Digital Filters & Spectral Analysis

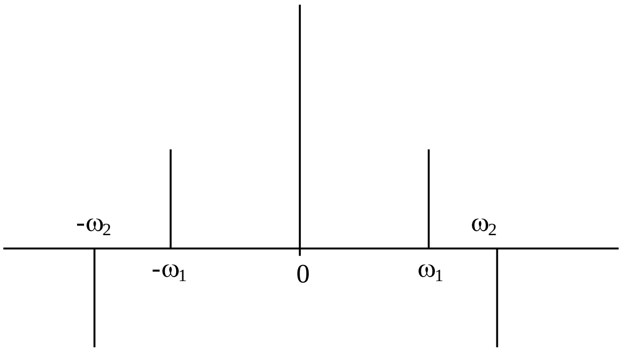
Lecture 4

Sampling

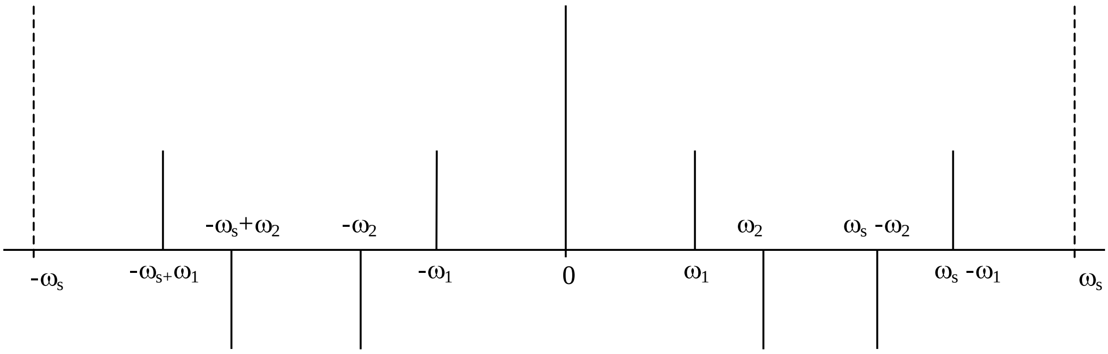
Problem sheet - Solutions

1. Consider the signal : $x(t) = \cos(\omega_1 t) - \cos(\omega_2 t)$ where $\omega_2 > \omega_1$
   a. Sketch the spectrum $X(\omega)$ of the signal
   b. The signal is sampled with a sampling period T_s resulting in the discrete time signal: $x[n] = x(nT_s)$. Sketch the spectrum for the sampled signal, assuming that the sampling frequency $\omega_s = 2\pi / T >> \omega_2$.
   c. What is the minimum sampling frequency $\omega_s = 2\pi / T$ required to avoid aliasing?
   d. Sketch the spectrum obtained when the sampling frequency is just below this value.
   e. Show that at a certain sampling frequency the sampled signal $x[n]$ will be 0.

1a. Original spectrum before sampling:

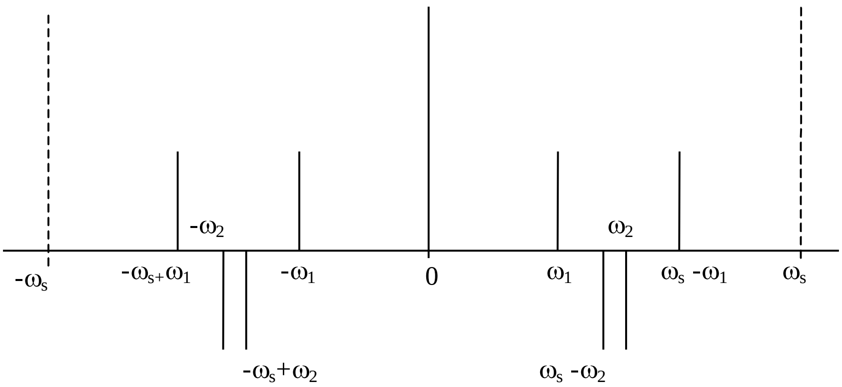


1b. For $\omega_s = 2\pi / T >> \omega_2$ the spectrum of the sampled signal looks like this:



1c. To avoid aliasing we require $\omega_s > 2\omega_2$.

1d. For $\omega_s < 2\omega_2$ (aliasing) the spectrum looks like this:



1d. If $\omega_s - \omega_2 = \omega_1$ then the aliased component of ω_2 will cancel the ω_1 component.

2. The signal $x_c(t) = \sin(2\pi 100t)$ was sampled with sampling period *T*=1/400 second to obtain a discrete-time signal *x[n].* What is the resulting sequence *x[n]*?

$$x[n] = x_c(nT) = \sin(2\pi(100)n\frac{1}{400}) = \sin(\frac{\pi}{2}n)$$

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3. The sequence $x[n] = \cos(\frac{\pi}{4}n)$ was obtained by sampling the continuous-time signal $x_c(t) = \cos(\omega_0 t)$ at a sampling rate of 1000 samples/sec. What are two possible positive values of ω_0 that could have resulted in the sequence *x[n]*

$$x[n] = \cos(\Omega_0 n) = \cos(\frac{\pi}{4}n) \Rightarrow \Omega_0 = \frac{\pi}{4}$$

$$\Omega_0 = \omega_0 T \Rightarrow \omega_0 = \frac{\Omega_0}{T} = \frac{\pi}{4} \times 1000 = 250\pi \ rads/\sec$$

$$or\ possibly \quad \omega_0 = (2\pi + \frac{\pi}{4}) \times 1000 = 2250\pi\, rads/\sec$$

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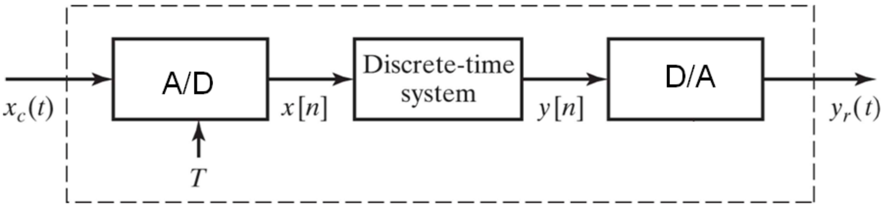
4. The continuous-time signal $x_c(t) = \cos(4000\pi t)$ is sampled with a sampling period T to obtain the discrete-time signal $x[n] = \cos(\frac{\pi n}{3})$
   a. Determine a choice of T consistent with this information
   b. Is your choice for T in part (a) unique? If so explain why. If not specify another choice of T consistent with the information given.

4a. $\Omega_0 = \omega_0 T \Rightarrow \frac{\pi}{3} = 4000\pi \times T \Rightarrow T = \frac{\pi}{12000\pi} = \frac{1}{12000}$ sec

4b. No. For example : $x[n] = \cos(\frac{\pi n}{3}) = \cos((\frac{\pi}{3} + 2\pi)n) = \cos(\frac{7\pi n}{3})$ in which case $T = \frac{7}{12000}$

5. Consider the system shown below with the discrete time system being an ideal low pass filter with cut-off frequency π/8 rads/sec

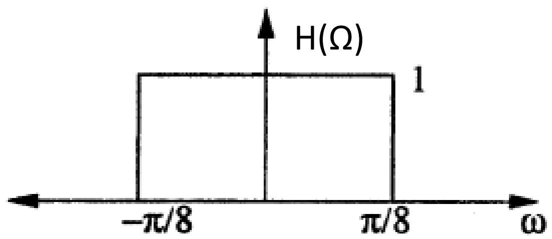


a. If $x_c(t)$ is band-limited to 5kHz what is the maximum value of *T* that will avoid aliasing in the A/D converter?

b. If 1/T = 10kHz what will the cut-off frequency of the effective continuous-time filter be?

c. Repeat part b for 1/T = 20kHz

Frequency response of ideal low pass filter with cut-off of π/8



5a. Nyquist rate is 2 x f_{max} => $\frac{1}{T} = 10000 \Rightarrow$ maximum period $T = \frac{1}{10000}$ sec

5b. $\Omega_c = \omega_c T \Rightarrow \omega_c = \frac{\Omega_c}{T} = 10000\frac{\pi}{8} = 1250\pi = 2\pi 625\, rad/\sec\ (2\pi f_c = 2\pi 625 \Rightarrow f_c = 625 Hz)$

5c. $\Omega_c = \omega_c T \Rightarrow \omega_c = \frac{\Omega_c}{T} = 20000\frac{\pi}{8} = 2500\pi = 2\pi 1250\, rad/\sec\ (2\pi f_c = 2\pi 1250 \Rightarrow f_c = 1250 Hz)$

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6. A continuous time signal $x_a(t)$ is composed of a linear combination of sinusoidal signals of frequencies 250 Hz, 450 Hz, 1.0 k Hz, 2.75 k Hz and 4.05 kHz. The signal $x_a(t)$ is sampled at a 1.5 kHz rate and the sampled sequence is passed through an ideal low pass filter with a cut-off frequency of 750Hz, generating a continuous time signal $y_a(t)$. What are the frequency components present in $y_a(t)$?

$X_c(t)$ consists of sinusoids at frequencies f_i and its spectrum $X_c(f)$ is given by:

$$X_c(f)=\frac{1}{2}\sum_{i=0}^{5}\delta(f-f_i)+\delta(f+f_i)$$, where f_0=250Hz, f_1=450Hz, f_2=1000Hz,….

The spectrum of the sampled signal $X_s(f)$ will contain multiple copies of the spectrum of the continuous-time signal $X_c(f)$ centred at ± f_s. f_s =1500 Hz.

$$X_s(f)=f_s\sum_{k=-\infty}^{\infty}X_c(f-kf_s)=\frac{f_s}{2}\sum_{k=-\infty}^{\infty}\sum_{i=0}^{5}\delta(f-f_i-kf_s)+\delta(f+f_i-kf_s)$$

f_0 (Hz)	k=0		k=-1		k=1		k=-2		k=2		k=-3		k=3	
250	250	-250	-1250	-1750	1750	1250	-2750	-3250	3250	2750	-4250	-4750	4750	4250
450	450	-450	-1050	-1950	1950	1050	-2550	-3450	3450	2550	-4050	-4950	4950	4050
1000	1000	-1000	-500	-2500	2500	500	-2000	-4000	4000	2000	-3500	-5500	5500	3500
2750	2750	-2750	1250	-4250	4250	-1250	-250	-5750	5750	250	-1750	-7250	7250	1750
4050	4050	-4050	2550	-5550	5550	-2550	1050	-7050	7050	-1050	-450	-8550	8550	450