# Chapter 3
# Momentum
# And
# The Law of Conservation of Momentum


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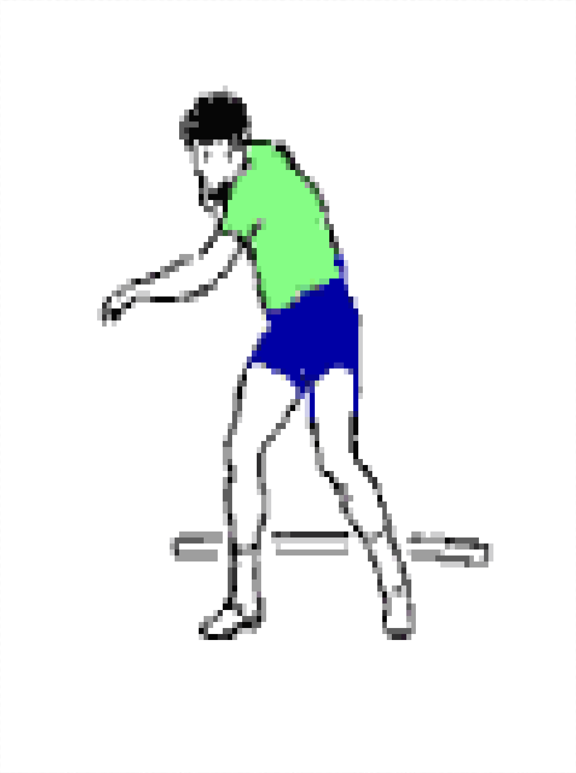


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# § 3.1 Theorem of Momentum for a Particle

**A slightly more general form of the Newton's second law:**

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t}$$

**Newton actually wrote this**

**Net force**

$$\rightarrow \vec{F}dt = d\vec{p}$$

**Differential form**

$$\rightarrow \vec{I} = \int_{t_1}^{t_2} \vec{F}dt = \vec{p}_2 - \vec{p}_1$$

**Integral form**

**Impulse**

**Momentum increment**

## Component equation

$$I_x = \int_{t_1}^{t_2} F_x dt = p_{2x} - p_{1x}$$

**Example3.1** **An object with mass of 2kg moves from rest. The force acting on this object is** $12t^2\vec{i}$N **Calculate the momentum of this object at the end of 2 second?**

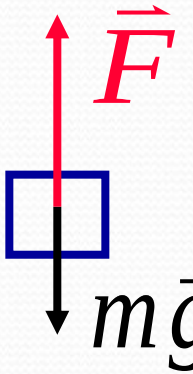
**Solution** **According to the theorem of momentum of a particle we have**

$$\int_{t_1}^{t_2} \vec{F} dt = \vec{p}_2 - \vec{p}_1 = \vec{p}_2$$

$$\rightarrow \vec{p}_2 = \vec{i}\int_0^2 12t^2 dt = 32\vec{i} \quad \text{kg} \cdot \text{m/s}$$

**Example3.2** **An object of mass 10kg is placed on the floor of a elevator. The elevator is going up with a acceleration $a=2+3t^2$(SI). Find the impulse on the object by the floor from $t=0$ to $t=1$s. What is the momentum increment of the object in this duration?**

**Solution According to Newton′s second law**




$$F-mg=ma$$

**We get**

$$F= m(g+a)= 118+30t^2$$

**So**

$$I = \int_0^1 F dt$$

$$= \int_0^1 (118 + 30t^2) dt$$

$$= 128 \quad N \cdot s$$

**The momentum increment of the object**

$$I = \int_0^1 madt$$

$$= \int_0^1 (20 + 30t^2) dt$$

$$= 30 \ N \cdot s$$

**Example3.3** **An object of mass 1.0kg， the equation of motion of which is $x$=2t+t$^3$ (m). What is the impulse on the object by the net force duration 0～2s?**

**Solution** $v = \mathrm{d}x/\mathrm{d}t = 2+3t^2$

$$I = \left| mv(2) - mv(0) \right| = 12 \quad N \cdot s$$

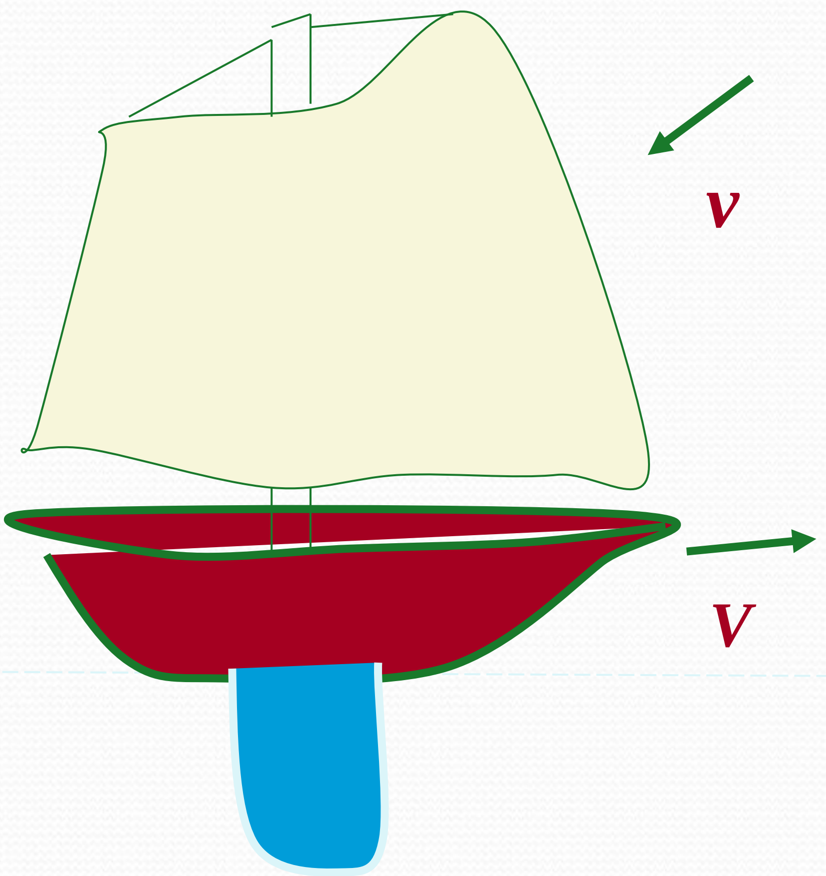
**Another method solving this problem is**

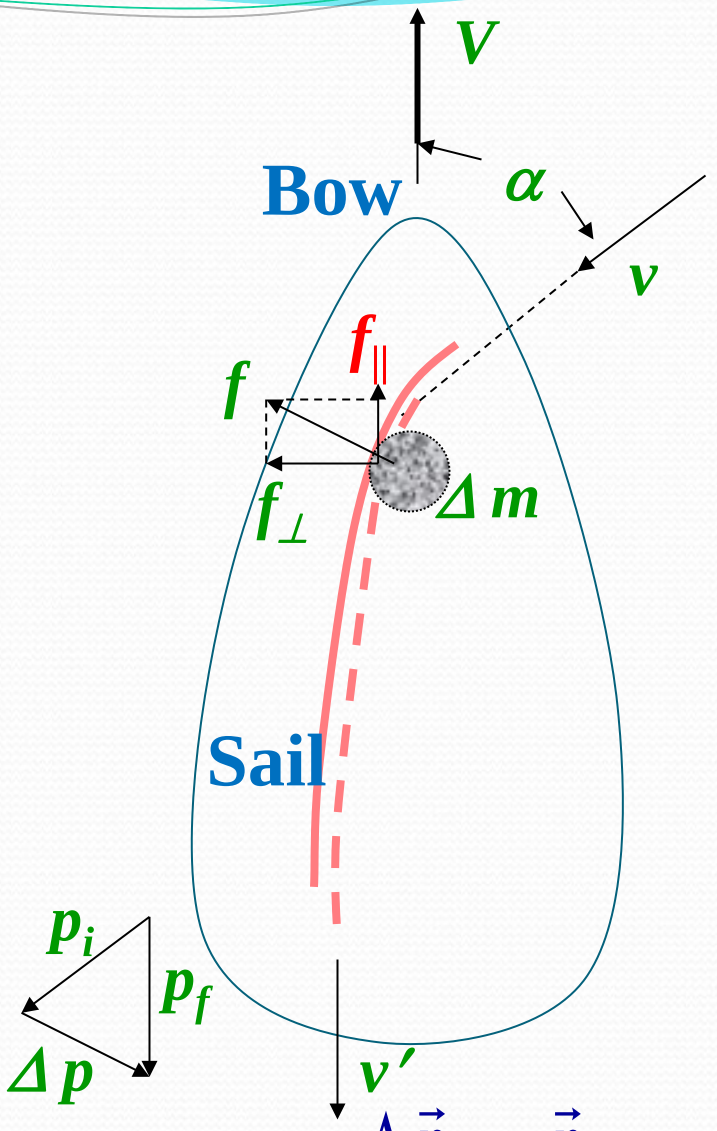
$$F = ma = 1.0 \times \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 6t$$

**So**

$$I = \int_0^2 F\mathrm{d}t = \int_0^2 6t\mathrm{d}t = 12\,\mathrm{N} \cdot \mathrm{S}$$

# Example3.4 Sailing against the wind

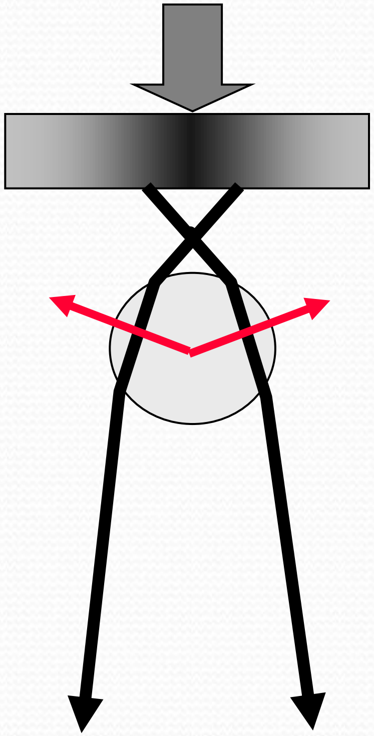



$$\Delta \vec{p} = \vec{p}_f - \vec{p}_i$$

e.g.

**Optical tweezers**



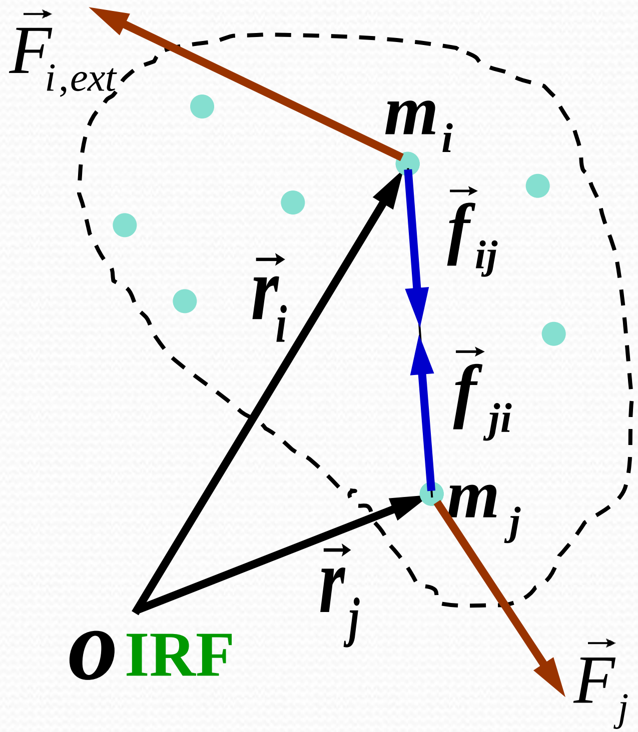
# § 3.2 Theorem of Momentum for a System

## For the *i*th particle

$$\vec{F}_i dt = d\vec{p}_i$$

$$\rightarrow \vec{F}_{i,ext} dt + \vec{F}_{i,\text{int}} dt$$

$$= d\vec{p}_i$$



## The summation is

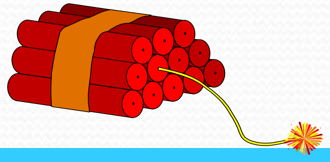
$$(\sum_i \vec{F}_{i,ext})dt + \boxed{(\sum_i \vec{F}_{i,\text{int}})dt} = d(\sum_i \vec{p}_i)$$

$$\rightarrow \ \vec{F}_{ext} dt = d\vec{p}_{total}$$

**Impulse of the sum of external forces**

**Momentum increment of the system**

**Note** **The total momentum of the system can not be changed by internal forces.**



# § 3.3 The Law of Conservation of Momentum

**For a particle system, if** $\vec{F}_{ext} = 0$

**Then** $\vec{p}_{total} = \text{constant vector}$

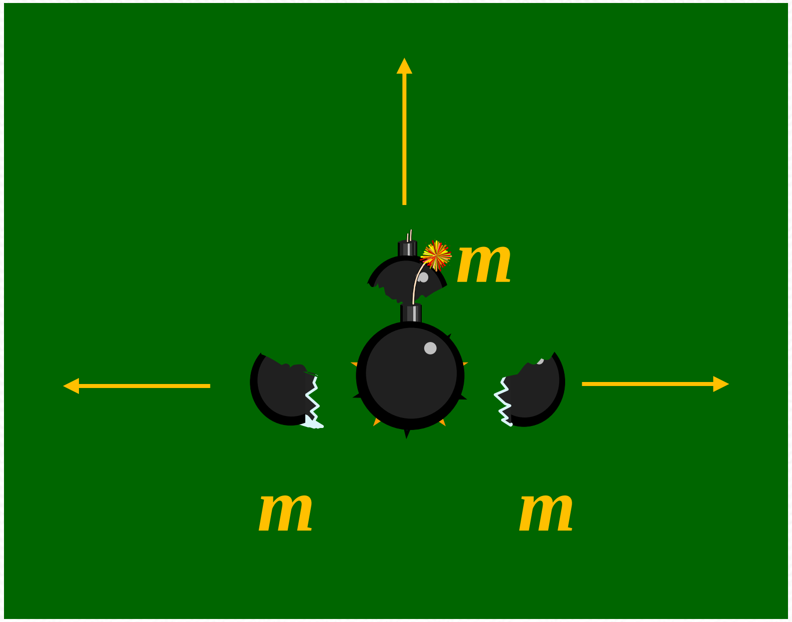
**Component equation**

$$\text{If } F_{ext,x} = 0 \rightarrow p_{total,x} = const.$$

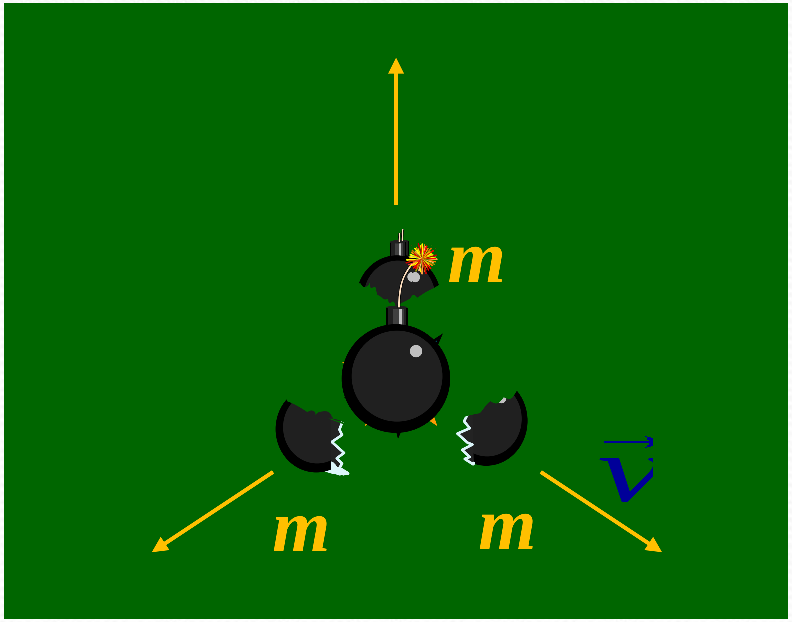
**e.g.**

**Example3.6** **A bomb explodes into 3 identical pieces. Which of the following configurations of velocities is possible?**

**(a) 1 (b) 2 (c) both**




(1)

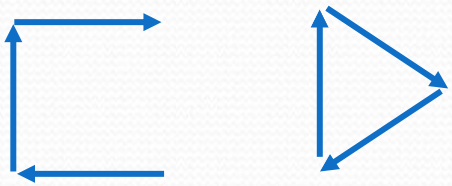



(2)

## Solution

- No external forces, so $P$ must be conserved. Initially: $P = 0$. In explosion (1) there is nothing to balance the upward momentum of the top piece, so $P_{final} \neq 0$.

- In explosion (2) all the momenta cancel out. So $P_{final} = 0$.



(1) (2)

**Why are fireworks spheroid ?**

◆ Newton's 2nd & 3rd laws → The sum of all the momenta of any number of mutually-interacting bodies making up a closed system(an isolated system ) (no external forces) is a constant throughout the motion.

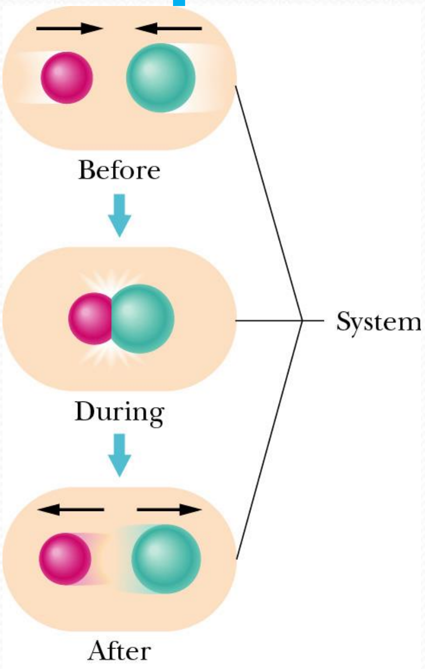
◆ Newton's second law is true in all inertial reference frames. It follows that if momentum is conserved in one inertial frame of reference, it will be conserved in all inertial reference frames. But the constant may not be the same in all inertial frames.

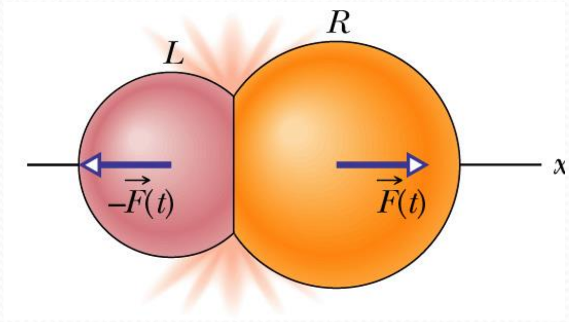
◆ The conservation of momentum does not hold in non-inertial frames of reference.

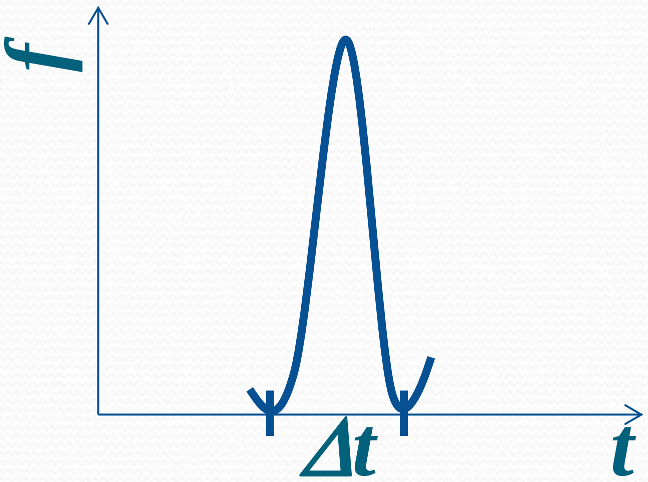
◆This is most useful in solving problems involving collisions.

## § 3.3.1 Collision（碰撞）

**In a collision, the force that alters the motion of two bodies is generally of short duration. If this is the case, we would tend to call such a force an impulsive force(冲力).**

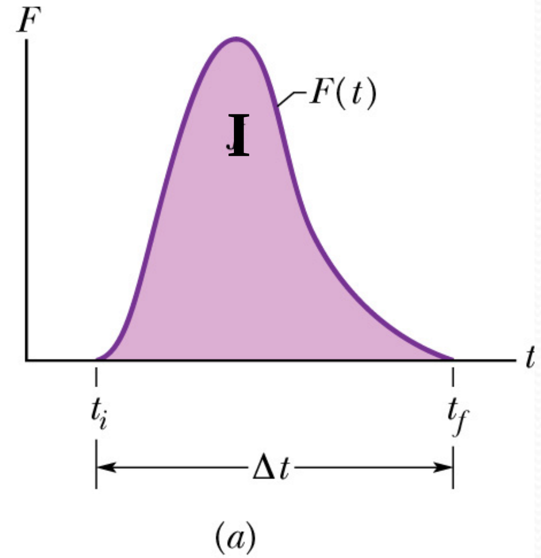


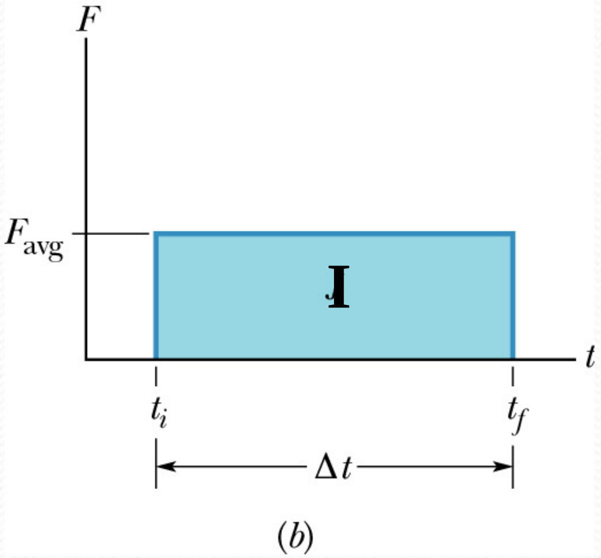



**The average force acting on the object over a time interval $\Delta t$**

$$\overline{\vec{F}} = \frac{\int_{t_1}^{t_2} \vec{F} dt}{t_2 - t_1} = \frac{\vec{p}_2 - \vec{p}_1}{t_2 - t_1}$$



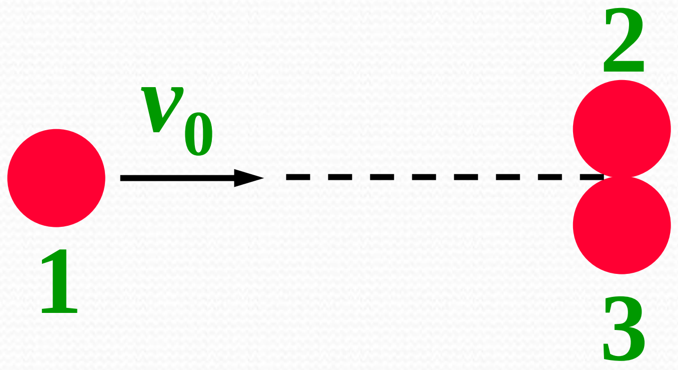
(a)

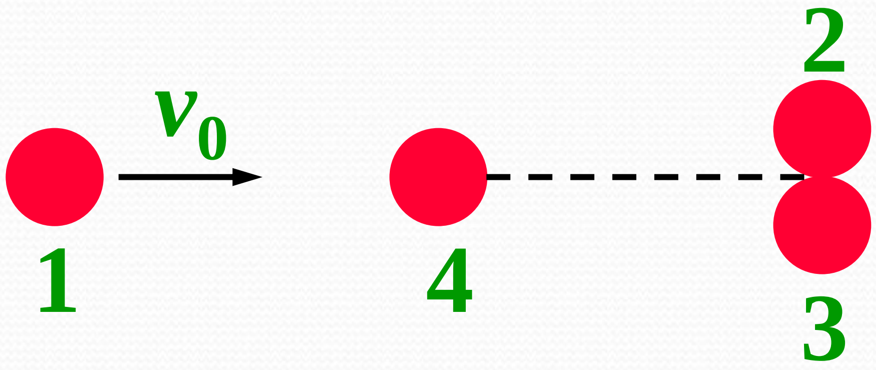


(b)

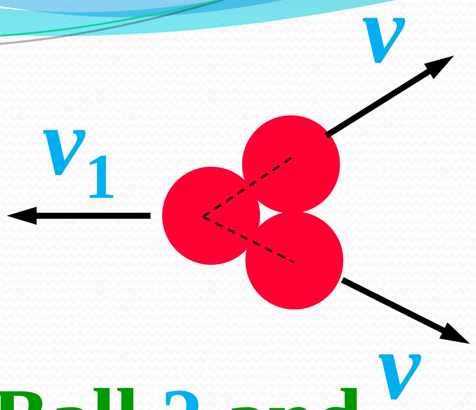
**During a collision, the impulses are equal and opposite: the gain in momentum for body 1 is equal to the loss of momentum for body 2. After collision, their momenta are different, but the total momentum of the system is unchanged.**

**Example3.5** **There are three identical little ball on the smooth horizontal surface, as in figure. At the beginning, Ball 2 and Ball 3 are at rest and the velocity of Ball 1 is $v_0$. Assume the collision between the balls are elastic collision, find the velocities of the three balls after collision. How about the velocities if there is the fourth ball?**



**Solution** **After collision the moving direction of the three balls shown in right figure.**




**The magnitude of the velocities of Ball 2 and Ball 3 are the same, $v$, and the magnitude of the velocity of Ball 1 is $v_1$.**

**Conservation of Momentum**

$$mv_0 = -mv_1 + 2mv\cos 30^\circ$$

**Conservation of Kinetic Energy**

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + 2\cdot\frac{1}{2}mv^2$$

**Solving above equations we have**

$$v_1 = \frac{v_0}{5} \qquad v = \frac{2\sqrt{3}}{5} v_0$$

**The fourth ball dose not change the collision process. Ball 1 transmits its mechanical energy and momentum to Ball 4 and stops at the position of ball 4. Ball 4 move at $v_0$ along the same direction of Ball 1. Ball 4 will collide with the other balls just like Ball 1. After the collision, ball 4 will move left with speed $v_0$/5 and collide with ball 1 again. Then Ball 1 will move with speed $v_0$/5 and Ball 4 stop again at the same position.**

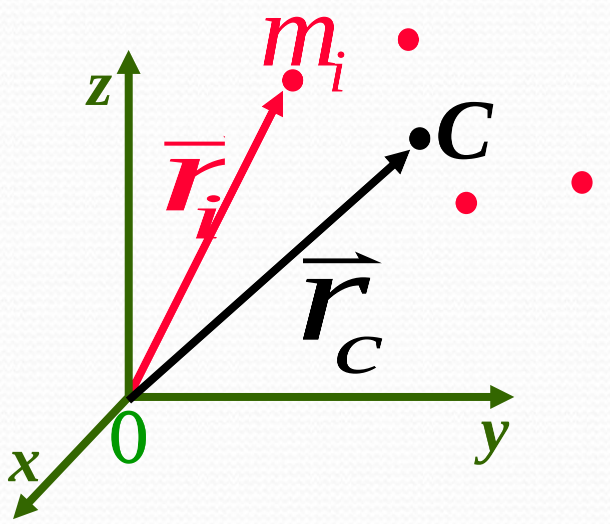
# § 3.4 The center of mass（质心）

## Define the Center of Mass

◆**For a collection of *N* individual point like particles whose masses and positions we know:**

$$\vec{r}_c = \frac{\sum_i m_i \vec{r}_i}{m}$$

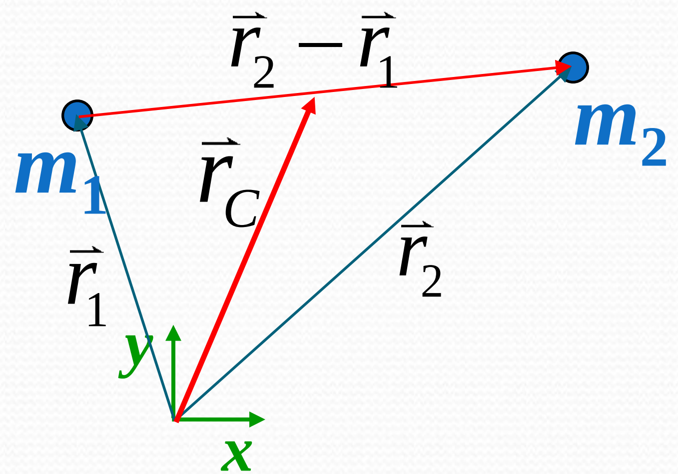
**Total mass**



◆**If the system is made up of only two particles**

$$\vec{R}_C = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$= \frac{(m_1 + m_2)\vec{r}_1 + m_2(\vec{r}_2 - \vec{r}_1)}{(m_1 + m_2)}$$




**So:** $$\vec{R}_C = \vec{r}_1 + \frac{m_2}{M}(\vec{r}_2 - \vec{r}_1)$$

**where** $M = m_1 + m_2$

If $m_1 = m_2$

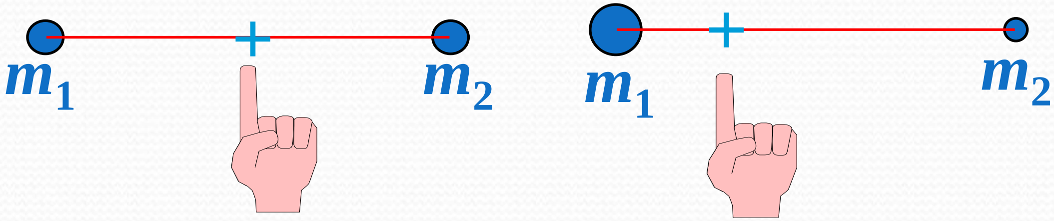
$$\vec{R}_C = \vec{r}_1 + \frac{1}{2}\left(\vec{r}_2 - \vec{r}_1\right)$$

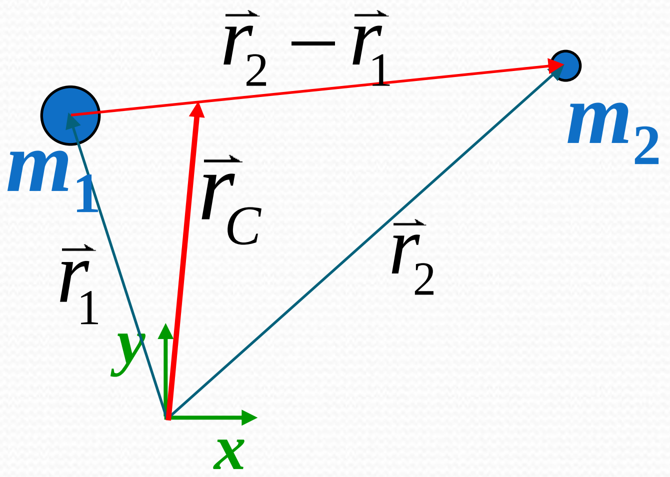
**the CM is halfway between the masses.**

If $m_1 = 3m_2$

$$\vec{R}_C = \vec{r}_1 + \frac{1}{4}\left(\vec{r}_2 - \vec{r}_1\right)$$

**the CM is now closer to the heavy mass.**




**The center of mass is where the system is balanced!**

**We can consider the components of $R_C$ separately:**

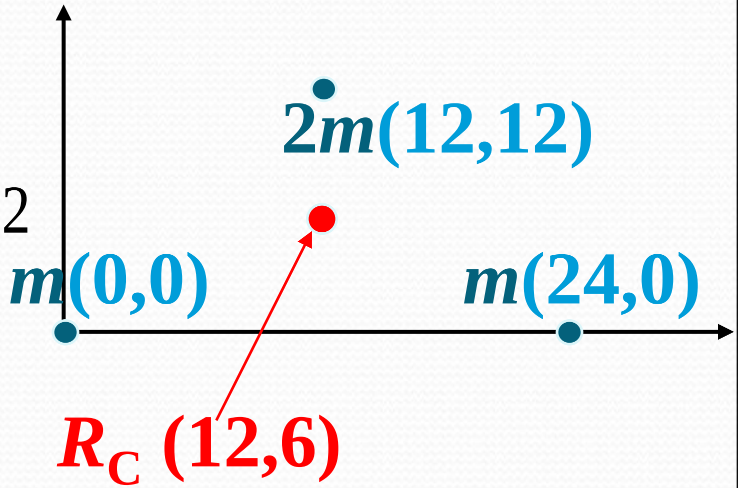
$$(X_C, Y_C, Z_C) = \left( \frac{\sum_i m_i x_i}{M}, \frac{\sum_i m_i y_i}{M}, \frac{\sum_i m_i z_i}{M} \right)$$

**Example3.7 Consider the following mass distribution, find the mass center.**

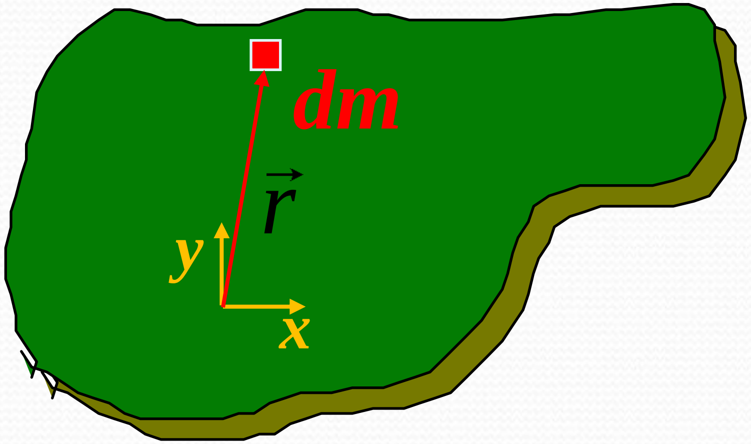
**Solution**

$$X_C = \frac{\sum_i m_i x_i}{M} = \frac{m\times 0 + (2m)\times 12 + m\times 24}{4m} = 12$$

$$Y_C = \frac{\sum_i m_i y_i}{M} = \frac{m\times 0 + 2m\times 12 + m\times 0}{4m} = 6$$



◆ **For a continuous solid, we have to do an integral.**

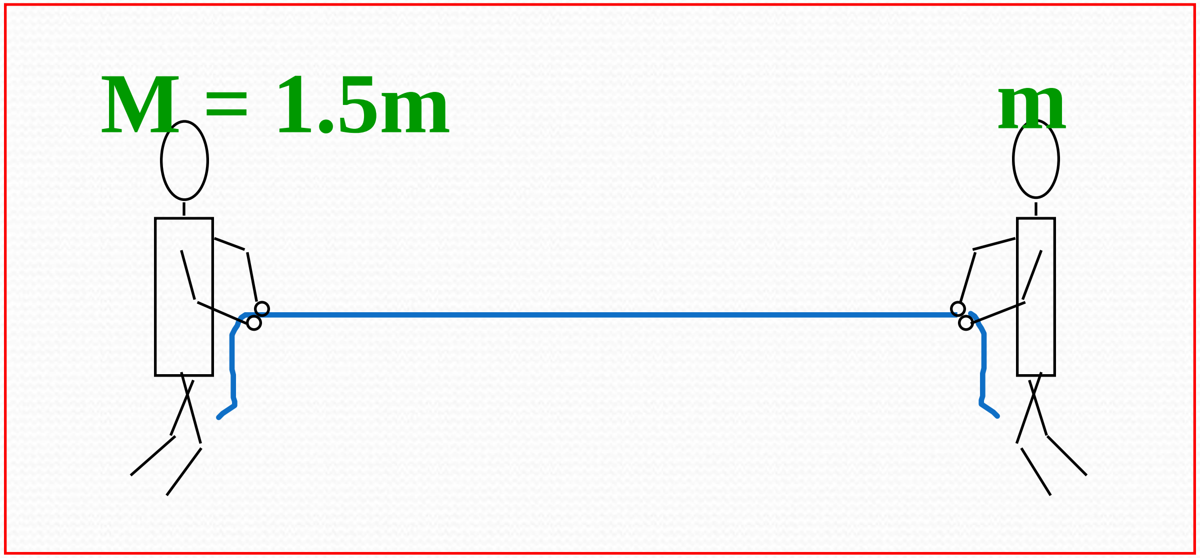



$$\vec{r}_c = \frac{\int \vec{r}\,\mathrm{d}m}{\int \mathrm{d}m}$$

**where d*m* is an infinitesimal mass element.**

**The location of the center of mass is an intrinsic property of the object! (it does not depend on where you choose the origin or coordinates when calculating it).**

**Example3.8 Two astronauts at rest in outer space are connected by a light rope. They begin to pull towards each other. Where do they meet?**



## Solution

- They start at rest, so $v_C = 0$.
- $v_C$ remains zero because there are no external forces.
- So, the CM does not move!
- They will meet at the CM.

**Finding the CM:**

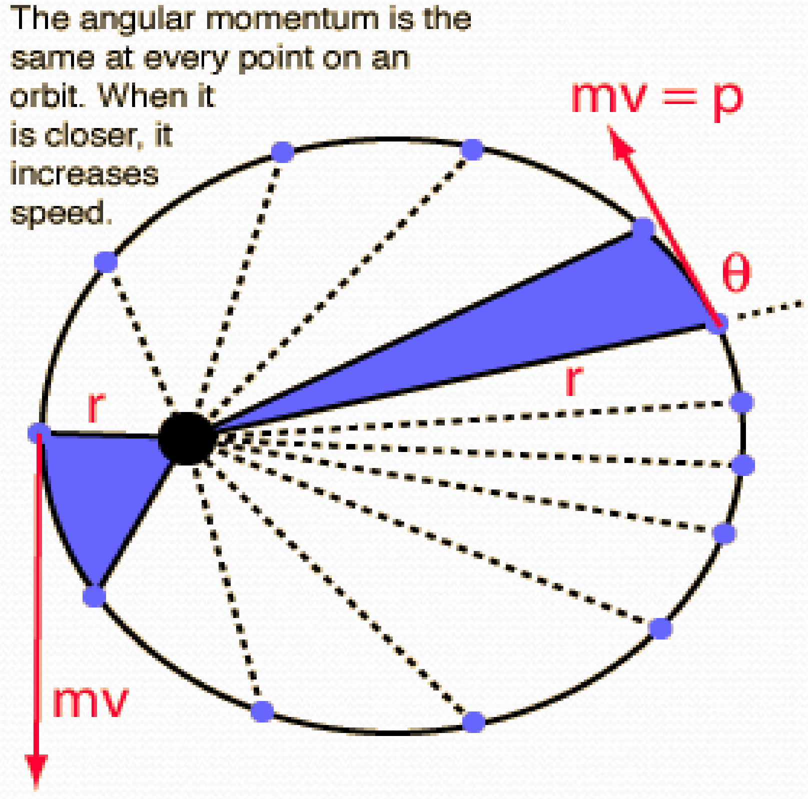
**If we take the astronaut on the left to be at $x = 0$:**

$$x_C = \frac{M(0) + m(L)}{M + m} = \frac{m(L)}{2.5m} = \frac{2}{5}L$$

# § 3.5 The Law of Conservation of Angular Momentum

**Kepler's second law:**

**A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.**

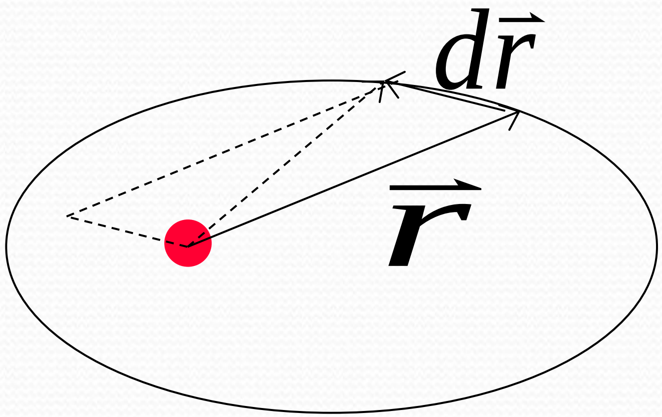


## Areal velocity (掠面速度):

$$\frac{dS}{dt} = \frac{\frac{1}{2}\left|\vec{r} \times d\vec{r}\right|}{dt}$$

$$= \frac{1}{2}\left|\vec{r} \times \vec{v}\right|$$

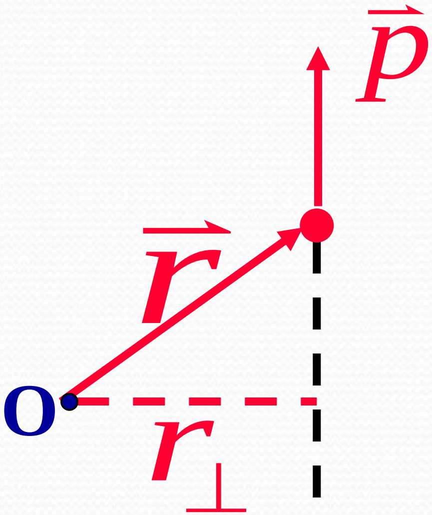
$$= \frac{1}{2m}\left|\boxed{\vec{r} \times \vec{p}}\right|$$



## ◆ The angular momentum of a mass point with respect to a fixed point

$$\vec{L} = \vec{r} \times \vec{p}$$

| Angular momentum | Position vector | momentum |
|---|---|---|

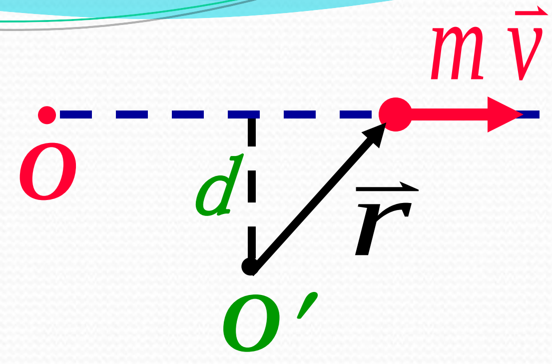



$\vec{L}$ { **magnitude:** $pr_\perp$ ; **direction:** **right-hand rule**

$\vec{L}$ **is also called moment of momentum**

**SI units:** **kg·m$^2$/s**

**Example3.9 Find the angular momentum of the mass point with respect to *O* or *O′*.**




**Solution**

**with respect to *O*:**

$$\vec{L} = 0$$

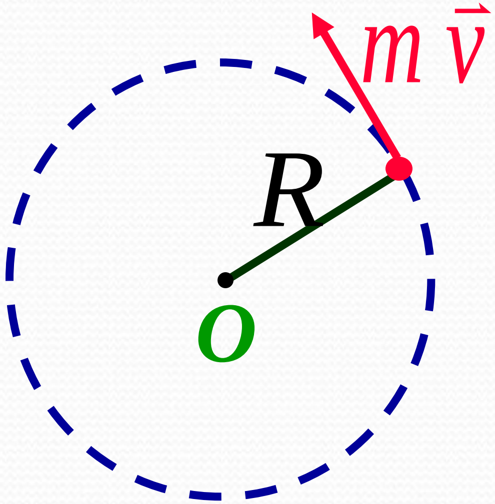
**with respect to *O′*:**

$\vec{L}$ { **magnitude:** $mvd$ ; **direction:** $\otimes$ }

**Example3.10 Find the angular momentum of the mass point with respect to the center *O*.**

**Solution**

$\vec{L}\begin{cases}\textbf{magnitude:}\quad m\,v\,R \\ \textbf{direction:}\quad \odot\end{cases}$



**Example3.11 The equation of motion of a mass point m is $\vec{r} = a\cos\omega t\vec{i} + b\sin\omega t\vec{j}$, where $a$, $b$ and $\omega$ are all constants. Find the angular momentum at $t$.**

**Solution**

$$\vec{L} = \vec{r} \times m\vec{v}$$

$$= (a\cos\omega t\vec{i} + b\sin\omega t\vec{j}) \times m(-\omega a\sin\omega t\vec{i} + \omega b\cos\omega t\vec{j})$$

$$= m\omega ab\vec{k}$$

**There is no time t in the expression of angular momentum $\vec{L}$. What does it indicate?**

## ◆ The theorem of angular momentum of a mass point

$$\vec{L} = \vec{r} \times \vec{p}$$

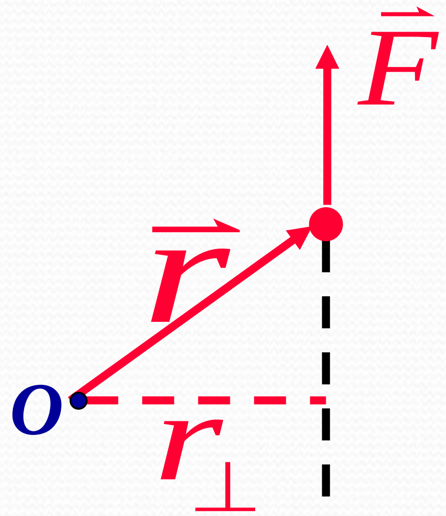
$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{p} + \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = 0 + \vec{r} \times \vec{F} = \vec{M}$$

$$\vec{M} = \frac{d\vec{L}}{dt}$$ ——the theorem of angular momentum of a mass point.

**It states that the torque acting on an object is equal to the time rate of change of the object′s angular momentum.**

## ◆ The torque of a mass point with respect to a fixed point

$$\vec{M} = \vec{r} \times \vec{F}$$

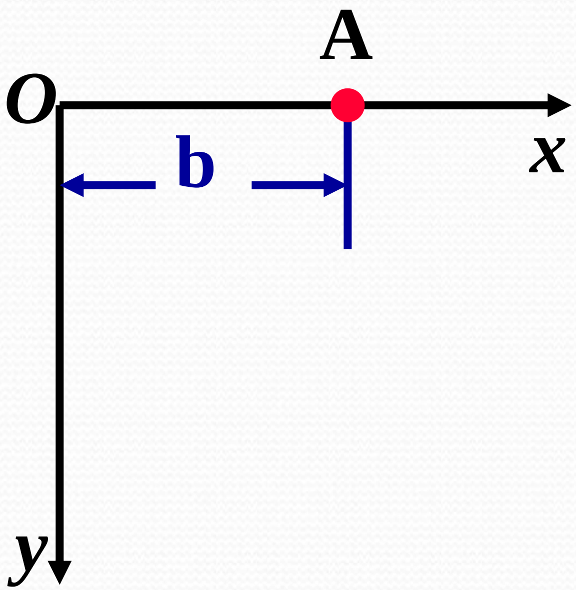



$\vec{M}$ :The torque of $\vec{F}$ with respect to $O$

$\vec{M}$ {
magnitude: $F\,r_\perp$
direction: right-hand rule

SI units: N·m

**Example3.11** **A mass point *m* rests at *A* initially. Then it falls freely. What are the torque and angular momentum of the mass point at *t* with respect to *O*?**




**Solution** **The position of the mass point at *t* is:**

$$\vec{r} = b\vec{i} + \frac{1}{2}gt^2\vec{j}$$

**The torque about *O* is**

$$\vec{M} = \vec{r} \times \vec{F} = (b\vec{i} + \frac{1}{2}gt^2\vec{j}) \times mg\vec{j} = mgb\vec{k}$$

**The velocity of the mass point at *t* is:**

$$\vec{v} = gt\vec{j}$$

**The angular momentum about *O* at *t* is :**

$$\vec{L} = \vec{r} \times m\vec{v}$$

$$= (b\vec{i} + \frac{1}{2}gt^2\vec{j}) \times mgt\vec{j}$$

$$= mgtb\vec{k}$$

**How about the torque and angular momentum if the initial velocity is $v_0\vec{i}$?**

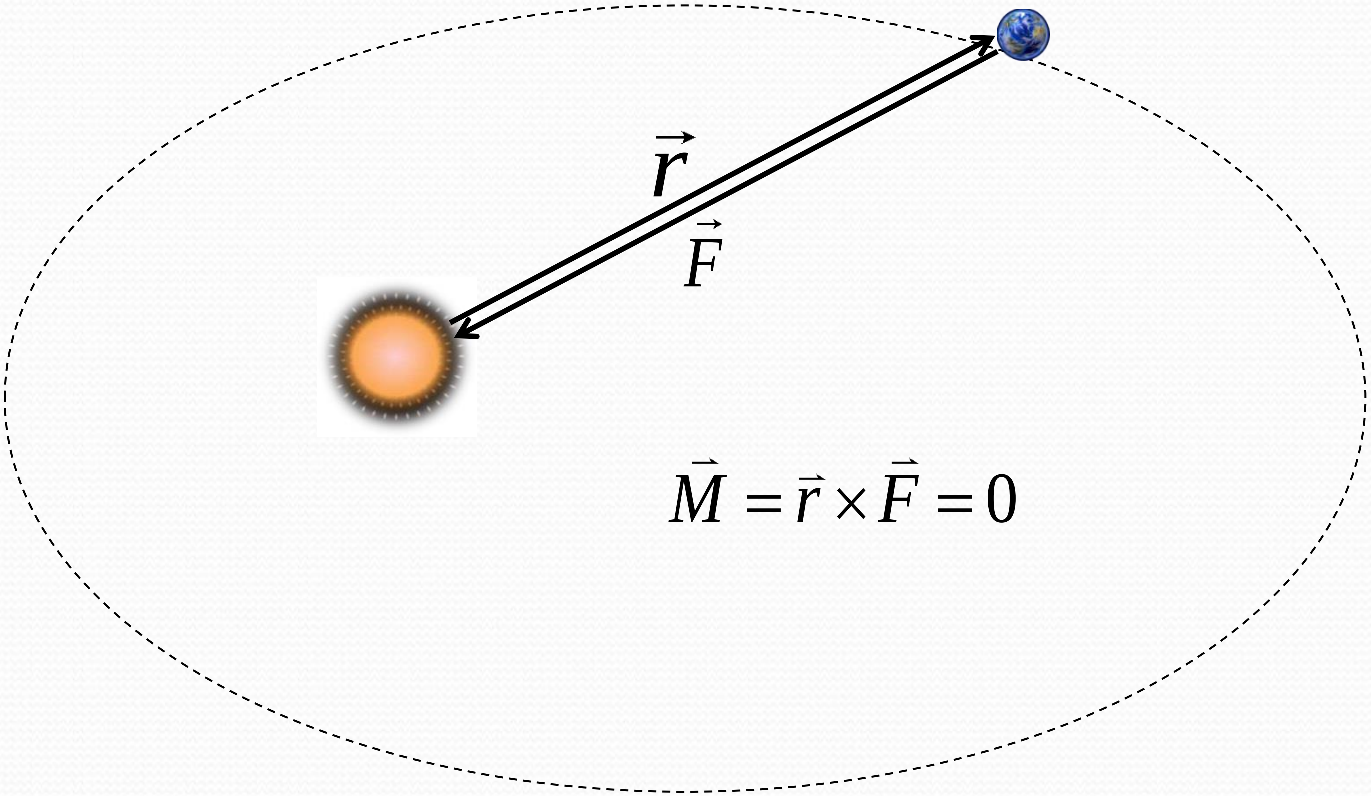
## ◆ The law of angular momentum conservation of a mass point

**For a mass point**

**If** $$\vec{M} = 0$$

**Then** $$\vec{L} = const.$$

**When the net torque is zero, the angular momentum remains constant in time.**

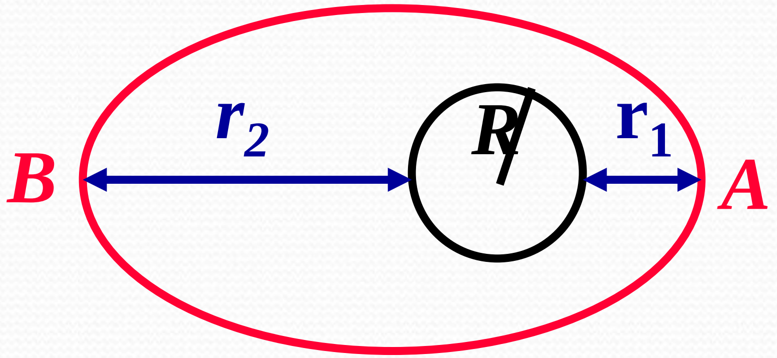



$$\vec{M} = \vec{r} \times \vec{F} = 0$$

**The Earth carries out an elliptical motion around the Sun.**

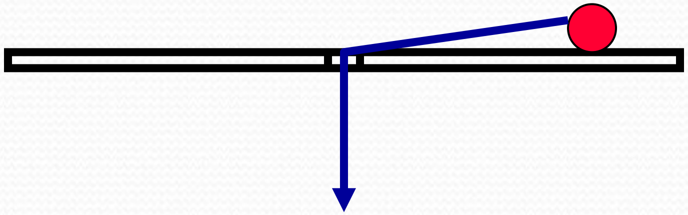
**Example3.12** A artificial satellite $m$ carries out an elliptical motion around the Earth. Point $A$ is at perihelion and point $B$ is at aphelion. The distances from points $A$ and $B$ to the center of the Earth is $r_1$ and $r_2$. If the speed of $m$ at point $A$ is $v_1$ , find the the speed of $m$ at point $B$.

**Solution** The angular momentum of the satellite about the Earth center is conservative.




$$m v_1 (R + L_1) = m v_2 (R + L_2) \quad \rightarrow v_2 = \frac{R + L_1}{R + L_2} v_1$$

**Example3.13** A small ball with mass $m$ tired to a rope carries out a circular motion with angular velocity $\omega_0$ and radius $r_0$ on a frictionless horizontal surface. if there is a vertically downward pulling force acting on the other end of the rope, the small ball will circle around with radius $r_0/3$. What is the new angular velocity of the ball?



**Solution** **The angular momentum of the small ball about the circle center is conservative.**

$$mv_0 r_0 = mv\frac{r_0}{3}$$

**Then we get**

$$v = 3v_0$$

**Can the small ball be pulled to the circle center？**

# For a mass point system

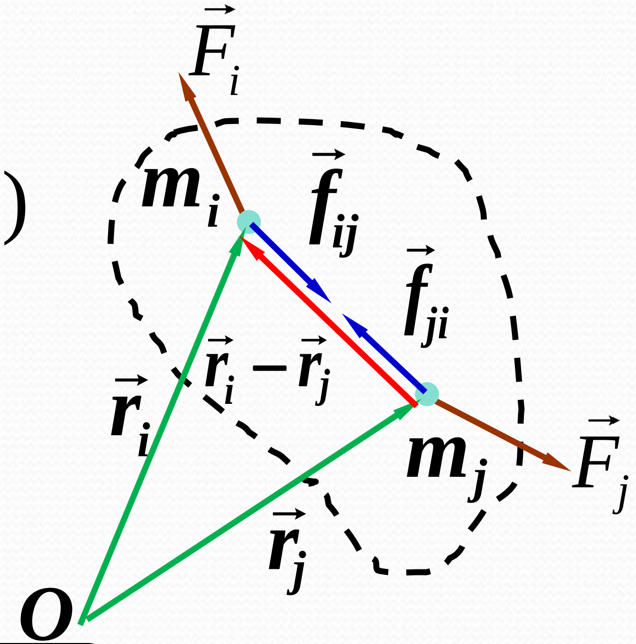
## The net torque acted on two arbitrary mass points：

$$\vec{M} = \vec{r}_i \times (\vec{F}_i + \vec{f}_{ij}) + \vec{r}_j \times (\vec{F}_j + \vec{f}_{ji})$$

$$= (\vec{r}_i \times \vec{F}_i + \vec{r}_j \times \vec{F}_j)$$

$$+ (\vec{r}_i \times \vec{f}_{ij} + \vec{r}_j \times \vec{f}_{ji})$$

$$= (\vec{r}_i \times \vec{F}_i + \vec{r}_j \times \vec{F}_j) + \boxed{(\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij}}$$



**The sum of the external torques is equal to the time rate of change of the total angular momentum.**

$$\vec{M}_{ext} = \frac{d\vec{L}}{dt}$$

**If** $\vec{M}_{ext} = 0$

**then** $\vec{L}_{total} = const.$

**When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).**

盘状星系

# See you next time!