+}1 : f.j.(n{+}1) \rangle$$

There are a number of other manipulation rules which we will introduce as we need them. For the time being these are sufficient.

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[2] An operator $\oplus$ is idempotent if for all x, $x \oplus x = x$

[3] Named after Gottfried Wilhelm Leibniz the law states

$$x = y \Rightarrow f.x = f.y$$

Often referred to as “substitution of equals for equals”.