Digital Filters & Spectral Analysis
Lecture 3

From the Fourier Series to the Fourier Transform

(From : J. Proakis and D. Manolakis, 'Digital Signal Processing: Principles, Algorithms and Applications', Macmillan)

4.1.3 The Fourier Transform for Continuous-Time Aperiodic Signals

In Section 4.1.1 we developed the Fourier series to represent a periodic signal as a linear combination of harmonically related complex exponentials. As a consequence of the periodicity, we saw that these signals possess line spectra with equidistant lines. The line spacing is equal to the fundamental frequency, which in turn is the inverse of the fundamental period of the signal. We can view the fundamental period as providing the number of lines per unit of frequency (line density), as illustrated in Fig 4.1.6.

With this interpretation in mind, it is apparent that if we allow the period to increase without limit, the line spacing tends toward zero. In the limit, when the period becomes infinite, the signal becomes aperiodic and its spectrum becomes continuous. This argument suggests that the spectrum of an aperiodic signal will be the envelope of the line spectrum in the corresponding periodic signal obtained by repeating the aperiodic signal with some period T_p.

Let us consider an aperiodic signal $x(t)$ with finite duration as shown in Fig 4.1.7(a). From this aperiodic signal, we can create a periodic signal $x_p(t)$ with period T_p, as shown in Fig 4.1.7(b). Clearly, $x_p(t) = x(t)$ in the limit as $T_p \to \infty$, that is,

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

This interpretation implies that we should be able to obtain the spectrum of $x(t)$ from the spectrum of $x_p(t)$ simply by taking the limit as $T_p \to \infty$.

We begin with the Fourier series representation of $x_p(t)$,

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}, \qquad F_0 = \frac{1}{T_p} \tag{4.1.20}$$

where

$$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x_p(t) e^{-j2\pi k F_0 t}\, dt \tag{4.1.21}$$

Since $x_p(t) = x(t)$ for $-T_p/2 \le t \le T_p/2$, (4.1.21) can be expressed as

$$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi k F_0 t}\, dt \tag{4.1.22}$$

It is also true that $x(t) = 0$ for $|t| > T_p/2$. Consequently, the limits on the integral in (4.1.22) can be replaced by $-\infty$ and ∞. Hence

$$c_k = \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) e^{-j2\pi k F_0 t}\, dt \tag{4.1.23}$$

Let us now define a function $X(F)$, called the *Fourier transform* of $x(t)$, as

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t}\, dt \tag{4.1.24}$$

$X(F)$ is a function of the continuous variable F. It does not depend on T_p or F_0. However, if we compare (4.1.23) and (4.1.24), it is clear that the Fourier coefficients c_k can be expressed in terms of $X(F)$ as

$$c_k = \frac{1}{T_p} X(kF_0)$$

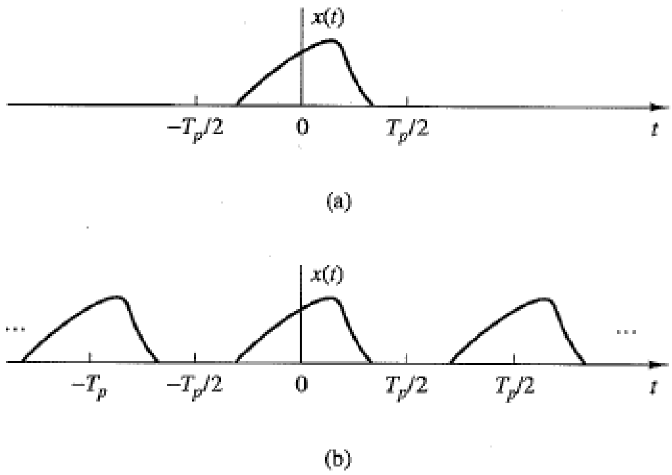

Figure 4.1.7
(a) Aperiodic signal $x(t)$ and (b) periodic signal $x_p(t)$ constructed by repeating $x(t)$ with a period T_p.

or equivalently,

$$T_p c_k = X(kF_0) = X\left(\frac{k}{T_p}\right) \tag{4.1.25}$$

Thus the Fourier coefficients are samples of $X(F)$ taken at multiples of F_0 and scaled by F_0 (multiplied by $1/T_p$). Substitution for c_k from (4.1.25) into (4.1.20) yields

$$x_p(t) = \frac{1}{T_p} \sum_{k=-\infty}^{\infty} X\left(\frac{k}{T_p}\right) e^{j2\pi k F_0 t} \tag{4.1.26}$$

We wish to take the limit of (4.1.26) as T_p approaches infinity. First, we define $\Delta F = 1/T_p$. With this substitution, (4.1.26) becomes

$$x_p(t) = \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k \Delta F t} \Delta F \tag{4.1.27}$$

It is clear that in the limit as T_p approaches infinity, $x_p(t)$ reduces to $x(t)$. Also, ΔF becomes the differential dF and $k\,\Delta F$ becomes the continuous frequency variable F. In turn, the summation in (4.1.27) becomes an integral over the frequency variable F. Thus

$$\begin{aligned} \lim_{T_p \to \infty} x_p(t) = x(t) &= \lim_{\Delta F \to 0} \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{-j2\pi k \Delta F t} \Delta F \\ x(t) &= \int_{-\infty}^{\infty} X(F) e^{j2\pi F t}\, dF \end{aligned} \tag{4.1.28}$$

This integral relationship yields $x(t)$ when $X(F)$ is known, and it is called the *inverse Fourier transform.*

This concludes our heuristic derivation of the Fourier transform pair given by (4.1.24) and (4.1.28) for an aperiodic signal $x(t)$. Although the derivation is not mathematically rigorous, it led to the desired Fourier transform relationships with relatively simple intuitive arguments. In summary, the frequency analysis of continuous-time aperiodic signals involves the following Fourier transform pair.

Frequency Analysis of Continuous-Time Aperiodic Signals

Synthesis equation (inverse transform)	$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi F t}\, dF$	(4.1.
Analysis equation (direct transform)	$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t}\, dt$	(4.1.

It is apparent that the essential difference between the Fourier series and the Fourier transform is that the spectrum in the latter case is continuous and hence the synthesis of an aperiodic signal from its spectrum is accomplished by means of integration instead of summation.