Chapter 7 Wave Motion

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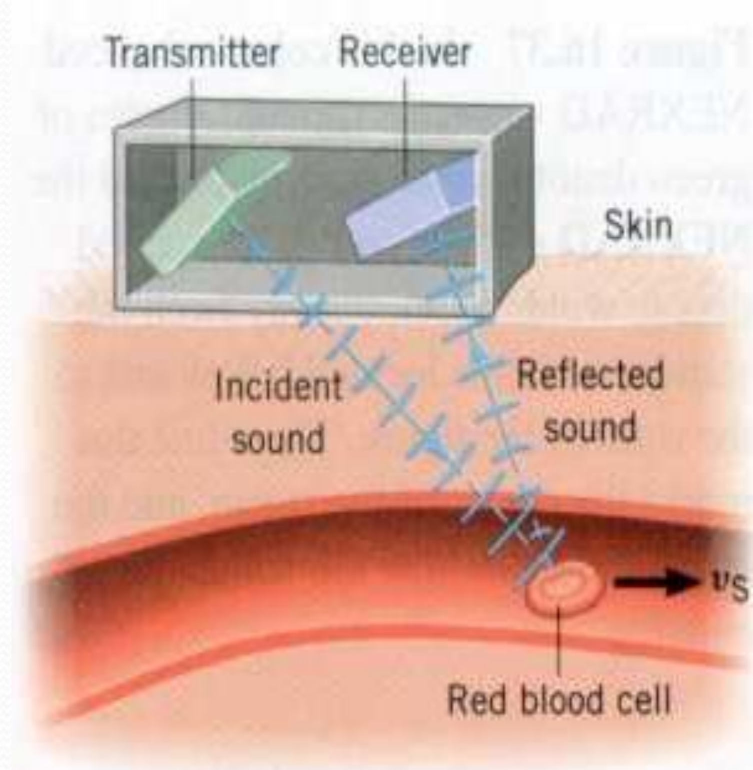

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§ 7.1 Conceptual discussion of wave motion

A wave is a traveling disturbance that transports energy but not matter.

Examples:

Sound waves (air moves back & forth)

Stadium waves (people move up & down)

Water waves (water moves up & down)

Light waves (what moves?)

electromagnetic field

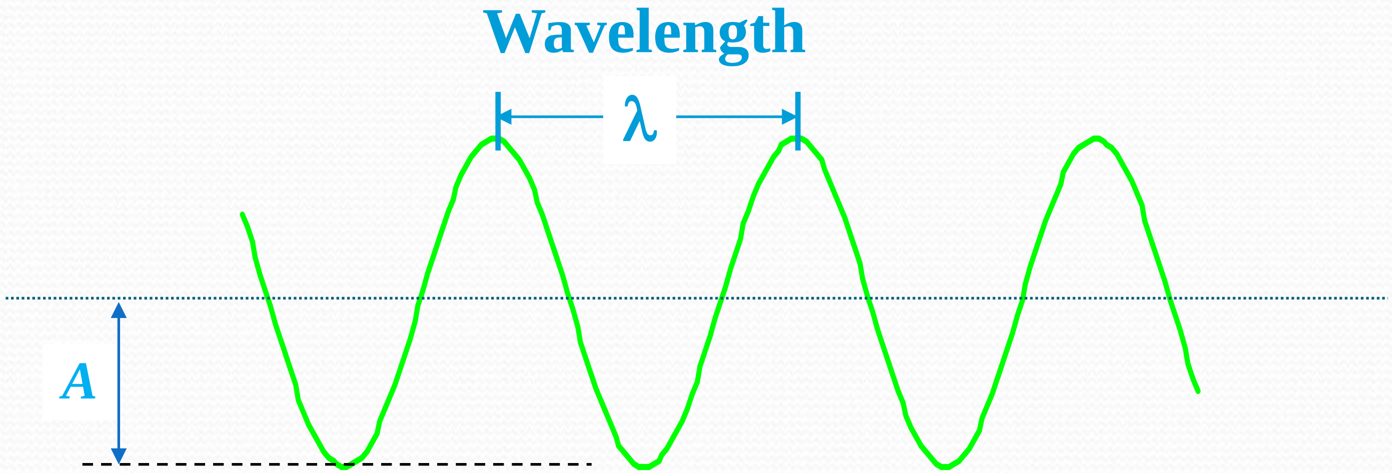
Types of Waves

- **Transverse:** **The medium oscillates perpendicular to the direction the wave is moving.**

- **Longitudinal:** **The medium oscillates in the same direction as the wave is moving.**

We will deal only with transverse waves in this course!

- **Wavelength:** The distance λ between identical points on the wave.
- **Amplitude:** The maximum displacement A of a point on the wave.
- **Period:** The time T for a point on the wave to undergo one complete oscillation.
- **Speed:** The wave moves one wavelength λ in one period T , so its speed is $v = \lambda / T$.

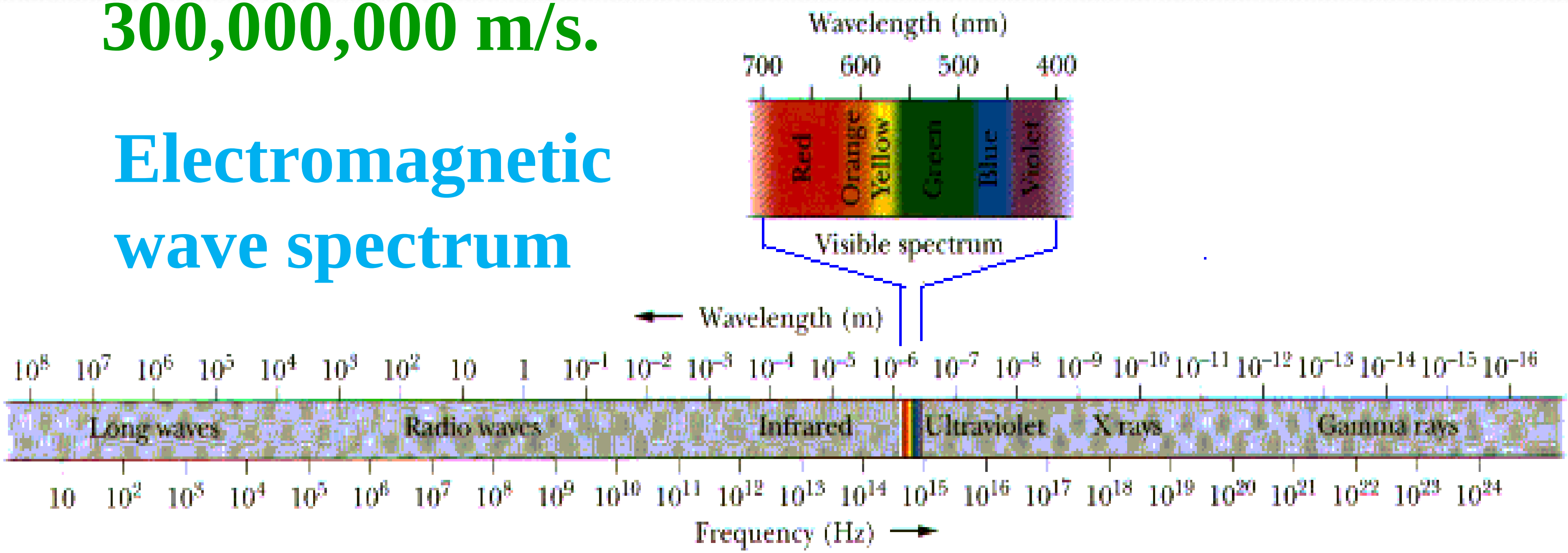


- **We will show that the speed of a wave is a *<u>constant</u>* that depends only on the medium, not on amplitude, wavelength, or period.**

What is the meaning of the following sentence?

The speed of light in air is about 300,000,000 m/s.

Electromagnetic wave spectrum



Example7.1 The speed of sound in air is a bit over 300 m/s, and the speed of light in air is about 300,000,000 m/s. Suppose we make a sound wave and a light wave that both have a wavelength of 3 meters.

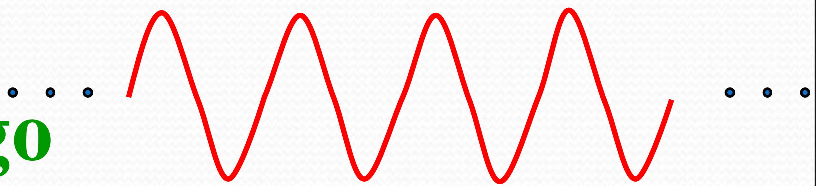
What is the ratio of the frequency of the light wave to that of the sound wave?

(a) About 10^6 (b) About 10^{-6} (c) About 10^3

(a)

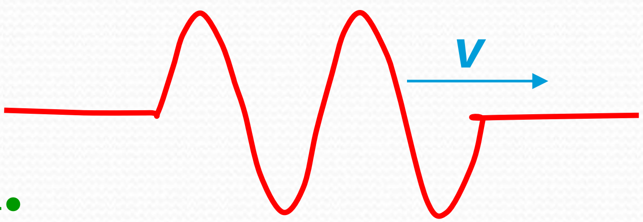
Wave Forms

- So far we have examined "continuous waves" that go on forever in each direction!



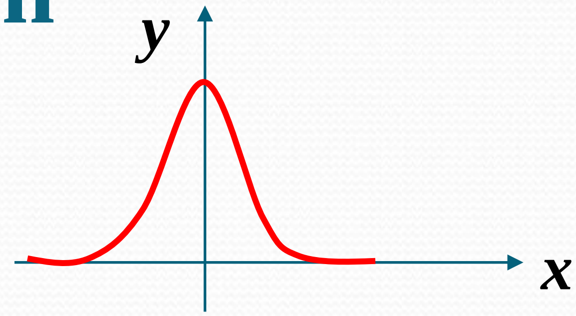
- We can also have "pulses" caused by a brief disturbance of the medium:

- And "pulse trains" which are somewhere in between.

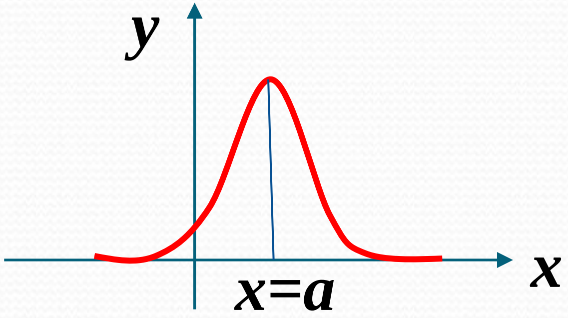


§ 7.2 Mathematical description

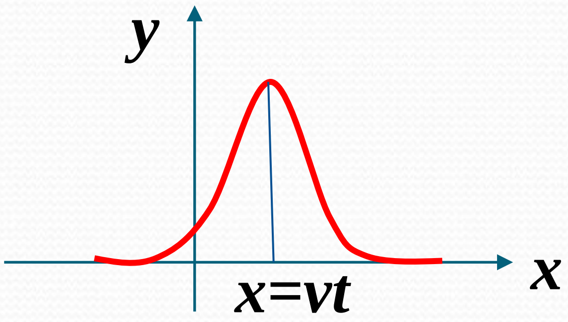
- Suppose we have some function $y = f(x)$:



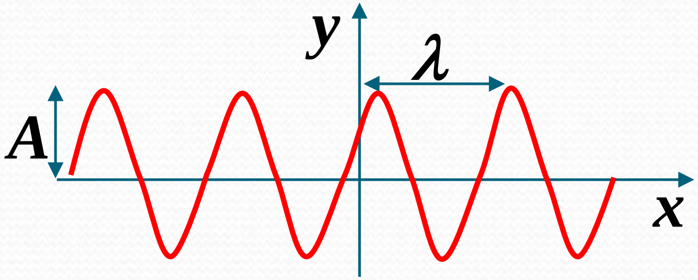
- $y=f(x - a)$ is just the same shape moved a distance a to the right:



- Let $a = vt$ Then $f(x - vt)$ will describe the same shape moving to the right with speed v.



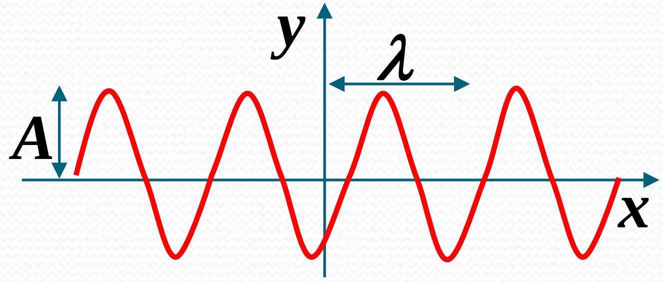
- **Consider a wave that is harmonic in *x* and has a wavelength of λ.**



- **If the wave can be described by this functional form:**

$$y(x) = A\cos\left(\frac{2\pi}{\lambda}x + \phi_0\right)$$

- **Now, if this is moving to the right with speed *v* it will be described by:**



$$y(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x - vt) + \phi_0\right) = A\cos\left(\omega t - \frac{2\pi}{\lambda}x + \phi\right)$$

- **So we see that a simple harmonic wave moving with speed *v* in the *x* direction is described by the equation:**

$$y(x,t) = A\cos\left(\omega t - \frac{2\pi}{\lambda}x + \phi\right)$$

- **By defining** $k \equiv \frac{2\pi}{\lambda}$ **we can write this as**

$$y(x,t) = A\cos\left(\omega t - \frac{2\pi}{\lambda}x + \phi\right)$$

$$= A\cos(\omega t - kx + \phi)$$

——Function of a simple harmonic wave

- **Formula describes a harmonic wave of amplitude *A* moving in the *+x* direction.**
- **Each point on the wave oscillates in the *y* direction with simple harmonic motion of angular frequency ω.**
- **The wavelength of the wave is** $\lambda = \frac{2\pi}{k}$
- **The speed of the wave is** $v = \frac{\lambda}{T} = \frac{\omega}{k}$
- **The quantity *k* is often called the "angular wave number"**

What about moving in the –*x* direction?

- **The sign of the term containing the *t* determines the direction of propagation.**
- **We change the sign to change the direction:**

$$y(x,t) = A\cos(\omega t - kx + \phi) \quad \textbf{moving toward } +x$$

$$y(x,t) = A\cos(\omega t + kx + \phi) \quad \textbf{moving toward } -x$$

Example7.2 **The function of a simple harmonic motion is $y=A\cos(Bt-Cx)$, where A, B, C are positive constants. Which one of the following description is right?**

(A)Wave speed is C

(B) Period is 1/B

(C)Wavelength is $2\pi/C$

(D) Angular frequency is $2\pi/B$

Solution: **The angular frequency is B, so the period is $T= 2\pi/B$. $C=2\pi/\lambda$, so we get $\lambda = 2\pi/C$. And then we know the wave speed is $v= \lambda /T=B/C$. (C)is right.**

Example7.3A harmonic wave moving in the positive x direction. Given the amplitude $A=0.1m$, period $T=0.5s$, wavelength $\lambda=10m$. The original point and the source of the wave coincide. The displacement of the wave source is positive maximum at $t=0$.

(1)Find the wave function.

(2)What is the displacement of the point $x=\lambda/4$ at $t_1=T/4$?

(3) What is the vibration speed of the point $x=\lambda/4$ at $t_2=T/2$?

Solution: **(1)The function of a harmonic wave moving toward +x direction can be written as**

$$y(x,t) = A\cos(\omega t - \frac{2\pi}{\lambda}x + \varphi)$$

We know the amplitude A=0.1m, wavelength λ=10m, period T=0.5s, so angular frequency ω=4π/s.

The original point is positive maximum at t=0, so φ=0. Substitute these four parameters into the above equation we get the wave function

$$y(x,t) = 0.1\cos(4\pi t - \frac{2\pi}{10}x)m$$

(2) Substituting x=λ/4 and t_1=T/4 into the wave function we get the displacement

$$y = 0.1\cos(4\pi\frac{1}{8} - \frac{2\pi}{10}\frac{10}{4}) = 0.1m$$

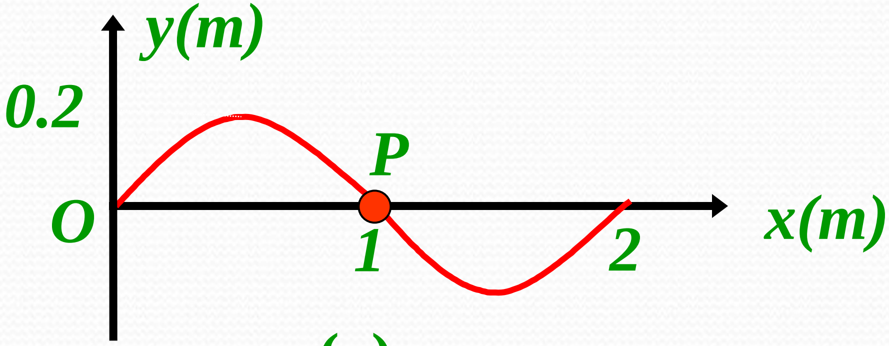
(3) The equation of the vibration speed is

$$\upsilon = -\omega A\sin(\omega t - \frac{2\pi x}{\lambda})$$

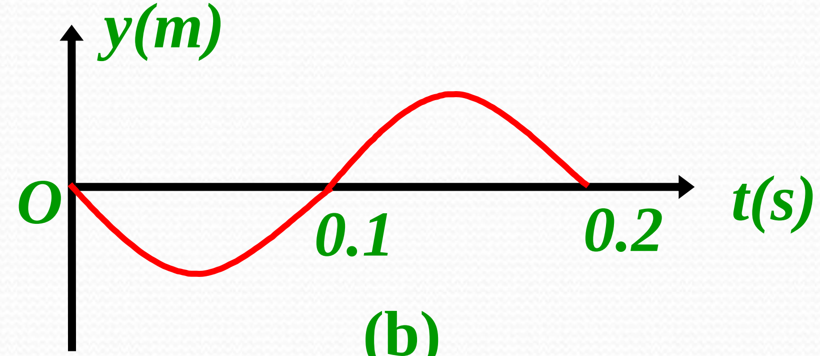
Substituting x=λ/4 and t_2=T/2 into this function we get

$$\upsilon = -0.4\pi\sin(\pi - \frac{\pi}{2}) = -1.26m/s$$

Example7.4 **The wave form of t=0 and the oscillation curve of point P are shown in the diagram below. What is the function of the wave?**



(a)



(b)

From (a): A=0.2m, λ=2m, y_p=0

From (b): T=0.2s, point P moves towards –y direction

ω=10π/s

u=10m/s,

The initial phase of point P is $\varphi_0=\pi/2$

So the wave function is

$$y(x,t)=0.2\cos[10\pi(t+x/10)+\pi/2]$$

Is it right?

Point P is NOT the original point

$$y(x,t)=0.2\cos[10\pi(\ t+(x-1)/10)+\pi/2]$$
$$= 0.2\cos[10\pi(\ t+x/10)-\pi/2]$$

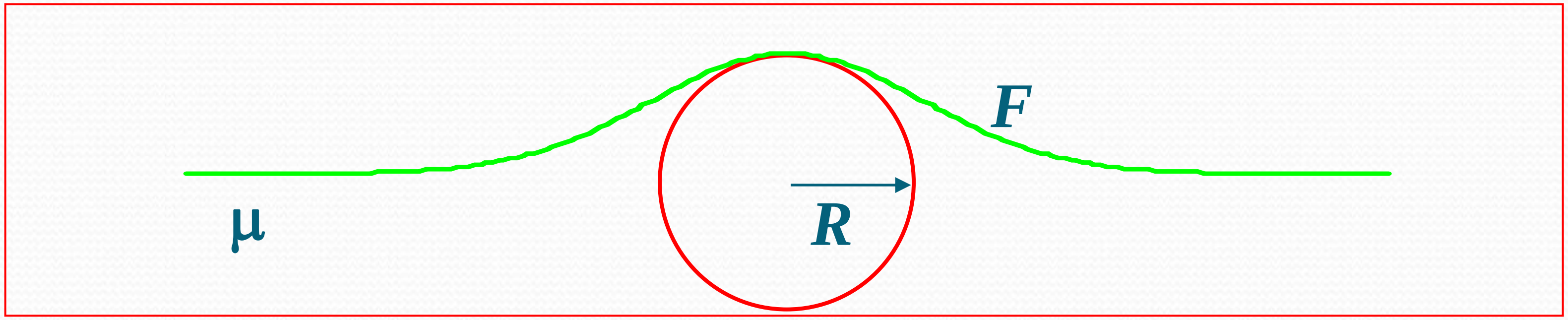
§ 7.3 Wave speed

What determines the speed of a wave?

Consider a pulse propagating along a string:

How can you make it go faster?

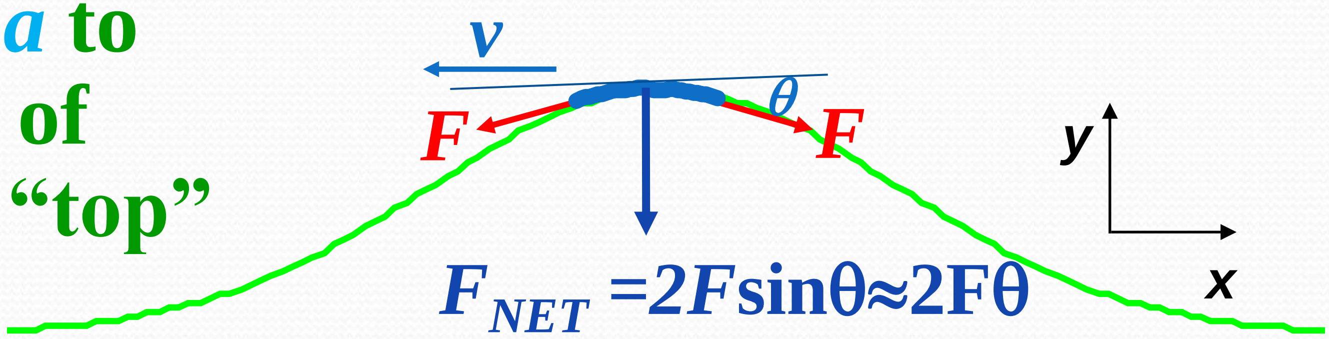
Suppose:



- **The tension in the string is F**
- **The mass per unit length of the string is μ (kg/m)**
- **The shape of the string at the pulse's maximum is circular and has radius R**

Consider moving along with the pulse

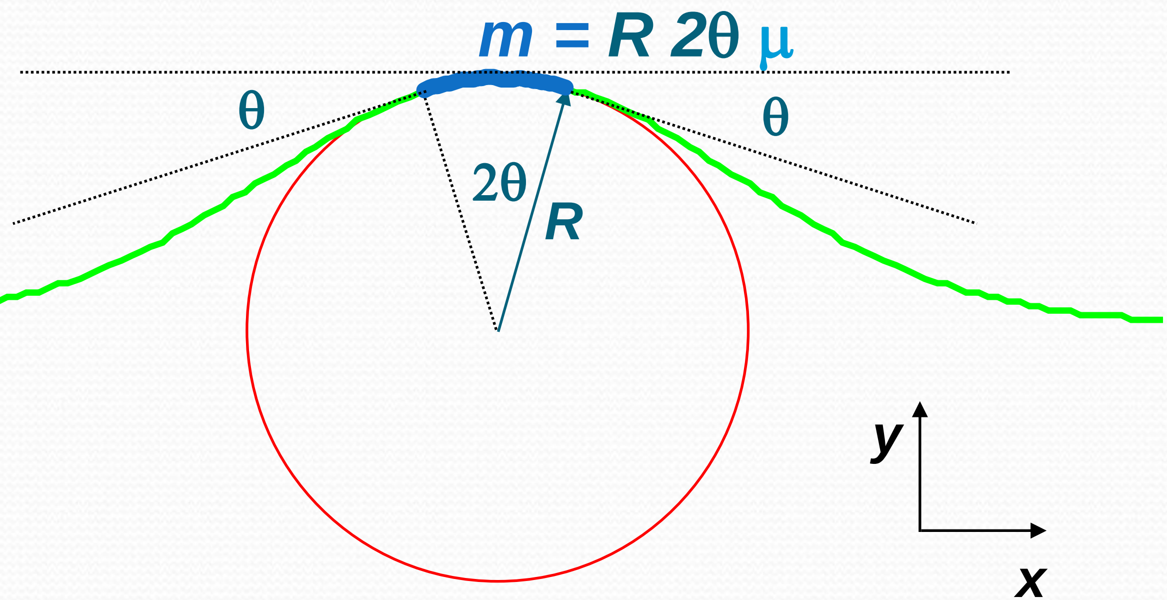
Apply $F = ma$ to the small bit of string at the "top" of the pulse



$F_{NET} = 2F\sin\theta \approx 2F\theta$

The mass m of the segment is its length ($R \times 2\theta$) times its mass per unit length μ. The acceleration a of the segment is v^2/R (centripetal) in the $-y$ direction.

$$\underbrace{2F\theta}_{F_{TOT}} = \underbrace{R2\theta\mu}_{m} \cdot \underbrace{\frac{v^2}{R}}_{a}$$



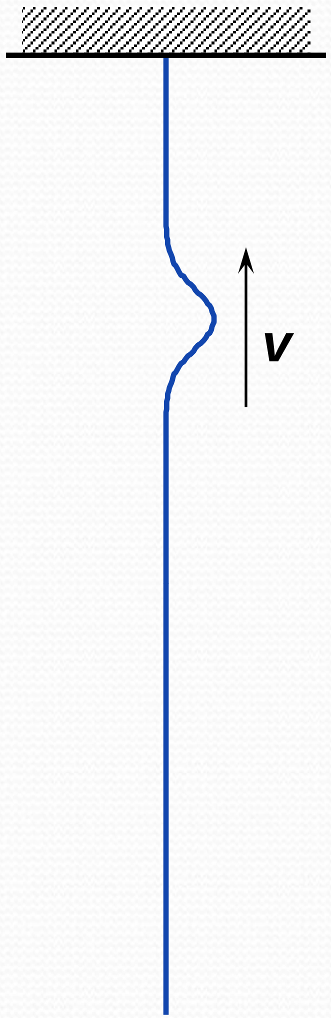
$$\rightarrow F = \mu v^2$$

$$\rightarrow v = \sqrt{\frac{F}{\mu}}$$

Note

- **Making the tension bigger increases the speed.**
- **Making the string heavier decreases the speed.**
- **As we asserted earlier, this depends only on the nature of the medium, not on amplitude, frequency, etc. of the wave.**

Example7.5 **A heavy rope hangs from the ceiling, and a small amplitude transverse wave is started by jiggling the rope at the bottom.**



As the wave travels up the rope, its speed will:

(a) increase

(b) decrease

(c) stay the same

Solution: **The tension *F* in the rope near the top is greater than the tension near the bottom since it has to support the weight of the rope beneath it! According to**

$$v = \sqrt{\frac{F}{\mu}}$$

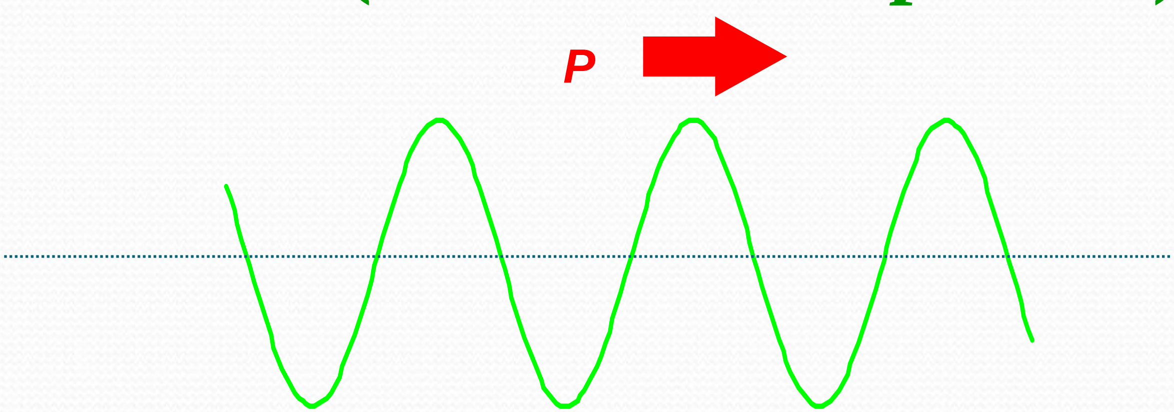
The speed of the wave will be greater at the top!

§ 7.4 Energy in wave motion and intensity of wave

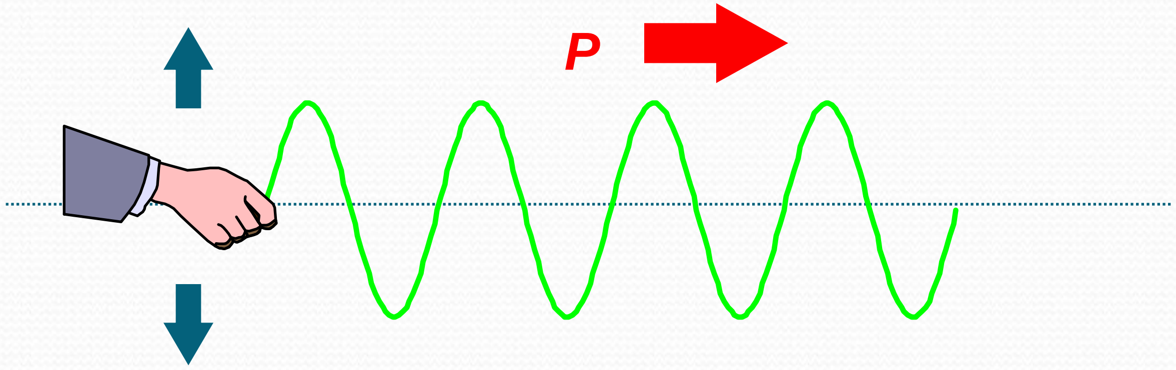
A wave propagates because each part of the medium communicates its motion to adjacent parts.

Energy is transferred since work is done!

How much energy is moving down the string per unit time. (i.e. how much *power*?)

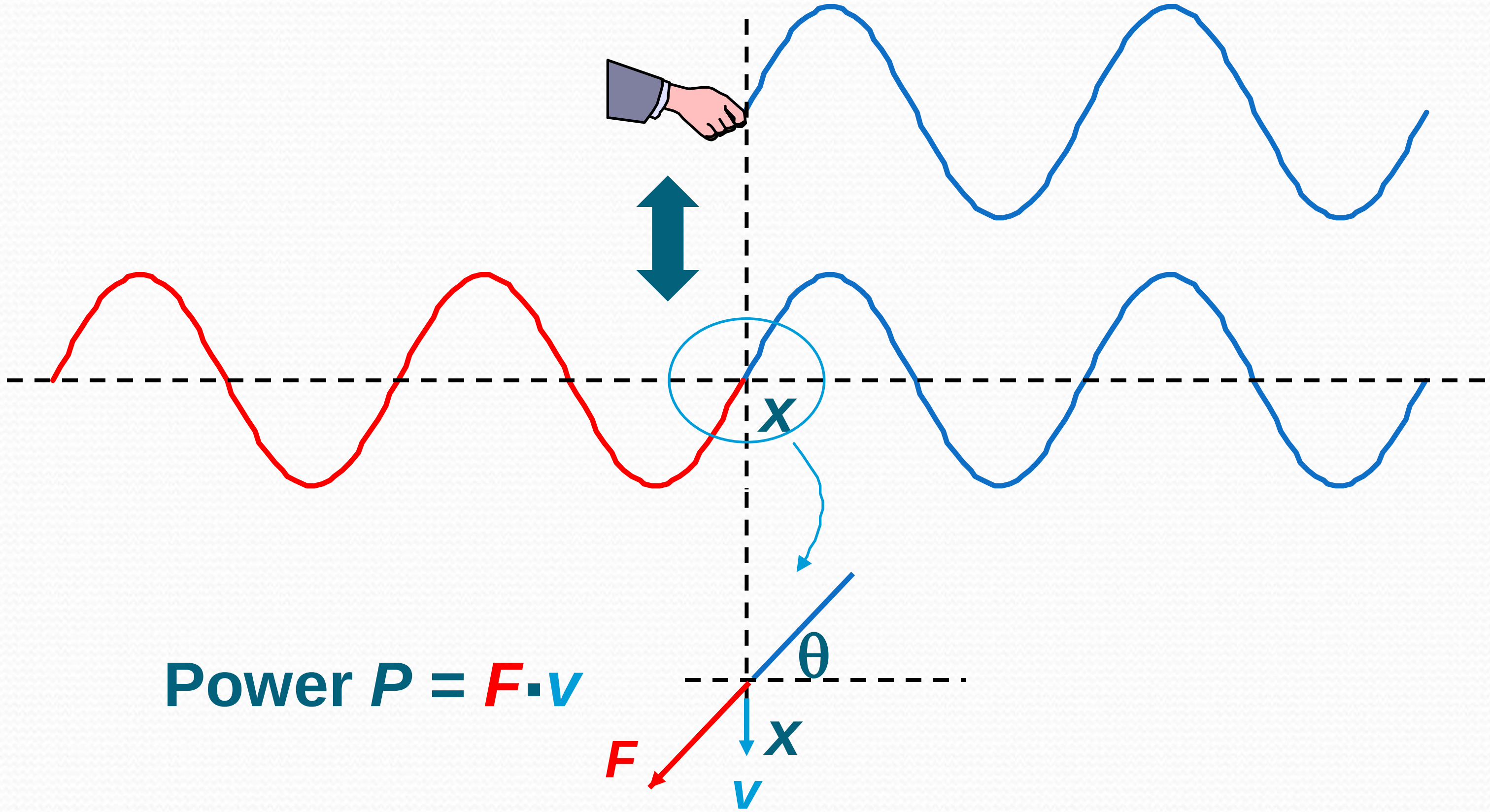


- **Think about grabbing the left side of the string and pulling it up and down in the *y* direction.**
- **You are clearly doing work since $F \cdot dr > 0$ as your hand moves up and down.**
- **This energy must be moving away from your hand (to the right).**



■ How is the energy moving?

Consider any position *x* on the string. The string to the left of *x* does work on the string to the right of *x*, just as your hand did:



Power $P = F \cdot v$

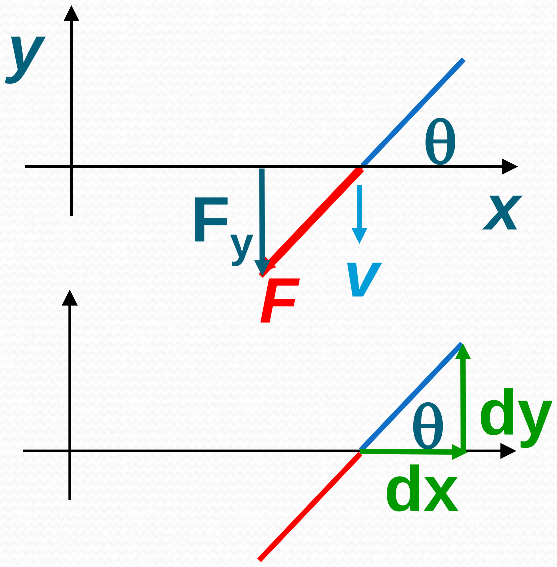
Since v is along the y axis only, to evaluate *Power* $= F \cdot v$ we only need to find $F_y = -F\sin\theta \approx -F\theta$ if θ is small.

We can easily figure out both the velocity v and the angle θ at any point on the string:

If $y(x,t) = A\cos(\omega t - kx)$

$$v_y(x,t) = \frac{dy}{dt} = -\omega A\sin(\omega t - kx)$$

$$\tan\theta = \frac{dy}{dx} = kA\sin(\omega t - kx) \approx \theta$$



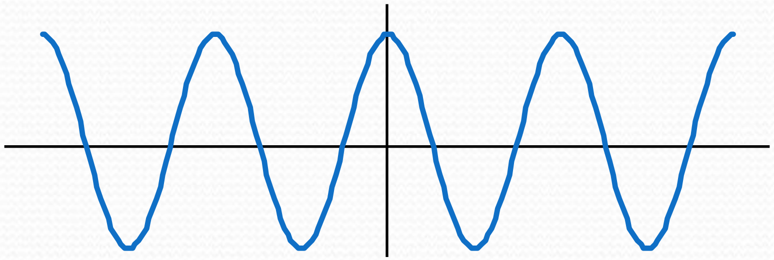
So: $P(x,t) \approx -F\theta v_y = \omega k F A^2 \sin^2(\omega t - kx)$

Which segment has the energy maximum in one period?

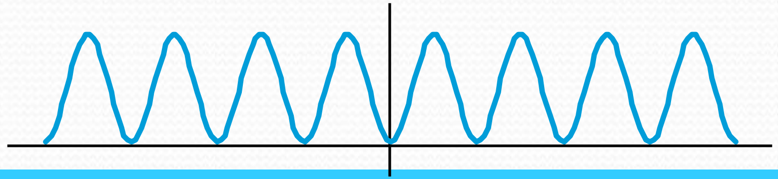
But last time we showed that

$$v = \frac{\omega}{k} \qquad F = \mu v^2$$

$$P\left(x,t\right) = \mu v \omega^2 A^2 \sin^2\left(\omega t - kx\right)$$



$\cos\left(\omega t - kx\right)$



$\sin^2\left(\omega t - kx\right)$

We just found that the power flowing past location x on the string at time t is given by:

$$P(x,t)=\mu v\omega^2 A^2 \sin^2(\omega t-kx)$$

We are often just interested in the average power moving down the string. To find this we recall that the average value of the function $\sin^2$ (kx-ωt) is 1/2 and find that:

$$\bar{P}=\frac{1}{2}\mu v\omega^2 A^2$$ **wave power or Average energy flow density**

It is generally true that wave power is proportional to the speed of the wave v and its amplitude squared A^2.

We have shown that energy "flows" along the string.

The source of this energy (in our picture) is the hand that is shaking the string up and down at one end.

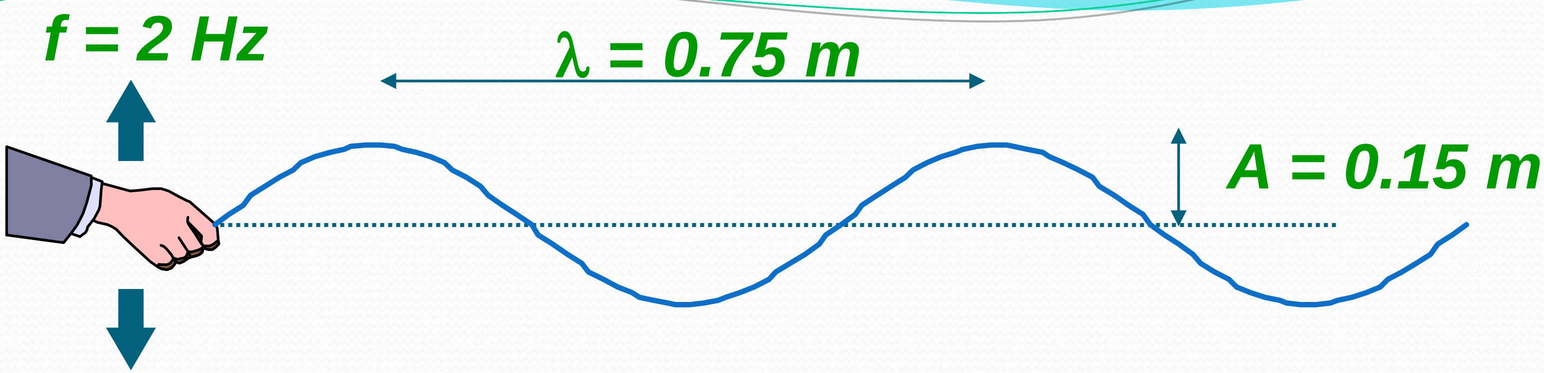
Each segment of string transfers energy to (does work on) the next segment by pulling on it, just like the hand.

We found that $\bar{P} = \frac{1}{2}\mu\omega^2 A^2 v$

$$\frac{d\bar{E}}{dt} = \frac{1}{2}\mu\omega^2 A^2 \frac{dx}{dt} \qquad d\bar{E} = \frac{1}{2}\mu\omega^2 A^2 dx$$

So $\frac{d\overline{E}}{dx}=\frac{1}{2}\mu\omega^2A^2$ is the average energy per unit length.

Example7.5 A rope with a linear density of mass μ=0.2kg/m lays on a frictionless floor. You grab one end and shake it from side to side twice per second with an amplitude of 0.15m. You notice that the distance between adjacent crests on the wave you make is 0.75m. What is the average power you are providing the rope? What is the average energy per unit length of the rope? What is the tension in the rope?



Solution: **We know A, μ and $\omega = 2\pi f$. We need to find v! Recall $v = \lambda f = (.75\text{ m})(2\text{ s}^{-1}) = 1.5\text{ m/s}$.**

So average power:

$$\overline{P} = \frac{1}{2}\mu v\omega^2 A^2$$

$$= \frac{1}{2}\times 0.2\times 1.5\times(2\pi\cdot 2)^2(0.15)^2$$

$$= 0.533\,W$$

$$\frac{d\overline{E}}{dx} = \frac{1}{2}\mu\omega^2 A^2$$

So $$\frac{d\overline{E}}{dx} = \frac{1}{2}\left(0.2\frac{kg}{m}\right)(2\pi \cdot 2Hz)^2(0.15m)^2$$

Average energy per unit length

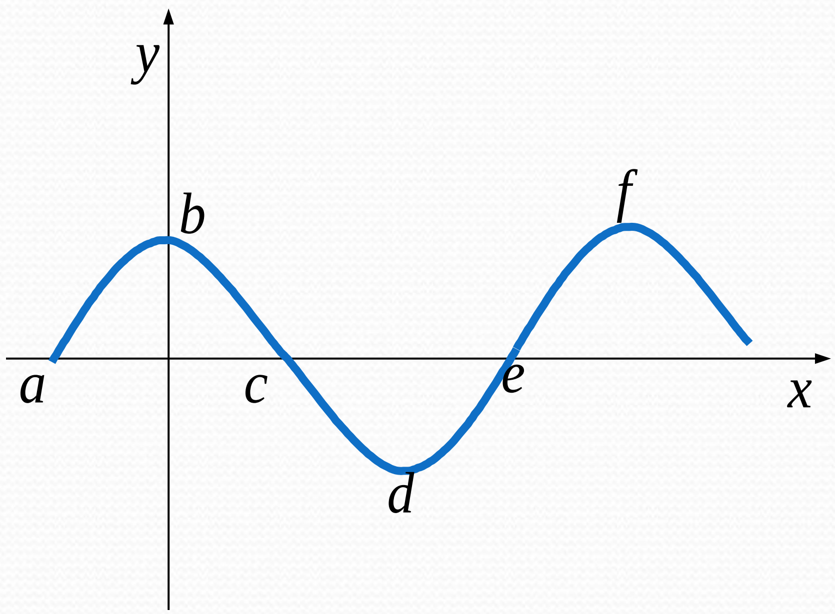
$$\frac{d\overline{E}}{dx} = 0.355\ J/m$$

We also know that the tension in the rope is related to speed of the wave and the mass density: $$F = \mu v^2 = \left(0.2\frac{kg}{m}\right)\left(1.5\frac{m}{s}\right)^2$$

Tension in rope: F = 0.45 N

Example7.6 A snapshot of a harmonic wave at time t is shown in the following diagram. Where does the kinetic energy have maximum value and where is it zero? Where does the potential energy have maximum value and where is it zero?

Solution: the kinetic energy and potential energy both have maximum values at positions *a, c, e*. The kinetic energy and potential energy are both zero at positions *b, d, f*.



Example7.7 A wave propagates on a string. If both the amplitude and the wavelength are doubled, by what factor will the average power carried by the wave change? (The velocity of the wave is unchanged).

(a) 1 (b) 2 (c) 4

Solution: We have shown that the average power

$$\overline{P} = \frac{1}{2}\mu\omega^2 A^2 v$$

So

$$\frac{\overline{P}_f}{\overline{P}_i} = \frac{\frac{1}{2}\mu\omega_f^2 A_f^2 v}{\frac{1}{2}\mu\omega_i^2 A_i^2 v} = \frac{\omega_f^2 A_f^2}{\omega_i^2 A_i^2}$$

But since $v = \lambda f = \lambda\omega / 2\pi$ is constant, $$\frac{\omega_f}{\omega_i} = \frac{\lambda_i}{\lambda_f}$$

i.e. doubling the wavelength halves the frequency.

$$\frac{\overline{P}_f}{\overline{P}_i} = \frac{\omega_f^2 A_f^2}{\omega_i^2 A_i^2} = \left(\frac{\lambda_i}{\lambda_f}\right)^2 \cdot \left(\frac{A_f}{A_i}\right)^2$$

$$= \left(\frac{1}{2}\right)^2 \cdot \left(\frac{2}{1}\right)^2 = 1$$

§ 7.5 The Principle of Superposition and Interference

Q: What happens when two waves "collide?"

A: They ADD together! demonstration

We say the waves are "superposed."

The principle of superposition asserts that when several waves combine at a point the displacement of any particle at any given time is simply the sum of the displacements that each individual wave acting alone would give it.

Suppose that two waves travel simultaneously along the same stretched string.

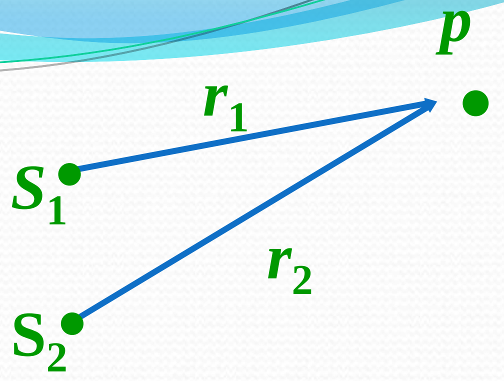
$$y(x) = y_1(x) + y_2(x)$$

When two waves which have same frequency, same oscillating direction and fixed difference of phase combined at a point, they are said to interfere, and the phenomenon is called interference(干涉). These two waves are also called interfering waves(相干波).

Suppose two interfering waves meet at point P

$$y_1 = A_1 \cos(\omega t - \frac{2\pi r_1}{\lambda} + \phi_1)$$

$$y_2 = A_2 \cos(\omega t - \frac{2\pi r_2}{\lambda} + \phi_2)$$



Using the synthesis of two simple harmonic motions of same frequency in same direction, we have

$$y = y_1 + y_2 = A\cos(\omega t + \phi)$$

where

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos(\phi_2 - \phi_1 - 2\pi \frac{r_2 - r_1}{\lambda})}$$

$$\mathrm{tg}\phi = \frac{A_1 \sin(\phi_1 - \frac{2\pi r_1}{\lambda}) + A_2 \sin(\phi_2 - \frac{2\pi r_2}{\lambda})}{A_1 \cos(\phi_1 - \frac{2\pi r_1}{\lambda}) + A_2 \cos(\phi_2 - \frac{2\pi r_2}{\lambda})}$$

The phase difference相位差

$$\Delta\phi = \phi_2 - \phi_1 - 2\pi \frac{r_2 - r_1}{\lambda}$$

- **Constructive interference(干涉加强,干涉极大,相长干涉)**

$$\Delta\phi = \phi_2 - \phi_1 - 2\pi \frac{r_2 - r_1}{\lambda} = \pm 2k\pi \quad (k = 0,1,2\cdots)$$

——in phase(同相)

- **Destructive interference (干涉减弱,干涉极小，相消干涉)**

$$\Delta\phi = \phi_2 - \phi_1 - 2\pi \frac{r_2 - r_1}{\lambda} = \pm 2\pi(k + \frac{1}{2}) \quad (k = 0,1,2\cdots)$$

——out of phase(反相)

If $\phi_1 = \phi_2$

$$\Delta\phi = 2\pi \frac{r_2 - r_1}{\lambda}$$

Constructive interference

$$\delta = r_1 - r_2 = \pm\lambda k \quad (k = 0,1,2\cdots)$$

Destructive interference

$$\delta = r_1 - r_2 = \pm(k + \frac{1}{2})\lambda \quad (k = 0,1,2\cdots)$$

δ — wave path difference

Example7.8 **The distance between two simple harmonic wave sources s_1 and s_2 is d=30m. The two waves move along x direction. Two Points x_1=9m and x_2=12m are adjacent points at rest because of interference. What are the wavelength and the minimum phase difference of the two wave source?**

Solution: **Suppose the two wave function are**

$$y_1 = A\cos(\omega t - 2\pi \frac{x}{\lambda} + \varphi_1)$$

$$y_2 = A\cos(\omega t + 2\pi \frac{x-d}{\lambda} + \varphi_2)$$

x_1=9m and x_2=12m are adjacent points of destructive interference, so

$$\varphi_2-\varphi_1+2\pi\frac{2x_1-d}{\lambda}=(2k+1)\pi \quad \text{———} \quad (1)$$

$$\varphi_2-\varphi_1+2\pi\frac{2x_2-d}{\lambda}=(2k+3)\pi \quad \text{———} \quad (2)$$

(2)-(1)

$$\frac{4\pi}{\lambda}(x_2-x_1)=2\pi \Rightarrow \lambda=2(x_2-x_1)=6m$$

Substituting λ into (1) or (2), we get

$$\varphi_2-\varphi_1=\pm\pi$$

§ 7.6 Standing Waves

When two interference waves with the same amplitude travel in opposite directions, they will produce standing wave —A special case of interference.

Represent the two waves by

$$y_1 = A\cos 2\pi(\frac{t}{T} - \frac{x}{\lambda})$$

$$y_2 = A\cos 2\pi(\frac{t}{T} + \frac{x}{\lambda})$$

The function of the resultant wave can be written as

$$y = y_1 + y_2 = A\left[\cos 2\pi(\frac{t}{T} - \frac{x}{\lambda}) + \cos 2\pi(\frac{t}{T} + \frac{x}{\lambda})\right]$$

$$= \underbrace{(2A\cos\frac{2\pi}{\lambda}x)}_{\text{amplitude}}\ \underbrace{\cos\frac{2\pi}{T}t}_{\text{oscillation}}$$

It does not describe a traveling wave because x and t are not in the combination form of $\omega t \pm kx$.

No propagation of oscillatory phase!

The distribution of amplitude

$$|\cos\frac{x}{\lambda}2\pi|=1 \rightarrow x=\pm k\frac{\lambda}{2}, \quad k=0,1,2\cdots$$ —antinodes 波腹

$$|\cos\frac{x}{\lambda}2\pi|=0 \rightarrow x=\pm(2k+1)\frac{\lambda}{4}$$ —nodes 波节

- **The adjacent antinodes(or nodes) are spaced one-half wavelength apart.** **demonstration**

$$x_{k+1}-x_{k+1}=\frac{\lambda}{2}$$

- **The amplitude of the other positions except nodes and antinodes are between 0 and 2A**

- **The oscillating particles between adjacent nodes are in phase, at two sides of a node out of phase.**
- **In a fixed reflection, there must be a node at the boundary—reflected and incident waves must be out of phase at the point.**

  —Half wavelength loss
- **No propagation of oscillatory phase; no traveling direction; no directed energy flow.**

Example7.9 **The function of incident wave $y_1 = A\cos\pi(x-4t)$ is reflected at fixed point *x*=0. Find (1)The function of reflected wave; (2)The function of resultant wave； (3)The positions of nodes and antinodes.**

Solution: **(1)The function of incident wave can be written as**

$$y_1 = A\cos 2\pi(2t - \frac{x}{2})$$

***x*=0 is a fixed point, there is a half wavelength loss at that point, so the reflected wave function is**

$$y_2 = A\cos 2\pi(2t + \frac{x}{2} + \pi)$$

(2)The resultant wave function is

$$y = y_1 + y_2 = 2A\sin\pi x \sin 4\pi t$$

(3)The positions of nodes and antinodes

nodes $\quad |\sin \pi x| = 0$

$$\Rightarrow \pi x = k\pi \Rightarrow x = k \quad k = 0, \pm 1, \pm 2, \cdots$$

antinodes $\quad |\sin \pi x| = 1$

$$\Rightarrow \pi x = (2k+1)\frac{\pi}{2} \Rightarrow x = k + \frac{1}{2} \quad k = 0, \pm 1, \pm 2, \cdots$$

See you next time!