The Fourier Series

Derivation of the Fourier Series coefficients of a triangle wave

(From : J From Aditya Ramamoorthy, Review Notes, "Signals and Systems I" course, Iowa State University)

Fourier Series

For a given periodic signal $x(t)$ with period T_0 it is possible to write $x(t)$ as a linear combination of complex exponentials. The coefficients can be computed as.

$$\boxed{a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j\omega_0 k t}\, dt} \quad \text{— (1) (ANALYIS)}$$

and $x(t)$ can be reconstructed as

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j\omega_0 k t}} \quad \text{— (2) (SYNTHESIS)}$$

where $\omega_0 = \frac{1}{T_0} \times 2\pi$.

Thus, the computation of Fourier series requires the computation of an integral where the integrand is the product of two functions. We often need to either find or simplify this integral by means of integration by parts.

Let us review the integration by parts rule.
Consider two functions $f(t)$ and $g(t)$, then by the differentiation rule for a product of functions, we have.

$$\frac{d}{dt}\left(f(t)\,g(t)\right) = f'(t)\,g(t) + f(t)\,g'(t)$$

On integrating both sides we have,

$$f(t)g(t) = \int f'(t)g(t)\,dt + \int f(t)g'(t)\,dt$$

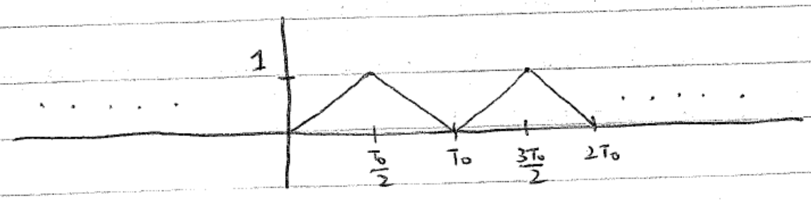
$$\Rightarrow \int f(t)g'(t)\,dt = f(t)g(t) - \int f'(t)g(t)\,dt$$

Thus, to integrate the product of two functions we should first identify an appropriate $f(t)$ and $g(t)$ such that the integrand can be expressed as $f(t)\,g'(t)$ and then apply the above rule. In the case when we have integrals with limits we can express the above rule as

$$\boxed{\int_a^b f(t)g'(t)\,dt = \left[f(t)g(t)\right]_a^b - \int_a^b f'(t)g(t)\,dt}$$

You can refer to any textbook on calculus to see more material on integration by parts.

We shall now demonstrate the computation of Fourier series for a periodic triangle wave drawn below,



that extends indefinitely in both directions. It has period T_0.

Let us define the signal in mathematical terms first. Let us call the signal between $[0, T_0]$, $f(t)$. Then,

$$f(t) = \begin{cases} \frac{2}{T_0} t & t \in [0, \frac{T_0}{2}) \\ \frac{2}{T_0}(T_0 - t) & , t \in [\frac{T_0}{2}, T_0) \end{cases}$$

Now

$$a_k = \frac{1}{T_0} \int_0^{T_0} f(t)\, e^{-j\omega_0 k t} dt$$

$$\Rightarrow T_0 a_k = \int_0^{T_0} f(t)\, e^{-j\omega_0 k t} dt$$

$$= \int_0^{T_0/2} \frac{2}{T_0} t\, e^{-j\omega_0 k t} dt + \int_{\frac{T_0}{2}}^{T_0} \frac{2}{T_0}(T_0 - t)\, e^{-j\omega_0 k t} dt$$

$$= I_1 + I_2$$

Let us first compute I_1

$$I_1 = \frac{2}{T_0} \int_0^{T_0/2} t\, e^{-j\omega_0 k t} dt = \frac{2}{T_0}\left[\left. \frac{t\, e^{-j\omega_0 k t}}{-j\omega_0 k} \right|_0^{T_0/2} - \int_0^{T_0/2} \frac{e^{-j\omega_0 k t}}{-j\omega_0 k} dt \right]$$

$$= \frac{2}{T_0}\left[\frac{T_0}{2} \cdot \frac{e^{-j\omega_0 k T_0/2}}{-j\omega_0 k} + \frac{1}{j\omega_0 k} \cdot \left. \frac{e^{-j\omega_0 k t}}{-j\omega_0 k} \right|_0^{T_0/2} \right]$$

$$= \frac{2}{T_0}\left[\frac{T_0}{2} \cdot \frac{e^{-j\omega_0 k T_0/2}}{-j\omega_0 k} + \frac{1}{\omega_0^2 k^2} \left(e^{-j\omega_0 k T_0/2} - 1 \right) \right]$$

Noting that $\frac{\omega_0 T_0}{2} = \pi$, we obtain

$$I_1 = \frac{e^{-j\pi k}}{-j\omega_0 k} + \frac{2}{T_0 \omega_0^2 k^2} \left(e^{-j\pi k} - 1\right)$$

Now let us find I_2.

$$I_2 = \int_{T_0/2}^{T_0} \frac{2}{T_0} (T_0 - t)\, e^{-j\omega_0 k t}\, dt$$

Observe that using a simple change of variables we can simplify I_2 and leverage our previous computation of I_1.

Let us substitute $\alpha = t - \frac{T_0}{2}$.

Then $d\alpha = dt$ and $t = \alpha + T_0/2$.

Therefore I_2 can be rewritten as,

$$I_2 = \int_0^{T_0/2} \frac{2}{T_0} \left(T_0 - (\alpha + T_0/2)\right) e^{-j\omega_0 k (\alpha + T_0/2)}\, d\alpha$$

$$= \frac{2}{T_0} e^{-j\pi k} \int_0^{T_0/2} \left(\frac{T_0}{2} - \alpha\right) e^{-j\omega_0 k \alpha}\, d\alpha$$

$$= \frac{2}{T_0} e^{-j\pi k} \left[\frac{T_0}{2} \int_0^{T_0/2} e^{-j\omega_0 k \alpha}\, d\alpha - \int_0^{T_0/2} \alpha\, e^{-j\omega_0 k \alpha}\, d\alpha \right]$$

$$= \frac{2}{T_0} e^{-j\pi k}\left[\frac{T_0}{2} \cdot \frac{e^{-j\omega_0 k\alpha}}{-j\omega_0 k}\Bigg|_0^{\frac{T_0}{2}} - \int_0^{\frac{T_0}{2}} \alpha\, e^{-j\omega_0 k\alpha}\, d\alpha \right]$$

$$= \frac{e^{-j\pi k}}{-j\omega_0 k}\left(e^{-j\omega_0 k \frac{T_0}{2}} - 1\right) - e^{-j\pi k} I_1$$

since $\frac{2}{T_0}\int_0^{\frac{T_0}{2}} \alpha\, e^{-j\omega_0 k\alpha}\, d\alpha = I_1$

$$\therefore\ I_2 = \frac{e^{-j\pi k}}{-j\omega_0 k}\left(e^{-j\pi k} - 1\right) - e^{-j\pi k} I_1$$

Now

$$T_0 a_k = I_1 + I_2$$

$$= I_1 + \frac{e^{-j\pi k}\left(e^{-j\pi k} - 1\right)}{-j\omega_0 k} - e^{-j\pi k} I_1$$

$$= \left(1 - e^{-j\pi k}\right) I_1 + e^{-j\pi k}\frac{\left(e^{-j\pi k} - 1\right)}{-j\omega_0 k}$$

Substituting for I_1, we have

$$T_0 a_k = \left[\frac{e^{-j\pi k}}{-j\omega_0 k} + \frac{2}{T_0\omega_0^2 k^2}\left(e^{-j\pi k} - 1\right)\right]\left(1 - e^{-j\pi k}\right) + \frac{e^{-j\pi k}\left(e^{-j\pi k} - 1\right)}{-j\omega_0 k}$$

$$= \cancel{\frac{e^{-j\pi k}}{-j\omega_0 k}\left(1 - e^{-j\pi k}\right)} + \frac{2}{T_0\omega_0^2 k^2}\left(e^{-j\pi k} - 1\right)^2 \times (-1) + \cancel{\frac{e^{-j\pi k}\left(e^{-j\pi k} - 1\right)}{-j\omega_0 k}}$$

$$= \frac{-2}{T_0\omega_0^2 k^2}\left(e^{-j2\pi k} + 1 - 2e^{-j\pi k}\right)$$

Noting that $e^{-j2\pi k} = 1$ and $e^{-j\pi k} = (-1)^k$, we have

$$a_k = \frac{-4}{T_0^2 \omega_0^2 k^2} \left(1 - (-1)^k \right)$$

Substituting for ω_0, we obtain

$$a_k = \frac{4\left((-1)^k - 1\right)}{T_0^2 \times \frac{4\pi^2}{T_0^2} k^2} = \frac{(-1)^k - 1}{\pi^2 k^2}$$

It remains to determine the value of a_0. Note that a substitution of $k=0$ in the above expression is not correct as it yields a mathematically undefined $0/0$ form.
In general a_0 always needs to be computed separately.
We have,

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t)\, e^{-j\omega_0 \times 0 \times t}\, dt$$

$$= \frac{1}{T_0} \int_0^{T_0} f(t)\, dt$$

$$= \frac{1}{T_0} \times \frac{1}{2} \cdot T_0 \times 1 \quad \text{(using the formula for the area of a triangle)}$$

$$= 1/2$$

We can now compactly express the result as:

$$a_k = \begin{cases} 0 & \text{if } k = \pm 2, \pm 4 \ldots \\ \frac{-2}{\pi^2 k^2} & \text{if } k = \pm 1, \pm 3 \ldots \\ \frac{1}{2} & \text{if } k = 0 \end{cases}$$

Note that the rate of decay of a_k with k is $\sim 1/k^2$. Thus, intuitively coefficients with high values of k are not likely to contribute significantly to the "synthesis" sum, i.e. we expect a few terms in the sum to result in a good reconstruction of the original signal.

The behavior of a_k for the periodic triangle wave should be compared with the behavior of a_k for the periodic square wave where the rate of decay is $\sim 1/k$. This should provide you an ~~an~~ insight that in the reconstruction of the periodic square wave you would probably need to sum more terms to have the same quality of reconstruction.