Chapter 1
The Kinematics of Mass Points

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- **Mechanics(力学)**

  The Science of motion and its causes.

- **Kinematics (运动学)**

  description of motion.

- **Dynamics(动力学)**

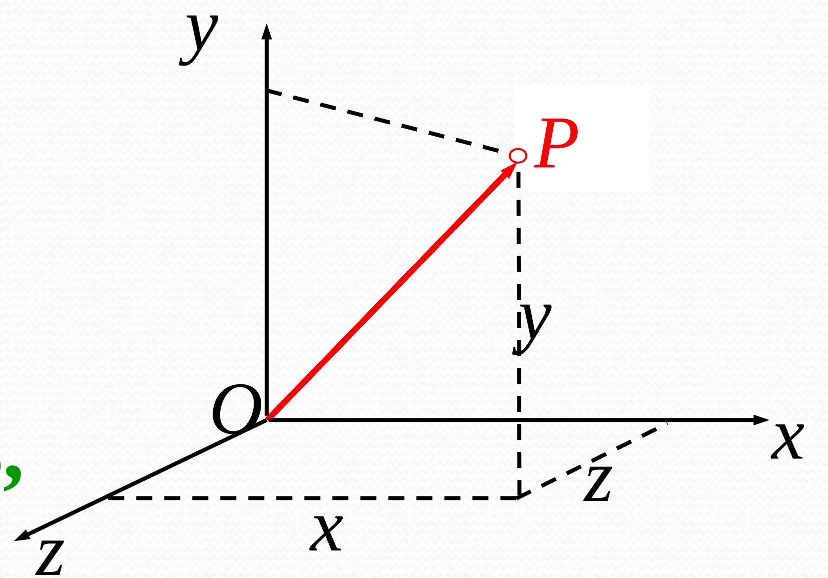
  causes of motion.

§ 1.1. Description of Motion

1. Position(位置)

The position vector

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$



Position vector

As the mass point moves, its position vector is a function of time

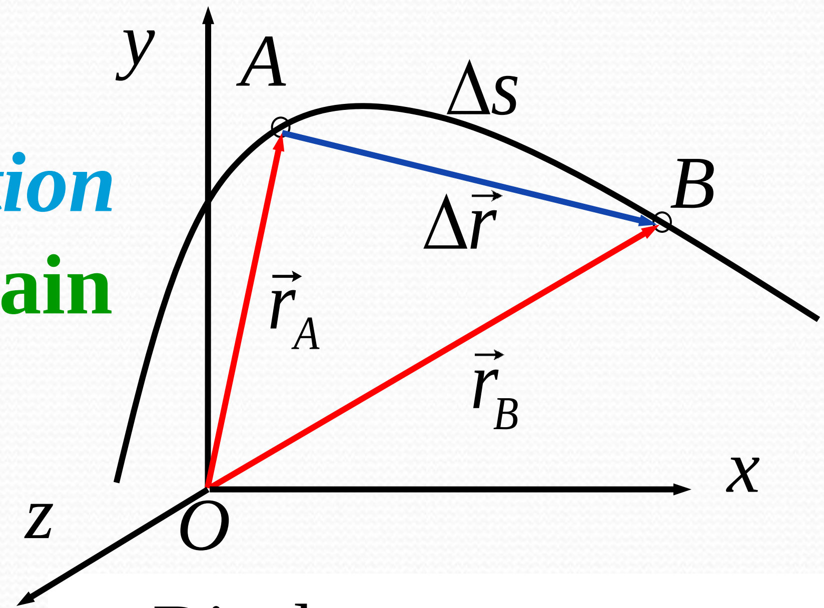
$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

The equation of motion of a mass point

质点的运动方程

2. Displacement(位移)

The change in the *position* of a body during a certain period of time Δt



Displacement vector

$$\Delta \vec{r} = \vec{r}_B - \vec{r}_A$$

Note

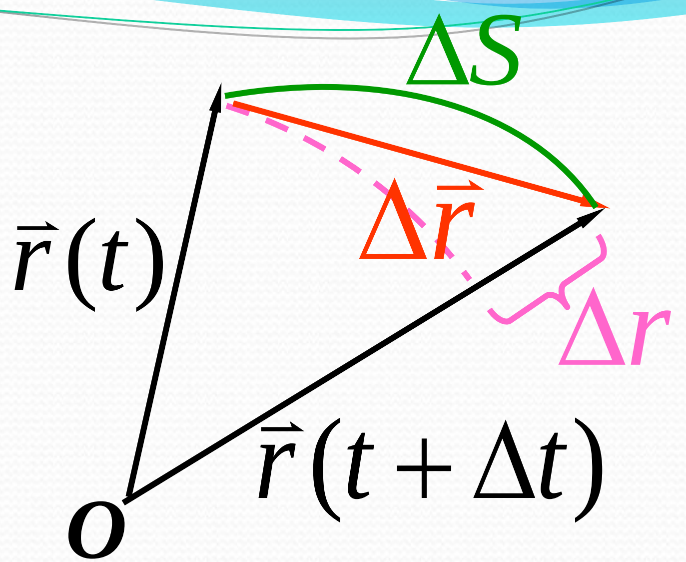
$$|\Delta \vec{r}| \neq \Delta s$$

the magnitude of displacement is not equal to the distance

But when Δt **approaches zero** $|\mathrm{d}\vec{r}| = \mathrm{d}s$

$$\left|\Delta\vec{r}\right| \neq \Delta r$$

And $\left|\mathrm{d}\vec{r}\right| \neq \mathrm{d}r$



Example1.1 **Assume** $\vec{r}(t) = \vec{i} + t\vec{j} + t^2\vec{k}\,(\mathrm{m})$

What is the displacement of the time interval from t=0 s to t=1s?

Solution: $\Delta\vec{r} = \vec{r}(1) - \vec{r}(0) = (\vec{i} + \vec{j} + \vec{k}) - \vec{i}$

$$= (\vec{j} + \vec{k})\ \ \mathrm{m}$$

Example1.2 **Assume** $\vec{r} = 4t^2\vec{i} + (2t+3)\vec{j}\,(\mathrm{m})$

Find the equation of the trajectory of the mass point.

Solution **From the equation of motion we know**

$$\begin{cases} x = 4t^2 \\ y = 2t+3 \end{cases}$$

Eliminating *t*, we get the equation of the trajectory

$$x = (y-3)^2$$

3.Velocity(速度)

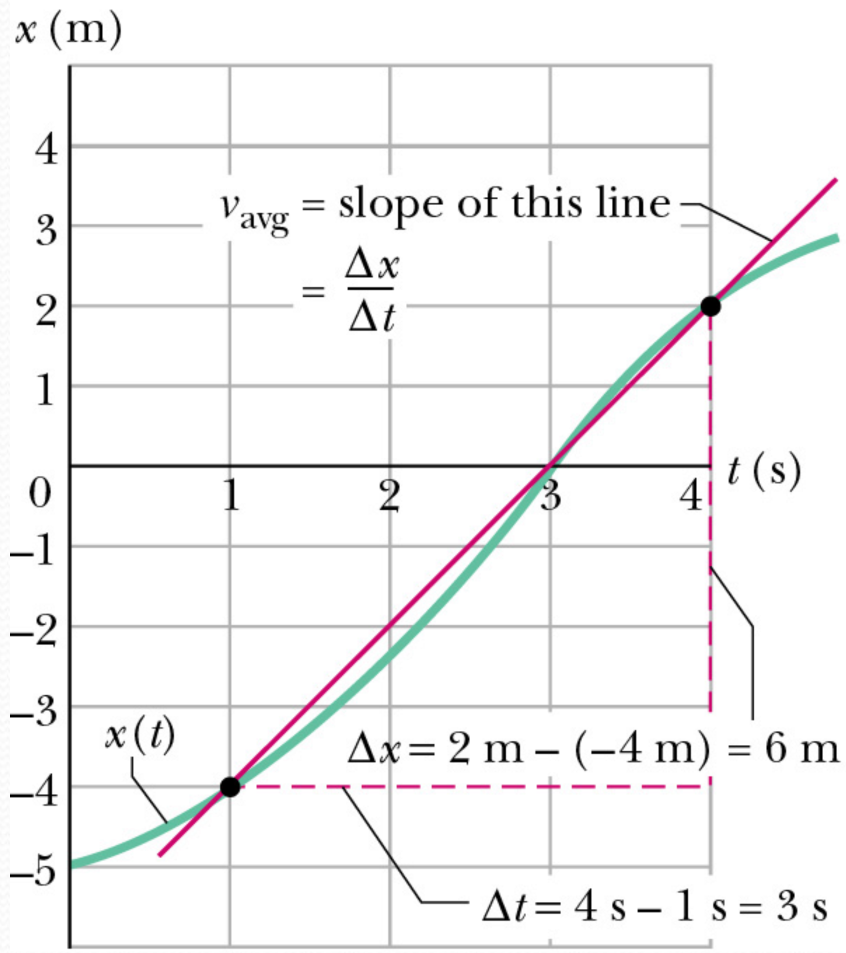
Velocity is how fast the displacement changes.

Average velocity

$$\vec{\bar{v}}_{ave} = \frac{\Delta \vec{r}}{\Delta t}$$

Instantaneous velocity (generally we will just call this the "velocity")

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



Speed(速率) describes just how fast the distance of a body changes.

$$\textbf{Average speed} \equiv \frac{\text{total distance}}{\text{time interval}}$$

Note

1. Speed is a non-negative scalar， velocity is a vector

2. We can simply define speed as the magnitude of the velocity:

$$v = |\vec{v}| = \left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right| = \frac{\mathrm{d}s}{\mathrm{d}t}$$

3. The velocity is given by the slope of the straight line tangent to the position curve at that point (in the limit where $\Delta t \to 0$).

4.Acceleration(加速度)

Acceleration describes just how fast the velocity of a body changes

Average acceleration $$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t}$$

Instantaneous acceleration (called acceleration in short)

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2 \vec{r}}{\mathrm{d}t^2}$$

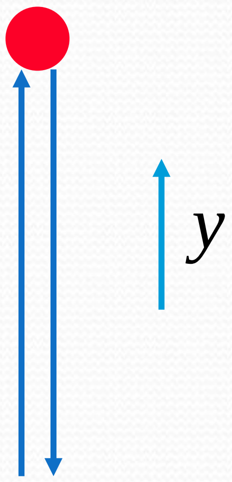
So acceleration is the second derivative of position vector. If the magnitude of the velocity is decreasing, then you can call it deceleration.

Example1.3 **When throwing a ball straight up, which of the following is true about its velocity v and its acceleration a at the highest point in its path?**

(a) Both $v = 0$ and $a = 0$.

(b) $v \neq 0$, but $a = 0$.

(c) $v = 0$, but $a \neq 0$.



Solution **In fact the acceleration is caused by gravity (g = 9.81 m/s^2). The answer is (c).**

Example1.4 **A mass point moves in a straight line. The movement is divided into four parts. Is the velocity more than zero, less than zero or equal to zero in every part respectively? How about the acceleration?**

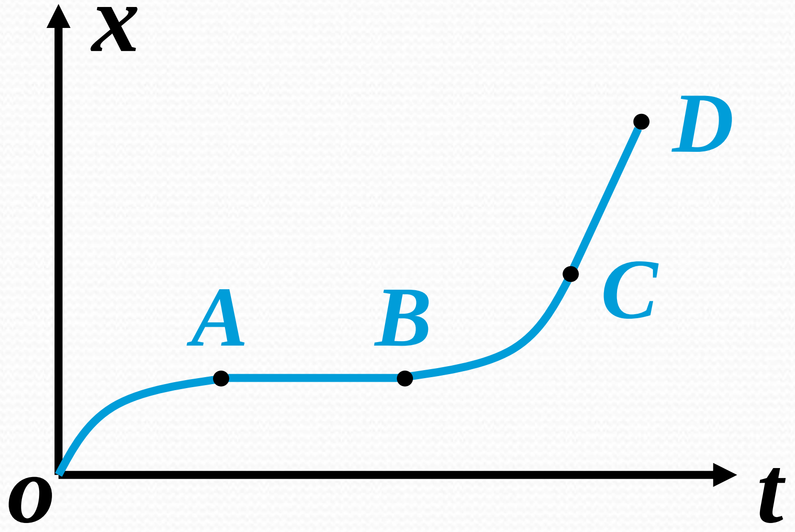
Solution $$v = \frac{\mathrm{d}x}{\mathrm{d}t},\ a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

OA: $v>0,\ a<0;$

AB: $v=0,\ a=0;$

BC: $v>0,\ a>0;$

CD: $v>0,\ a=0.$



Example1.5 **Assume the equation of motion of a mass point is $\vec{r}(t)=\vec{i}+t\vec{j}+t^2\vec{k}$ (m), what are the velocity and acceleration of that mass point?**

Solution: **The velocity of the mass point is**

$$\vec{v}(t)=\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}=\vec{j}+2t\vec{k}\ \text{(SI)}$$

And the acceleration is

$$\vec{a}(t)=\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}=2\vec{k}\ \mathrm{m/s^2}$$

Example 1.6 **An object moves along a straight line from v_0 with $\frac{\mathrm{d}v}{\mathrm{d}t} = -kvt$ (k is a constant). What is the function of $v(t)$?**

Solution: $\frac{\mathrm{d}v}{\mathrm{d}t} = -kvt$ **can be written as** $\frac{\mathrm{d}v}{v} = -kt\mathrm{d}t$

Take the integral of both sides of this equation

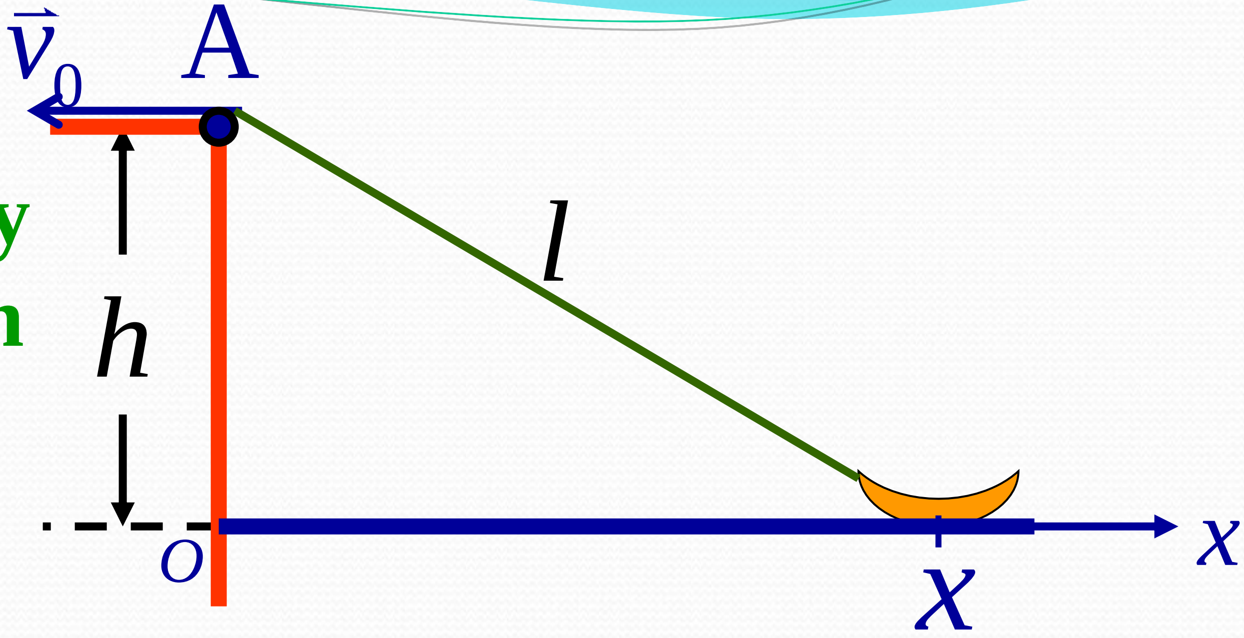
$$\int_{v_0}^{v} \frac{\mathrm{d}v}{v} = -\int_0^t kt\mathrm{d}t$$

Carrying out the integrals, we obtain

$$v = v_0 e^{-\frac{1}{2}kt^2}$$

Example1.7

Find the velocity and acceleration of the boat at arbitrary point *x*.



Solution: The length of the rope between the boat and point A is *l*, then x can be written as

$$x = \sqrt{l^2 - h^2}$$

And we know $\dfrac{\mathrm{d}l}{\mathrm{d}t} = -v_0$

then

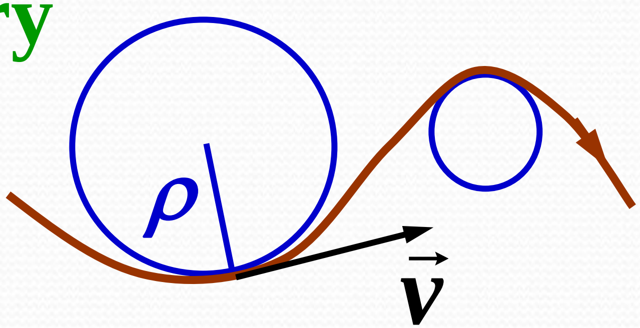
$$v=\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2l}{2\sqrt{l^2-h^2}}\cdot\frac{\mathrm{d}l}{\mathrm{d}t}=-v_0\sqrt{1+(\frac{h}{x})^2}$$

And the acceleration is

$$a=\frac{\mathrm{d}v}{\mathrm{d}t}=-v_0\cdot\frac{2(\frac{h}{x})(-\frac{h}{x^2})}{2\sqrt{1+(\frac{h}{x})^2}}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{h^2v_0^2}{x^3}$$

§ 1.2 Curvilinear Motion

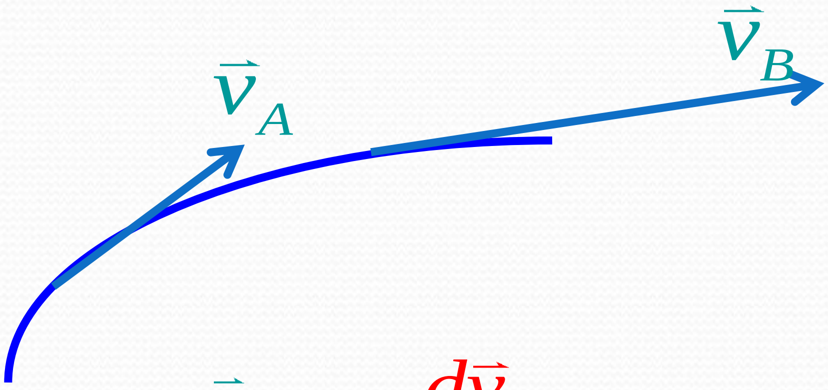
$\vec{v}$ **is tangent to the trajectory**

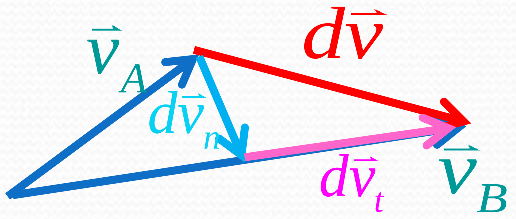


$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d\vec{v}_n}{dt} + \frac{d\vec{v}_t}{dt}$$

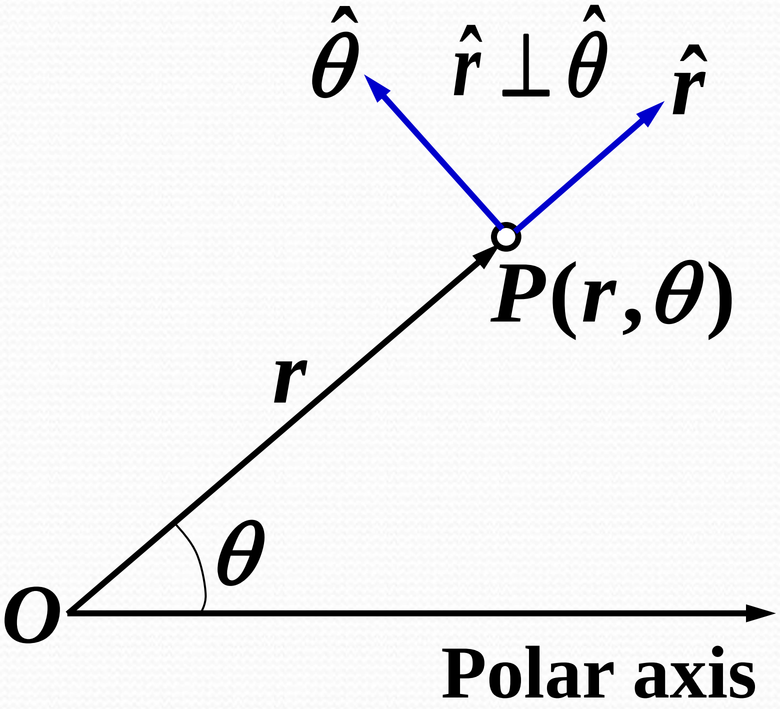
$$= \vec{a}_n + \vec{a}_t$$





1. In plane polar coordinates (平面极坐标)*

(1) Definition

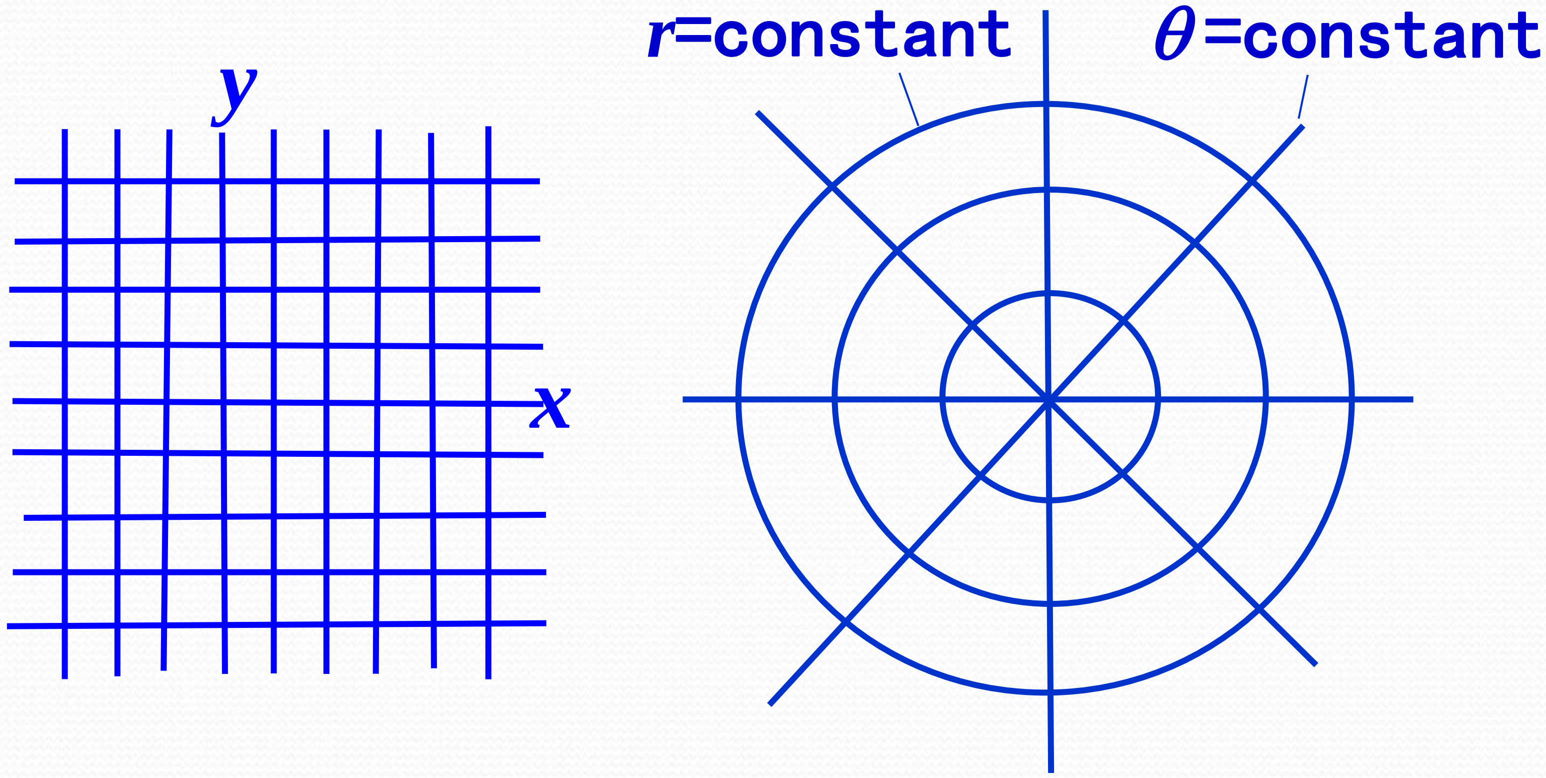


r: radial coordie

θ: angular coordinate(rad)

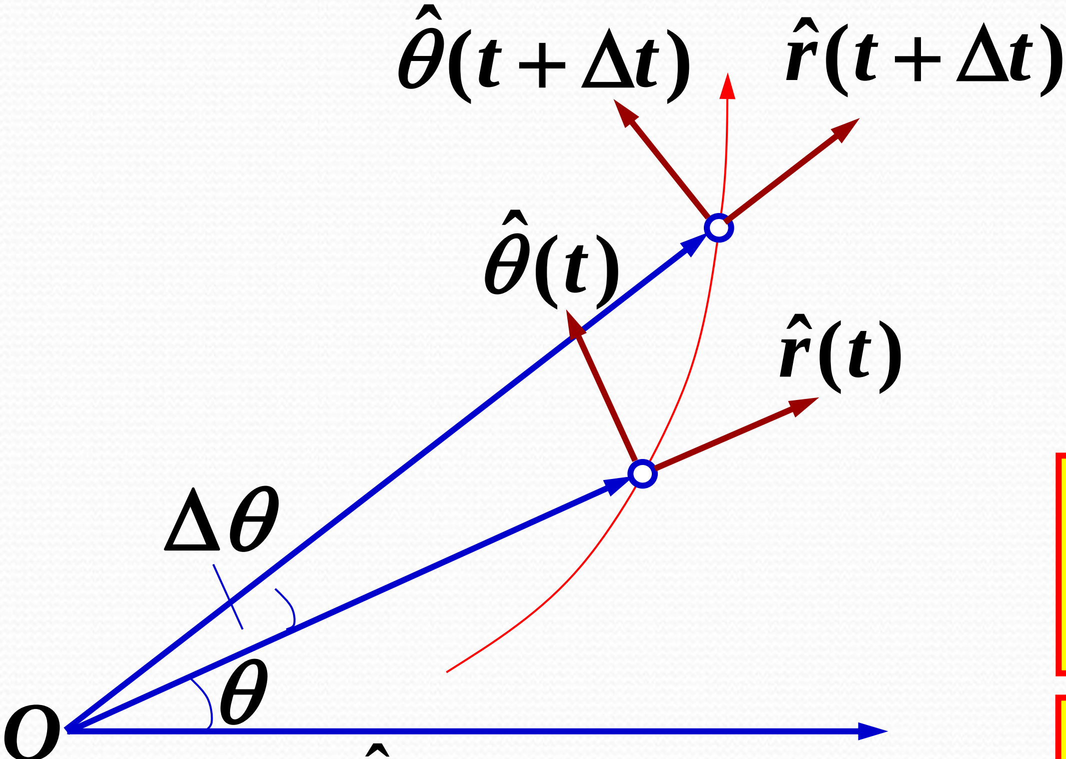
Note that radians (or degrees) is a dimensionless unit.

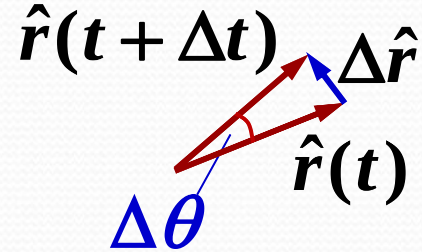
$\hat{r}$ and $\hat{\theta}$ are unit vectors, but they are not constants.



lines of equal coordinates in rectangular coordinate system and plane polar coordinates

(2) $\hat{r}$ and $\hat{\theta}$ 对时间的导数

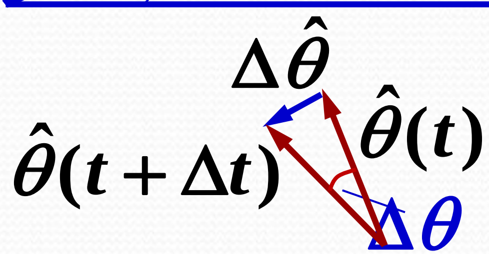




$$|\Delta\hat{r}| \approx |\hat{r}| \cdot \Delta\theta = \Delta\theta$$

$$\Delta\hat{r} \approx \Delta\theta \cdot \hat{\theta}$$

$$\boxed{\dot{\hat{r}} = \frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta} = \dot{\theta}\hat{\theta}}$$



$$\Delta\hat{\theta} \approx -\Delta\theta \cdot \hat{r}$$

$$\boxed{\dot{\hat{\theta}} = \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{r} = -\dot{\theta}\hat{r}}$$

(3) Position, velocity and acceleration in plane polar coordinates

① **Position** $\vec{r}(t) = r(t)\hat{r}(t)$ $\boxed{\vec{r} = r\,\hat{r}}$

② **velocity** $\boxed{\dot{\hat{r}} = \dot{\theta}\hat{\theta}}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\,\hat{r})}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\boxed{\vec{v} = \vec{v}_r + \vec{v}_\theta}$$

where $\vec{v}_r = \dot{r}\hat{r}$

$\vec{v}_\theta = r\dot{\theta}\hat{\theta}$

How about a circular motion?

③ Acceleration

$$\vec{a}=\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}=\frac{\mathrm{d}(\vec{v}_r+\vec{v}_\theta)}{\mathrm{d}t}=\frac{\mathrm{d}(\dot{r}\hat{r})}{\mathrm{d}t}+\frac{\mathrm{d}(r\dot{\theta}\hat{\theta})}{\mathrm{d}t}$$

$$\frac{\mathrm{d}(\dot{r}\hat{r})}{\mathrm{d}t}=\ddot{r}\hat{r}+\dot{r}\dot{\hat{r}}=\ddot{r}\hat{r}+\dot{r}\dot{\theta}\hat{\theta}$$

$$\boxed{\begin{aligned}\dot{\hat{r}}&=\dot{\theta}\hat{\theta}\\ \dot{\hat{\theta}}&=-\dot{\theta}\,\hat{r}\end{aligned}}$$

$$\frac{\mathrm{d}(r\dot{\theta}\hat{\theta})}{\mathrm{d}t}=\dot{r}\dot{\theta}\hat{\theta}+r\ddot{\theta}\hat{\theta}-r\dot{\theta}^2\hat{r}$$

$$\vec{a}=(\underline{\ddot{r}-r\dot{\theta}^2})\hat{r}+(\underline{2\dot{r}\dot{\theta}+r\ddot{\theta}})\hat{\theta}$$

$$\boxed{\vec{a} = \vec{a}_r + \vec{a}_\theta}$$

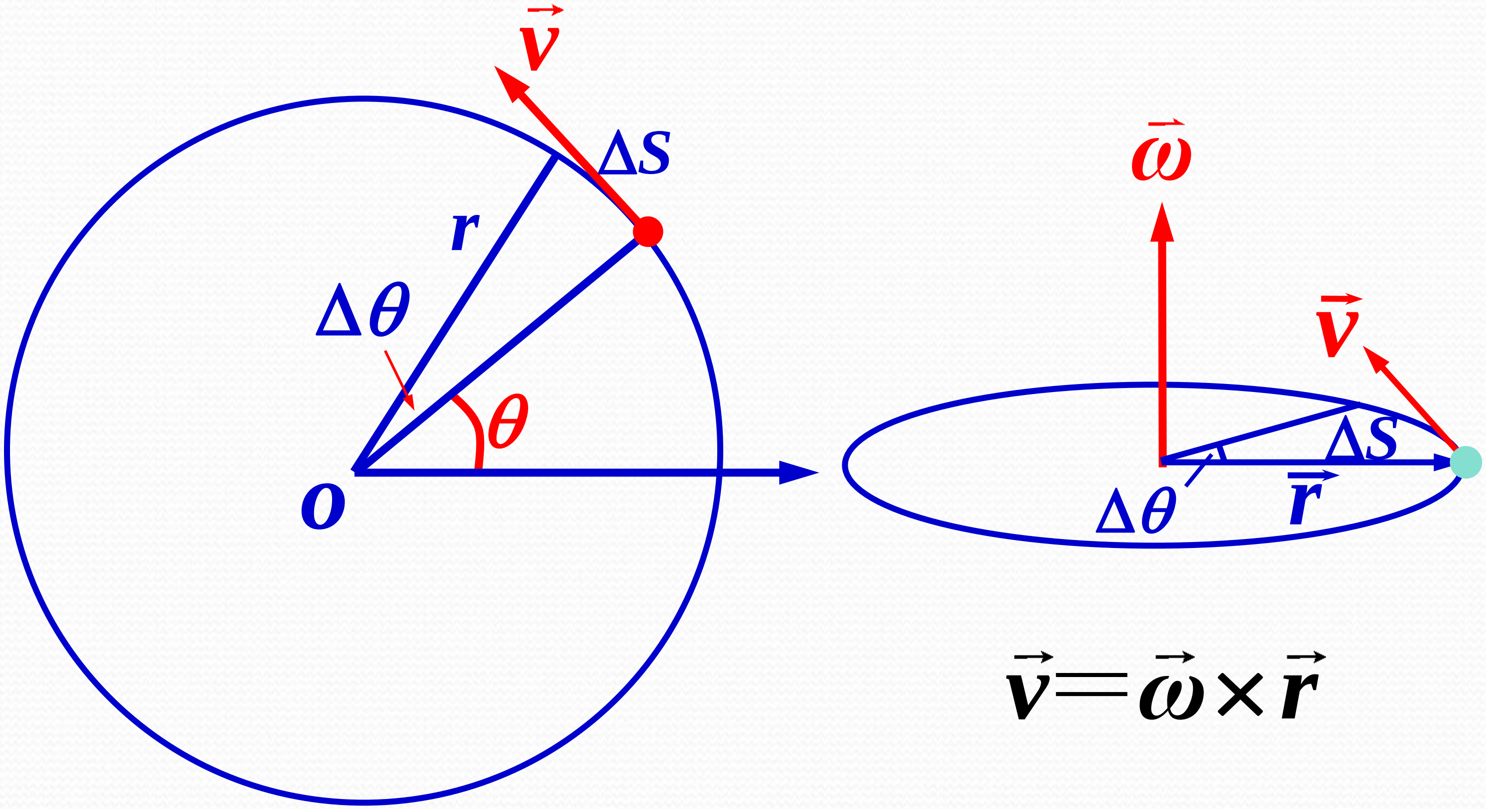
Where

$$\boxed{\vec{a}_r = (\ddot{r} - r\dot{\theta}^2)\hat{r}}$$

$$\boxed{\vec{a}_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}}$$

How about a circular motion?

2. Circular motion



$$\vec{v} = \vec{\omega} \times \vec{r}$$

We have

$\Delta\theta$ —angular displacement (rad)

$\Delta s = r\Delta\theta$ —linear position (arc length)

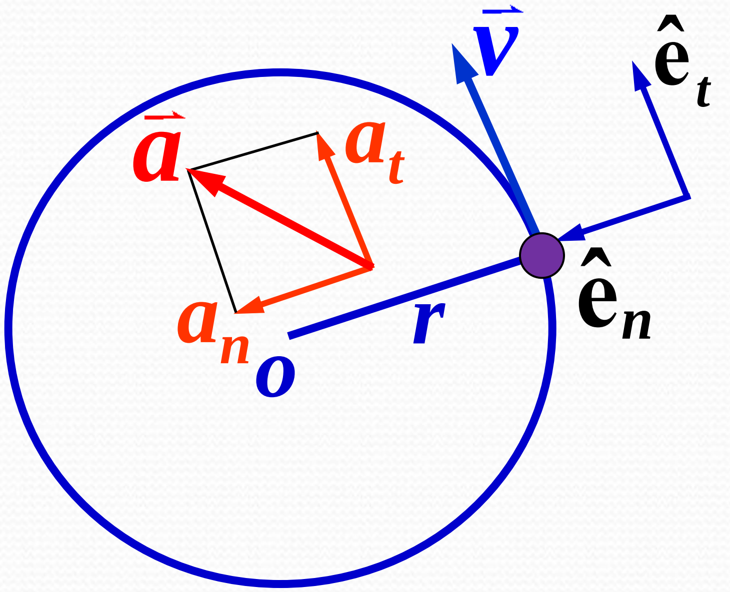
$\vec{\omega} = \frac{d\theta}{dt}\hat{\omega}$ —angular velocity(rad/s)

$v = \frac{ds}{dt} = \frac{r d\theta}{dt} = \omega r$ —linear velocity

$\vec{\beta} = \frac{d\vec{\omega}}{dt}$ —angular acceleration(rad/s^2)

$\vec{a} = \frac{d\vec{v}}{dt}$ —acceleration(m/s^2)

The tangential and centripetal acceleration



Here $\hat{e}_t = \hat{\theta}$

$\hat{e}_n = -\hat{r}$

$\vec{e}_t$ is the base unit vectors along the tangential direction

$\vec{e}_n$ is the base unit vectors along the normal direction

$$\vec{a} = a_t \hat{e}_t + a_n \hat{e}_n$$

a_t —tangential acceleration

a_n —centripetal acceleration

Centripetal acceleration

$$\boxed{\dot{r} = 0}$$

$$\vec{a}_r = (\ddot{r} - r\dot{\theta}^2)\hat{r} = r\dot{\theta}^2(-\hat{r})$$

$$= r\dot{\theta}^2\,\hat{\mathbf{e}}_n = a_n\,\hat{\mathbf{e}}_n$$

$$\boxed{a_n = r\omega^2 = v\omega = \frac{v^2}{r}}$$

——directed radially inward

Tangential acceleration

$$\boxed{\dot{r} = 0}$$

$$\vec{a}_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

$$= r\ddot{\theta}\hat{\theta} = r\dot{\omega}\hat{\mathbf{e}}_t = a_t\,\hat{\mathbf{e}}_t$$

$$a_t = \frac{\mathrm{d}(r\dot{\theta})}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t}$$

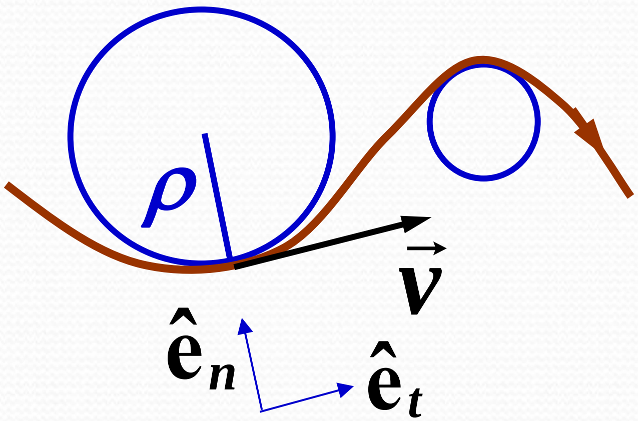
$$\boxed{a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = r\dot{\omega} = r\beta}$$

——tangent to the trajectory

3. Plane curvilinear motion

Here

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$



$$\vec{a}_t = a_t \vec{e}_t$$

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t}$$

The time rate of change of the magnitude of velocity

$$\vec{a}_n = a_n \vec{e}_n$$

$\vec{e}_\mathrm{n}$ **is the base unit vectors along the normal direction**

$$a_n = \frac{v^2}{\rho}$$

radius of curvature

Example1.8 A wheel with a fixed axle is rotating and the angular velocity is given as a function of time by $\omega = 2t + 4t^2$. What is the angular acceleration at $t = 0.3\mathrm{s}$?

Solution the angular acceleration α is

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = 2 + 8t$$

Substitute $t = 0.3\mathrm{s}$, we get

$$\alpha = 4.4/\mathrm{s}^2$$

Example1.9 A mass point moves on a circle of radius R according to the rule $s = v_0 t - ct^2/2$ .What are the magnitude of centripetal acceleration and tangential acceleration at any time t? (c is a constant)

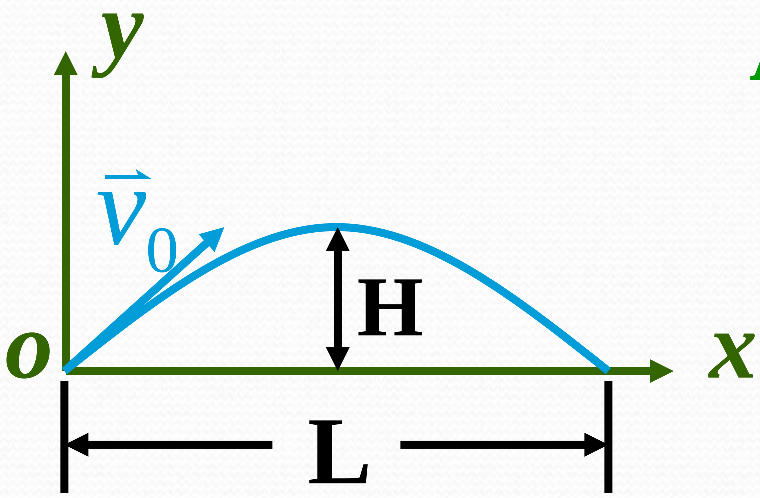
Solution The value of linear velocity is

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = v_0 - ct$$

the tangential acceleration $a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = -c$

the centripetal acceleration $a_n = \frac{v^2}{R} = \frac{(v_0 - ct)^2}{R}$

§ 1.4 Projectile Motion



Acceleration： $\vec{a} \equiv \vec{g}$

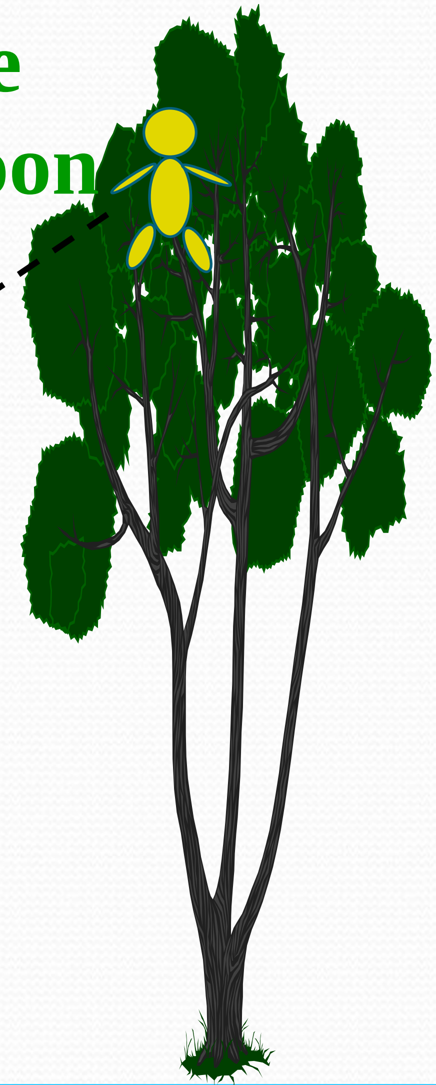
velocity：

$$\begin{cases} v_x = v_{0x} & ① \\ v_y = v_{0y} - gt & ② \end{cases}$$

The equation of motion of the mass point

$$\begin{cases} x = v_{0x}t & ③ \\ y = v_{0y}t - \frac{1}{2}gt^2 & ④ \end{cases}$$

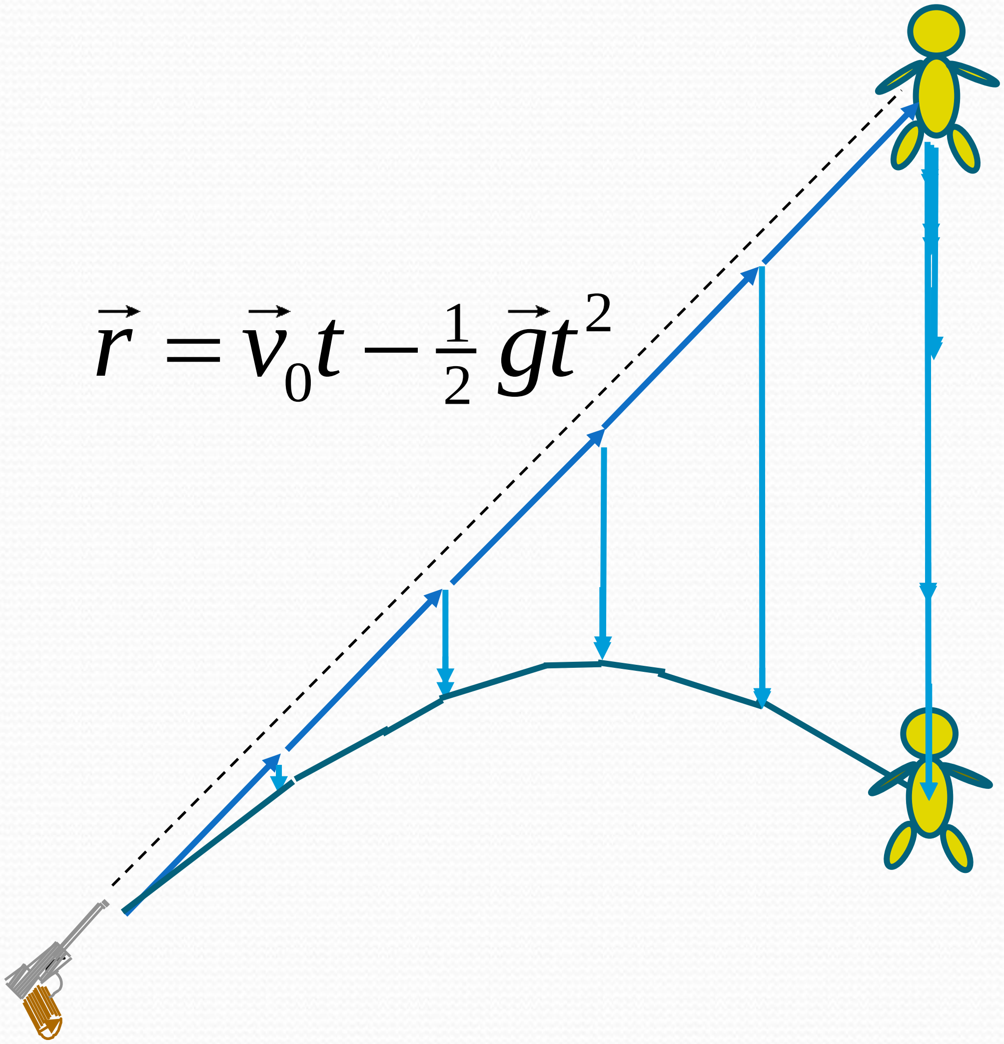
Example1.10 Where does the zookeeper aim if he wants to hit the monkey?(He knows the monkey will fall freely as soon as he shoots !)



Solution If there were no gravity, simply aim at the monkey.

$$\vec{r} = \vec{v}_0 t$$

With gravity, still aim at the monkey

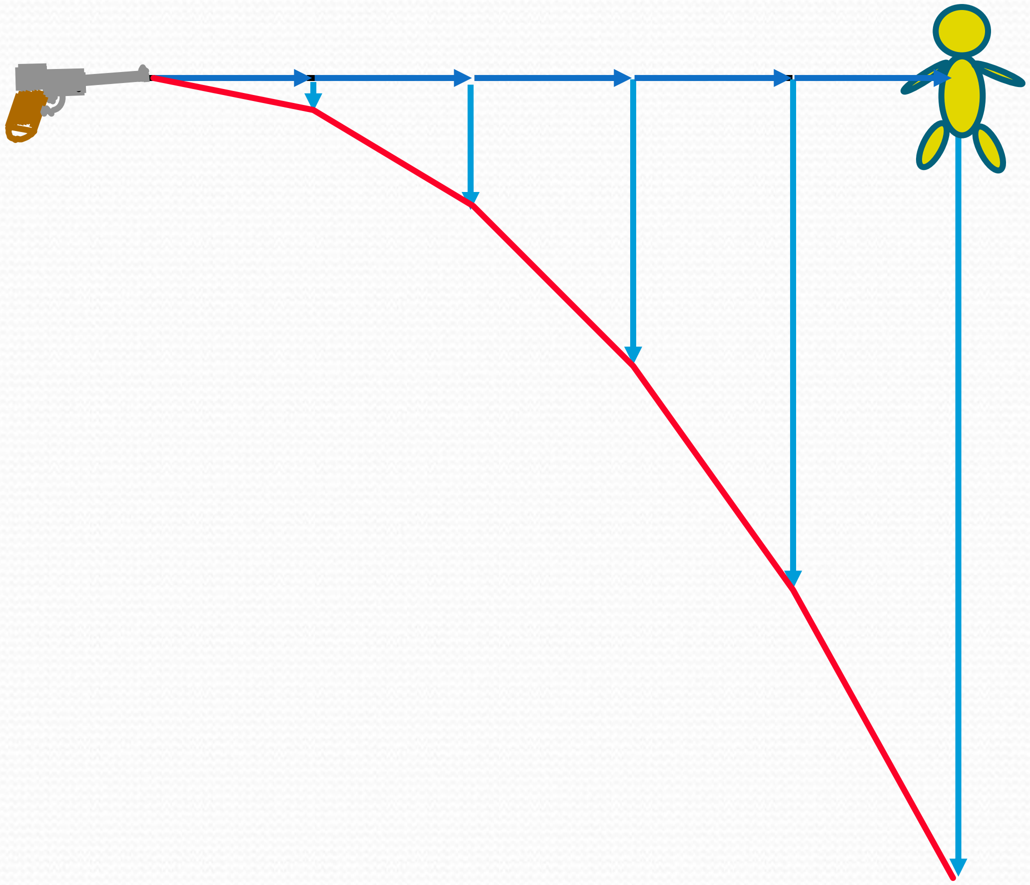


$$\vec{r} = \vec{v}_0 t - \tfrac{1}{2}\vec{g}t^2$$

Dart hits the monkey!

This may be easier to think about.

It's exactly the same idea!!

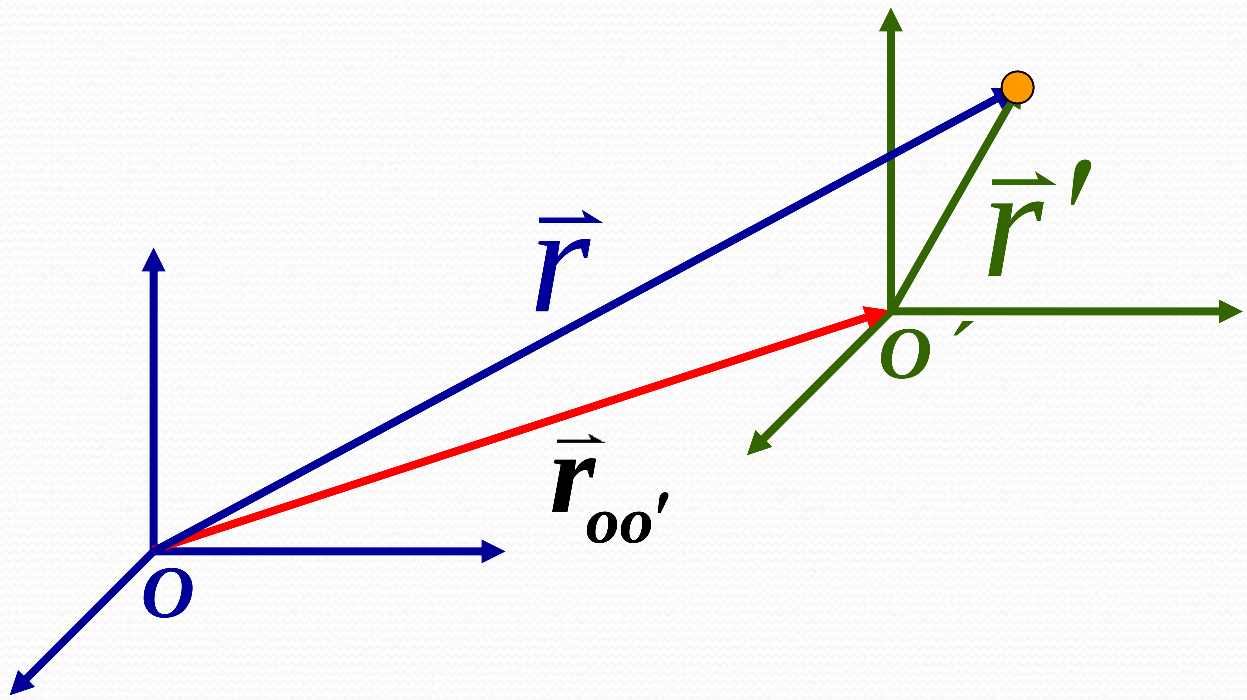


§ 1.5 Relative Motion

Newton′s space and time

Absolut space, in its own nature, without relation to anything external, remains always similar and immovable·· ·· ··

Absolut, true and mathmatical time of itself and from it own nature, flows equally without relation to anything external·· ·· ··



O' **moves along the *x* axis with a constant velocity** $\vec{v}_{oo'}$ **relative to** O.

The relations between the position and time of the mass point in two coordinate systems are

$$\vec{r}\,' = \vec{r} - \vec{r}_{oo'}$$

Taking the derivative with respect to time of above equation gives

$$\vec{u}\,' = \boxed{\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t'} = \frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}} = \frac{\mathrm{d}(\vec{r} - \vec{r}_{oo'})}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - \frac{\mathrm{d}\vec{r}_{oo'}}{\mathrm{d}t}$$

$$= \vec{u} - \vec{v}_{oo'}$$ **Galilean transformation**

Galilean transformation proves effective if the speed of the mass point is far less than the speed of light.

$$\vec{a} = \vec{a}'$$

If O' moves along the x axis with a changing velocity $\vec{v}$ relative to O, then

$$\begin{cases} \vec{r} = \vec{r}' + \vec{r}_{oo'} \\ \vec{u} = \vec{u}' + \vec{v}_{oo'} \\ \vec{a} = \vec{a}' + \vec{a}_{oo'} \end{cases}$$

$$\rightarrow \vec{F} = \vec{F}' + \vec{F}_{oo'}$$

This force is ?

See you next time!