Chapter 6
Simple Harmonic Motion

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安徽卫视
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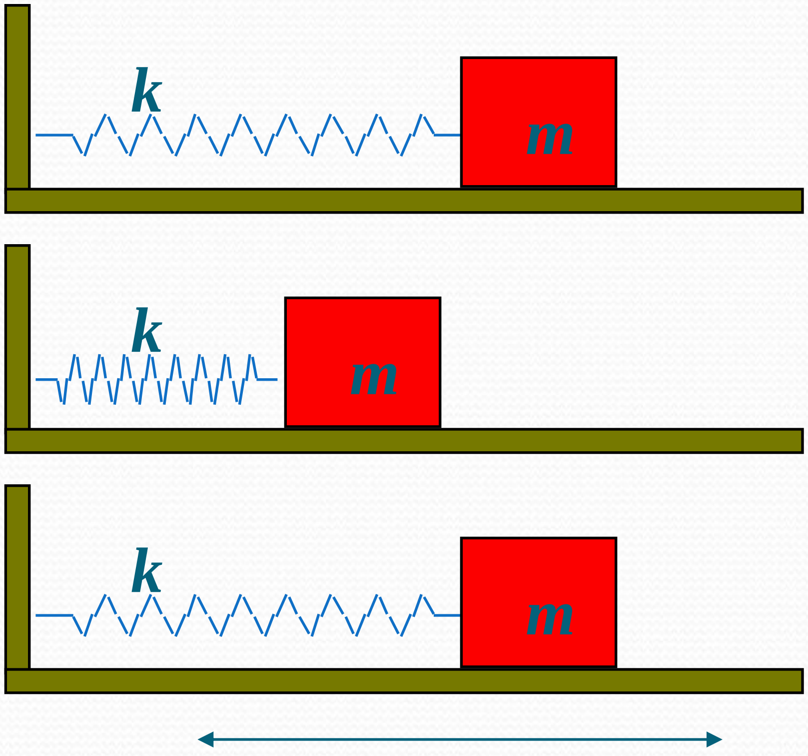
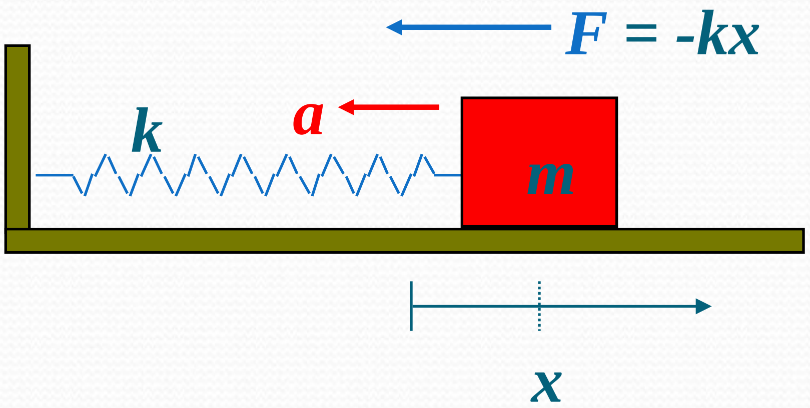
We know that if we stretch a spring with a mass on the end and let it go, the mass will oscillate back and forth (if there is no friction).

This oscillation is called Simple Harmonic Motion.



§ 6.1 The Dynamic Equation of Simple Harmonic Motion

At any given instant we know that $F = ma$ must be true.



But in this case $F = -kx$

And

$$ma = m\frac{d^2x}{dt^2}$$

So

$$-kx = ma = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$ **a differential equation for *x(t)*!**

define $\omega = \sqrt{\frac{k}{m}}$ $\Rightarrow$ $$\frac{d^2x}{dt^2} = -\omega^2 x$$

Try the solution $x = A\cos(\omega t)$

$$\frac{dx}{dt} = -\omega A\sin(\omega t)$$

Where ω is the angular frequency of motion.

$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t) = -\omega^2 x$$ ✓

This works, so it must be a solution!

We just showed that $\frac{d^2x}{dt^2} = -\omega^2 x$ has the solution

$$x = A\cos(\omega t)$$

This is not a unique solution, $x = A\sin(\omega t)$ **is also a solution.**

The most general solution is a linear combination of these two solutions!

$$x = B\sin(\omega t) + C\cos(\omega t)$$

$$\frac{dx}{dt} = \omega B\cos(\omega t) - \omega C\sin(\omega t)$$

$$\frac{d^2x}{dt^2} = -\omega^2 B\sin(\omega t) - \omega^2 C\cos(\omega t) = -\omega^2 x$$ ✓

We want to use the most general solution:

$x = A\cos(\omega t + \phi)$

$x = B\sin(\omega t) + C\cos(\omega t)$

equivalent to each other

$x = A\cos(\omega t + \phi)$

$= A\cos(\omega t)\cos\phi - A\sin(\omega t)\sin\phi$

$= C\cos(\omega t) + B\sin(\omega t)$

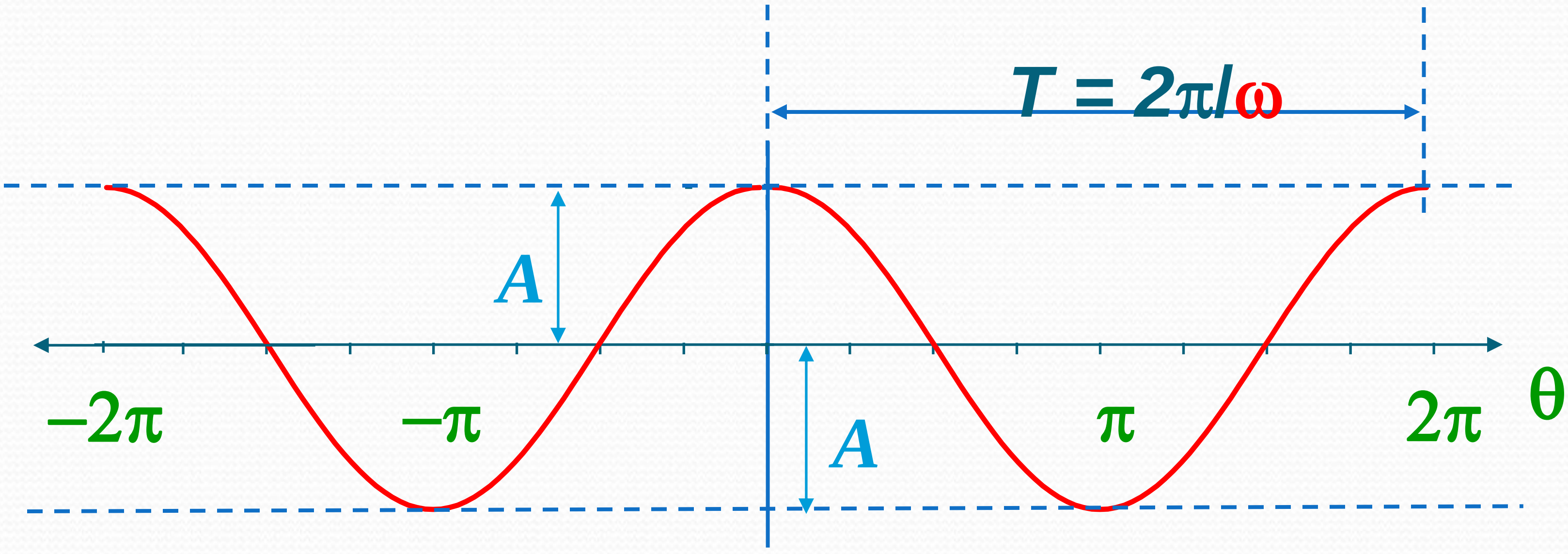
where $C = A\cos(\phi)$ **and** $B = -A\sin(\phi)$

It works!

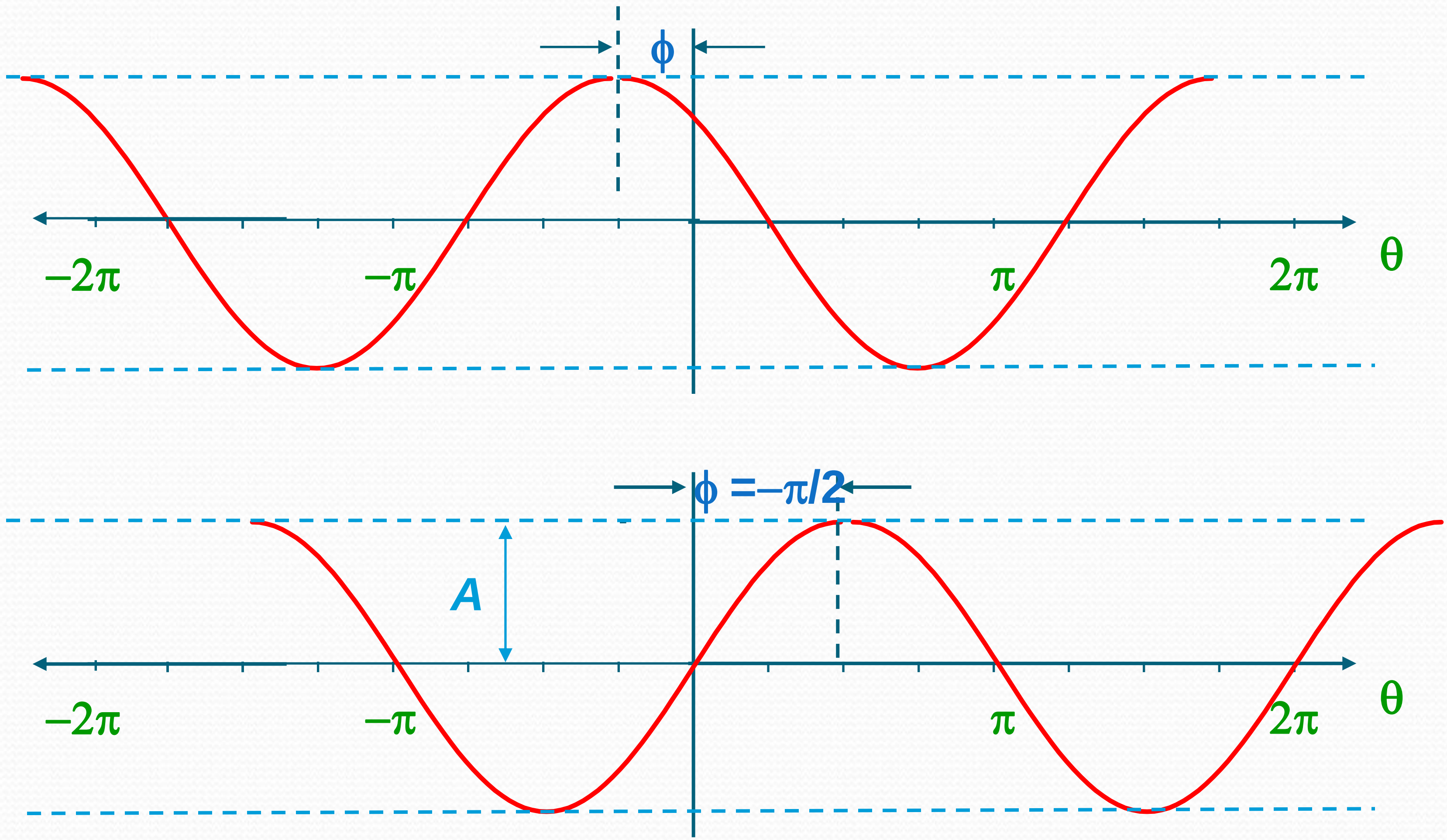
So we can use $x = A\cos(\omega t + \phi)$ as the most general solution!

Drawing of $A\cos(\omega t)$

A=**amplitude of oscillation**

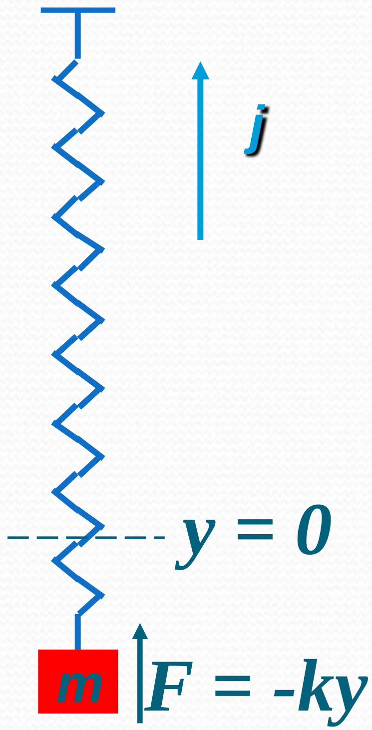


Drawing of $A\cos(\omega t + \phi)$



For a vertical spring

$$E_P = \frac{1}{2}ky^2$$ if y is measured from the equilibrium position



The force of the spring is the negative derivative of this function:

$$F = -\frac{dE_P}{dy} = -ky$$

So this will be just like the horizontal case:

$$-ky = ma = m\frac{d^2y}{dt^2}$$

Which has solution $y = A\cos(\omega t + \phi)$

$$\omega = \sqrt{\frac{k}{m}}$$

Note

- **The most general solution is** $x = A\cos(\omega t + \phi_o)$

  where A **= amplitude**

  ω **= angular frequency**

  $\phi = \omega t + \phi_o$ **phase**

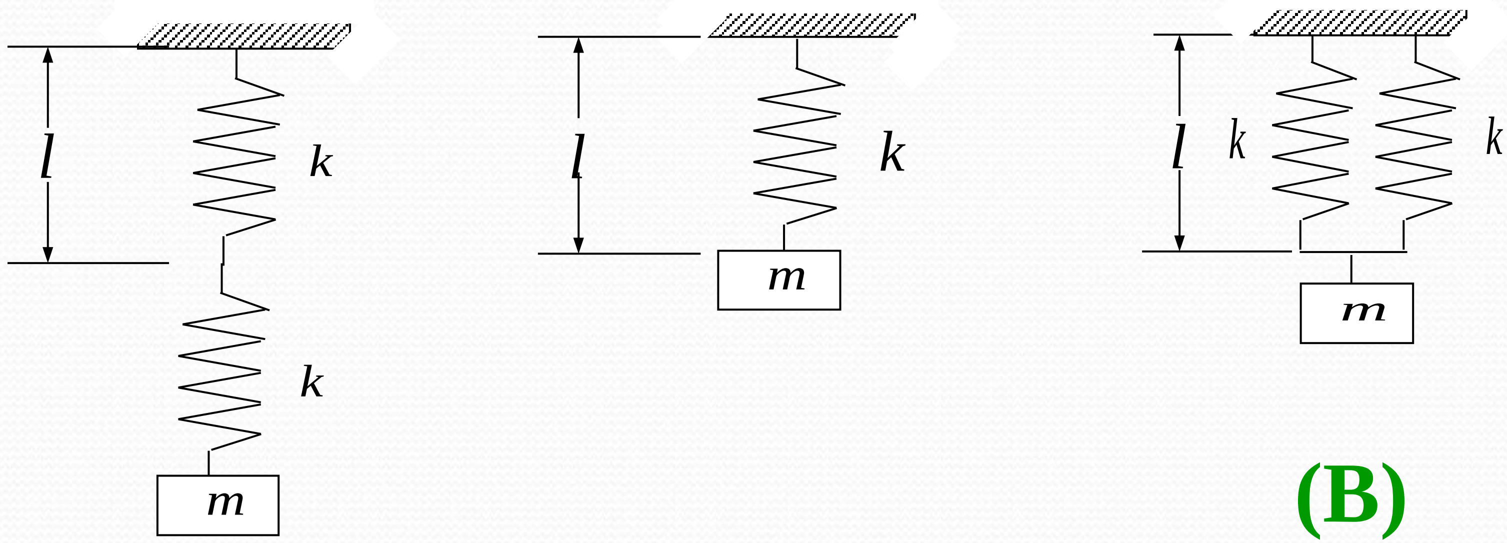
  For a mass on a spring $$\omega = \sqrt{\frac{k}{m}}$$

  - **The frequency does *<u>not</u>* depend on the amplitude!**
  - **We will see that this is true of all simple harmonic motion!**

- **The oscillation occurs around the equilibrium point where the net force is zero!**

Example6.1 **As shown in figure, (a), (b) and (c) are three oscillating systems. Their angular frequencies are ω_a, ω_b and ω_c, respectively. The ration of them $\omega_a:\omega_b:\omega_c$ is**

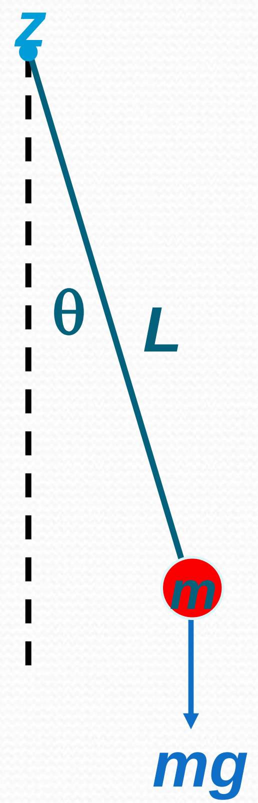
(A) 2:1:1/2 (B)1:√2:2 (C)2:√2:1 (D)1:1:2



(B)

The Simple Pendulum

A pendulum is made by suspending a mass *m* at the end of a string of length *L*. Find the angular frequency of oscillation for small displacements.



The torque due to gravity about the rotation (*z*) axis is

$$M = -mgL\sin\theta \approx -mgL\theta \text{ for small } \theta$$

A Taylor expansion of *sin*θ about θ → 0 gives:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

So for θ << *1*, $\sin\theta \approx \theta$

The inertial moment is $J = mL^2$

So we have $M = -mgL\theta = J\alpha = J\dfrac{d^2\theta}{dt^2}$

That is $-\dfrac{g}{L}\theta = \dfrac{d^2\theta}{dt^2}$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\omega^2\theta$$ **where** $\omega = \sqrt{\dfrac{g}{L}}$

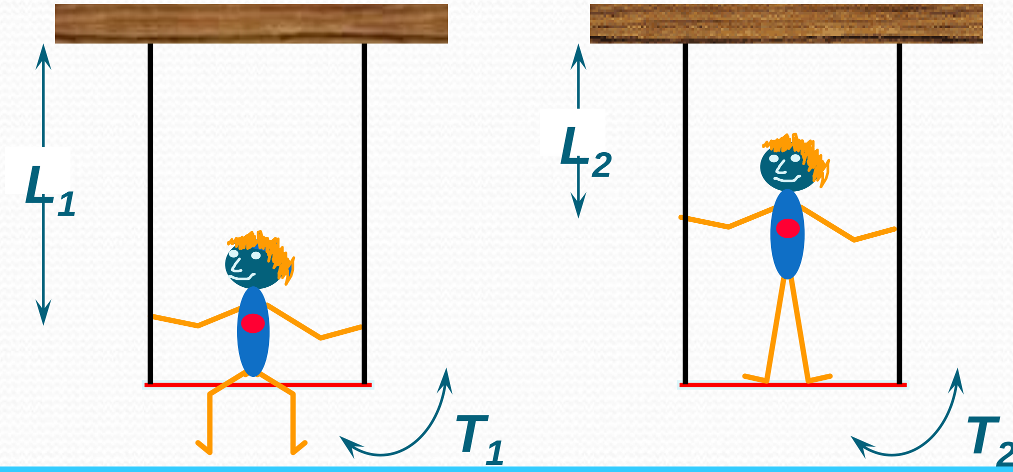
$$\theta = \theta_0 \cos(\omega t + \phi_0)$$

Example6.2 **You are *sitting* on a swing. A friend gives you a small push and you start swinging back & forth with period T_1. Suppose you were *standing* on the swing rather than sitting. When given a small push you start swinging back & forth with period T_2. Which of the following is true?**

(a) $T_1 = T_2$

(b) $T_1 > T_2$

(c) $T_1 < T_2$



Solution We have shown that for a simple pendulum

$$\omega = \sqrt{\frac{g}{L}}$$

Since $$T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\frac{L}{g}}$$

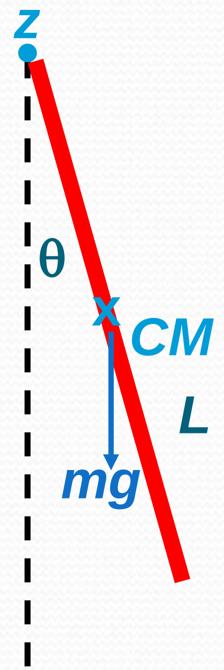
If we make a pendulum shorter, it oscillates faster (smaller period)

Standing up raises the CM of the swing, making it shorter!

Since $L_1 > L_2$ we see that $T_1 > T_2$.

➢ The Rod Pendulum

A pendulum is made by suspending a thin rod of length *L* and mass *m* at one end. Find the angular frequency of oscillation for small displacements.



The torque about the rotation (*z*) axis is

$$M = -mg\frac{L}{2}\sin\theta \approx -mg\frac{L}{2}\theta$$ **for small θ**

In this case $J = \frac{1}{3}mL^2$

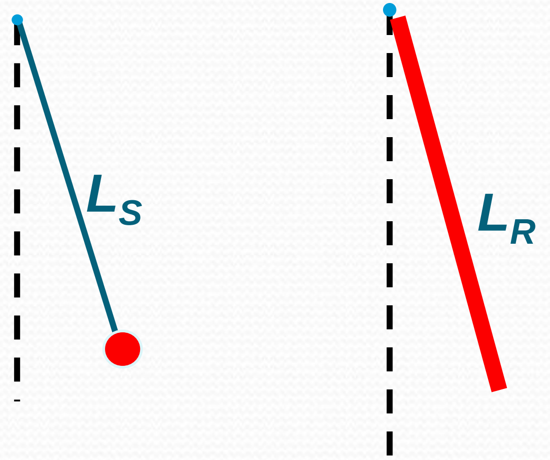
$M = J\alpha$ **becomes** $-mg\frac{L}{2}\theta = \frac{mL^2}{3}\frac{d^2\theta}{dt^2}$

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta \quad \textbf{where} \quad \omega = \sqrt{\frac{3g}{2L}}$$

Example6.3 What length do we make the simple pendulum so that it has the same period as the rod pendulum?

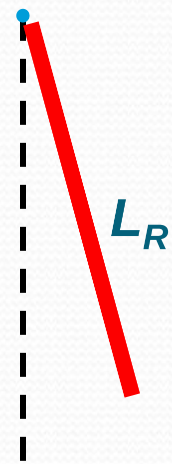
(a) $L_S = \frac{3}{2}L_R$ **(b)** $L_S = \frac{2}{3}L_R$

(c) $L_S = L_R$



Solution

$$\omega_S = \sqrt{\frac{g}{L_S}}$$



$$\omega_R = \sqrt{\frac{3g}{2L_R}}$$

$$\boxed{\boldsymbol{\omega_S = \omega_P} \quad \textbf{if} \quad L_S = \frac{2}{3}L_R}$$

§ 6.2 Kinematics of Simple Harmonic Motion

➢ The equation of simple harmonic motion

$$\boxed{x = A\cos(\omega t + \phi_0)}$$

by taking derivatives, since:

$$v(t) = \frac{dx}{dt} \qquad a(t) = \frac{dv}{dt}$$

Velocity $\quad v = dx/dt = -\omega A\sin(\omega t + \phi_0)$

Acceleration

$$a = d^2x/dt^2 = -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x$$

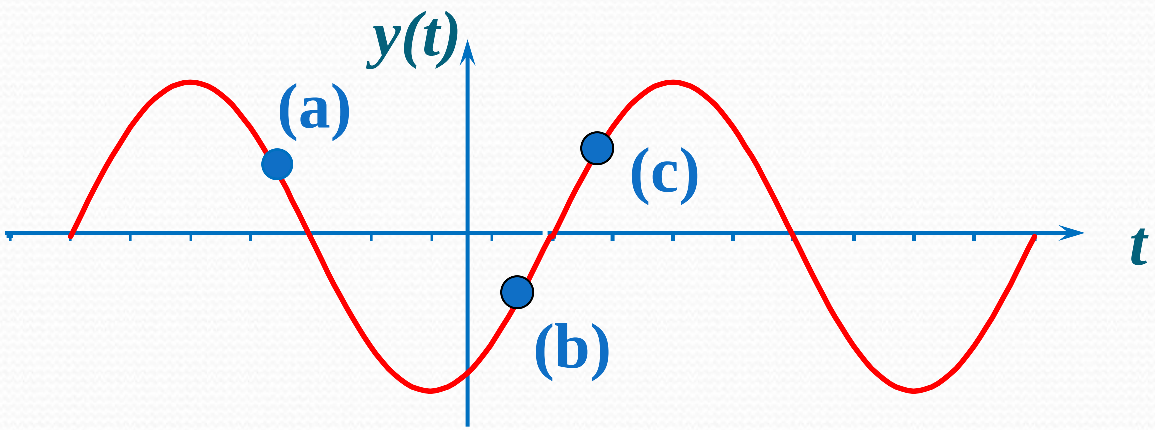
And we have

$$x_{\max} = A$$

$$v_{\max} = A\omega$$

$$a_{\max} = \omega^2 A$$

Example6.4 A mass oscillates up & down on a spring. Its position as a function of time is shown below. At which of the points shown does the mass have positive velocity and negative acceleration?

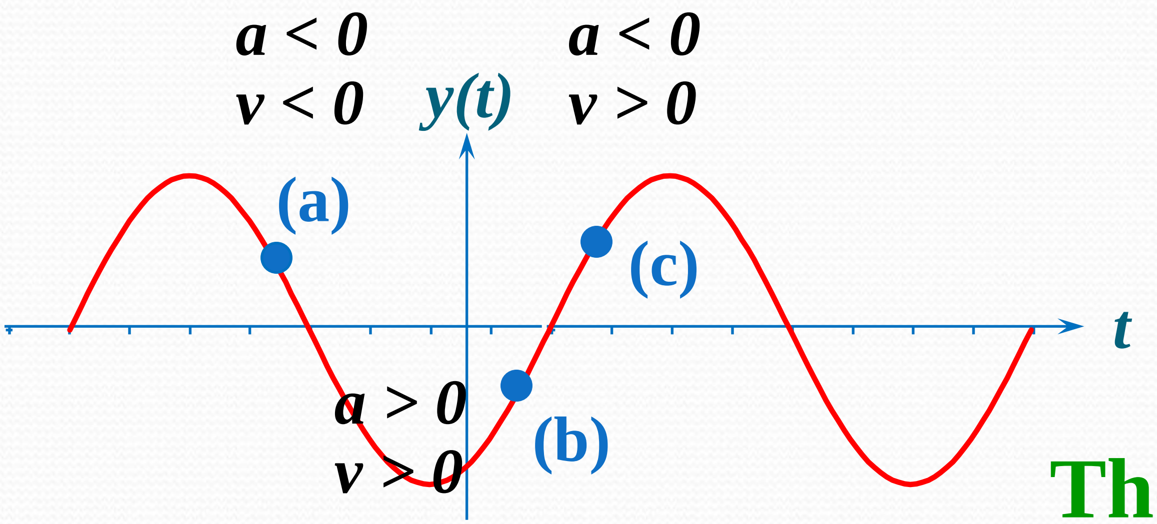


The slope of *y(t)* tells us the sign of the velocity since

$$v = \frac{dy}{dt}$$

***y(t)* and *a(t)* have the opposite sign since**

$$a = -\omega^2 A$$



The answer is (c).

Example6.5 **A mass m=2kg on a spring oscillates with amplitude A=10cm. At $t = 0$ its speed is maximum, and is v_{max} = +2 m/s.**

What is the angular frequency of oscillation ω?

What is the spring constant k?

Solution

$$v_{\max} = A\omega$$

$$\omega = \frac{v_{MAX}}{A} = \frac{2\,m/s}{10\,cm} = 20\,s^{-1}$$

$$\omega = \sqrt{\frac{k}{m}} \rightarrow k = m\omega^2$$

So k = (2 kg)×(20 s^{-1})^2 = 800 kg/s^2 = 800 N/m

- **Use "initial conditions" to determine phase ϕ_0!**

Example6.6 Suppose we are told *$x(0) = 0$* , and *x* is initially increasing (i.e. *$v(0) =$ positive*). Find the initial phase ϕ_0.

Solution

$$x(0) = A\cos(\phi_0) = 0$$

So

$$\phi_0 = \frac{\pi}{2} \quad \text{or} \quad \phi_0 = -\frac{\pi}{2}$$

And

$$v(0) = -\omega A\sin(\phi_0) > 0$$

So

$$\phi_0 = -\frac{\pi}{2}$$

Example6.7 **A mass hanging from a vertical spring is lifted a distance *d* above equilibrium and released at $t = 0$. Which of the following describes its velocity and acceleration as a function of time?**

(a) $v(t) = -v_{max} \sin(\omega t)$

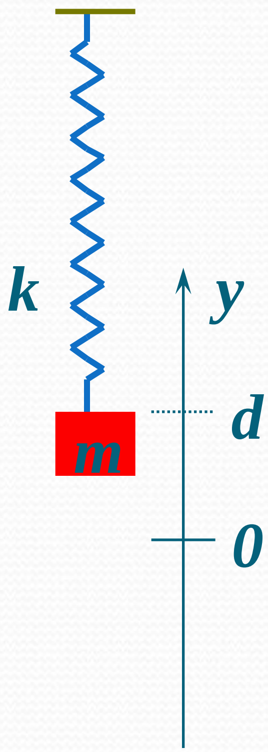
$a(t) = -a_{max} \cos(\omega t)$

(b) $v(t) = v_{max} \sin(\omega t)$

$a(t) = a_{max} \cos(\omega t)$

(c) $v(t) = v_{max} \cos(\omega t)$

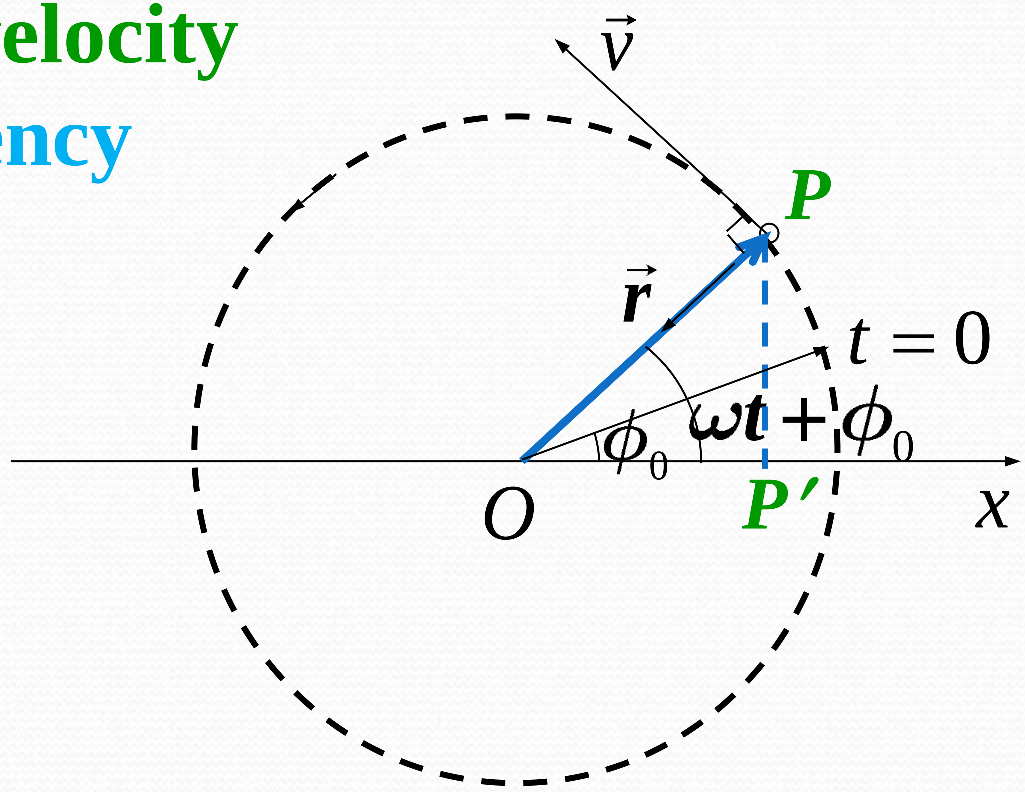
$a(t) = -a_{max} \cos(\omega t)$



(a)

phasor diagram

- **constant angular velocity**
  ω = angular frequency
- **The radius $|\vec{r}|$ is**
  ***A* = amplitude**
- **The angle ϕ_0 at**
  ***t*=0 is initial phase**
- **The angle $\omega t+\phi_0$**
  at *t* is phase



- **The particle travels anticlockwise**

The *x* component

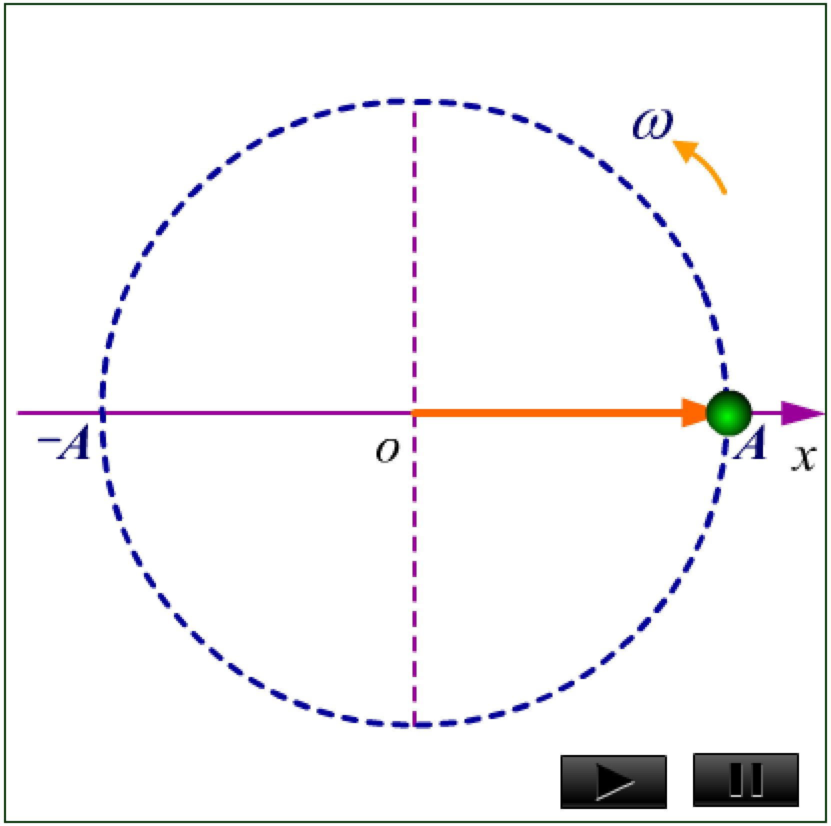
$$[\vec{A}]_x = A\cos(\omega t + \phi_0)$$

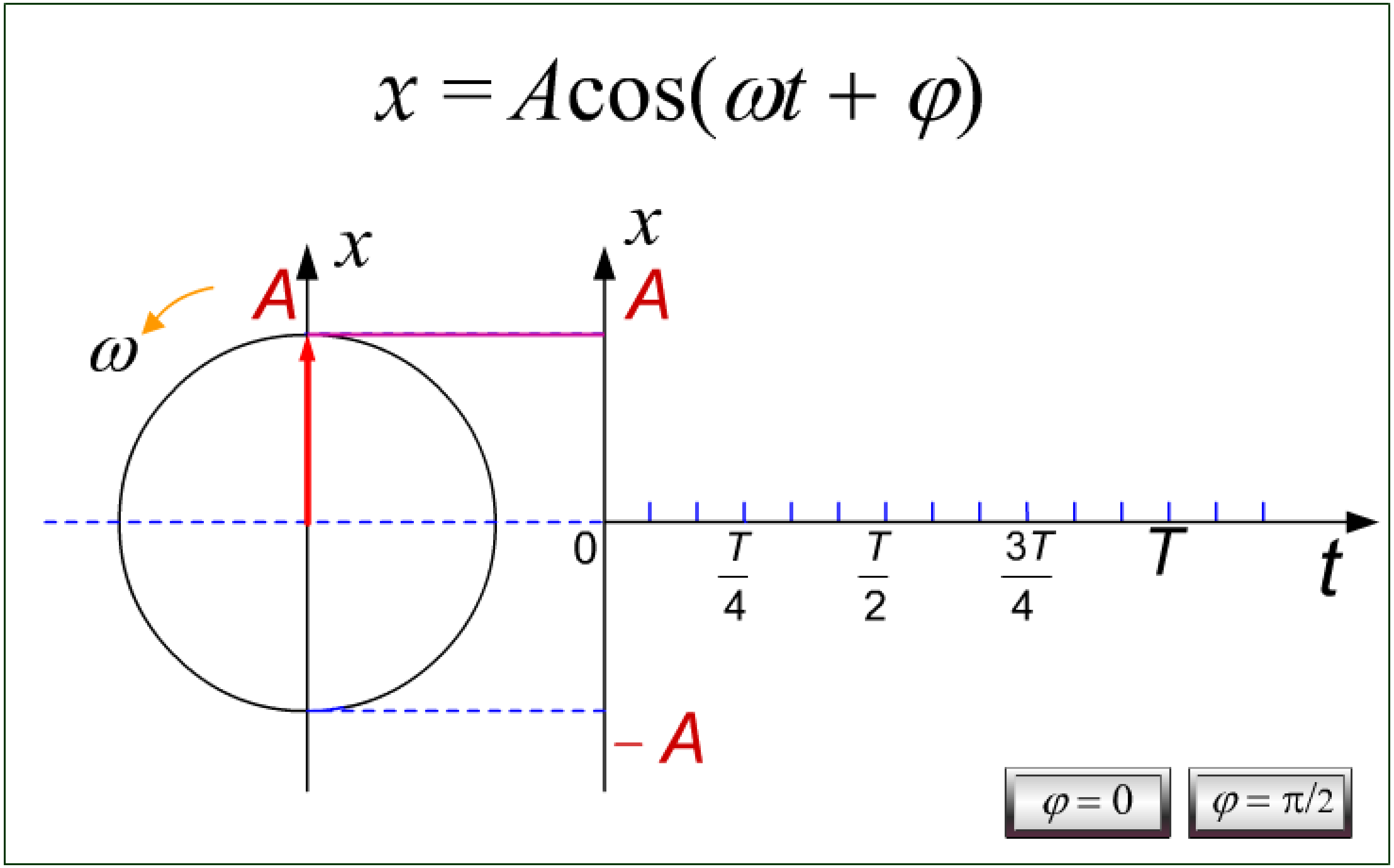
$$[d\vec{A}/dt]_x$$

$$= -\omega A\sin(\omega t + \phi_0)$$

$$[d^2\vec{A}/dt^2]_x$$

$$= -\omega^2 A\cos(\omega t + \phi_0)$$



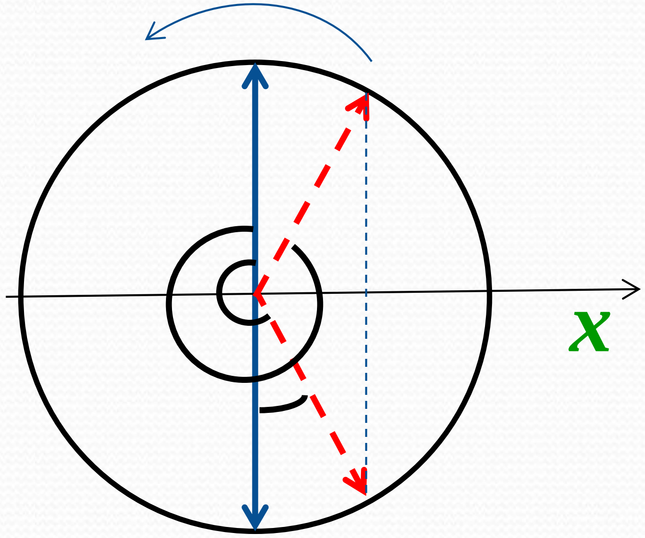


Simple harmonic motion can be described as the projection of uniform circular motion along a diameter of the circle. This method is called rotating vector diagram.

Example 6.8 **A particle on a spring executes simple harmonic motion with a period *T*. From the rotating vector diagram, determine the least time required for the particle to come in positive halfway position from the equilibrium location.**

Solution

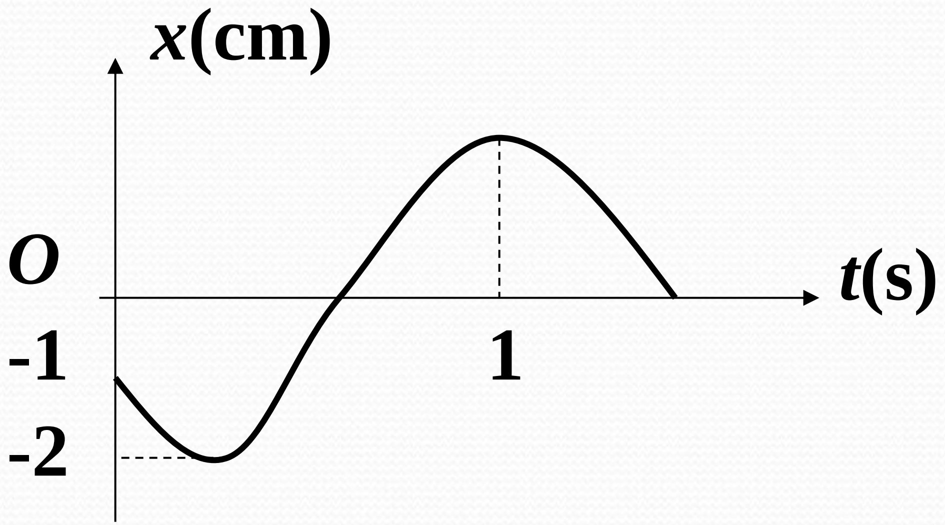
$$t = \frac{\pi/6}{\omega 2\pi} = \frac{\pi/6}{2\pi/T} = \frac{T}{12}$$



Example 6.9 **A oscillation curve of a simple harmonic oscillator is as shown in figure. Find the equation of simple harmonic motion.**

Solution **Assume**

$$x = A\cos(\omega t + \phi_0)$$

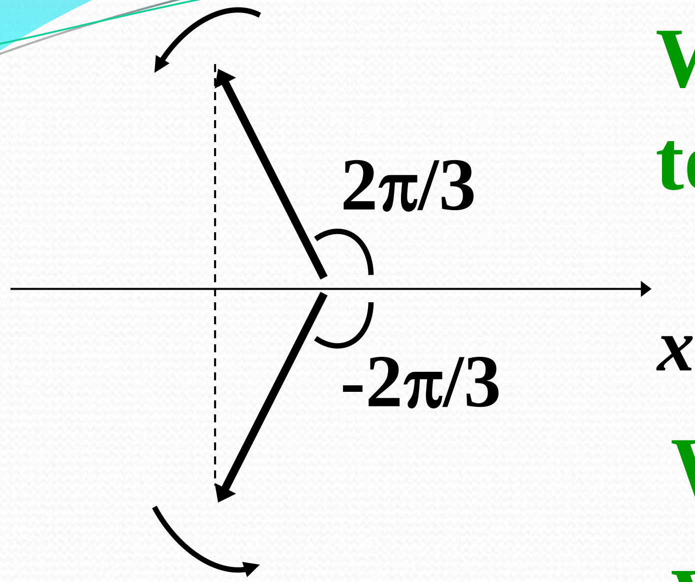


Given ***A*=2 cm**

Let t=0, we have

$$-1 = 2\cos\phi_0$$

$$\rightarrow \phi_0 = 2\pi/3 \ \textbf{or} \ -2\pi/3$$



When t>0 , the particle moves towards –*x* direction, so

$$\phi_0 = 2\pi/3$$

When t=1s, *x*=*A*,

We know $\phi = 2\pi$

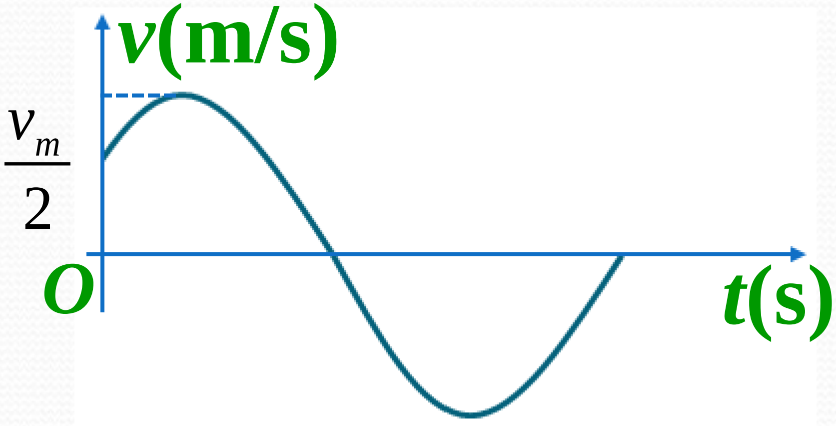
Since $\phi = \omega t + \phi_0$

$$\omega = \frac{\phi - \phi_0}{t} = \frac{2\pi - 2\pi/3}{1} = 4\pi/3 \ \ (s^{-1})$$

So we get $x = 2\cos(4\pi t/3 + 2\pi/3) \ \ cm$

Example 6.10 **The *v-t* curve of a simple harmonic motion is shown in figure. What is the initial phase.**

Solution



$$x = A\cos(\omega t + \phi_0)$$

$$v = \frac{dx}{dt}$$

$$= -\omega A \sin(\omega t + \phi_0)$$

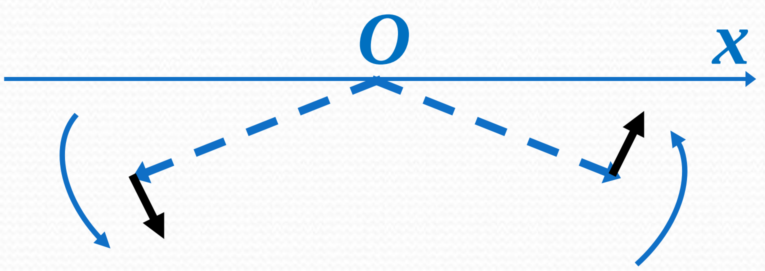
$$= -v_m \sin(\omega t + \phi_0)$$

Let *t*=0, then $v_m / 2 = -v_m \sin \phi_0$

$$\rightarrow \sin \phi_0 = -1/2$$

$$\rightarrow \phi_0 = -\pi / 6 \quad or \quad -5\pi / 6$$

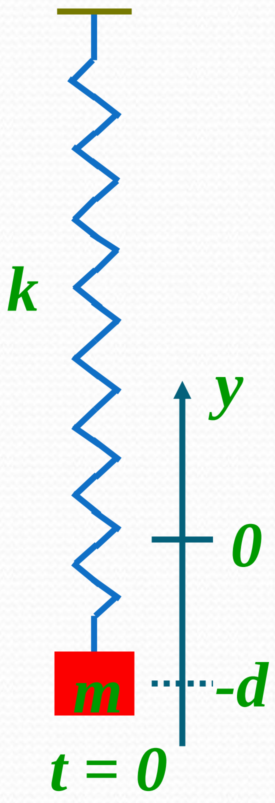
The rotating vector diagram



When *t*>0, the *v* increases with *t*， then

$$\phi_0 = -5\pi / 6$$

Example 6.11 A mass m=102g is hung from a vertical spring. The equilibrium position is at y=0. The mass is then pulled down a distance d=10cm from equilibrium and released at t=0. The measured period of oscillation is T=0.8s. What is the spring constant k? Write down the equations for the position, velocity, and acceleration of the mass as functions of time. What is the maximum velocity? What is the maximum acceleration?



Solution

What is *k*

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow k = \omega^2 m$$

$$\omega = \frac{2\pi}{T} = 7.85 s^{-1}$$

So: $$k = \left(7.85 s^{-1}\right)^2 0.102 kg = 6.29 \frac{N}{m}$$

What are the equations of motion?

At $t = 0$

$$y = -d = -y_{max}$$

$$v = 0$$

$$y(t) = -d\cos(\omega t)$$

So we conclude:

$$v(t) = \omega d \sin(\omega t)$$

$$a(t) = \omega^2 d \cos(\omega t)$$

And

$$y_{\max} = d = 0.1m$$

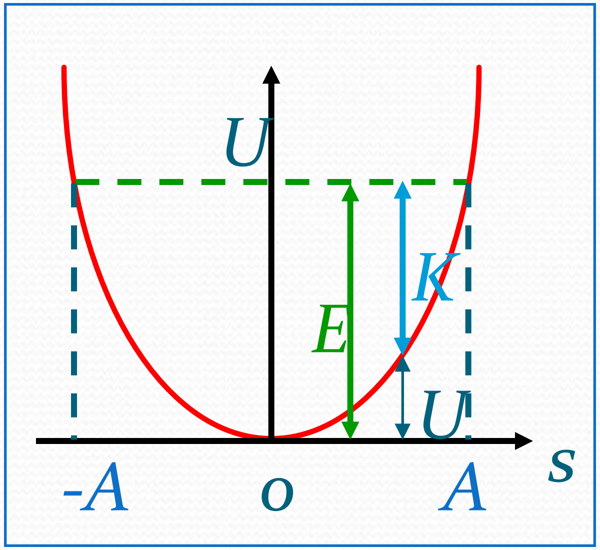
$$v_{\max} = \omega d = 0.78 m/s$$

$$a_{\max} = \omega^2 d = 6.2 m/s^2$$

§ 6.3 Energy in Simple Harmonic Motion

- For both the spring and the pendulum, we can derive the SHM solution by using energy conservation.

- The total energy (K + U) of a system undergoing SHM will always be constant!
- This is not surprising since there are only conservative forces present, hence K+U energy is conserved.



Example 6.12 Calculate the mechanical energy of a spring oscillator whose motion can be described with cosine function. Let Ep(x=0)=0.

Solution

$$E_p = \int_x^0 F dx = \int_x^0 -m\omega^2 x dx = \frac{1}{2} m\omega^2 x^2$$

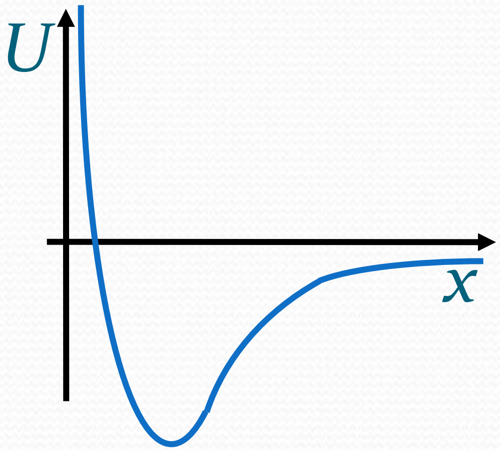
$$= \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi_0)$$

$$E_k = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi_0)$$

$$\boxed{E = E_p + E_k = \frac{1}{2} m\omega^2 A^2}$$

Energy conservation

- **SHM will occur whenever the potential is quadratic.**
- **Generally, this will not be the case: For example, the potential between H atoms in an H_2 molecule looks something like this:**



- **However, if we do a Taylor expansion of this function about the minimum, we find that for small displacements, the potential IS quadratic:**

$$U(x) = U(x_0) + U'(x_0)(x - x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2 + \cdots$$

$U'(x_0) = 0$ (since x_0 is minimum of potential)

Define $x' = x - x_0$

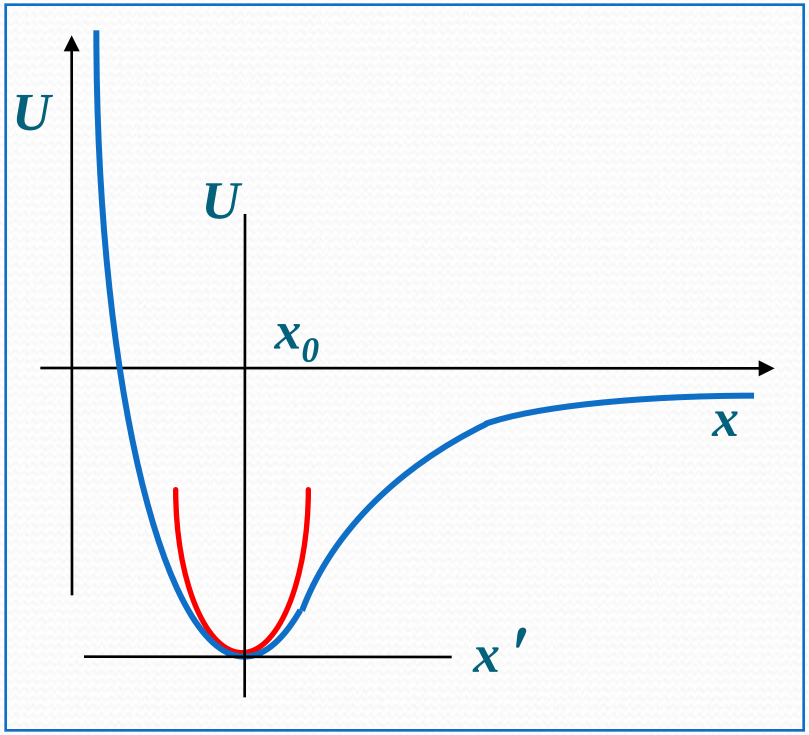
$U(x_0) = 0$

Then

$$U(x') = \frac{1}{2}U''(x_0)x'^2$$

Let $k = U''(x_0)$

$$U(x') = \frac{1}{2}kx'^2$$



Example 6.13 The equations of motion of a simple harmonic motion is $x=A\cos\omega t$. When dose the kinetic energy equal to the potential energy in the interval $0 \le t \le T/2$?

Solution

$$\begin{cases} x = A\cos\omega t \\ E_p = E/2 \end{cases} \qquad \begin{cases} E_p = \dfrac{1}{2}m\omega^2 x^2 \\ E = \dfrac{1}{2}m\omega^2 A^2 \end{cases}$$

$$\rightarrow \frac{x}{A} = \pm\frac{\sqrt{2}}{2} \quad \rightarrow \cos\omega t = \pm\frac{\sqrt{2}}{2}$$

$$0 \le t \le T/2 \quad \rightarrow 0 \le \omega t \le \pi$$

$$\rightarrow \omega t = \pi/4 \;\; or \;\; 3\pi/4$$

So

$$t_1 = \frac{\pi/4}{\omega} = \frac{\pi/4}{2\pi/T} = \frac{T}{8}$$

$$t_2 = \frac{3\pi/4}{\omega} = \frac{3\pi/4}{2\pi/T} = \frac{3T}{8}$$

§ 6.4 Synthesis of Simple Harmonic Motion

- Synthesis of Two Simple Harmonic Motions of Different Frequency in Same Direction

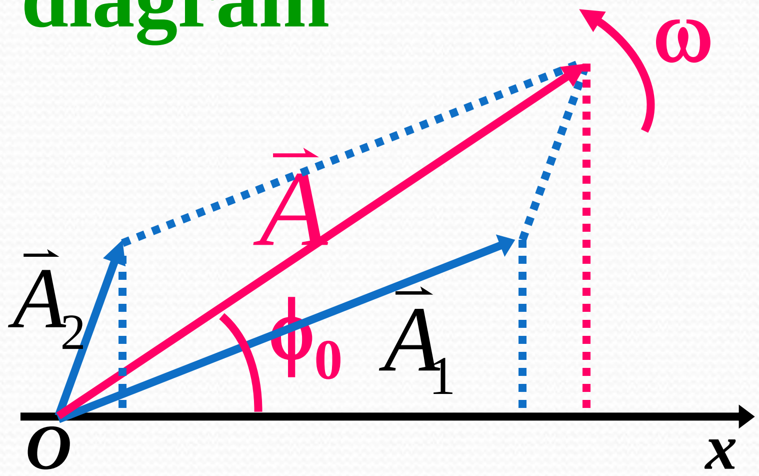
An animation

- Synthesis of Two Simple Harmonic Motions of Same Frequency in Same Direction

$$\begin{cases} x_1 = A_1 \cos(\omega t + \phi_{10}) \\ x_2 = A_2 \cos(\omega t + \phi_{20}) \end{cases}$$

$$\rightarrow x = x_1 + x_2 = A\cos(\omega t + \phi_0)$$

Rotating vector diagram



$$\rightarrow \begin{cases} A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_{20} - \phi_{10})} \\ tg\phi_0 = \dfrac{A_1\sin\phi_{10} + A_2\sin\phi_{20}}{A_1\cos\phi_{10} + A_2\cos\phi_{20}} \end{cases}$$

If $\phi_{20} - \phi_{10} = 2k\pi \quad (k=0, \pm1, \pm2, \ldots)$

Then $A = A_1 + A_2$ ——**max.**

If $\phi_{20} - \phi_{10} = (2k+1)\pi \quad (k=0, \pm1, \pm2, \ldots)$

Then $A = |A_1 - A_2|$ ——**min.**

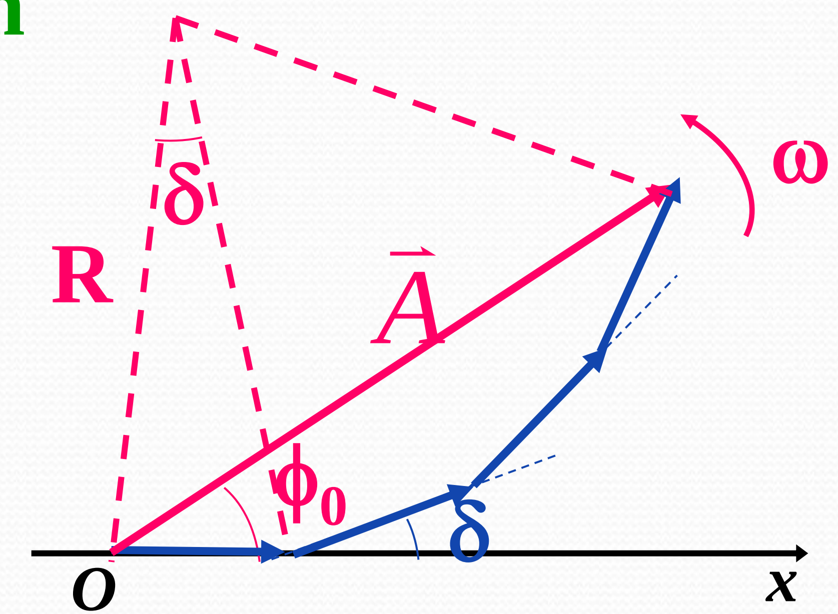
- **Synthesis of n Simple Harmonic Motions of Same Frequency in Same Direction**

$$x_i = a\cos[\omega t + (i-1)\delta] \quad (i = 1, 2, \cdots, n)$$

$$\rightarrow x = \sum_{i=1}^{n} x_i = A\cos(\omega t + \phi_0)$$

Rotating vector diagram

$$\rightarrow \begin{cases} A = 2R\sin\dfrac{n\delta}{2} \\ R = \dfrac{a}{2} / \sin\dfrac{\delta}{2} \end{cases}$$



$$\rightarrow \boxed{A = a\frac{\sin\dfrac{n\delta}{2}}{\sin\dfrac{\delta}{2}}}$$

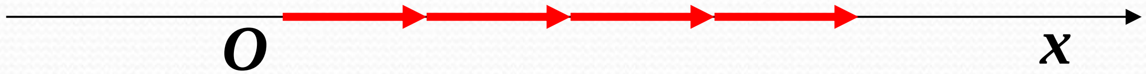
$$\phi_0 = (\frac{\pi}{2} - \frac{\delta}{2}) - (\frac{\pi}{2} - \frac{n\delta}{2})$$

$$= \frac{n-1}{2}\delta$$

Notes: If $\delta = 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots)$

Then $A = na$ ——max.

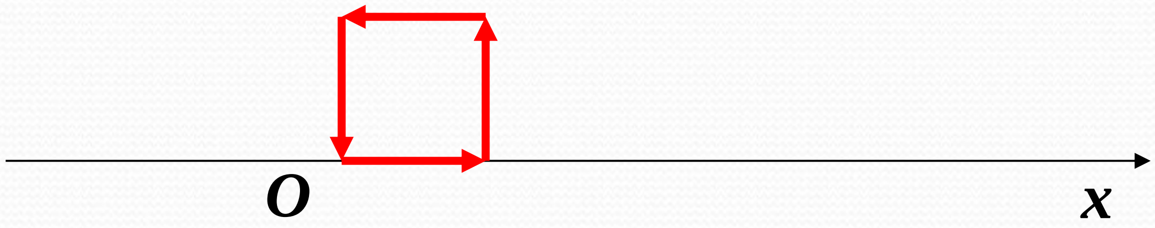
e.g. $n=4, \delta=0$



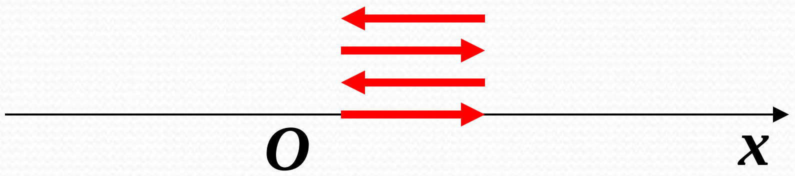
If $\delta \neq 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots)$

But $n\delta = 2k'\pi \quad (k'$ is integer$)$

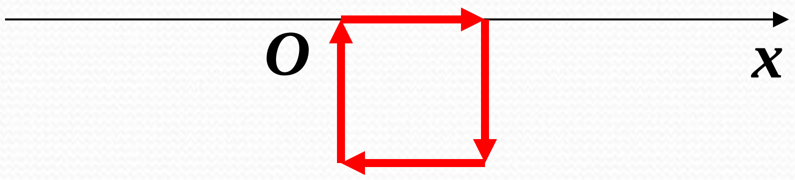
Then $A = 0$ ——min.



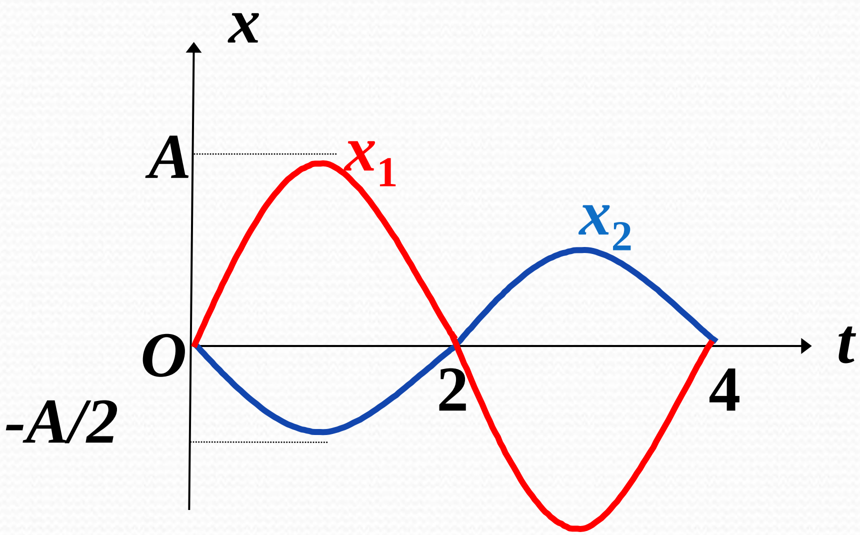
n=4, $\delta=\pi$

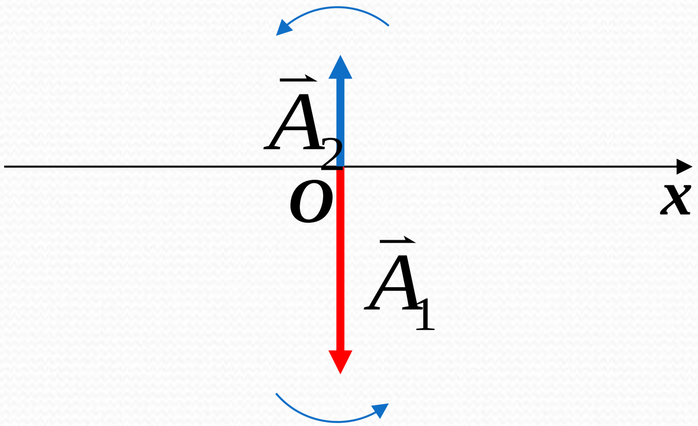


n=4, $\delta=3\pi/2$



Example 6.14 **The curves of two simple harmonic motions are shown in the figure. What is the initial phase of the resultant motion?**





So the initial phase of the resultant motion

$$\phi_0 = -\pi/2$$

- **Synthesis of Two Simple Harmonic Motions of Different Frequency in two perpendicular Directions**

Animation 1

Animation 2

See you next time!