Chapter 27 : Another appeal to monotonicity.

In which we once again use that nice property.

We are given f[0..M), g[0..N) of int. We are told that f is ascending and g is descending. We are asked to construct a program to compute the number of pairs f.i and g.j whose sum exceeds 37.

$$r = \langle + i,j : 0 \le i < M \wedge 0 \le j < N : h.(f.i).(g.j) \rangle$$

where

* (0) h.x.y = 1 ⇐ $x + y > 37$
* (1) h.x.y = 0 ⇐ $x + y \le 37$

We begin by modelling our domain.

* (2) C.m.n = $\langle + i, j : m \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle$

Emptying the components of the range in turn lead us to the following

- (3) C.M.n = 0 , $0 \le n \le N$

- (4) C.m.N = 0 , $0 \le m \le M$

We observe,

C.m.n
= {(2)}
$\langle + i,j : m \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle$
= { Split off i = m term }
$\langle + i,j : m +1 \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle + \langle + j : n \le j < N : h.(f.m).(g.j) \rangle$
= {(2), (7)}
C.(m+1).n + D.n

- (5) C.m.n = C.(m+1).n + D.n , $0 \le m < M$

We observe,

$$
\begin{aligned}
& C.m.n \\
= \quad & \{(2)\} \\
& \langle + i, j : m \le i < M \wedge n \le j < N : h.(f.i).(g.j) \rangle \\
= \quad & \{ \text{Split off } j = n \text{ term} \} \\
& \langle + i, j : m \le i < M \wedge n+1 \le j < N : h.(f.i).(g.j) \rangle + \langle + i : m \le i < M : h.(f.i).(g.n) \rangle \\
= \quad & \{(2), (8)\} \\
& C.m.(n+1) + E.m
\end{aligned}
$$

- (6) $C.m.n \quad = \quad C.m.(n+1) + E.m \qquad , 0 \le n < N$

* (7) $D.n \quad = \quad \langle + j : n \le j < N : h.(f.m).(g.j) \rangle$

-*(8) $E.m \quad = \quad \langle + i : m \le i < M : h.(f.i).(g.n) \rangle$

We now turn our attention to investigating D and E

- (9) $D.n \quad = \quad ? \qquad \Leftarrow f.m + g.n > 37$

- (10) $D.n \quad = \quad 0 \qquad \Leftarrow f.m + g.n \le 37$

- (11) $E.m \quad = \quad M-m \qquad \Leftarrow f.m + g.n > 37$

- (12) $E.m \quad = \quad ? \qquad \Leftarrow f.m + g.n \le 37$

Our postcondition can now be written as

Post : $r = C.0.0$

Invariants.
As invariants we choose

P0 : $r + C.m.n = C.0.0$
P1 : $0 \le m \le M \wedge 0 \le n \le N$

Establish Invariants.

$$R, m, n := 0, 0, 0$$

Upon termination

$$P0 \wedge P1 \wedge (m=M \vee n=N) \Rightarrow Post$$

Guard.

$$m \neq M \wedge n \neq N$$

Loop body

$P0$
$= \quad$ {definition of $P0$}
$r + C.m.n = C.0.0$
$= \quad$ {(5)}
$r + C.(m+1).n + D.n = C.0.0$
$= \quad$ {case analysis $f.m + g.n \leq 37$ (10)}
$r + C.(m+1).n + 0 = C.0.0$
$= \quad$ {WP}
$(r, m := r+0, m+1).P0$

Giving us

$$\text{if } f.m + g.n \leq 37 \rightarrow r, m := r+0 , m+1$$

We also observe

$P0$
$= \quad$ {definition of $P0$}
$r + C.m.n = C.0.0$
$= \quad$ {(11)}
$r + C.m.(n+1) + E.m = C.0.0$
$= \quad$ {case analysis $f.m + g.n > 37$ (11)}
$r + C.m.(n+1) + M-m = C.0.0$
$= \quad$ {WP}
$(r, n := r +(M-m), n+1).P0$

Giving us

$$\text{if } f.m + g.n > 37 \rightarrow r, n := r +(M-m), n+1$$

There is no point in appealing to (12) so we ignore that case

Finished Algorithm.

```
r, n, m := 0, 0, 0 {P0 ∧ P1}
; do m≠M ∧ n≠N →    {P0 ∧ P1 ∧ m≠M ∧ n≠N}

                    if f.m + g.n ≤ 37 → r, m := r+0 , m+1
                    []f.m + g.n > 37 → r,  n := r +(M-m),  n+1
                    fi

                     {P0 ∧ P1}
od
{P0 ∧ P1 ∧ (m=M ∨ n=N)}
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