Chapter 4
Work and Energy

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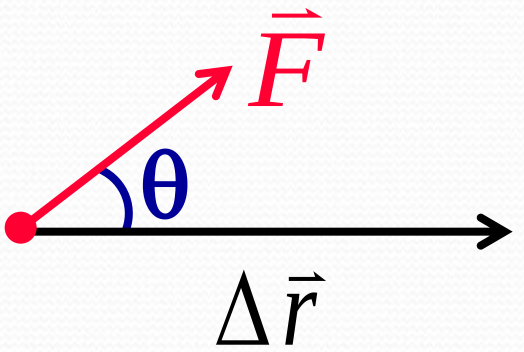
§ 4.1 Work

The work on a mass point by a force is equal to the product of the component of the force along the direction of the displacement and the magnitude of the displacement.

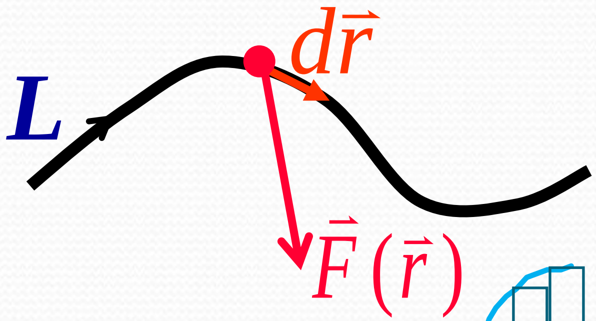
Work, W, of a constant force $\vec{F}$ acting through a displacement $\Delta\vec{r}$ is:

$$W = F\left|\Delta\vec{r}\right|\cos\theta$$

$$= \vec{F}\cdot\Delta\vec{r}$$



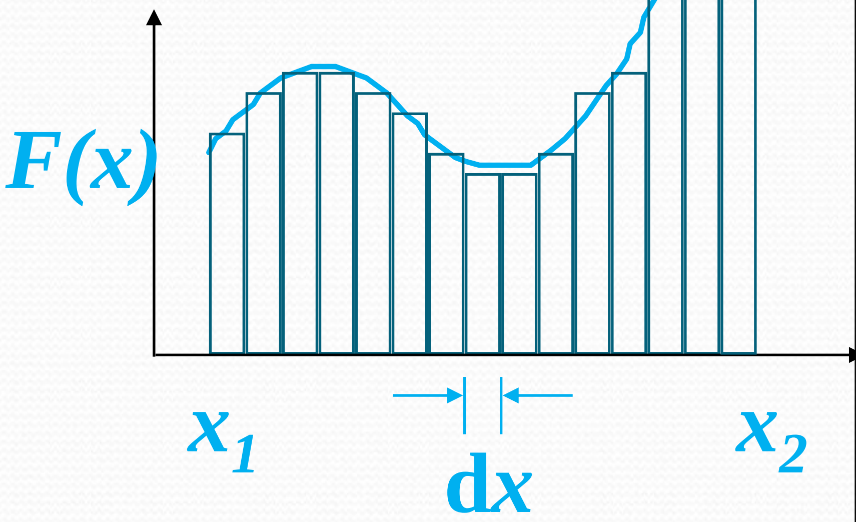
Assume a mass point moves along a curve under a variable force.



Element of the work by the force

$$\mathrm{d}W = \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r}$$

So the work of this process is the area under the curve of the force *F(x)* plotted vs. *x*



$$W = \int_L \vec{F}(\vec{r}) \cdot d\vec{r}$$

Technique: divide the path into small segments, so that force in each segment can be regarded as constant.

The net work done by the component forces

$$W = \int_L (\sum_i \vec{F}_i) \cdot d\vec{r}$$

$$= \sum_i \int_L \vec{F}_i \cdot d\vec{r} = \sum_i W_i$$

In Cartesian coordinates

$$W = \int_L F_x dx + F_y dy + F_z dz$$

Note

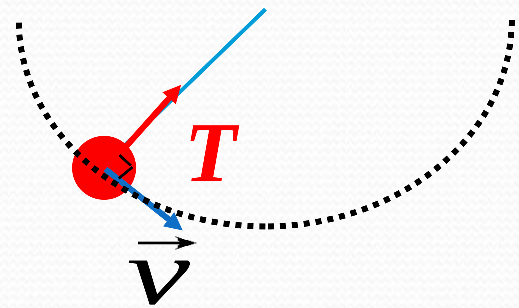
- **The work, dot product of a force vector and a displacement vector, is a scalar;**
- **The work can be negative or positive;**
- **The unit and dimension of work**

Force × Distance = Work

Newton × Meter = Joule

$[M][L]/[T]^2 \times [L] \qquad [M][L]^2/[T]^2$

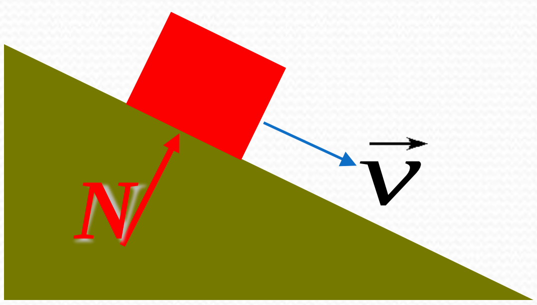
Example 4.1 **Find the work done by forces *T* and *N* in a time Δt in the right figures.**



Solution

Since the force is perpendicular to the displacement, i. e. $\theta = 90^{o}$, no work is done.

No work done by *T*



No work done by *N*

Example 4.2 **A mass point moves along the** **+*x*** **axis from** ***x*****=1m to** ***x*****=2m with the force** $\vec{F} = 4x^3\vec{i}$ (N). **Find the work done by the force in this process.**

Solution

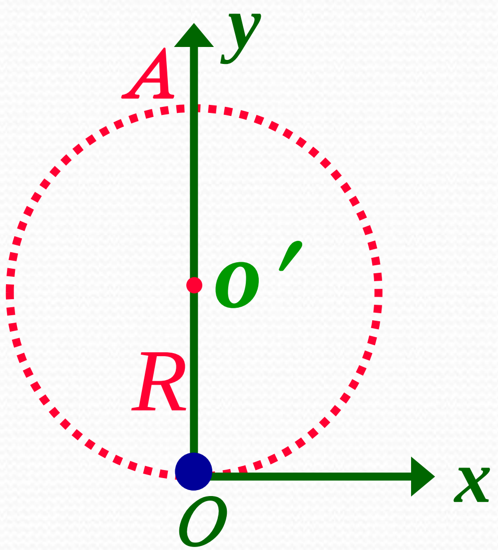
$$W = \int_{x_1}^{x_2} F_x(x)\mathrm{d}x$$

$$= \int_1^2 4x^3 \mathrm{d}x$$

$$= 15\ \mathrm{J}$$

Can we get the work by average force?

Example 4.3 A mass point moves on a circle. $\vec{F} = k(x\vec{i} + y\vec{j})$. What is the work done by the force in the process of the object moving from point O to point A?



Solution

$$W = \int_L F_x \mathrm{d}x + F_y \mathrm{d}y$$

$$= \int_0^0 kx\mathrm{d}x + \int_0^{2R} ky\mathrm{d}y$$

$$= 2kR^2$$

Example 4.4 **Which one of the following statements is right?**

(A) If the momentum of a mass point changes, the kinetic energy definitely changes.

(B) If the kinetic energy doesn′t change, its momentum doesn′t change.

(C) If the impulse of the external force is zero, the work done by this force is definitely zero.

(D) If the work done by the external force is zero, the impulse of the external force is definitely zero.

Solution (C)

Some orders of magnitude for kinetic energies

System	Kinetic energy (J)
Electron orbiting around nucleus	10^{-18}
Molecule at room temperature	10^{-17}
Electron in TV tube	10^{-15}
Walking ant	10^{-8}
Falling raindrop	10^{-3}
Baseball pitch	10^{2}
Running human	10^{3}
Automobile on a highway	10^{5}
Cruising airplane	10^{11}
Large earthquake	10^{17}
Earth revolving around the Sun	10^{33}

§ 4.2 Theorem of Kinetic Energy for a Particle

$$\mathrm{d}W = \vec{F}\cdot d\vec{r} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot \mathrm{d}\vec{r} = md\vec{v}\cdot\vec{v} = \mathrm{d}(\frac{1}{2}mv^2)$$

Integral the both sides we get

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

Net work

Kinetic energy increment

The net work done by the forces acting on a mass point is equal to the change in the object′s kinetic energy. ——Kinetic energy theorem of a particle.

Example 4.5 **A 1kg object initially at rest moves along *x* axis from the origin *O* under the force** $\vec{F} = (7 + 4x^3)\vec{i}\ (\mathrm{N})$ **What is the velocity of the object when *x*=1m?**

Solution

$$W = \int_0^1 (7 + 4x^3)\mathrm{d}x = 8\ J$$

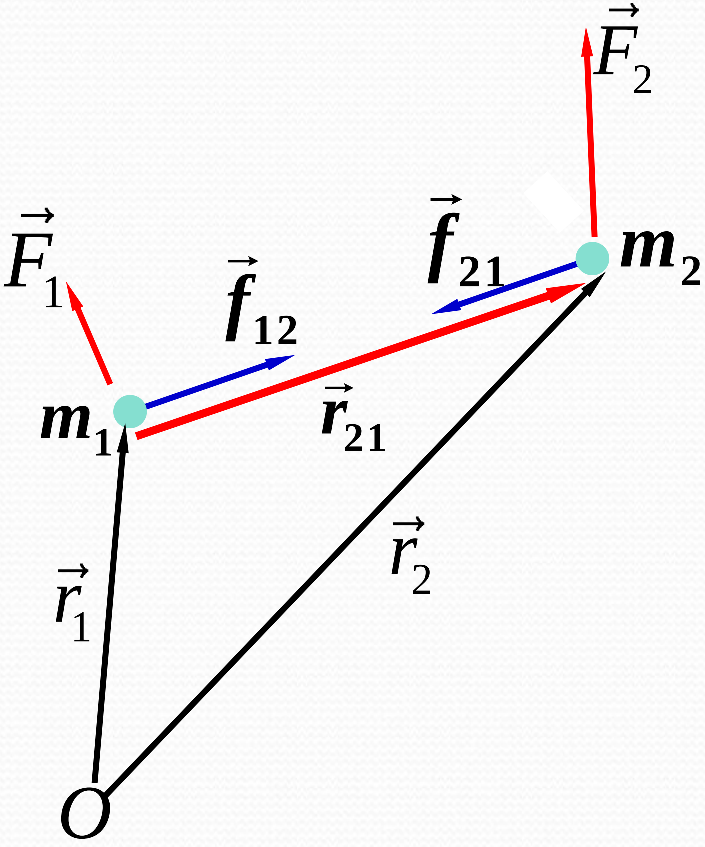
According to the kinetic energy theorem of a particle.

$$W = \tfrac{1}{2}mv^2 - 0$$

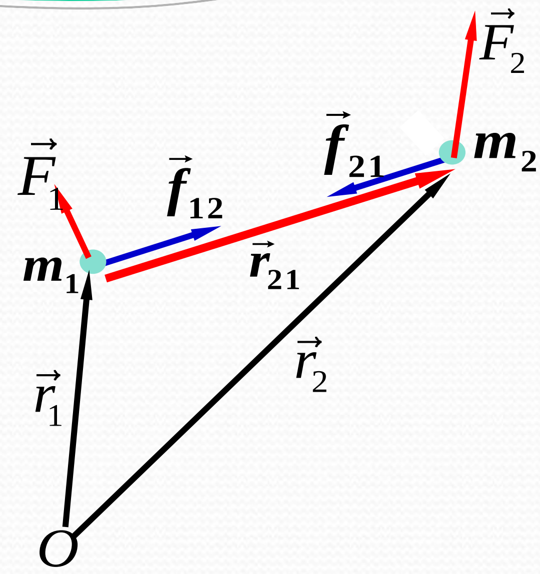
$$\rightarrow v = \sqrt{\frac{2A}{m}} = 4\ \ \mathrm{m/s}$$

§ 4.3 The Kinetic Energy Theorem of a System of Particles

Can the work done by all the internal forces change the total kinetic energy of the mass point system ?



$$\mathrm{d}A=\mathrm{d}A_1+\mathrm{d}A_2$$



$$=(\vec{F}_1+\vec{f}_{12})\cdot\mathrm{d}\vec{r}_1+(\vec{F}_2+\vec{f}_{21})\cdot\mathrm{d}\vec{r}_2$$

$$=(\vec{F}_1\cdot\mathrm{d}\vec{r}_1+\vec{F}_2\cdot\mathrm{d}\vec{r}_2)+(\vec{f}_{12}\cdot\mathrm{d}\vec{r}_1+\vec{f}_{21}\cdot\mathrm{d}\vec{r}_2)$$

$$=\mathrm{d}A_{ext}+\boxed{\vec{f}_{12}\cdot(\mathrm{d}\vec{r}_1-\mathrm{d}\vec{r}_2)}$$

Is zero?

For a n mass points system

$$\sum_{i=1}^{n} W_i = \sum_{i=1}^{n} E_{\mathrm{k}i} - \sum_{i=1}^{n} E_{\mathrm{k}i0}$$

the sum of the work done on the system	the sum of the final kinetic energy	the sum of the initial kinetic energy

$\sum_{i=1}^{n} W_i$ should be the sum of the work done by all the external forces and the work done by all the internal forces of the mass points system.

$$\sum_{i=1}^{n} W_i = \sum_{i=1}^{n} W_i^{ex} + \sum_{i=1}^{n} W_i^{\text{in}} = W^{\text{ex}} + W^{\text{in}}$$

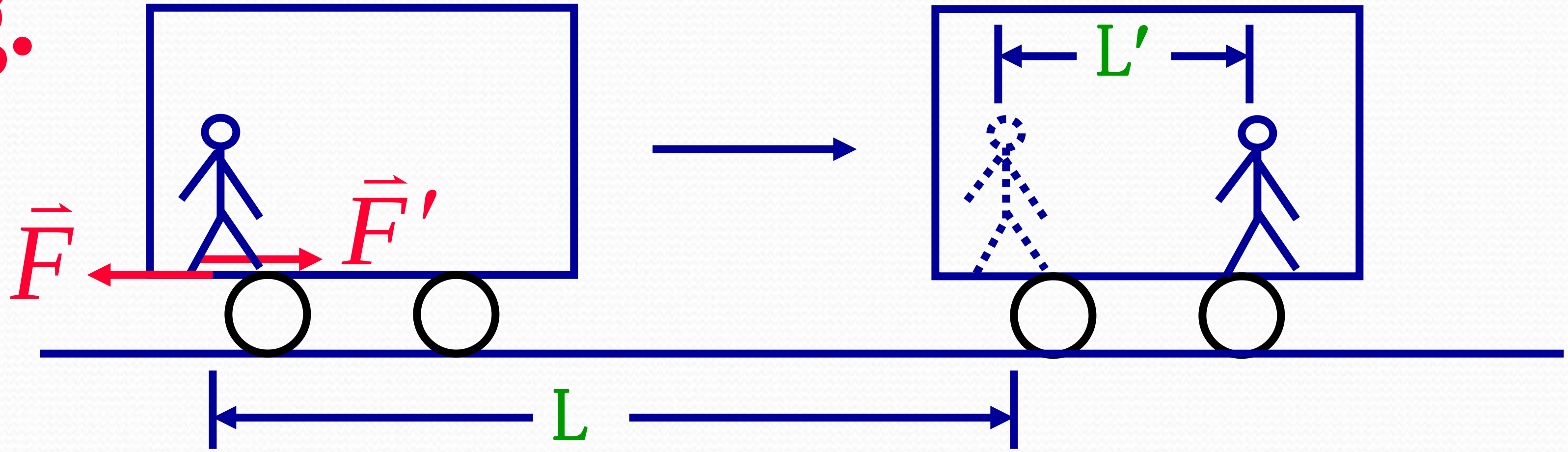
So we have

$$W^{\text{ex}} + W^{\text{in}} = \sum_{i=1}^{n} E_{\text{k}i} - \sum_{i=1}^{n} E_{\text{k}i0}$$

——the kinetic energy theorem of a system of mass points. It indicates that the change of the kinetic energy of a system of mass points is equal to the sum of the work done by the external forces on the system and the work done by the internal forces of the mass points in the system.

The work done by a pair of forces

e.g.



In the reference frame of the ground

$$W_{\vec{F}} + W_{\vec{F}'} = (-FL) + F'(L + L') = F'L'$$

In the reference frame of the vehicle

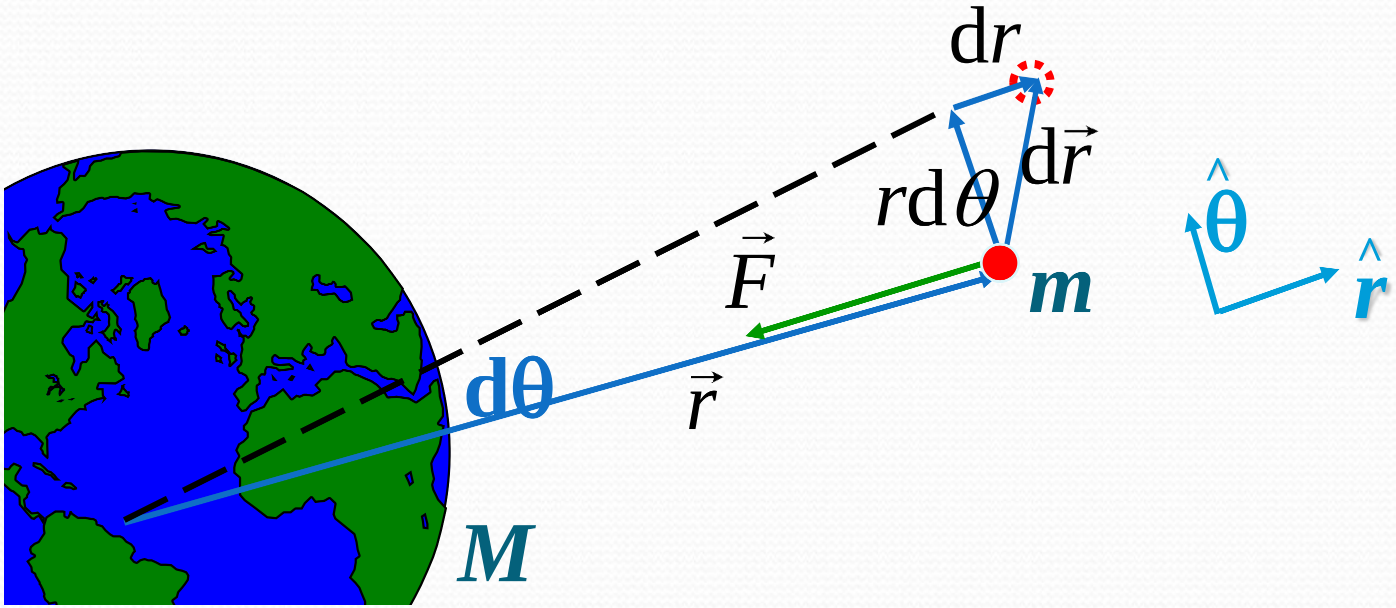
$$W'_{\vec{F}} + W'_{\vec{F}'} = 0 + F'L' = F'L'$$

$$\rightarrow \quad W_{\vec{F}} + W_{\vec{F}'} = W'_{\vec{F}} + W'_{\vec{F}'}$$

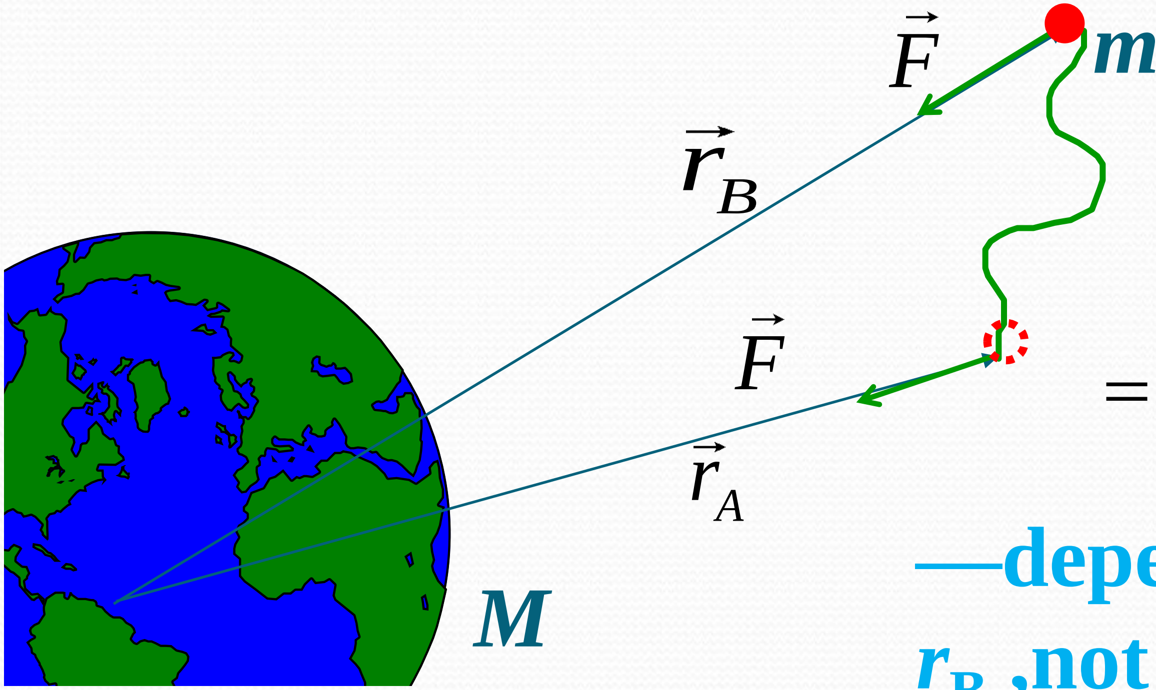
◆ Work done by Gravitational Force(a pair of force)

$$\mathrm{d}W = \vec{F}\cdot\mathrm{d}\vec{r} = -\frac{GMm}{r^3}\vec{r}\cdot\mathrm{d}\vec{r}$$

$$= -\frac{GMm}{r^2}\mathrm{d}r$$



When the mass point *m* moves from *A* to *B* along an arbitrary path, we can get the work done by the universal gravitational force using integral is



$$W=\int_{r_A}^{r_B}-\frac{GMm}{r^2}dr$$

$$=GmM\left(\frac{1}{r_B}-\frac{1}{r_A}\right)$$

$$=\left(-\frac{GmM}{r_A}\right)-\left(-\frac{GmM}{r_B}\right)$$

—depends upon r_A and r_B ,not on the path taken.

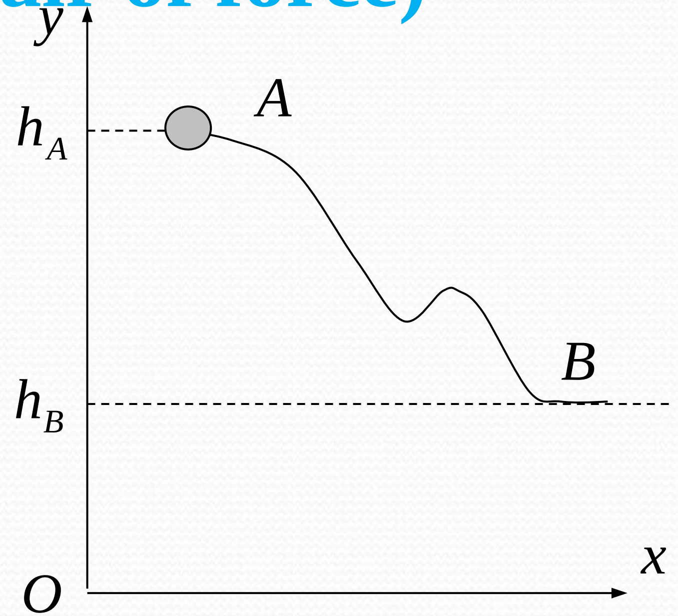
§ 4.4 Conservative Force

In general, if the work done does not depend on the path taken (only depends on the initial and final distances between objects), the force involved is said to be *conservative*.

◆ **Work done by gravity(a pair of force)**

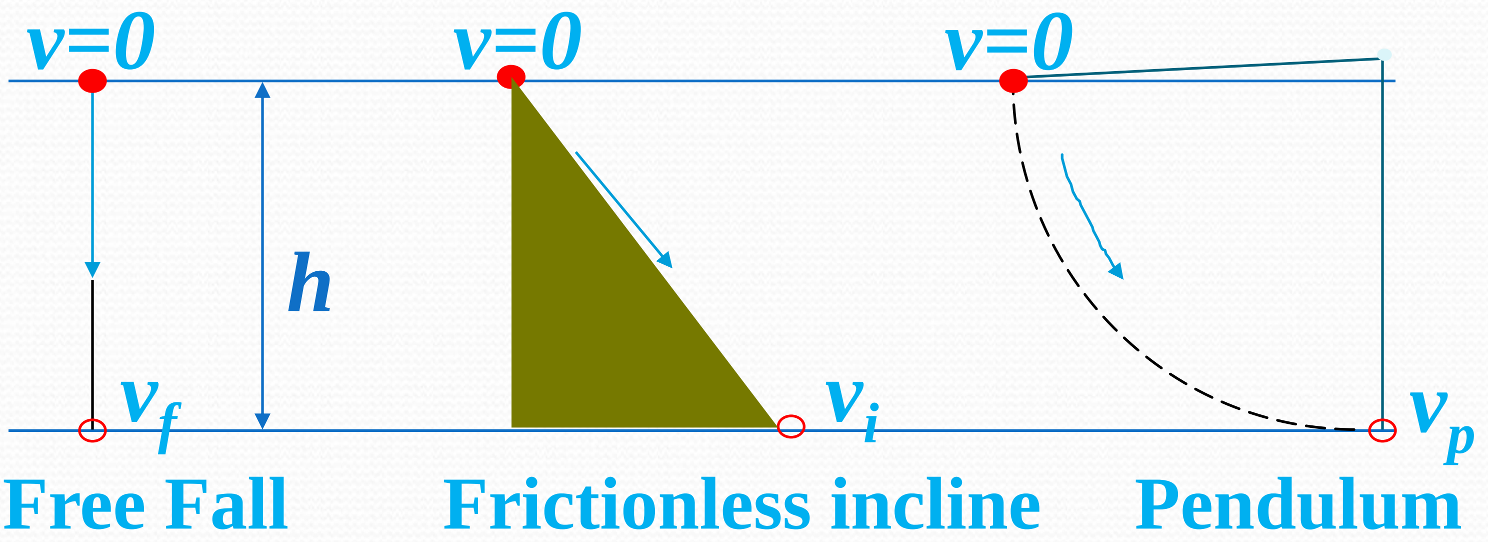
$$W = \int_{h_A}^{h_B} m\vec{g} \cdot \mathrm{d}\vec{r}$$

$$= -(mgh_B - mgh_A)$$



Depends only on the initial and final positions, i.e. Δy, not on the path taken.

Example 4.6 Three objects of mass m begin at height h with velocity 0. One falls straight down, one slides down a frictionless inclined plane, and one swings on the end of a pendulum. What is the relationship between their velocities when they have fallen to height 0?



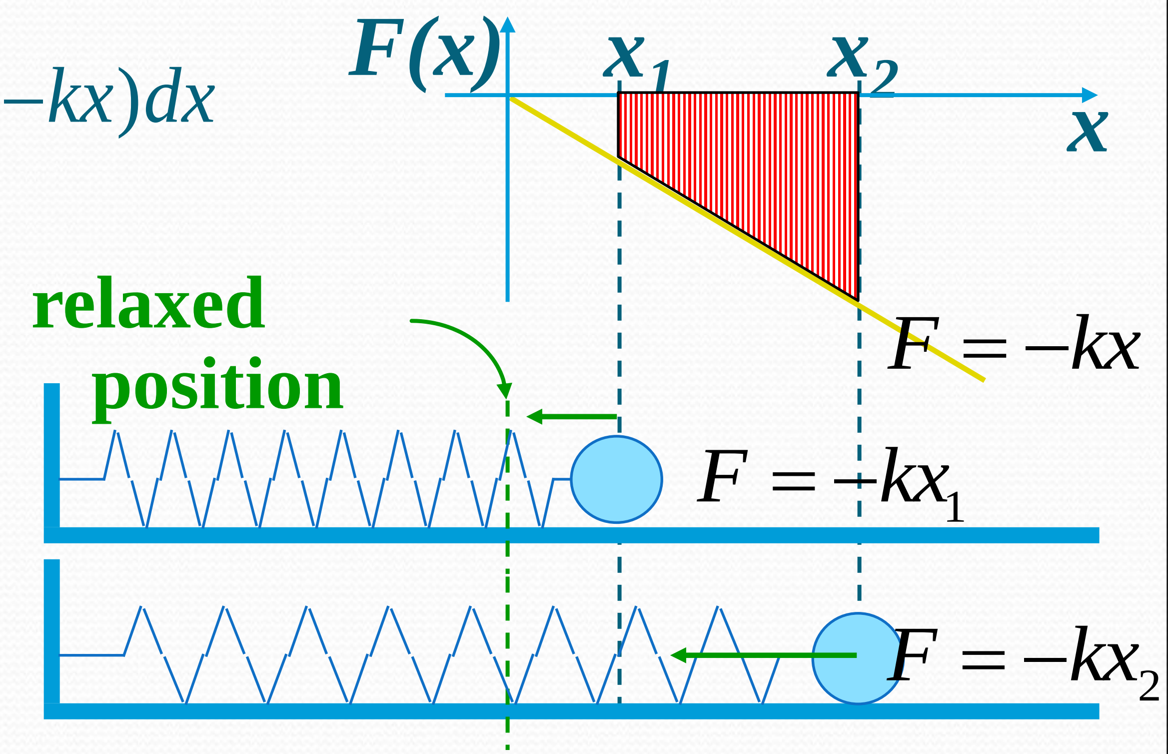
(a) $v_f > v_i > v_p$ (b) $v_f > v_p > v_i$ (c) $v_f = v_p = v_i$

◆ Work done by elastic force(a pair of force)

$$W_s = \int_{x_1}^{x_2} F(x)dx = \int_{x_1}^{x_2} (-kx)dx$$

$$= -\frac{1}{2}kx^2 \Big|_{x_1}^{x_2}$$

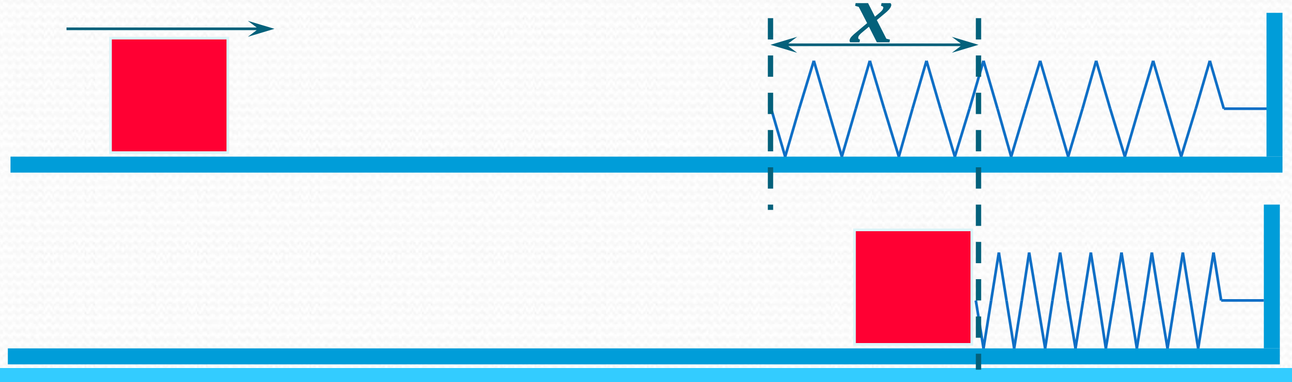
$$W_s = -\frac{1}{2}k\left(x_2^2 - x_1^2\right)$$



The work depends on the initial and final positions of the object, and is independent of the process of the elastic deformation.

Example 4.7 **A box sliding on a horizontal frictionless surface runs into a fixed spring, compressing it a distance *x_1* from its relaxed position while momentarily coming to rest. If the initial *speed* of the box were *doubled* and its *mass* were *halved*, how far *x_2* would the spring compress ?**

$(a)\, x_2 = x_1 \qquad (b)\, x_2 = \sqrt{2}\, x_1 \qquad (c)\, x_2 = 2x_1$



Solution

In the case of x_1

$$\frac{1}{2}kx_1^2 = \frac{1}{2}m_1v_1^2$$

We get

$$x_1 = v_1\sqrt{\frac{m_1}{k}}$$

So if $v_2 = 2v_1$ and $m_2 = m_1/2$

$$x_2 = 2v_1\sqrt{\frac{m_1/2}{k}} = v_1\sqrt{\frac{2m_1}{k}}$$

$$= \sqrt{2}x_1$$

In general, if the work done does not depend on the path taken (only depends on the initial and final distances between objects), the force involved is said to be *conservative*.

- **Gravitational force is a conservative force:**

$$W = -\frac{GmM}{r_A} - \left(-\frac{GmM}{r_B}\right)$$

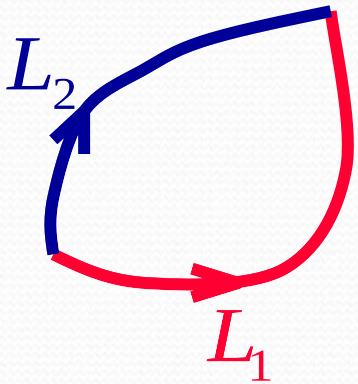
- **Gravity near the Earth's surface:**

$$W = mgh_A - mgh_B$$

- **A spring produces a conservative force:**

$$W = \frac{1}{2}kx_A^2 - \frac{1}{2}kx_B^2$$

Because the work done by a conservative force does not depend on the path taken, then



$W_1 = W_2$

$$\int_{L_1} \vec{F} \cdot d\vec{r} = \int_{L_2} \vec{F} \cdot d\vec{r}$$

$$\rightarrow \int_{L_1} \vec{F} \cdot d\vec{r} + \int_{-L_2} \vec{F} \cdot d\vec{r} = 0$$

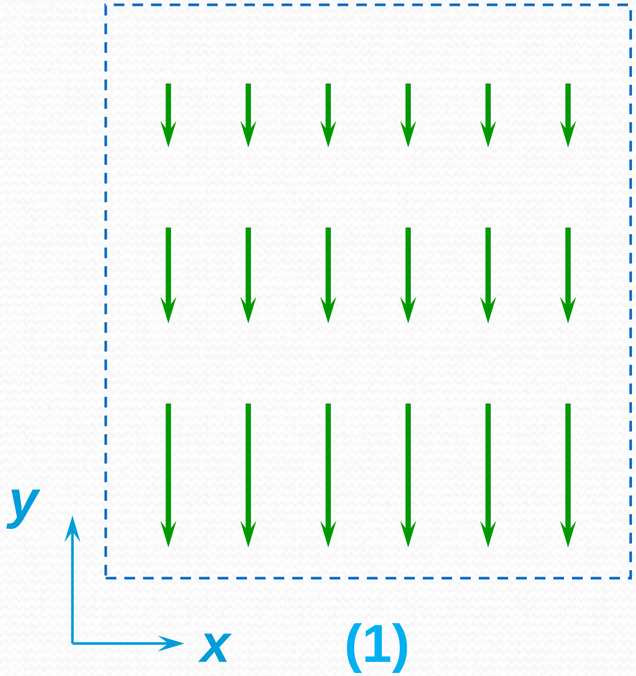
$$\rightarrow \oint_{L} \vec{F} \cdot d\vec{r} = 0$$

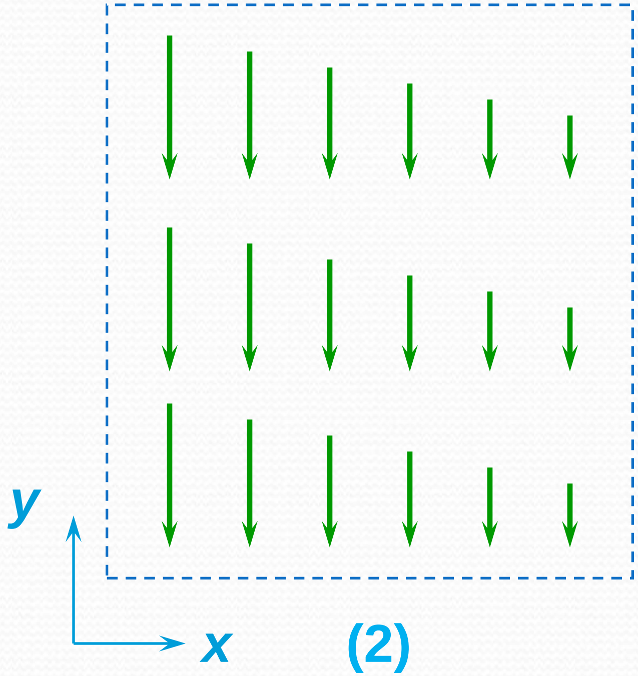
Therefore the work done in a closed path is 0.

Not all forces have this property, e.g., the work done by the often encountered frictional force depends on the path. The force whose work done depends on the path is a non-conservative force.

Example4.8 **The pictures below show force vectors at different points in space for two forces. Which one is conservative ?**

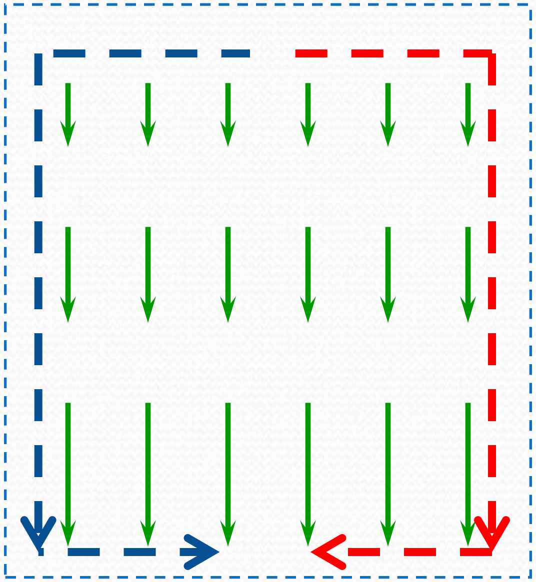
(a) 1 (b) 2 (c) both





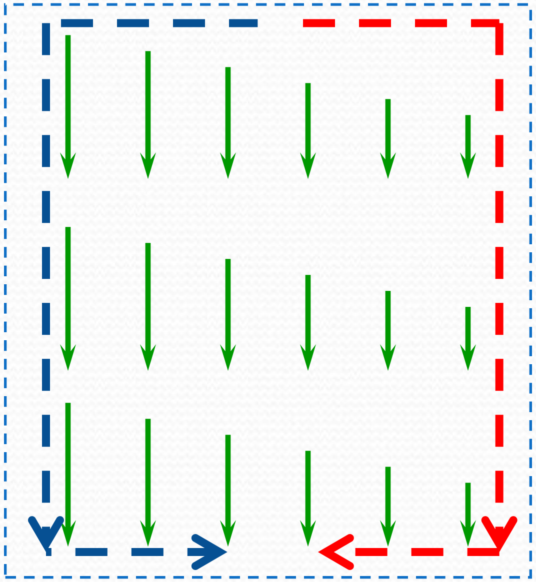
Solution **Consider the work done by force when moving along different paths in each case.**

$W_A = W_B$



(1)

$W_A > W_B$



(2)

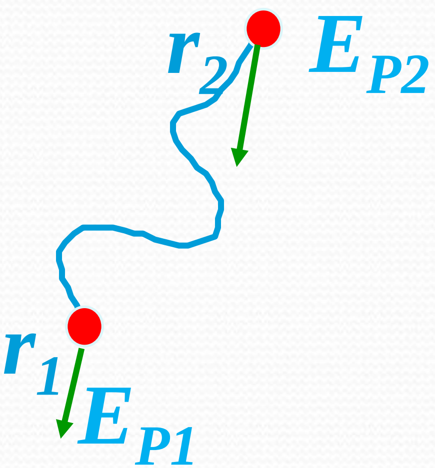
You could make money on type (2) if it ever existed!

§ 4.5 Potential Energy

For any conservative force we can define a potential energy function E_P in the following way:

$$W=\int_1^2 \vec{F}\cdot \mathrm{d}\vec{r}=-(E_{\mathrm{P2}}-E_{\mathrm{P1}})=-\Delta E_{\mathrm{P}}$$

The work done by a conservative force is equal and opposite to the change in the potential energy function.



Which point,1 or 2, can be selected as zero point if we want to find the potential energy of a position?

Note

- **The potential energy belongs to the system;**
- **The value of the potential energy depends on the selection of the zero point. We can choose the location where $E_P = 0$ at our convenience;**
- **The potential energy only has a relative meaning. But the difference of the potential energy between any two points has an absolute meaning.**
- **The potential energy is a scalar, its unit is Joule.**

Choosing the gravity potential to be zero on the ground, we define the gravity potential energy of a system with mass *m* is

$$E_{\mathrm{P}} = mgh$$

If the zero point of the gravitational potential is chosen at infinity, the universal gravitational potential of two objects with distance *r* is

$$E_{\mathrm{P}} = -G\frac{mM}{r}$$

When the horizontally placed spring is at its equilibrium, its elastic potential energy is taken to be zero. Then the elastic potential of the elongation *x* is

$$E_{\mathrm{P}} = \frac{1}{2}kx^2$$

Conservative Forces & Potential Energies

Force F	Work $W(1\text{-}2)$	Change in P.E $\Delta E_P = E_{P2} - E_{P1}$	P.E. function E_P
$\vec{F}_g = -mg\vec{j}$	$-mg(y_2 - y_1)$	$mg(y_2 - y_1)$	$mgy + C$
$\vec{F}_g = -\frac{GMm}{R^2}\hat{r}$	$GMm\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$	$-GMm\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$	$-\frac{GMm}{R} + C$
$F_s = -kx$	$-\frac{1}{2}k\left(x_2^2 - x_1^2\right)$	$\frac{1}{2}k\left(x_2^2 - x_1^2\right)$	$\frac{1}{2}kx^2 + C$

(R is the center-to-center distance, x is the spring stretch)

Example4.9 When the distance between two objects with masses of m_1 and m_2 is shorten from a to b, what is the work done by the gravitational force?

Solution

$$W = E_{pa} - E_{pb}$$

$$= \left(-\frac{Gm_1m_2}{a}\right) - \left(-\frac{Gm_1m_2}{b}\right)$$

$$= \frac{Gm_1m_2(a-b)}{ab}$$

§ 4.6 The principle of Work and Energy

The work done by all the internal forces of the system can be expressed as

$$W^{\mathrm{in}} = W_{\mathrm{c}}^{\mathrm{in}} + W_{\mathrm{nc}}^{\mathrm{in}}$$

$W_{\mathrm{c}}^{\mathrm{in}}$ **the sum of the work done by the conservative internal forces of the system**

$W_{\mathrm{nc}}^{\mathrm{in}}$ **the sum of the work done by the non-conservative internal forces of the system**

The work done by all the conservative internal forces is

$$W_{\mathrm{c}}^{\mathrm{in}} = -\left(\sum_{i=1}^{n} E_{\mathrm{p}i} - \sum_{i=1}^{n} E_{\mathrm{p}i0}\right)$$

So the work done by all the external and non-conservative internal forces on the system is

$$W^{\mathrm{ex}}+W_{\mathrm{nc}}^{\mathrm{in}}=(\sum_{i=1}^{n}E_{\mathrm{k}i}+\sum_{i=1}^{n}E_{\mathrm{p}i})-(\sum_{i=1}^{n}E_{\mathrm{k}i0}+\sum_{i=1}^{n}E_{\mathrm{p}i0})$$

$$=E-E_0$$

The final mechanical energy

The initial mechanical energy

——the principle of work and energy. It shows that the change of the mechanical energy of the system is a sum of the work done by the external forces and the non-conservative internal forces on the system.

§ 4.7 The Law of Conservation of mechanical Energy

If $$W^{\mathrm{ex}} + W_{\mathrm{nc}}^{\mathrm{in}} = 0$$

Then $$\sum_{i=1}^{n} E_{\mathrm{k}i} + \sum_{i=1}^{n} E_{\mathrm{p}i} = \sum_{i=1}^{n} E_{\mathrm{k}i0} + \sum_{i=1}^{n} E_{\mathrm{p}i0}$$

That is $$E = E_0$$

——the law of conservation of mechanical energy. **If the sum of the work done by the external forces and the non-conservative internal forces on the system is zero, the total mechanical energy of the system remains constant.**

Notes

- **In an isolated system in which only conservative forces act, the total mechanical energy remains constant.**
- **The kinetic energy theorem of a system of mass points, the principle of work and energy, the law of conservation of mechanical energy are valid only in inertial frames of reference.**

Example4.10 **An object of mass *m* moves in uniform circular motion with radius *r*. The centripetal force is $F=-k/r^2$. Let $E_{p\infty}=0$. What is the mechanical energy of the system?**

Solution **According to the Newton's second law we have**

$$\frac{k}{r^2}=m\frac{v^2}{r}$$

So the kinetic energy is

$$E_k=\frac{k}{2r}$$

The potential energy is

$$E_p = \int_r^{\infty} -\frac{k}{r^2} dr = -\frac{k}{r}$$

Then the mechanical energy is

$$E = E_k + E_p = -\frac{k}{2r}$$

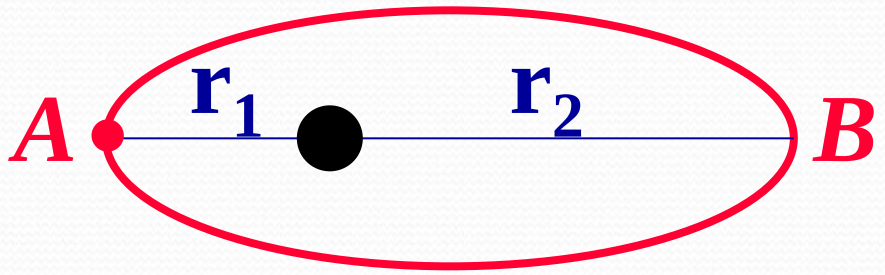
Example4.11 **The distance between a mass point *m* and the center of the Earth is *r*. Let $E_p(r=4R)=0$. What is the potential energy of the system when $r=2R$?(R, M are the radius and mass of the Earth).**

Solution

$$E_p = \left(-\frac{GMm}{2R}\right) - \left(-\frac{GMm}{4R}\right)$$

$$= -\frac{GMm}{4R}$$

Example4.12 A artificial satellite m carries out an elliptical motion around the Earth. Point A is at perihelion and point B is at aphelion. The distances from points A and B to the center of the Earth is r_1 and r_2. Find the difference of kinetic energy E_{kB}-E_{kA}. (M is the mass of the Earth, G is gravitational constant)



Solution The mechanical energy of the satellite and the Earth is conservative.

$$E_{pA} + E_{kA} = E_{pB} + E_{kB}$$

$$E_{kB} - E_{kA} = E_{pA} - E_{pB}$$

$$= \left(-\frac{GMm}{r_1}\right) - \left(-\frac{GMm}{r_2}\right)$$

$$= GMm\frac{r_1 - r_2}{r_1 r_2}$$

See you next time!