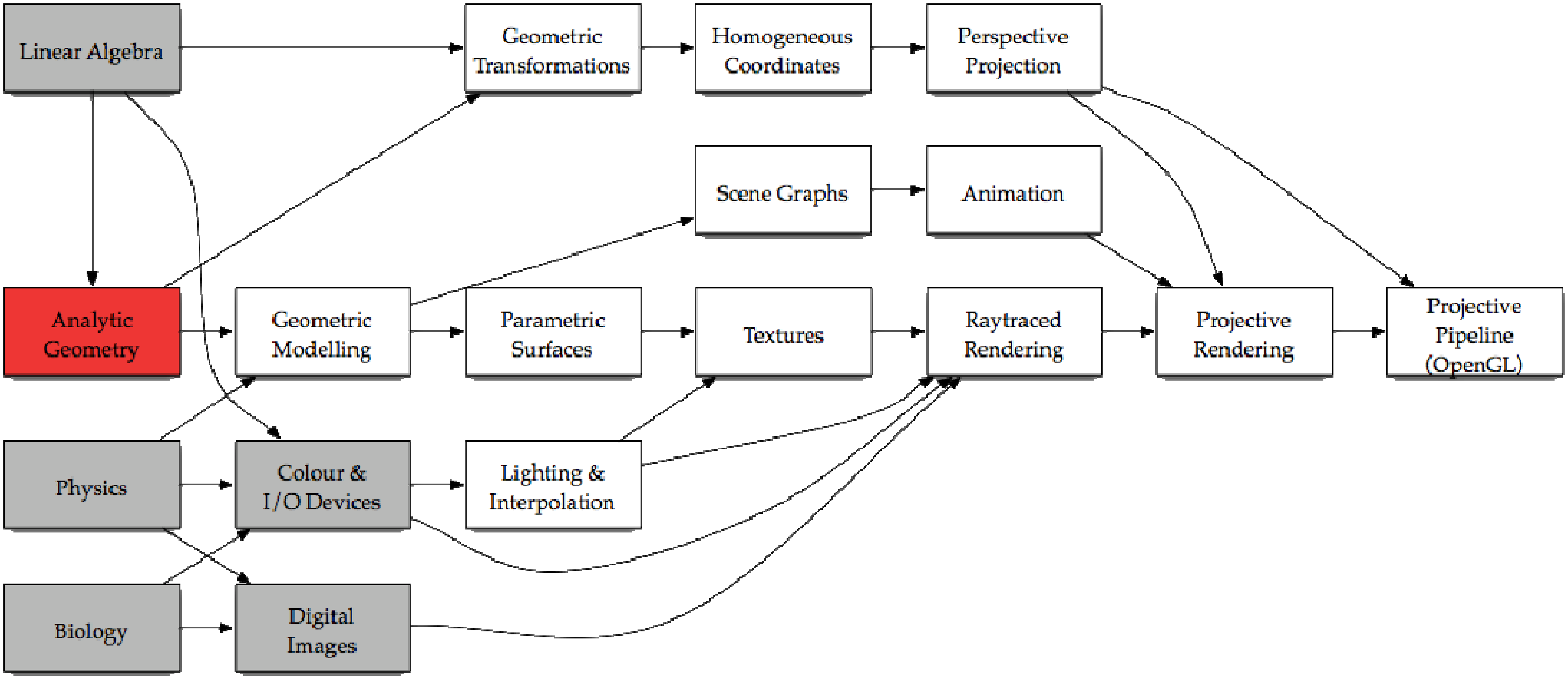
Lines in 2-D

Where we Are


Linear Algebra
Geometric Transformations
Homogeneous Coordinates
Perspective Projection
Scene Graphs
Animation
Analytic Geometry
Geometric Modelling
Parametric Surfaces
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Projective Rendering
Projective Pipeline (OpenGL)
Physics
Colour & I/O Devices
Lighting & Interpolation
Biology
Digital Images

Geometric Results

- The dot product gives us:
  - length of a vector
  - angle between two vectors
  - normal (perpendicular) vectors
  - projection of vector onto line
  - distance from point to line
  - intersection of two lines

Short-cut for Length

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{v_x v_x + v_y v_y}$$

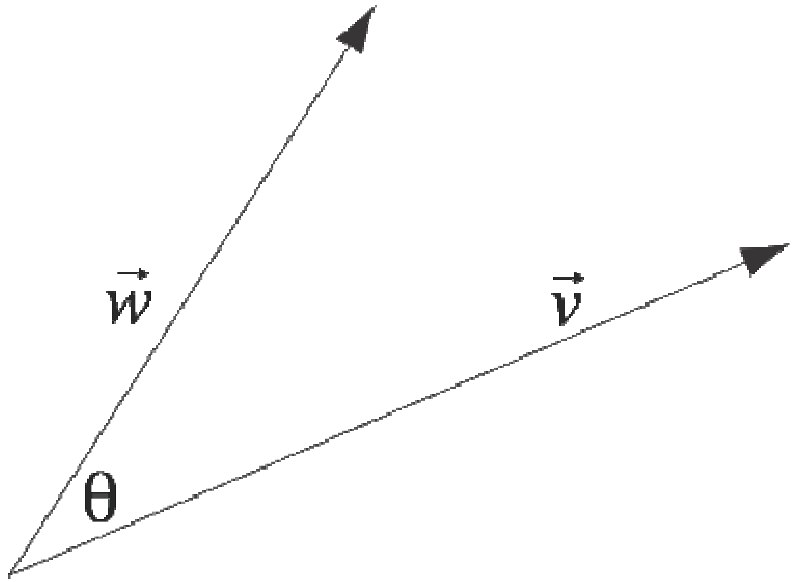
$$= \sqrt{\vec{v} \cdot \vec{v}}$$

or

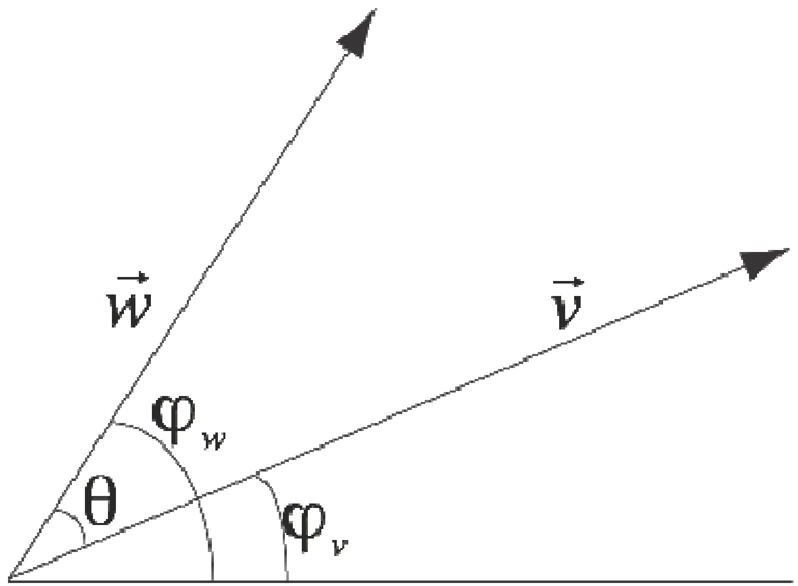
$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$$

Angle Between Vectors

- What's the angle between two vectors?



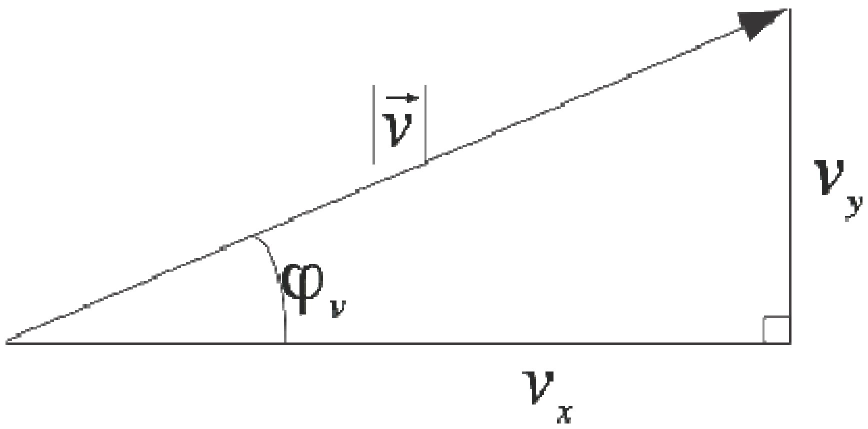
First Step

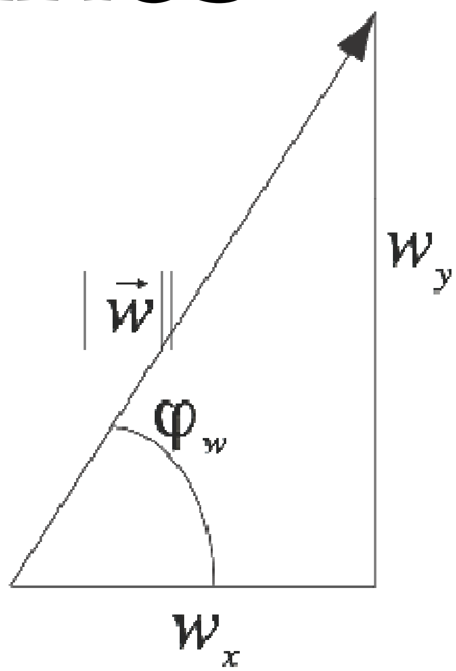


$$\theta = \varphi_w - \varphi_v$$

$$\cos\theta = \cos\varphi_v \cos\varphi_w + \sin\varphi_v \sin\varphi_w$$

Sines & Cosines





$$\sin\varphi_v = \frac{v_y}{\|\vec{v}\|} \qquad \sin\varphi_w = \frac{w_y}{\|\vec{w}\|}$$

$$\cos\varphi_v = \frac{v_x}{\|\vec{v}\|} \qquad \cos\varphi_w = \frac{w_x}{\|\vec{w}\|}$$

Dot Product Form

$$\theta = \varphi_w - \varphi_v$$

$$\cos\theta = \cos\varphi_v \cos\varphi_w + \sin\varphi_v \sin\varphi_w$$

$$= \frac{v_x}{\|\vec{v}\|}\frac{w_x}{\|\vec{w}\|} + \frac{v_y}{\|\vec{v}\|}\frac{w_y}{\|\vec{w}\|}$$

$$= \frac{v_x w_x + v_y w_y}{\|\vec{v}\|\|\vec{w}\|}$$

$$= \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$$

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$$

Normal Vectors

- A normal vector is perpendicular to v

$$\frac{\vec{v} \cdot \vec{n}}{\|\vec{v}\|\|\vec{n}\|} = \cos 90^\circ$$

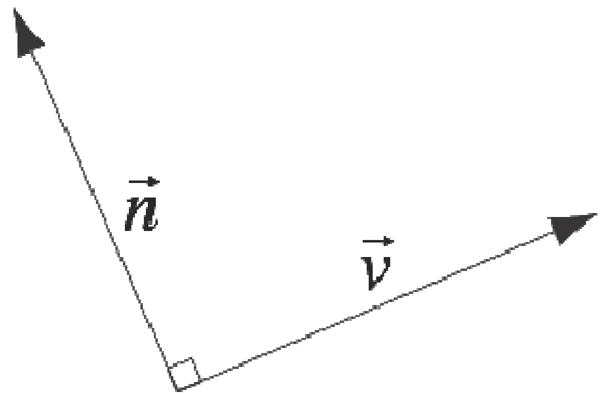
$$= 0$$

$$\vec{v} \cdot \vec{n} = 0$$

$$v_x n_x + v_y n_y = 0$$

$$\text{Let } \vec{n} = \begin{bmatrix} -v_y \\ v_x \end{bmatrix}$$

$$v_x\left(-v_y\right) + v_y\left(v_x\right) = 0$$



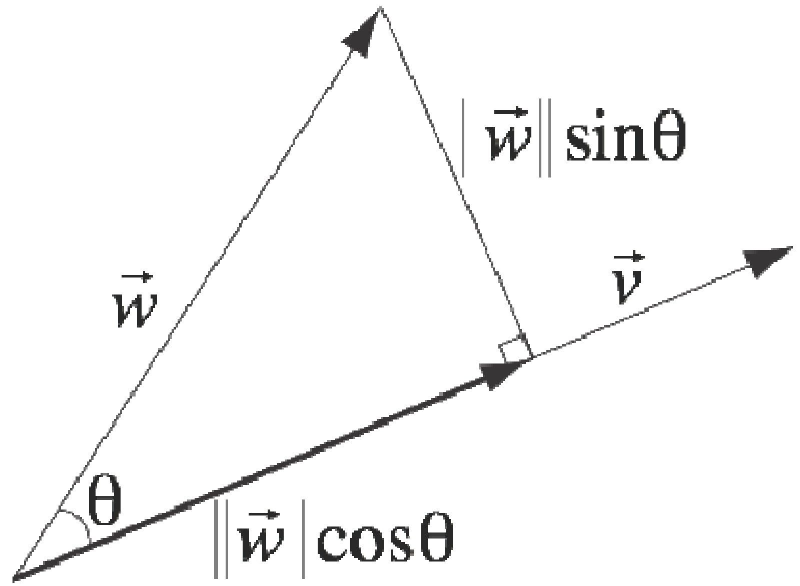
Projection

$$\Pi_{\vec{v}}\left(\vec{w}\right) = \left(\|\vec{w}\|\cos\theta\right)\frac{\vec{v}}{\|\vec{v}\|}$$

$$= \left(\cancel{\|\vec{w}\|}\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cancel{\|\vec{w}\|}}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|^2}\vec{v}$$

$$= \frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\vec{v}$$



Equations of Lines

- How do we describe a line?
  - Explicit form: y depends on x
  - Implicit form: x, y satisfy an equation
  - Normal form: defined by normal vector
  - Parametric form: x, y depend on t

Explicit Form

- y is a function of x:

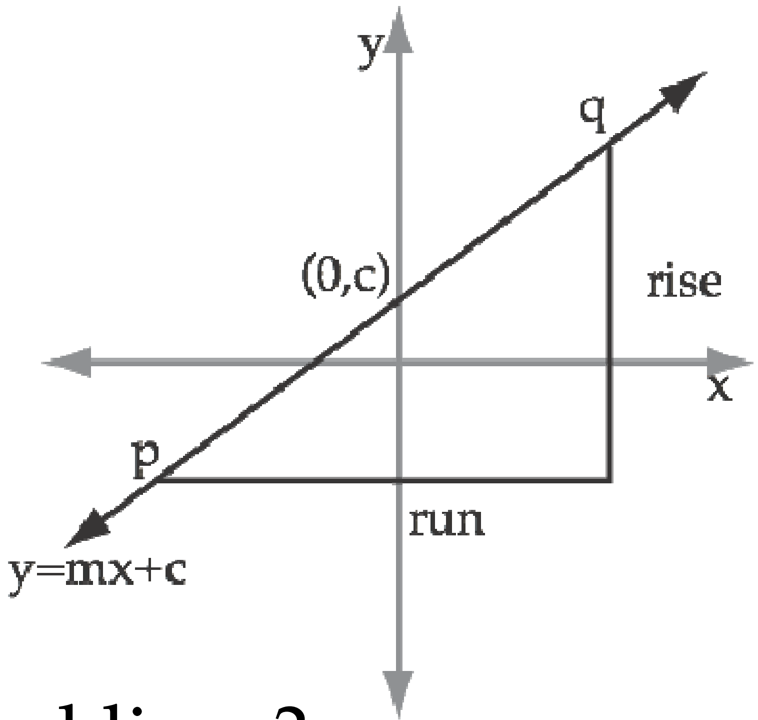
$$y = mx + c$$

$$slope\ m = \frac{rise}{run} = \frac{q_y - p_y}{q_x - p_x}$$

y intercept:

$$p_y = mp_x + c$$

$$c = p_y - mp_x$$



- Does this work for vertical lines?

Implicit Form

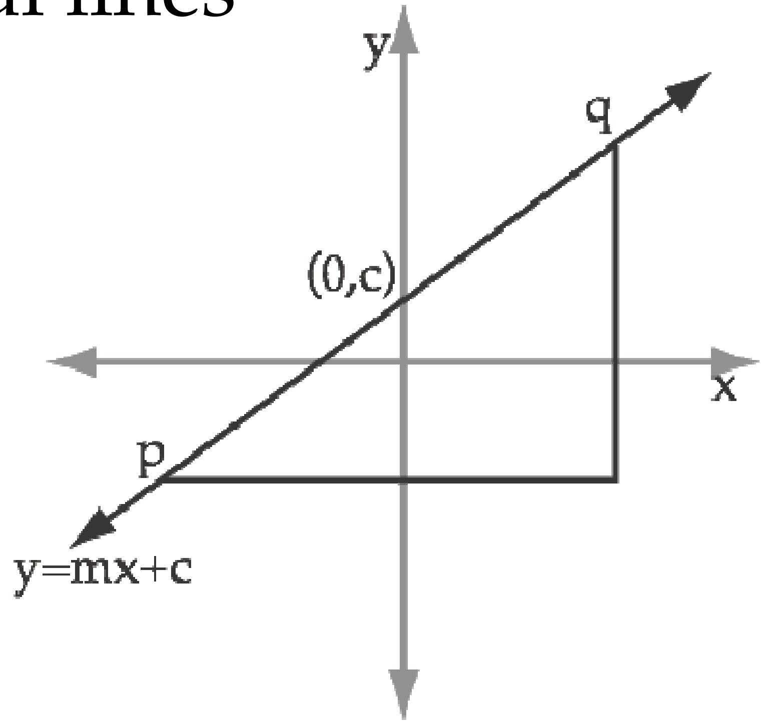
- A form that allows vertical lines
  - but harder to draw
- Any (x,y) that satisfies:

$$y = mx + c$$

$$1y - mx = c$$

$$Ax + By = c$$

  - is on the line



Normal Form

- Rewrite to use a dot product

$$Ax + By = c$$

$$\begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = c$$

$$\vec{n} \cdot p = c$$

$$\vec{n} \cdot p - c = 0$$

$\vec{n}$ is the *normal* vector for the line

- We'll see a simpler variation later

Parametric Form

- Imagine a fly flying along the line
- Line is a function of t
  - t (parameter) is time

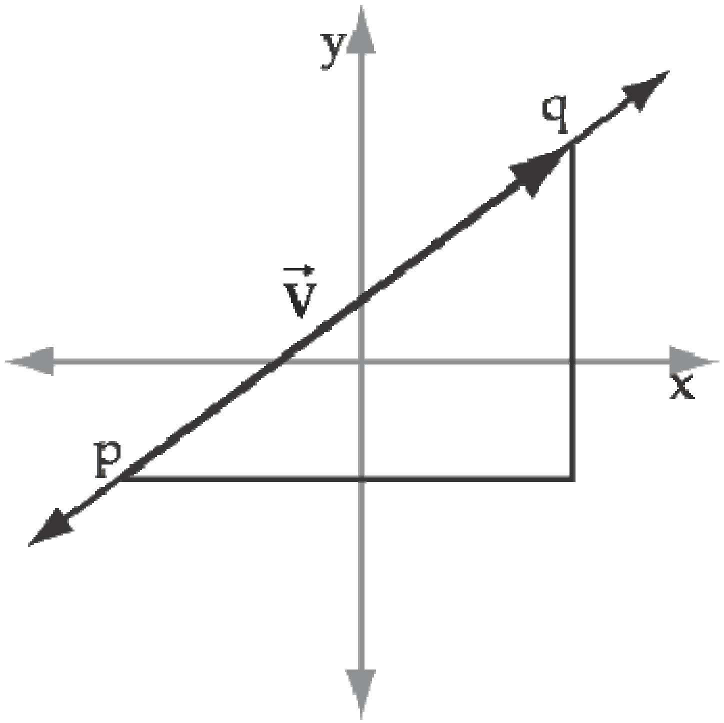
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \end{bmatrix} t$$

$$\overleftrightarrow{l} = p + \overrightarrow{vt}$$

p is any point on $\overleftrightarrow{l}$

$\vec{v}$ is any vector along $\overleftrightarrow{l}$

$$e.g.\ \vec{v} = \begin{bmatrix} q_x - p_x \\ q_y - p_y \end{bmatrix}$$



Points & Lines

- Several questions:
  - Is a point p on a line?
  - What is the distance from p to a line?
  - Which side of a line is p on?
  - What is the closest point to p on a line?
  - What is the intersection of two lines?

Point on a Line

- A point p is on the line l when:

$$p_x = mp_y + c \qquad \text{(explicit)}$$

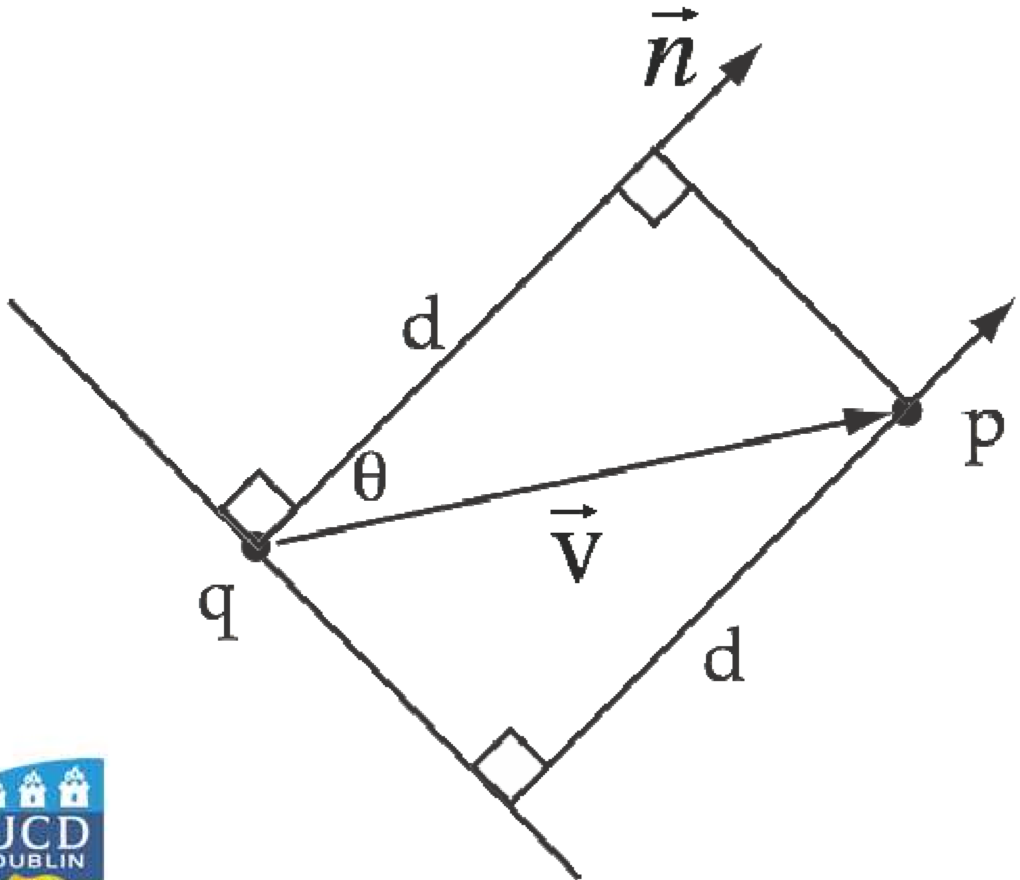
$$Ap_x + Bp_y = c \qquad \text{(implicit)}$$

$$\vec{n} \cdot p = c \qquad \text{(normal)}$$

$$p = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \text{ for some } t \qquad \text{(parametric)}$$

Distance to Line

- How far (d) is a point p from a line?



$$d = \|\vec{v}\|\cos\theta = \frac{\vec{n}\cdot(p-q)}{\|\vec{n}\|}$$

$$= \|\vec{v}\|\left(\frac{\vec{n}\cdot\vec{v}}{\|\vec{n}\|\|\vec{v}\|}\right) = \frac{\vec{n}\cdot p - \vec{n}\cdot q}{\|\vec{n}\|}$$

$$= \frac{\vec{n}\cdot\vec{v}}{\|\vec{n}\|} = \frac{\vec{n}\cdot p - c}{\|\vec{n}\|}$$

Distance to a Line

- Easier if normal is normalized $\left(\|\vec{n}\| = 1\right)$
- We'll simplify it more later
- But d is actually signed distance along n
- This also solves some other problems
  - which side of a line we are on
  - what the closest point on a line is

Which side of a line

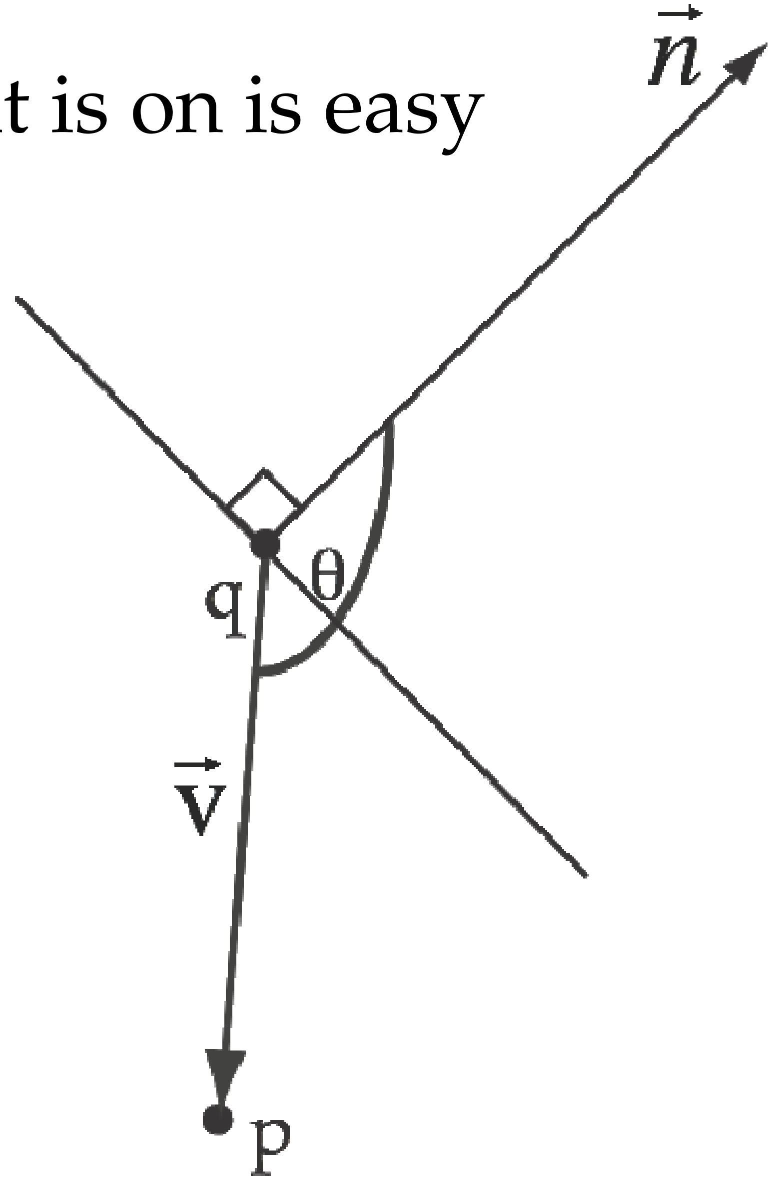
- Testing which side a point is on is easy with the normal form

$$\frac{\vec{n} \cdot p - c}{\|\vec{n}\|} = \vec{n} \cdot \vec{v}$$

$$= \|\vec{n}\| \|\vec{v}\| \cos\theta$$

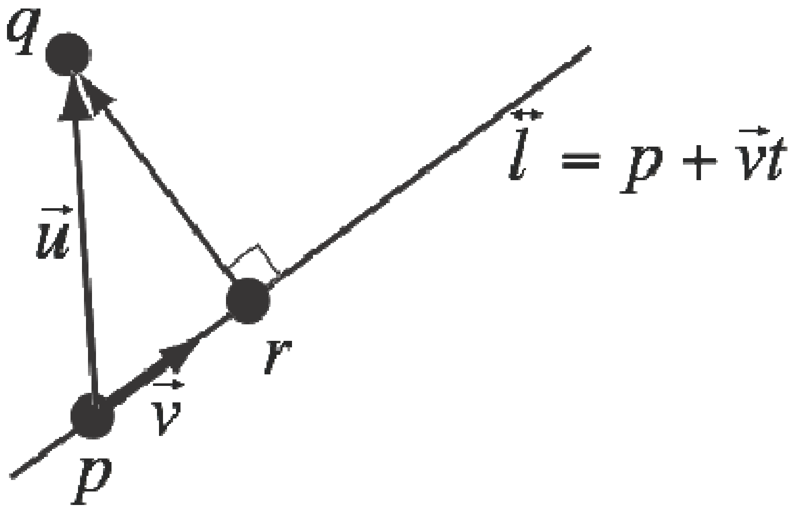
$$\cos\theta < 0 \text{ when } \vec{n} \cdot p - c < 0$$

- +: in direction of normal
- 0: on line
- -: away from normal



Closest Point on a Line

- Given p+vt, q
- Find closest point r
  - drop a perpendicular
  - What is r-p?



Projection to the Line

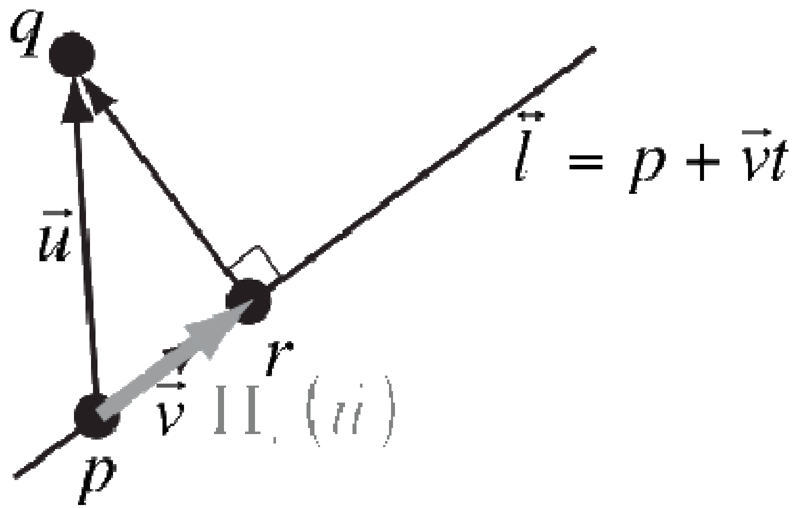
We've seen $r - p$ before:

the projection of $\vec{u}$ onto $\vec{v}$

$$\Pi_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

$$\text{So } r = p + \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

$$\text{i.e. } t = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$



Intersection of 2 Lines

- Explicit form:

$$y = m_1 x + c_1 = m_2 x + c_2$$

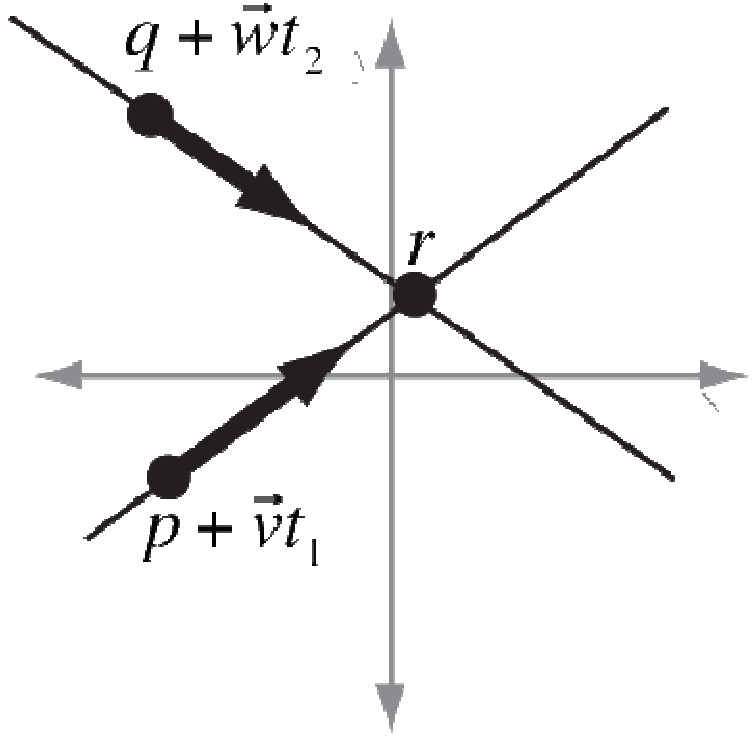
- Implicit / normal forms: $(m_1 - m_2)x = c_1 - c_2$
  - convert to explicit & solve

$$x = \frac{c_1 - c_2}{m_1 - m_2}$$

- Parametric form:
  - messy to set up, easy to do

Two Parametric Lines

- Each has a separate parameter
- We want to find r
    - by finding t1 at r
    - or t2 at r



System of Equations

$$p + \vec{v}t_1 = q + \vec{w}t_2$$

$$or:$$

$$p_x + v_x t_1 = q_x + w_x t_2$$

$$p_y + v_y t_1 = q_y + w_y t_2$$

$$or:$$

$$v_x t_1 - w_x t_2 = q_x - p_x$$

$$v_y t_1 - w_y t_2 = q_y - p_y$$

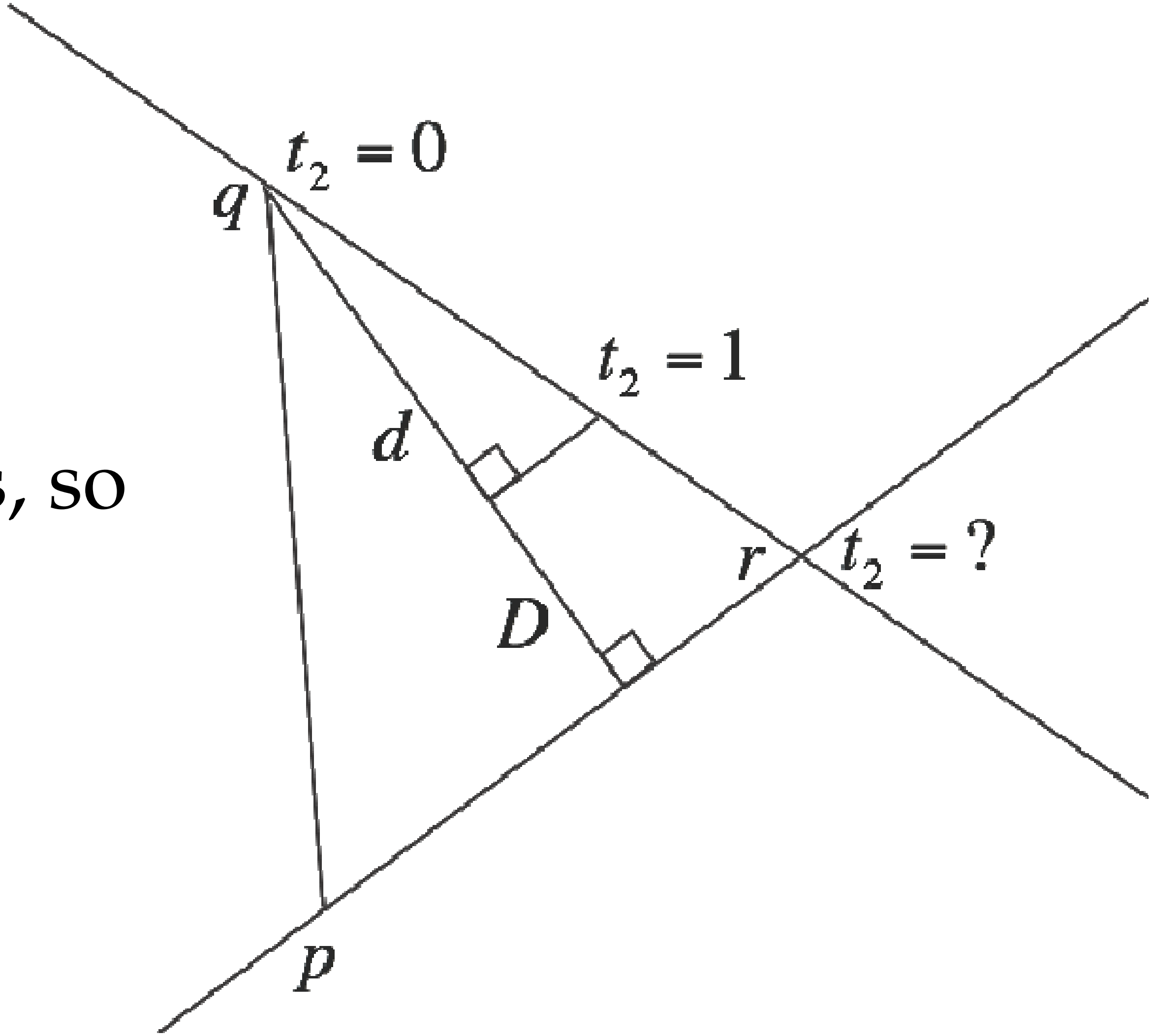
Two equations in two unknowns,
but there's a "simpler" way

Similar Triangles

- Find t2 at r
- Similar triangles, so

$$\frac{t_2}{1} = \frac{D}{d}$$

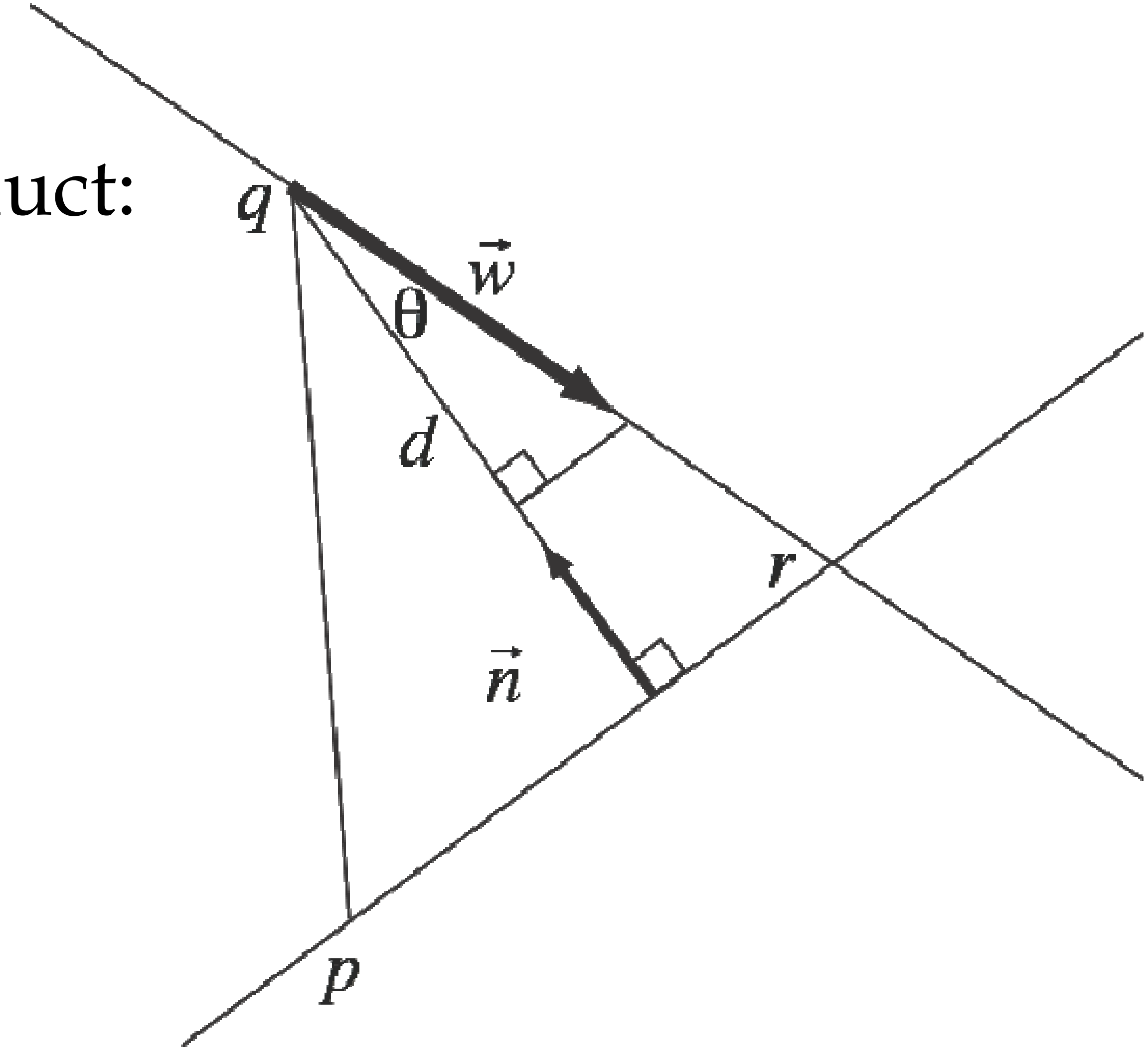
- find d and D



Finding d

- Use the dot product:

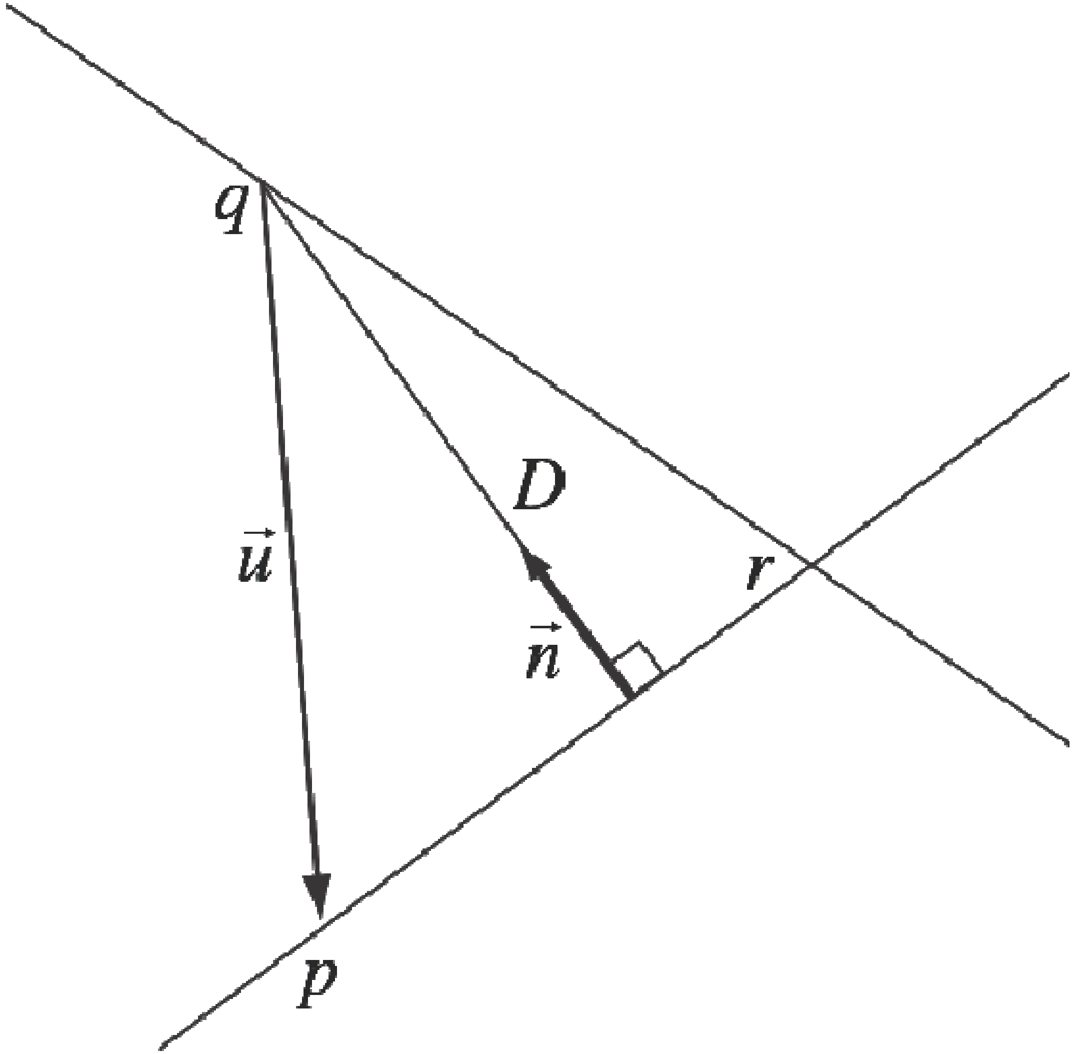
$$
\begin{aligned}
d &= \left\|\Pi_{\vec{n}}\left(\vec{w}\right)\right\| \\
&= \left\|\vec{w}\right\|\cos\theta \\
&= \left\|\vec{w}\right\|\frac{\vec{n}\cdot\vec{w}}{\left\|\vec{w}\right\|\left\|\vec{n}\right\|} \\
&= \frac{\vec{n}\cdot\vec{w}}{\left\|\vec{n}\right\|}
\end{aligned}
$$



Finding D

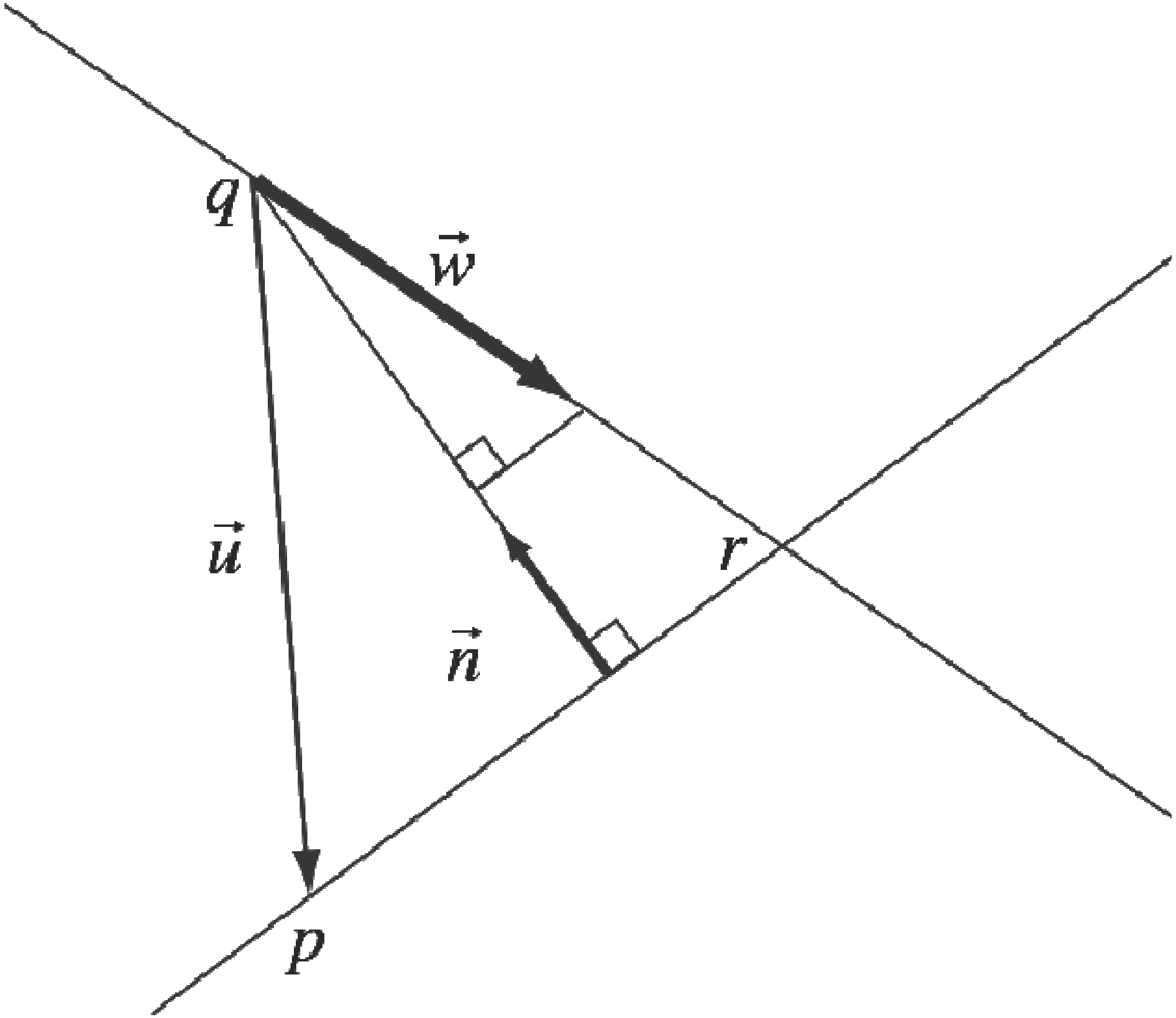
- Similarly,

$$
\begin{aligned}
D &= \left\|\Pi_{\vec{n}}\left(\vec{u}\right)\right\| \\
&= \left\|\vec{u}\right\|\cos\theta \\
&= \left\|\vec{u}\right\|\frac{\vec{n}\cdot\vec{u}}{\left\|\vec{u}\right\|\left\|\vec{n}\right\|} \\
&= \frac{\vec{n}\cdot\vec{u}}{\left\|\vec{n}\right\|}
\end{aligned}
$$



Solution

$$
\begin{aligned}
r &= q + \vec{w}t_2 \\
&= q + \vec{w}\left(\frac{D}{d}\right) \\
&= q + \vec{w}\left(\frac{\dfrac{\vec{n}\cdot\vec{u}}{\|\vec{n}\|}}{\dfrac{\vec{n}\cdot\vec{w}}{\|\vec{n}\|}}\right) \\
&= q + \frac{\vec{n}\cdot\vec{u}}{\vec{n}\cdot\vec{w}}\vec{w}
\end{aligned}
$$



Why Bother?

- Example of the use of dot products
  - especially their geometric meaning
  - geometry is easier than algebra
  - because we can see it
- In the next assignment, we will use this stuff