Chapter 29: The Minimum distance.

In which we again use monotonicity.

Given f[0..M), g[0..N) of int, which contain values. $\{0 \le M \wedge 0 \le N\}$
We are also given that both of these are ascending.

We are asked to establish

$$\text{Post} : r = \langle \downarrow i,j : 0 \le i < M \wedge 0 \le j < N : |f.i - g.j| \rangle$$

Where $|x|$ means the absolute value of x.

Domain modelling.

$$* \ (0)\ C.m.n = \langle \downarrow i,j : m \le i < M \wedge n \le j < N : |f.i - g.j| \rangle$$

Appealing to emptying or falsifying the range we immediately get the following theorems

$$- \ (1)\ C.M.n \quad = \quad Id\downarrow \qquad ,0 \le n \le N$$
$$- \ (2)\ C.m.N \quad = \quad Id\downarrow \qquad ,0 \le m \le M$$

We observe

$$\begin{array}{ll}
 & C.m.n \\
= & \quad \{(0)\} \\
 & \langle \downarrow i,j : m \le i < M \wedge n \le j < N : |f.i - g.j| \rangle \\
= & \quad (\text{ split off } i{=}m \text{ term } \} \\
 & \langle \downarrow i,j : m+1 \le i < M \wedge n \le j < N : |f.i - g.j| \rangle \downarrow \langle \downarrow j : n \le j < N : |f.m - g.j| \rangle \\
= & \quad \{(0)\ (5)\ \} \\
 & C.(m+1).n \downarrow D.n
\end{array}$$

$$- \ (3)\ C.m.n \quad = \quad C.(m+1).n \downarrow D.n \qquad , 0 \le m < M$$

We observe

$$\begin{array}{ll}
 & C.m.n \\
= & \quad \{(0)\} \\
 & \langle \downarrow i,j : m \le i < M \wedge n \le j < N : |f.i - g.j| \rangle \\
= & \quad (\text{ split off } j{=}n \text{ term } \} \\
 & \langle \downarrow i,j : m \le i < M \wedge n+1 \le j < N : |f.i - g.j| \rangle \downarrow \langle \downarrow i : m \le i < M : |f.i - g.n| \rangle \\
= & \quad \{(0)\ (6)\ \} \\
 & C.(m+1).n \downarrow E.m
\end{array}$$

- (4) $C.m.n \quad = \quad C.m.(n+1) \downarrow E.m \quad , 0 \le n < N$

* (5) $D.n \quad = \quad \langle \downarrow j : \ n \le j < N : \ |f.m - g.j| \rangle$

* (6) $E.m \quad = \quad \langle \downarrow i : \ m \le i < M : \ |f.i - g.n| \rangle$

Now lets explore the monotonicity of f and g.

- (7) $D.n \quad = \quad g.n - f.m \quad \Leftarrow \quad f.m \le g.n$

- (8) $D.n \quad = \quad ? \quad \Leftarrow \quad f.m > g.n$

- (9) $E.m \quad = \quad ? \quad \Leftarrow \quad f.m < g.n$

- (10) $E. \quad = \quad f.m - g.n \quad \Leftarrow \quad g.n \le f.m$

Now let us return to our program. We can now rephrase our postcondition as

$\text{Post} : r = C.0.0$

Choose invariants.

$P0 : r \downarrow C.m.n = C.0.0$
$P1 : 0 \le m \le M \wedge 0 \le n \le N$

Establish invariants.

$r, m, n := Id\downarrow, 0, 0$

Termination.

(1) and (2) allow us to conclude that

$P0 \wedge P1 \wedge (m=M \vee n=N) \Rightarrow \text{Post}$

Guard.

$m \ne M \wedge n \ne N$

Loop body.

We observe

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition } P0\} \\
 & r \downarrow C.m.n = C.0.0 \\
= & \quad \{\text{case analysis } f.m \le g.n\ , (7)\} \\
 & r \downarrow C.(m+1).n \downarrow (g.n - f.m) = C.0.0 \\
= & \quad \{\text{WP}) \\
 & (r, m := r \downarrow (g.n - f.m), m+1).P0
\end{array}
$$

Giving us

$$\text{if } f.m \le g.n \rightarrow r, m := r \downarrow (g.n - f.m), m+1$$

We observe

$$
\begin{array}{ll}
 & P0 \\
= & \quad \{\text{definition } P0\} \\
 & r \downarrow C.m.n = C.0.0 \\
= & \quad \{\text{case analysis } g.n \le f.m\ , (10)\} \\
 & r \downarrow C.m.(n+1) \downarrow (f.m - g.n) = C.0.0 \\
= & \quad \{\text{WP}\} \\
 & (r, n := r \downarrow (f.m - g.n), n+1).P0
\end{array}
$$

which gives us

$$\text{if } g.n \le f.m \rightarrow r, n := r \downarrow (f.m - g.n), n+1$$

Finished program.

```
r, m, n := Id↓, 0, 0 {P0 ∧ P1}
;do m≠M ∧ n≠N →     {P0 ∧ P1 ∧ m≠M ∧ n≠N}

                    if f.m ≤ g.n → r, m := r ↓ (g.n – f.m), m+1
                    [] g.n ≤ f.m → r, n := r ↓(f.m – g.n), n+1
                    fi

                    {P0 ∧ P1}
od
{P0 ∧ P1 ∧ (m=M ∨ n=N)}
```