Digital Filters & Spectral Analysis

Lecture 2

The Fourier Series

Derivation of the Fourier Series forward transform (analysis function)

(From : J. Proakis and D. Manolakis, 'Digital Signal Processing: Principles, Algorithms and Applications', Macmillan)

From Chapter 1 we recall that a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \tag{4.1.1}$$

is a periodic signal with fundamental period $T_p = 1/F_0$. Hence we can think of the exponential signals

$$\{e^{j2\pi k F_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots\}$$

as the basic "building blocks" from which we can construct periodic signals of various types by proper choice of the fundamental frequency and the coefficients $\{c_k\}$. F_0 determines the fundamental period of $x(t)$ and the coefficients $\{c_k\}$ specify the shape of the waveform.

Suppose that we are given a periodic signal $x(t)$ with period T_p. We can represent the periodic signal by the series (4.1.1), called a *Fourier series*, where the fundamental frequency F_0 is selected to be the reciprocal of the given period T_p. To determine the expression for the coefficients $\{c_k\}$, we first multiply both sides of (4.1.1) by the complex exponential

$$e^{-j2\pi F_0 l t}$$

where l is an integer and then integrate both sides of the resulting equation over a single period, say from 0 to T_p, or more generally, from t_0 to $t_0 + T_p$, where t_0 is an arbitrary but mathematically convenient starting value. Thus we obtain

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi l F_0 t}\, dt = \int_{t_0}^{t_0+T_p} e^{-j2\pi l F_0 t} \left(\sum_{k=-\infty}^{\infty} c_k e^{+j2\pi k F_0 t} \right) dt \tag{4.1.2}$$

To evaluate the integral on the right-hand side of (4.1.2), we interchange the order of the summation and integration and combine the two exponentials. Hence

$$\sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_p} e^{j2\pi F_0 (k-l) t}\, dt = \sum_{k=-\infty}^{\infty} c_k \left[\frac{e^{j2\pi F_0 (k-l) t}}{j2\pi F_0 (k-l)} \right]_{t_0}^{t_0+T_p} \tag{4.1.3}$$

For $k \neq l$, the right-hand side of (4.1.3) evaluated at the lower and upper limits, t_0 and $t_0 + T_p$, respectively, yields zero. On the other hand, if $k = l$, we have

$$\int_{t_0}^{t_0+T_p} dt = t \Big|_{t_0}^{t_0+T_p} = T_p$$

Consequently, (4.1.2) reduces to

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi l F_0 t}\, dt = c_l T_p$$

and therefore the expression for the Fourier coefficients in terms of the given periodic signal becomes

$$c_l = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x(t)e^{-j2\pi l F_0 t}\, dt$$

Since t_0 is arbitrary, this integral can be evaluated over any interval of length T_p, that is, over any interval equal to the period of the signal $x(t)$. Consequently, the integral for the Fourier series coefficients will be written as

$$c_l = \frac{1}{T_p} \int_{T_p} x(t)e^{-j2\pi l F_0 t}\, dt \tag{4.1.4}$$