Digital Filters & Spectral Analysis
Lecture 5

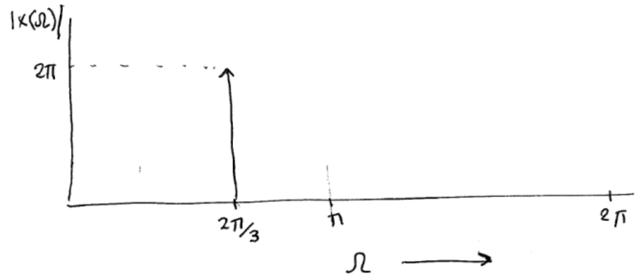
DTFT

Problem sheet - Solutions

1. The analogue signal $x(t) = \cos(2\pi f_0 t) - j\sin(2\pi f_0 t)$ comprising a cosine with frequency f_0 = 200*Hz* is sampled at a frequency $f_s = 300Hz$ to give a discrete time signal $x[n] = x(n/f_s)$. Sketch, over the range $0 \le \Omega < 2\pi$ the magnitude $|X(\Omega)|$ of the DTFT of the discrete time signal.

$x(t) = e^{-j2\pi f_0 t}$

$$\Omega_0 = 2\pi \frac{f_0}{f_s} = -\frac{4\pi}{3} \; or \; 2\pi - \frac{4\pi}{3} = \frac{2\pi}{3}$$



2. Find the DTFT for the following signal and sketch its magnitude and phase response:

$$x[n]=a^{|n|} \quad 0<a<1$$

$$X(\Omega)=\sum_{n=-\infty}^{\infty}a^{n}u[n]e^{-j\Omega n}=\sum_{n=0}^{\infty}a^{n}e^{-j\Omega n}$$

This is a geometric series with $|ae^{-j\Omega}|<1$

The sum of an infinite geometric series converges to

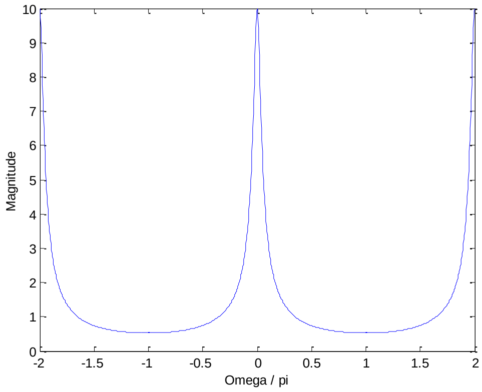
$S=\sum_{i=0}^{\infty}\alpha^{i}=\frac{1-\alpha^{\infty}}{1-\alpha}=\frac{1}{1-\alpha}$ providing that $|\alpha|<1$

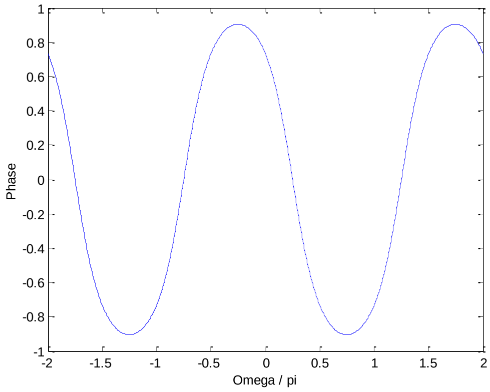
$$\Rightarrow X(\Omega)=\frac{1}{1-ae^{-j\Omega}}=\frac{1}{1-a\cos\Omega+ja\sin\Omega}$$

$$|X(\Omega)|=\frac{1}{\sqrt{(1-a\cos\Omega)^{2}+(a\sin\Omega)^{2}}}=\frac{1}{\sqrt{1-2a\cos\Omega+a^{2}\cos^{2}\Omega+a^{2}\sin^{2}\Omega}}=\frac{1}{\sqrt{1+a^{2}-2a\cos\Omega}}$$

$$\angle X(\Omega)=-\tan^{-1}\left(\frac{a\sin\Omega}{1-a\cos\Omega}\right)$$

The magnitude response has peaks at $\Omega=k2\pi$ of magnitude $|X(k2\pi)|=\frac{1}{(1-a)}$ and minima at $\Omega=k2\pi+\pi$ with magnitude $|X(k2\pi+\pi)|=\frac{1}{(1+a)}$





3. Find an expression for the DTFT of a discrete time signal:

$$x[n] = \begin{cases} 1 & 0 \le n < N_1 \\ -1 & N_1 \le n < 2N_1 \\ 0 & elsewhere \end{cases}$$

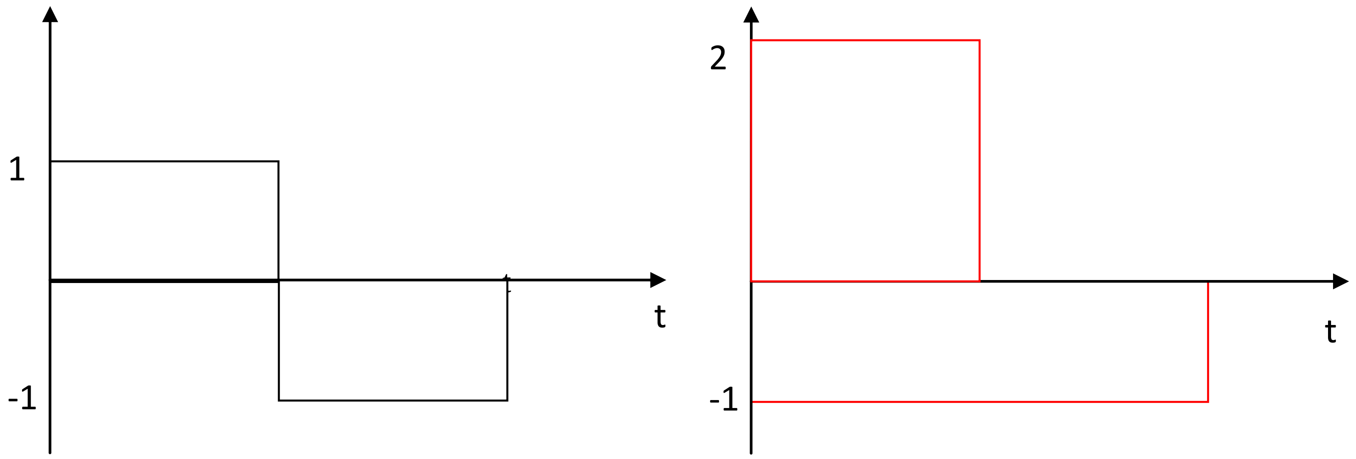
You may use the knowledge that:

$$x[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & elsewhere \end{cases} \Rightarrow X(\Omega) = e^{-j\Omega(N-1)/2} \frac{\sin(\Omega N/2)}{\sin(\Omega/2)}$$

We need to use the given result i.e. express the new signal in terms of a single square pulse the DTFT of which we know

$$x_1[n] = \begin{cases} 2 & 0 \le n < N_1 \\ 0 & \text{elsewhere} \end{cases}, \quad x_2[n] = \begin{cases} -1 & 0 \le n < 2N_1 \\ 0 & \text{elsewhere} \end{cases}$$

$$x_n = x_1[n] + x_2[n] \Rightarrow DTFT\{x[n]\} = DTFT\{x_1[n]\} + DTFT\{x_2[n]\}$$



$$X(\Omega) = 2e^{-j\Omega(N_1-1)/2} \frac{\sin(\Omega N_1/2)}{\sin(\Omega/2)} - e^{-j\Omega(2N_1-1)/2} \frac{\sin(\Omega N_1)}{\sin(\Omega/2)}$$

Find the DTFT for the following signal and sketch its magnitude and phase response: $x[n] = a^{|n|} \quad 0 < a < 1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} + \sum_{n=0}^{\infty} a^n e^{j\Omega n} - 1$$

Both summations are geometric series thus:

$$X(\Omega)=\frac{1}{1-ae^{-j\Omega}}+\frac{1}{1-ae^{j\Omega}}-1=\frac{\left(1-ae^{j\Omega}\right)+\left(1-ae^{-j\Omega}\right)-\left(1-ae^{-j\Omega}-ae^{j\Omega}+a^2\right)}{1-ae^{-j\Omega}-ae^{j\Omega}+a^2}=\frac{1-a^2}{1-2a\cos\Omega+a^2}$$

Thus $X(\Omega)$ is real with a peak at $\Omega=0$ of height: $X(0)=\dfrac{1+a}{1-a}$