Communication Channels, Mutual Information, COVID test

with MATLAB example

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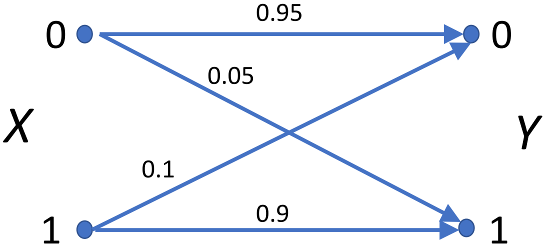
Simple Communication Channel

A simple binary communication channel carries messages by using two signals 0 and 1. We assume that for 40% of the time 1 is transmitted; the probability that a transmitted 0 is received correctly is 0.95; transmitted 1 is received correctly is 0.9.

- Determine: (a) the probability that 0 is received, (b) given that 0 is received, the probability that 0 was transmitted.

i.e.
$P(X = 1) = 0.4$
$P(X = 0) = 0.6$
$P(Y = 0|X = 0) = 0.95$
$P(Y = 1|X = 1) = 0.9$

Simple Communication Channel - the task



$P(X = 1) = 0.4,\ P(X = 0) = 0.6,$
$P(Y = 0|X = 0) = 0.95, P(Y = 1|X = 0) = 0.05$
$P(Y = 1|X = 1) = 0.9, P(Y = 0|X = 1) = 0.1$
but the task really is:
$P(X = 0|Y = 0)$

Bayes Theorem

Let X and Y be two arbitrary RV with $P(x) \neq 0$ and $P(y) \neq 0$. Then:

$$P(x|y) = \frac{P(y|x)\,P(x)}{P(y)}$$

Which can also be expressed as:

$$P(x|y) = \frac{P(y|x)\,P(x)}{\sum_{j=1}^{n} P(y|X = x_j)\,P(X = x_j)}$$

Where:

$P(y) = \sum_{j=1}^{n} P(y, X = x_j) = \sum_{j=1}^{n} P(y|X = x_j)\,P(X = x_j)$.

This result is very useful in evaluating causal relationships between events (an outcome and a cause). We can evaluate *a posteriori* probability $P(x|y)$ in terms of *a priori* probability $P(x)$ and *likelihood* $P(y|x)$.

Bayes Theorem example

- Calculate $P(Y = 0) =?$

$P(Y = 0) = P(Y = 0|X = 1)P(X = 1) + P(Y = 0|X = 0)P(X = 0) = 0.1 \cdot 0.4 + 0.95 \cdot 0.6 = 0.61$

- Calculate $P(X = 0|Y = 0) =?$

From Bayes theorem:
$P(X|Y) = \frac{P(Y=0|X=0)P(X=0)}{P(Y=0)} = \frac{0.95 \cdot 0.6}{0.61} = 0.9344$

The Twist: COVID test

A new coronavirus test is marketed. The test has *Sensitivity* of 98% and *Specificity* of 99%.

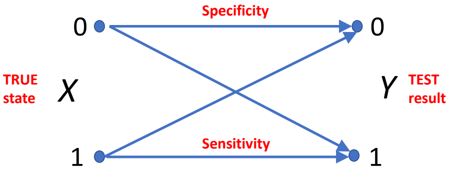
- Is it a good test?
- What is the probability that a person has the virus given the positive test result?

Assume prevalence of the virus in the population of 1%.

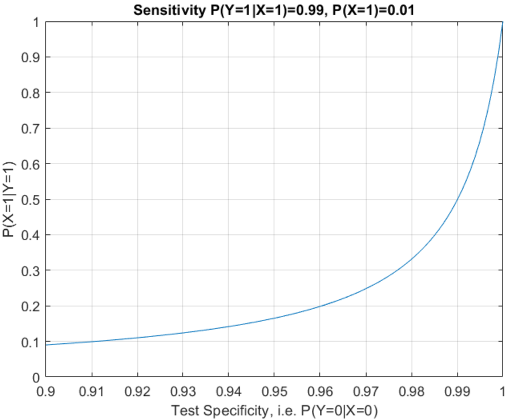
Example: COVID test as Communication channel

We will model the uncertainty of the test as a communication channel and use the Bayes Theorem.

- Specificity is a true negative rate (TNR) i.e. P(Y=0|X=0)
- Sensitivity is a true positive rate (TPR) i.e. P(Y=1|X=1)



The probability that a person has the virus given the positive test result is: $P(X=1|Y=1) = \frac{P(Y|X)P(X)}{P(Y)} = 0.4975$



Let's do some MATLAB coding and compute that probability! (and investigate a range of parameters)

Mutual Information of the COVID test

Recall that Mutual Information is defined as:

$$I(X;Y) = H(Y) - H(Y|X)$$

Therefore, we need to compute both entropy terms individually.

- To compute $H(Y)$, we need the marginal $P(y)$:

  $$P(y) = \sum_{j=1}^{n} P(y, X = x_j) = \sum_{j=1}^{n} P(y|X = x_j)\, P(X = x_j)\,.$$

  which gives:
  $P(Y = 1) = 0.0197$, and
  $P(Y = 0) = 1 - P(Y = 1) = 0.9803$.
- Y is a binary variable, for which the entropy is given by:

  $$H(p) = p \log \frac{1}{p} + (1-p) \log \frac{1}{(1-p)}.$$

  with $p = 0.0197$ we obtain: $H(Y) = 0.1398 \mathit{bits}$

Mutual Information of the COVID test

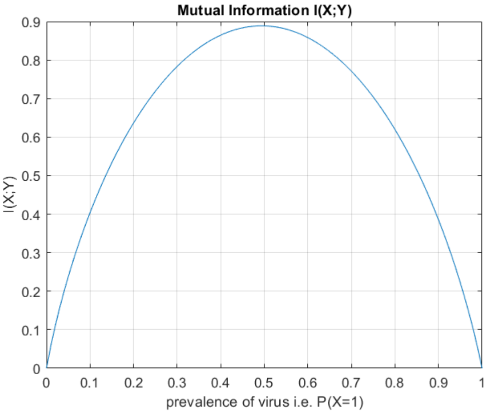
Now compute $H(Y|X)$

- $H(Y|X=0)$; $P(Y|X=0)$ is a binary RV, and hence the entropy is given by $H(p)$, where, $p = \mathit{specificity}$.
- $H(Y|X=1)$; $P(Y|X=1)$ is a binary RV, and the entropy is given by $H(p)$, where, $p = \mathit{sensitivity}$.

Collecting the terms:
$H(Y|X) = H(Y|X=0)P(X=0) + H(Y|X=1)P(X=1) =$
$0.0808 \cdot 0.99 + 0.1414 \cdot 0.01 = 0.0814$
Finally,

$$I(X;Y) = H(Y) - H(Y|X) = 0.1398 - 0.0814 = 0.0584\mathit{bits}$$

Mutual Information of the COVID test



Mutual Information as a function of the virus prevalence.