第 1.1 节集合与初等函数

一、内容提要（contents）

Composite function（复合函数）、函数的奇偶性（odd function 奇函数& even function 偶函数）、基本初等函数（Basic elementary function）：常函数、幂函数（power function）、指数函数（exponential function）、对数函数（logarithm function）、三角函数（trigonometric function）、反三角函数（Inverse trigonometric function）。

初等函数：由基本初等函数经过有限次四则运算和有限次复合运算得到的由一个表达式表达的函数称为初等函数。

二、习题解答（answers）

Exercise 1.1

1. Determine whether the function is even, old, or neither.

(1) $f(x) = \ln\ (x + \sqrt{1 + x^2})$;

Solution:

Firstly, the domain of $f(x) = \ln\ (x + \sqrt{1 + x^2}\,)$ is $D_f = R = (-\infty, +\infty)$; the all real number set, which is symmetric about the origin.

Secondly, for all $x \in D_f$,

$f(-x) = \ln\left(-x + \sqrt{1 + x^2}\right)$

$=\ln\ (\frac{\sqrt{1+x^2}-x}{1})$ (make transformation)

$=\ln\ (\frac{\left(\sqrt{1+x^2}-x\right)(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}+x})$ (the numerator rationalization by using Difference of two squares)

$=\ln \frac{\sqrt{1+x^2}^2-x^2}{\sqrt{1+x^2}+x}$

$=\ln \frac{1}{\sqrt{1+x^2}+x}$

$=\ln 1-\ln (x+\sqrt{1+x^2})$

$=0-\ln (x+\sqrt{1+x^2})$

$=-f(x)$

So, $f(x)$ is odd.

(2) $f(x)=\frac{2^x+2^{-x}}{2}$;

Solution:

The domain of $f(x)$ is $D_f=R=(-\infty,+\infty)$, which is symmetric about the origin. And for all $x \in D_f$, we have

$$f(-x)=\frac{2^{-x}+2^{-(-x)}}{2}=\frac{2^x+2^{-x}}{2}=f(x)$$

So, $f(x)$ is even

(3) $f(x)=\frac{a^x-a^{-x}}{2}$;

Solution:

The domain of $f(x)$ is $D_f=R=(-\infty,+\infty)$, which is symmetric about the origin. And for all $x \in D_f$, we have

$f(-x)=\frac{a^{-x}-a^{-(-x)}}{2}=-\frac{a^x-a^{-x}}{2}=-f(x)$

So $f(x)$ is odd

(4) $f(x)=x(x-1)(x+1)$;

Solution:

The domain of $f(x)$ is $\mathrm{D_f} = \mathrm{R} = (-\infty, +\infty)$, which is symmetric about the origin. And for all $\mathrm{x} \in \mathrm{D_f}$, we have

$$f(-x) = -x(-x-1)(-x+1)$$
$$= x(x+1)[-(x-1)]$$
$$= -x(x-1)(x+1)$$
$$= -f(x)$$

So, $f(x)$ is odd

(5) $f(x) = \sin x - \cos x + 2;$

Solution.

The domain of $f(x)$ is $\mathrm{D_f} = \mathrm{R} = (-\infty, +\infty)$, but

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} + 2 = 1 - 0 + 2 = 3$$

$$f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) - \cos\left(-\frac{\pi}{2}\right) + 2 = -1 - 0 + 2 = 1$$

So $f\left(\frac{\pi}{2}\right) \neq f\left(-\frac{\pi}{2}\right)$ and $f\left(\frac{\pi}{2}\right) \neq -f\left(-\frac{\pi}{2}\right)$

Which means that $f(-x) = f(x)$ is not valid, whereas $f(-x) = -f(x)$ is not valid, too. So $f(x)$ is neither even nor odd.

2. Prove the any function $f(x)$ with the real number set R (real number set) as its domain can be expressed as a sum of an even and an odd function.

Proof.

Suppose that $f(x)$ is a function with R (real number set)as its domain, then $f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}$, where $\frac{f(x)+f(-x)}{2}$ is an even function whereas $\frac{f(x)-f(-x)}{2}$ is an odd function, so $f(x)$ is a sum of an even and an odd function.

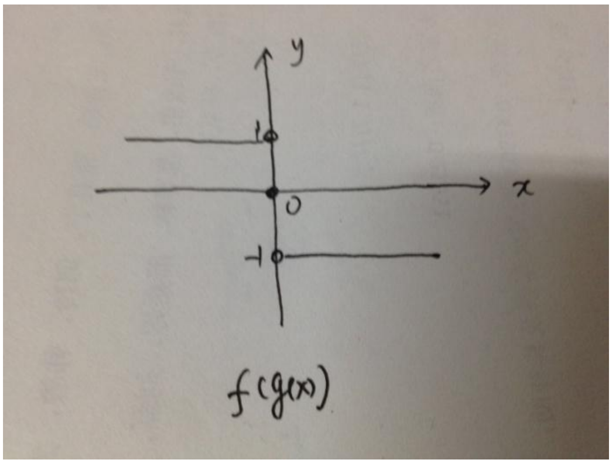
3. Let $f(x)=\begin{cases} 1, |x|<1; \\ 0, |x|=1; \\ -1, |x|>1 \end{cases}$, $g(x)=e^x$. Find $f(g(x))$ and $g(f(x))$, and shetch the graph of the two functions.

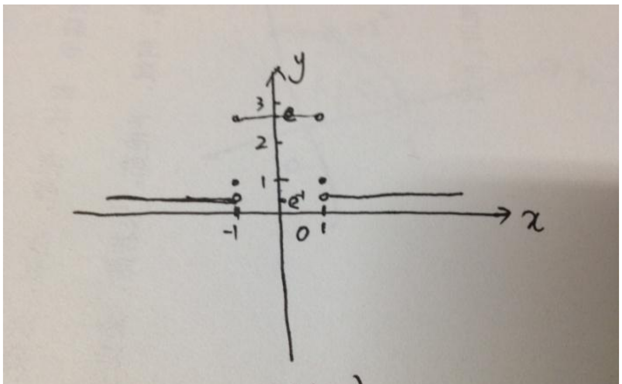
Solution.

$$f(g(x))=f(e^x)=\begin{cases} 1, e^x<1; \\ 0, e^x=1; \\ -1, e^x>1 \end{cases}=\begin{cases} 1, x<0; \\ 0, x=0; \\ -1, x>0 \end{cases}$$

$$g(f(x))=\begin{cases} e, |x|<1; \\ 1, |x|=1; \\ e^{-1}, |x|>1 \end{cases}$$

The graphs of $f(g(x))$ are as follows.





$g(f(x))$