Digital Filters & Spectral Analysis
Lecture 8

Spectral Smearing

Problem sheet

Solutions

1. An audio signal sampled at 44.1KHz is to be analysed with the DFT using a rectangular window.
   a. Estimate the minimum length of DFT which could be used in order to be able to distinguish between different frequency components separated by at least 10 Hz.
   b. How would your answer change if a smoother window function (e.g. Hamming) was used?

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a) Frequency samples are spaced at f_s/N: $\frac{f_s}{N} \leq 10\text{Hz} => N \geq \frac{f_s}{10} = 4.41\text{KHz}$

b) A smoother window will have a wider main lobe in the frequency domain than the rectangular window. This means we will need a larger value of N

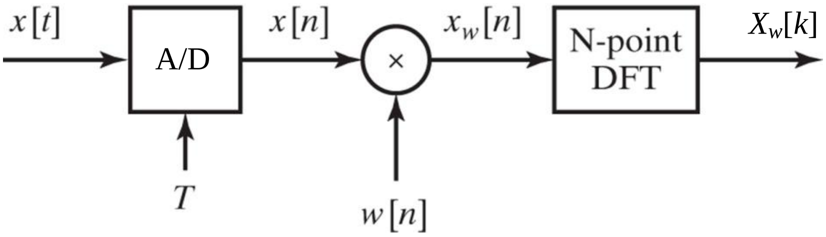
2. A signal $x[n] = a_1 \cos(n\Omega_1) + a_2 \cos(n\Omega_2)$ is analysed using an N point DFT. Under what circumstances will the two cosine functions be clearly distinguishable in the frequency domain?

- Ω_1 and Ω_2 must be separated by a frequency of at least $2\pi / N$ in order to appear in different frequency bins.
- The relative magnitudes a_1 and a_2 must be of the same order of magnitude in order to avoid the smaller component being lost from the power leakage of the larger larger component.

3. Consider the system shown in the figure below. The input signal $x(t) = e^{-j\left(\frac{3\pi}{8}\right)10^4 t}$, is sampled with a period $T = 10^{-4}$ and windowed with

$$w[n] = \begin{cases} 1, 0 \leq nN - 1 \\ 0, \ otherwise \end{cases}$$

What is the smallest nonzero value of *N* such that $X_w[k]$ is non-zero at exactly one value of *k*?



(a) Sampling the continuous-time input signal

$$x(t) = e^{j(3\pi/8)10^4 t}$$

with a sampling period $T = 10^{-4}$ yields a discrete-time signal

$$x[n] = x(nT) = e^{j3\pi n/8}$$

In order for $X_w[k]$ to be nonzero at exactly one value of k, it is necessary for the frequency of the complex exponential of $x[n]$ to correspond to that of a DFT coefficent, $w_k = 2\pi k/N$. Thus,

$$\begin{aligned} \frac{3\pi}{8} &= \frac{2\pi k}{N} \\ N &= \frac{16k}{3} \end{aligned}$$

The smallest value of k for which N is an integer is $k = 3$. Thus, the smallest value of N such that $X_w[k]$ is nonzero at exactly one value of k is

$$N = 16$$

4. We want to estimate the spectrum of a discrete-time signal *x*[*n*] using the DFT with a Hamming window applied to *x*[*n*]. We wish to be able to resolve sinusoidal signals that are separated by as little as π/100 in frequency. The window length *L* is constrained to be a power of 2. What is the minimum length $L=2^v$ that will meet our resolution requirement?

The Hamming window's mainlobe is $\Delta\omega_{ml} = \frac{8\pi}{L-1}$ radians wide. We want

$$\begin{aligned} \Delta\omega_{ml} &\leq \frac{\pi}{100} \\ \frac{8\pi}{L-1} &\leq \frac{\pi}{100} \\ L &\geq 801 \end{aligned}$$

Because the window length is constrained to be a power of 2, we see that

$$L_{\min} = 1024$$