Digital Filters & Spectral Analysis
Lecture 9

FFT

Problem sheet

Solutions

1. A time domain signal, $x(t)=\cos(2\pi ft)$, where f=4Hz, is sampled at 16Hz for 0.25s. Draw the appropriate decimation in time (DIT) FFT flowgraph, showing all complex coefficients and use this to compute the DFT for this signal. Comment on the following:
   a. Is there any smearing in the resulting spectrum? Explain your answer.
   b. What is the complexity of your FFT?

Period of cosine:
T = 1/f = 1/4 = 0.25s

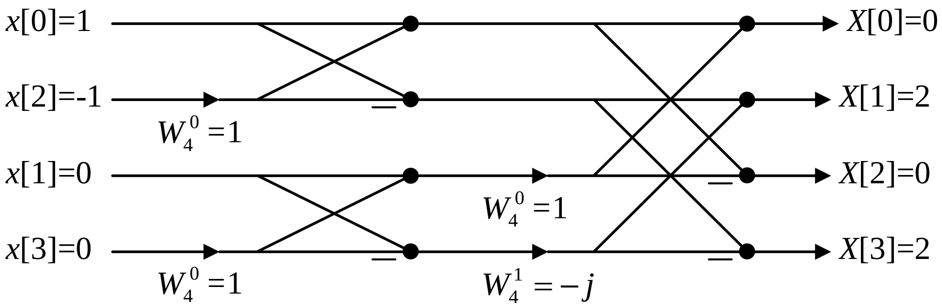
Sampling time of 0.25s equals one period of the cosine function.

Sampling period:
T_s = 1/fs = 1/16 =T/4

Sampling at 16 Hz gives 4 samples at $nT_s : n = 0,1,2,3 \quad T_s = 1/16$. =>

$$x[n] = x(nT_s) = \cos(2\pi f nT_s) = \cos(\frac{\pi}{2}n) = \{1, \quad 0, \quad -1, \quad 0\}$$

We need a N=4 point FFT, which has 2 stages.
Twiddle factor : $W_4 = e^{-2\pi/4} = -j$



$$X[k] = \{0, \quad 2, \quad 0, \quad 2\}$$

Normalised frequency is $\Omega = 2\pi f/f_s$,= $2\pi/4$. We sample the spectrum at k $2\pi/N$ where N =4 here. The frequency of the cosine function coincides with an FFT sample and thus there is no smearing.

The FFT requires 4 trivial multipliers 3 of which are by W^0=1 and the remaining one is by W^1=-j. The FFT also uses 8 complex additions.

2. Show that we can calculate the DFT $G[k]$ of a sequence $g[n]$ of length 2N using an N-point DFT and some additional computation. Use the following properties of the DFT/FFT to derive the solution:
   a. The FFT algorithm is designed to perform complex multiplications and additions
   b. The DFT is a linear operation
   c. Relevant properties of the DFT
   d. The FFT follows a decimation and merge approach

A. Define two real valued sequences $x_1[n]$ and $x_2[n]$ of length N as:
   $x_1[n]=g[2n],\ x_2[n]=g[2n+1],\ n=0,1,..,N-1$

B. Define a complex-valued sequence x[n] as :
   $$x[n]=x_1[n]+jx_2[n] \quad \Rightarrow \quad x_1[n]=\frac{x[n]+x^*[n]}{2}\ ,\ x_2[n]=\frac{x[n]-x^*[n]}{2j}$$

C. Linearity of DFT means that: $X_1[k]=\frac{DFT\{x[n]\}+DFT\{x^*[n]\}}{2},\quad X_2[k]=\frac{DFT\{x[n]\}-DFT\{x^*[n]\}}{2j}$
   $$X_1[k]=\frac{X[k]+X^*[N-k]}{2} \qquad X_2[k]=\frac{X[k]-X^*[N-k]}{2j}$$

D. Express the 2N-point DFT G[k] in terms of the two N-point DFTs $X_1[k]$ and $X_2[k]$ and proceed as in the DIT FFT algorithm:
   $$G[k]=\sum_{n=0}^{N-1}g[2n]w_{2N}^{2nk}+\sum_{n=0}^{N-1}g[2n+1]w_{2N}^{(2n+1)k}=\sum_{n=0}^{N-1}x_1[n]w_N^{nk}+w_{2N}^{k}\sum_{n=0}^{N-1}x_2[n]w_N^{nk}$$

$$\begin{aligned} G[k] &= X_1[k]+w_{2N}^{k}X_2[k] \\ G[k+N] &= X_1[k]-w_{2N}^{k}X_2[k] \end{aligned} \qquad k=0,1,\ldots,N-1$$

2N-DFT for the price of a N-DFT
(plus some additional computation)