

Digital Filters & Spectral Analysis

Lecture 3

On Symmetry

Symmetrical signals, Symmetry of Fourier Transform, Hermitian Symmetry

(From : Lance R. Williams, "Digital Image Processing" Course Notes, University of New Mexico)

Even and Odd Functions

A function, f , is *even* (or *symmetric*) when

$$f(x) = f(-x).$$

A function, f , is *odd* (or *antisymmetric*) when

$$f(x) = -f(-x).$$

Even and Odd Functions (contd.)

Theorem 5.1 Any function can be written as a sum of even and odd functions.

$$\begin{aligned} f(t) &= \frac{1}{2} \left[f(t) + \underbrace{f(-t) - f(-t)}_0 + f(t) \right] \\ &= \underbrace{\frac{1}{2} [f(t) + f(-t)]}_{f_e} + \underbrace{\frac{1}{2} [f(t) - f(-t)]}_{f_o} \end{aligned}$$

f_e is even because $f_e(t) = f_e(-t)$:

$$\begin{aligned} f_e(t) &= f(t) + f(-t) \\ &= f(-t) + f(t) \\ &= f_e(-t). \end{aligned}$$

f_o is odd because $f_o(t) = -f_o(-t)$:

$$\begin{aligned} f_o(t) &= f(t) - f(-t) \\ &= -[f(-t) - f(t)] \\ &= -f_o(-t). \end{aligned}$$

Even and Odd Functions (contd.)

Theorem 5.2 The integral of the product of odd and even functions is zero.

$$\int_{-\infty}^{\infty} f_e(x)f_o(x)dx = \int_{-\infty}^0 f_e(x)f_o(x)dx + \int_0^{\infty} f_e(x)f_o(x)dx.$$

Substituting $-x$ for x and $-dx$ for dx in the first term yields:

$$\begin{aligned} & \int_{\infty}^0 -f_e(-x)f_o(-x)dx + \int_0^{\infty} f_e(x)f_o(x)dx \\ &= \int_0^{\infty} f_e(-x)f_o(-x)dx + \int_0^{\infty} f_e(x)f_o(x)dx \\ &= \int_0^{\infty} [f_e(-x)f_o(-x) + f_e(x)f_o(x)] dx. \end{aligned}$$

Substituting $f_e(-x)$ for $f_e(x)$ and $-f_o(-x)$ for $f_o(x)$ yields:

$$\int_0^{\infty} \underbrace{[f_e(-x)f_o(-x) - f_e(-x)f_o(-x)]}_0 dx.$$

Fourier Transform Symmetry

The Fourier transform of $f(t)$ is defined to be:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt.$$

This can be rewritten as follows:

$$F(s) = \int_{-\infty}^{\infty} f(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} f(t) \sin(2\pi st) dt.$$

Substituting $f_e(t) + f_o(t)$ for $f(t)$ yields:

$$F(s) = \int_{-\infty}^{\infty} f_e(t) \cos(2\pi st) dt + \int_{-\infty}^{\infty} f_o(t) \cos(2\pi st) dt - \\ j \int_{-\infty}^{\infty} f_e(t) \sin(2\pi st) dt - j \int_{-\infty}^{\infty} f_o(t) \sin(2\pi st) dt.$$

Fourier Transform Symmetry (contd.)

However, the second and third terms are zero (Theorem 5.2):

$$F(s) = \int_{-\infty}^{\infty} f_e(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} f_o(t) \sin(2\pi st) dt.$$

It follows that:

$$F(s) = F_e(s) + F_o(s).$$

Even Functions

Theorem 5.3 The Fourier transform of a real even function is real.

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \\ &= \int_{-\infty}^{\infty} f(t) [\cos(2\pi st) - j \sin(2\pi st)] dt \\ &= \int_{-\infty}^{\infty} f(t) \cos(2\pi st) dt \end{aligned}$$

which is real.

Odd Functions

Theorem 5.4 The Fourier transform of a real odd function is imaginary.

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \\ &= \int_{-\infty}^{\infty} f(t) [\cos(2\pi st) - j \sin(2\pi st)] dt \\ &= j \int_{-\infty}^{\infty} f(t) \sin(2\pi st) dt \end{aligned}$$

which is imaginary.

Even Functions (contd.)

Theorem 5.5 The Fourier transform of an even function is even.

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st} dt$$

Substituting $f(-t)$ for $f(t)$ yields:

$$F(s) = \int_{t=-\infty}^{t=\infty} f(-t)e^{-j2\pi st} dt.$$

Substituting u for $-t$ and $-du$ for dt yields:

$$\begin{aligned} &= \int_{u=\infty}^{u=-\infty} -f(u)e^{-j2\pi s(-u)} du \\ &= \int_{-\infty}^{\infty} f(u)e^{-j2\pi(-s)u} du \\ &= F(-s). \end{aligned}$$

Odd Functions (contd.)

Theorem 5.6 The Fourier transform of an odd function is odd.

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st} dt$$

Substituting $-f(-t)$ for $f(t)$ yields:

$$F(s) = \int_{t=-\infty}^{t=\infty} -f(-t)e^{-j2\pi st} dt.$$

Substituting u for $-t$ and $-du$ for dt yields:

$$\begin{aligned} &= \int_{u=\infty}^{u=-\infty} f(u)e^{-j2\pi s(-u)} du \\ &= \int_{-\infty}^{\infty} -f(u)e^{-j2\pi(-s)u} du \\ &= -F(-s). \end{aligned}$$

Fourier Transform Symmetry (contd.)

- The Fourier transform of the even part (of a real function) is real (Theorem 5.3):

$$\mathcal{F}\{f_e\}(s) = F_e(s) = \operatorname{Re}(F_e(s)).$$

- The Fourier transform of the even part is even (Theorem 5.5):

$$\mathcal{F}\{f_e\}(s) = F_e(s) = F_e(-s).$$

- The Fourier transform of the odd part (of a real function) is imaginary (Theorem 5.4):

$$\mathcal{F}\{f_o\}(s) = F_o(s) = \operatorname{Im}(F_o(s)).$$

- The Fourier transform of the odd part is odd (Theorem 5.6):

$$\mathcal{F}\{f_o\}(s) = F_o(s) = -F_o(-s).$$

Hermitian Symmetry

We can summarize all four symmetries possessed by the Fourier transform of a real function as follows:

$$\begin{aligned} F(s) &= F_e(s) + F_o(s) \\ &= F_e(-s) - F_o(-s) \\ &= \overline{F_e(-s) + F_o(-s)} \\ &= \overline{F(-s)}. \end{aligned}$$

Hermitian Symmetry (contd.)

This symmetry matches the symmetry of the functions which comprise the Fourier basis:

$$e^{j2\pi st} = \overline{e^{j2\pi s(-t)}}.$$