Chapter 16: The minimal segment sum.

In which we begin to show the power of calculation.

We are given f[0..N) of int which contains values and we are asked to construct an algorithm to compute the minimal segment sum. Our specification is as follows (note, this already introduces a formalisation of what we mean by segment sum, but more of this later).

$\{f[0..N) \text{ has values}\}$

S

$\{r = \langle \downarrow i,j : 0 \le i \le j \le N : SS.i.j \rangle\}$

Domain modelling.

We begin with the definition of SS

* (0) $SS.i.j \quad = \quad \langle + k : i \le k < j : f.k \rangle \quad , 0 \le i \le j \le N$

From which the following theorems emerge quite easily[1]

- (1) $SS.i.i \quad = \quad 0 \quad , 0 \le i \le N$

- (2) $SS.i.(j+1) \quad = \quad SS.i.j + f.j \quad , 0 \le i \le j < N$

Now let us return to considering the postcondition. Its shape suggests that we parameterise it using "name and conquer".

* (3) $C.n \quad = \quad \langle \downarrow i,j : 0 \le i \le j \le n : SS.i.j \rangle \quad , 0 \le n \le N$

Now what theorems can we derive from this? We might be tempted to try to empty the range, and indeed we could, but the presence of the "≤" at both ends of the range mean that to empty it we would have to make n negative. This isn't very elegant as so far n has been taken to be natural (in fact we have defined it so).

On the other hand, we could set n to 0 which would shrink the range to a single point.

[1] Recall the empty range and the law concerning splitting off a term

We observe,

$(n := 0).C.n$
$= \quad \{\text{textual substitution}\}$
$C.0$
$= \quad \{(3)\}$
$\langle \downarrow i,j : 0 \le i \le j \le 0 : SS.i.j \rangle$
$= \quad \{\text{1-point rule}\}$
$SS.0.0$
$= \quad \{(1)\}$
0

So we have

- (4) $C.0 \quad = \quad 0$

Now lets consider associativity. Lets look at C.(n+1), where $0 \le n < N$

$C.(n+1)$
$= \quad \{ (3) \}$
$\langle \downarrow i,j : 0 \le i \le j \le n+1 : SS.i.j \rangle$
$= \quad \{ \text{split off } j = n+1 \text{ term } \}$[2]

[2] We have already seen "splitting off a term when we were quantifying over a single variable. This is just a generalisation of it. It goes like this;

$\langle \oplus\, i,j : 0 \le i \le j \le n+1 : SS.i.j \rangle$
$= \quad \{ \text{split off } j = n+1 \text{ term } \}$
$\langle \oplus\, i,j : 0 \le i \le j \le n : SS.i.j \rangle \oplus \langle \oplus\, i : 0 \le i \le n+1 : SS.i.(n+1) \rangle$

To see that this is a generalisation observe the following;

$\langle \oplus\, i : 0 \le i \le n+1 : f.i \rangle$
$= \quad \{\text{split off } i = n+1 \text{ term } \}$
$\langle \oplus\, i : 0 \le i \le n : f.i \rangle \oplus \langle \oplus\, i : \; i = n+1 : f.i \rangle$
$= \quad \{ \text{1-point rule } \}$
$\langle \oplus\, i : 0 \le i \le n : f.i \rangle \oplus f.(n+1)$

$\langle\downarrow i,j : 0 \le i \le j \le n : SS.i.j\rangle \downarrow \langle\downarrow i : 0 \le i \le n+1 : SS.i.(n+1)\rangle$

= { (3) }

$C.n \downarrow \langle\downarrow i : 0 \le i \le n+1 : SS.i.(n+1)\rangle$

= { name and conquer [3]}

$C.n \downarrow D.(n+1)$

Thus we have (5)

- (5) $C.(n+1) = C.n \downarrow D.(n+1)$, $0 \le n < N$

This has introduced D, so lets see what theorems we can get from its definition.

* (6) $D.n = \langle\downarrow i : 0 \le i \le n : SS.i.n\rangle$,$0 \le n \le N$

Shrinking the range to a single point gives us

- (7) $D.0 = 0$

Now lets look at associativity. Consider D where $0 \le n < N$

$D.(n+1)$

= { (6) }

$\langle\downarrow i : 0 \le i \le n+1 : SS.i.(n+1)\rangle$

= { split off i = n+1 term }

$\langle\downarrow i : 0 \le i \le n : SS.i.(n+1)\rangle \downarrow SS.(n+1).(n+1)$

= { (1) }

$\langle\downarrow i : 0 \le i \le n : SS.i.(n+1)\rangle \downarrow 0$

= { (2) }

$\langle\downarrow i : 0 \le i \le n : SS.i.n + f.n\rangle \downarrow 0$

= { $+/\downarrow$ for non-empty ranges [4]}

$(\langle\downarrow i : 0 \le i \le n : SS.i.n\rangle + f.n) \downarrow 0$

= { (6) }

$(D.n + f.n) \downarrow 0$

[3] C was introduced parameterised with n so it seemed reasonable to introduce D in like manner which is why the term introduced here is D.(n+1)

[4] We note that for non-empty ranges

$$X + \langle\downarrow i : 0 \le i \le n : f.n\rangle = \langle\downarrow i : 0 \le i \le n : X + f.n\rangle$$

For X an int, and f[0..N) an array of int. There are lots of nice distribution laws like this which we will encounter later.

So we have

- (8) $D.(n+1) = (D.n + f.n) \downarrow 0$

We probably have a rich enough model to proceed. Before we do lets just sum up the model.

The minimal segment sum prefix calculus.

* (0) $SS.i.j$	=	$\langle \downarrow k : i \le k < j : f.k \rangle$	$, 0 \le i \le j \le N$
- (1) $SS.i.i$	=	0	$, 0 \le i \le N$
- (2) $SS.i.(j+1)$	=	$SS.i.j + f.j$	$, 0 \le i \le j < N$
* (3) $C.n$	=	$\langle \downarrow i,j : 0 \le i \le j \le n : SS.i.j \rangle$	$, 0 \le n \le N$
- (4) $C.0$	=	0	
- (5) $C.(n+1)$	=	$C.n \downarrow D.(n+1)$	$, 0 \le n < N$
* (6) $D.n$	=	$\langle \downarrow i : 0 \le i \le n : SS.i.n \rangle$	$,0 \le n \le N$
- (7) $D.0$	=	0	
- (8) $D.(n+1)$	=	$(D.n + f.n) \downarrow 0$	$, 0 \le n < N$

Now lets return to our programming task.

Rewrite the postcondition using the model

Post : $r = C.N$

Strengthening this gives us

Post' : $r = C.n \wedge n=N$

Choosing invariants.

We now choose our invariants

$P0 : r = C.n \wedge d = D.n$ [5]
$P1 : 0 \le n \le N$

Guard.

$n \neq N$

Establishing the invariants.

Both r and d should be bound to known values of C and D. Our model suggests that the appropriate ones are C.0 and D.0 respectively. Setting n to 0 also establishes P1. Thus the invariants are established by

$n, r, d := 0, 0, 0$

Termination.

We observe that

$P0 \wedge P1 \wedge n=N \Rightarrow Post'$

V*ariant.*

We choose N-n as our variant

Calculate the Loop body.

Now let us calculate the loop body. The motivation for decreasing the variant by the assignment n := n+1 which maintains P1 has been dealt with before so we will not touch on it here.

We solve for the two unknowns E and E'

[5] Why dear reader did we include a variable bound to D.n? Well our little model shows that in some cases C is defined in terms of D so it is perhaps reasonable to ensure that we have a value for D in case we need it. If, in our development, it turns out not to be needed then we will weaken our invariants and remove it.

$(n, r, d := n+1, E, E').P0$

$=$ $\{$textual substitution $\}$

$E = C.(n+1) \wedge E' = D.(n+1)$

$=$ $\{ (5)\}$

$E = C.n \downarrow D.(n+1) \wedge E' = D.(n+1)$

$=$ $\{(8)$ twice $\}$

$E = C.n \downarrow (D.n + f.n) \downarrow 0 \wedge E' = (D.n + f.n) \downarrow 0$

$=$ $\{P0 \}$

$E = r \downarrow (d + f.n) \downarrow 0 \wedge E' = (d + f.n) \downarrow 0$

Final program.

And our final program is therefore

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$n, r, d := 0, 0, 0$ $\{P0 \wedge P1\}$
;do $n \neq N \rightarrow$    $\{P0 \wedge P1 \wedge n \neq N\}$

        $n, r, d := n+1, r \downarrow (d + f.n) \downarrow 0, (d + f.n) \downarrow 0$

        $\{ P0 \wedge P1\}$
od
$\{P0 \wedge P1 \wedge n = N\}$
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We observe that it has temporal complexity of O(N) which is nice.

We have separated the modelling of the domain, which is of course where we come to understand the domain, from the programming task. The modelling involved precise, yet very shallow mathematics, but once it was done the programming task became trivial; a mere calculation. Our intuition, experience, and cleverness were never called upon; perhaps we might continue to give them a well earned rest. But what resulted from our calculation was a beautiful solution. Perhaps we can continue to calculate such beautiful solutions. ☺️

Exercises.

Given the same array, calculate a program to compute the maximal segment sum.

Given the same array, calculate a program to compute the minimal segment product[6]. This will be the homework for the week.

[6] By way of a hint, that is to say a piece of knowledge that you need !!!! we provide you with some new laws about quantified expressions.

* (Q1) $X * \langle\downarrow i : 0 \le i \le n : f.i\rangle = \langle\downarrow i : 0 \le i \le n : X * f.i\rangle \Leftarrow 0 \le X \wedge 0 \le n$

* (Q2) $X * \langle\uparrow i : 0 \le i \le n : f.i\rangle = \langle\downarrow i : 0 \le i \le n : X * f.i\rangle \Leftarrow X \le 0 \wedge 0 \le n$

* (Q3) $X * \langle\uparrow i : 0 \le i \le n : f.i\rangle = \langle\uparrow i : 0 \le i \le n : X * f.i\rangle \Leftarrow 0 \le X \wedge 0 \le n$

* (Q4) $X * \langle\downarrow i : 0 \le i \le n : f.i\rangle = \langle\uparrow i : 0 \le i \le n : X * f.i\rangle \Leftarrow X \le 0 \wedge 0 \le n$