第 1.5 节：闭区间上连续函数的性质（Properties of continuous functions on a close dinterval）

一、内容提要（contents）

①最值定义（Maximum and minimum）:如果函数 $f(x)$ 在区间 I 上有定义，且 $x_0 \in I$，满足 $f(x_0) \le f(x)$ 对于任意 $x \in I$，则 $f(x_0)$ 称为函数 $f(x)$ 在区间 I 上的最小值；如果 $f(x_0) \ge f(x)$ 对于任意 $x \in I$，则 $f(x_0)$ 称为函数 $f(x)$ 在区间 I 上的最大值。

②函数的零点（zero point）：如果存在 x_0 使得 $f(x_0)=0$，则 x_0 称为函数 $f(x)$ 的一个零点。

③闭区间上连续函数的性质定理：

（1）、最值定理（Maximum and minimum Theorem）：如果函数 $f(x)$ 在 $[a,b]$ 上连续，则 $f(x)$ 一定能取得最大值和最小值。即，存在 $x_0 \in [a,b]$ 使得 $f(x_0) \ge f(x)$ 对于任意 $x \in [a,b]$ 均成立；存在 $x_1 \in [a,b]$ 使得 $f(x_1) \le f(x)$ 对于任意 $x \in [a,b]$ 均成立；其中 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最大值(Maximum)，$f(x_1)$ 为 $f(x)$ 在 $[a,b]$ 上的最小值（minimum）。

(2)、有界性定理（Bounded theorem）：如果函数 $f(x)$ 在 $[a,b]$ 上连续，则 $f(x)$ 在 $[a,b]$ 上是有界函数。

（3）、零点定理（Zero point Theorem）：如果函数 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)\cdot f(b)<0$，则至少存在一个 $x_0 \in (a,b)$ 使得 $f(x_0)=0$。

（4）、介值定理（Intermediate Value Theorem）：如果函数 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a) \ne f(b)$，则对于任意一个给定的介于 $f(a)$ 和 $f(b)$ 之间的值 μ，一定存在至少一个 $x_0 \in (a,b)$ 使得 $f(x_0)=\mu$。

（5）、推论：闭区间上连续函数的值域是一个闭区间。

二、习题解答（answers）

Exercise 1.5

1. If $f(x)=x^3-x^2+x$, use the Intermediate Value Theorem to show that

   There is a number ξ such that $f(\xi)=10$.

   Proof. It is clear that $f(x)=x^3-x^2+x$ is continuous on the closed interval $[-3,3]$, and $f(-3)=-27-9-3=-39<0$, $f(3)=27-9+3=21>0$ .According to the Intermediate Value Theorem, there is a $\xi\in(-3,3)$ such that $f(\xi)=10$, because $f(-3)=-39<10<21=f(3)$, which means that 10 lies just between $f(-3)$, and $f(3)$.

That is the end of the proof.

2. Use the Intermediate Value Theorem to show that there is a root of the given equations in the specified interval.

(1) $x^3+x-3=0$, $(1,2)$

Proof. Let $f(x)=x^3+x-3$, which is continuous on the closed interval $[1,2]$.

And $f(1)=1+1-3=-1<0, f(2)=8+2-3=7>0$. $f(1)<0<f(2)$, that is 0 is just an intermediate value, so , according to the Intermediate Value Theorem, there is a number $\xi\in(1,2)$ such that $f(\xi)=0$, which means that ξ is a root of the given equation.

(2). $\cos x=x$, $(0,1)$

Proof. Let $f(x)=x-\cos x$, which is continuous on the closed interval $[0,1]$.

And $f(0)=0-\cos 0=-1<0, f(1)=1-\cos 1>0$. $f(0)<0<f(1)$, that is 0 is just an intermediate value, so , according to the Intermediate Value Theorem, there is a number $\xi\in(0,1)$ such that $f(\xi)=0$, which means that ξ is a root of the given equation.

3. Prove that the equation has at least one real root.

(1) $e^x=3-x$

Proof. Let $f(x)=e^x-3+x$, which is continuous on the closed interval $[-3,3]$,and

$$f(-3)=e^{-3}-3+(-3)=-6+\frac{1}{e^3}<0, f(3)=e^3-3+3=e^3>0.$$

According to the Zero Point Theorem, there is at least a number $\xi\in(-3,3)$ such that $f(\xi)=0$, which means that ξ is a real root of the given equation $e^x=3-x$.

That is all for the proof.

(2). $x^5+x^2-5x+6=0$

Proof. Let $f(x)=x^5+x^2-5x+6$, which is continuous on the closed interval $[-2,2]$, and $f(-2)=-12<0, f(2)=32>0$.

According to the Zero Point Theorem, there is at least a number $\xi\in(-2,2)$ such that $f(\xi)=0$, which means that ξ is a real root of the given equation $x^5+x^2-5x+6=0$.

That is all for the proof.

4. At which number $g(x)$ is continuous? Where $g(x)$ is given as follows.

$$g(x)=\begin{cases}0, & x \text{ is a rational number}\\ x, & x \text{ is a irrational number}\end{cases}$$

Solution. $g(x)$ is continuous only at the point $x=0$.

Because $\lim\limits_{\substack{x\to 0\\ x \text{ are all rational numbers}}} g(x)=\lim\limits_{\substack{x\to 0\\ x \text{ are all rational numbers}}} x=0=g(0)$,

$$\lim_{\substack{x\to 0\\ x \text{ are all irrational numbers}}} g(x)=\lim_{\substack{x\to 0\\ x \text{ are all irrational numbers}}} 0=0=g(0),$$

So, $\lim\limits_{x\to 0} g(x)=0=g(0)$, which means that $g(x)$ is continuous at $x=0$.

If x_0 is any irrational number, then $g(x)$ is discontinuous at x_0.

For example, we take $x_0=\sqrt{2}$.

$$\lim_{\substack{x\to\sqrt{2}\\ x \text{ are all irrational numbers}}} g(x)=\lim_{\substack{x\to\sqrt{2}\\ x \text{ are all irrational numbers}}} x=\sqrt{2},$$

Whereas

$$\lim_{\substack{x\to\sqrt{2}\\ x \text{ are all rational numbers}}} g(x)=\lim_{\substack{x\to\sqrt{2}\\ x \text{ are all rational numbers}}} 0=0,$$

So $\lim\limits_{x\to\sqrt{2}} g(x)$ does not exist, so $g(x)$ is discontinuous at $x=\sqrt{2}$.

Similarly, we take x_0 be any rational number not equal to 0. Then $g(x)$ is discontinuous at x_0.