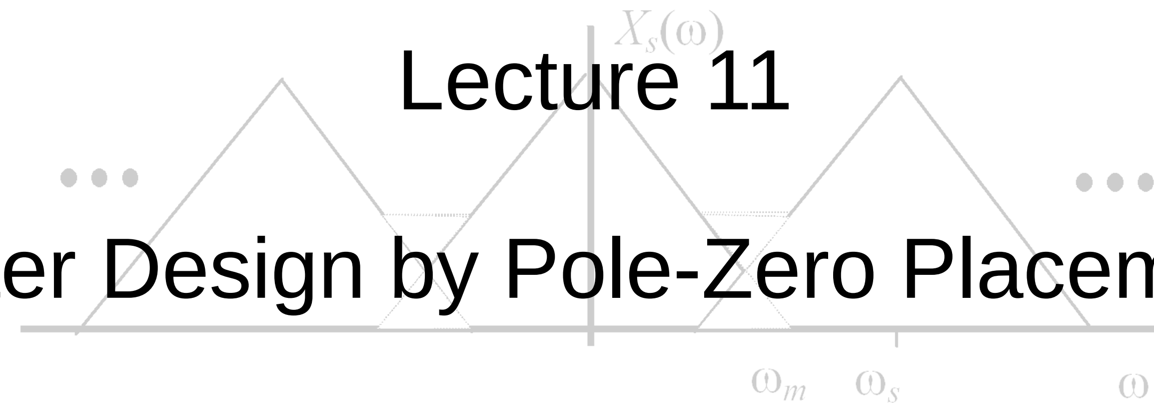
Lecture 11



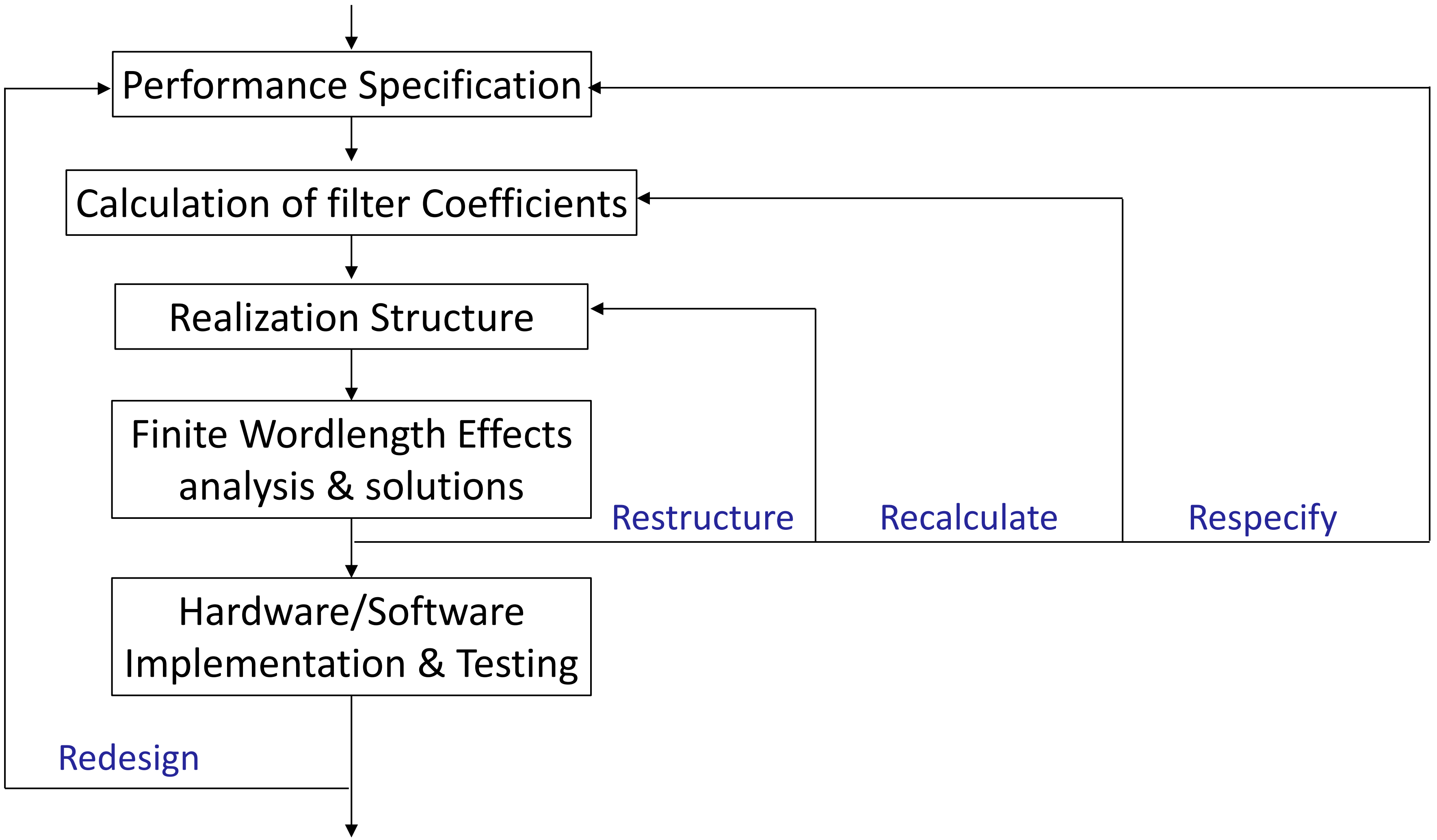
Filter Design by Pole-Zero Placement

Design of simple filters by placing poles and zeros on the z-plane

Dr. Dimitris Agrafiotis, Dept. Of Electrical & Electronic Engineering

Digital Filter Design Procedure

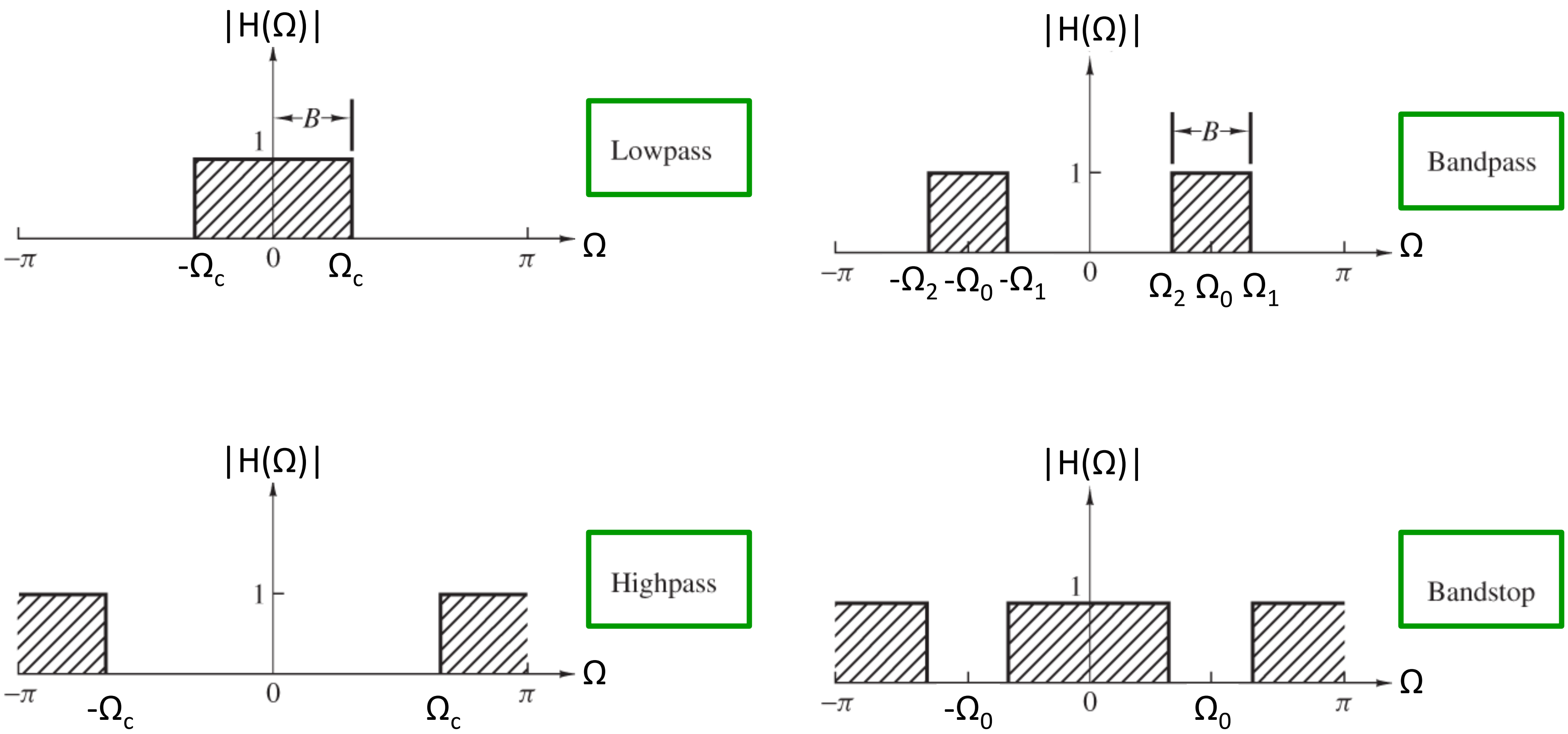
Design stages for Digital Filters



Filter Design by Pole-Zero Placement

Performance Specification

Ideal Frequency Response Specification of Digital Filters

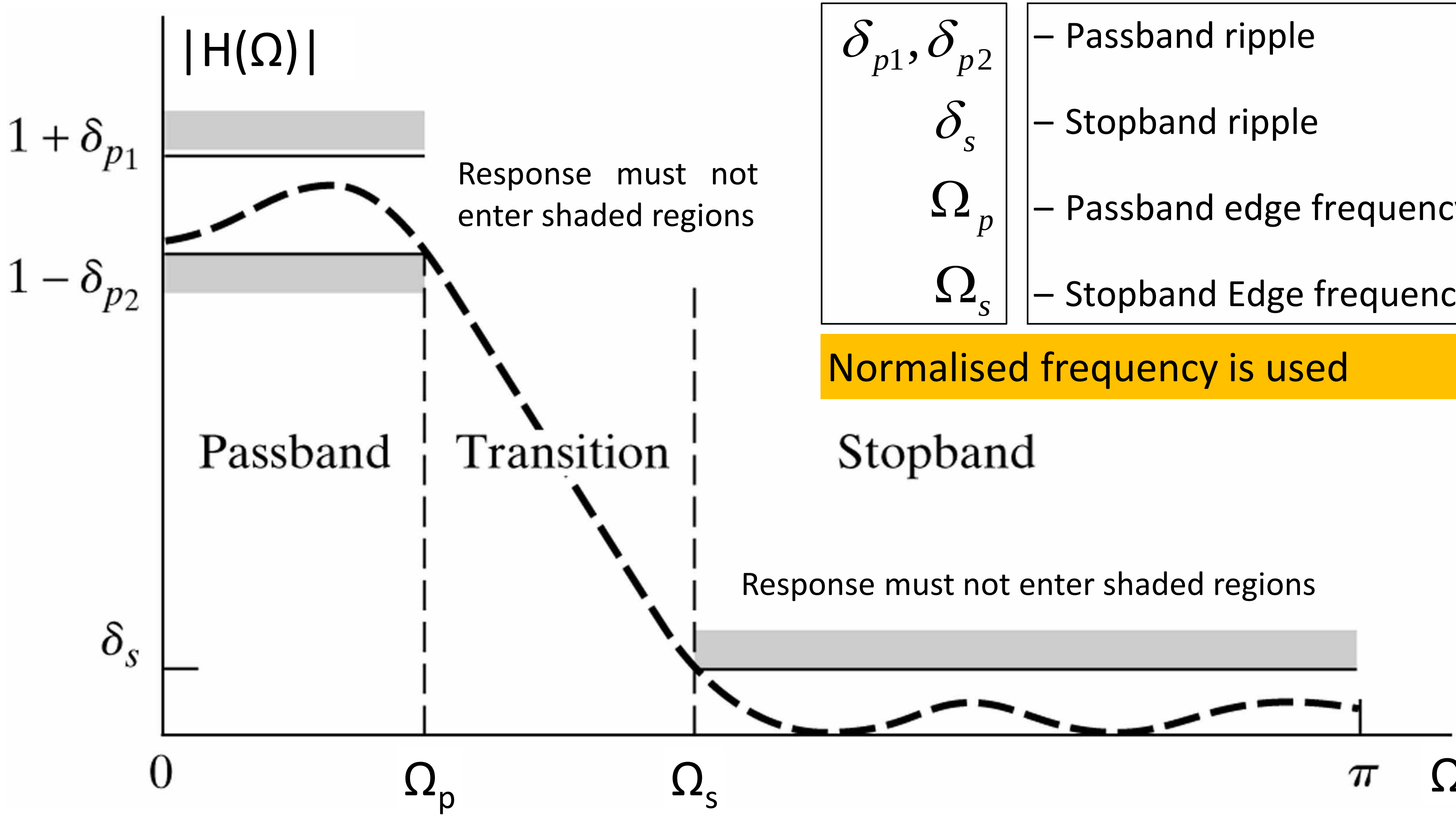


Ideal frequency selective filters

Filter Design by Pole-Zero Placement

Performance Specification

Practical Frequency Response Specification - Tolerance Scheme

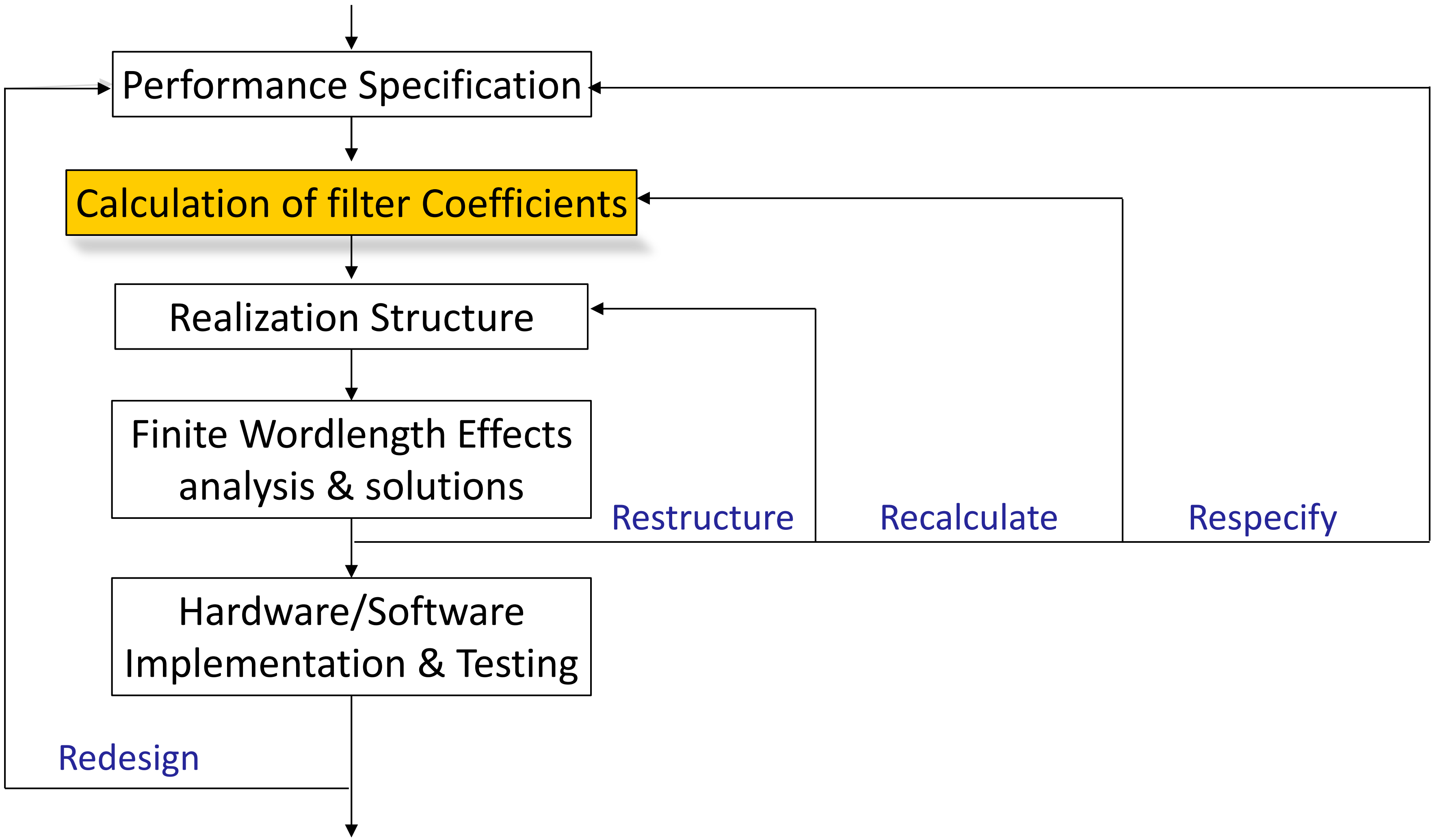


δ_{p1}, δ_{p2}	– Passband ripple
δ_s	– Stopband ripple
Ω_p	– Passband edge frequency
Ω_s	– Stopband Edge frequency

Normalised frequency is used

Digital Filter Design Procedure

Design Stages for Digital Filters



Filter Design by Pole-Zero Placement

Filter Design Methods

Methods for calculating coefficients b_k & α_k of difference equation/transfer function

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \longrightarrow$$

δ_{p1}, δ_{p2}	– Passband ripple
δ_s	– Stopband ripple
Ω_p	– Passband edge frequency
Ω_s	– Stopband Edge frequency

Pole-Zero Placement

Place pole & zeros on the z-plane so that the resulting filter has the desired response

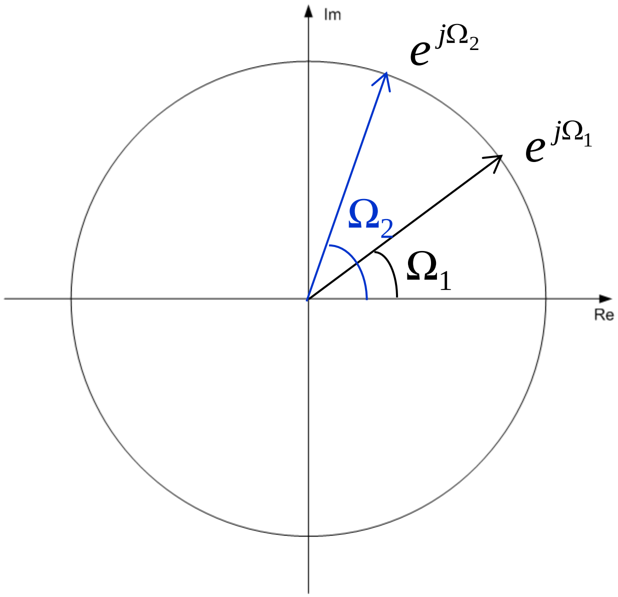
$$H(e^{j\Omega})=b_0\frac{\prod_{k=1}^{N}(1-z_k e^{-j\Omega})}{\prod_{k=1}^{N}(1-p_k e^{-j\Omega})}=b_0\frac{\prod_{k=1}^{N}(e^{j\Omega}-z_k)}{\prod_{k=1}^{N}(e^{j\Omega}-p_k)} \overset{r=1}{\Leftrightarrow} H(z)=b_0\frac{\prod_{k=1}^{N}(1-z_k z^{-1})}{\prod_{k=1}^{N}(1-p_k z^{-1})}=b_0\frac{\prod_{k=1}^{N}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

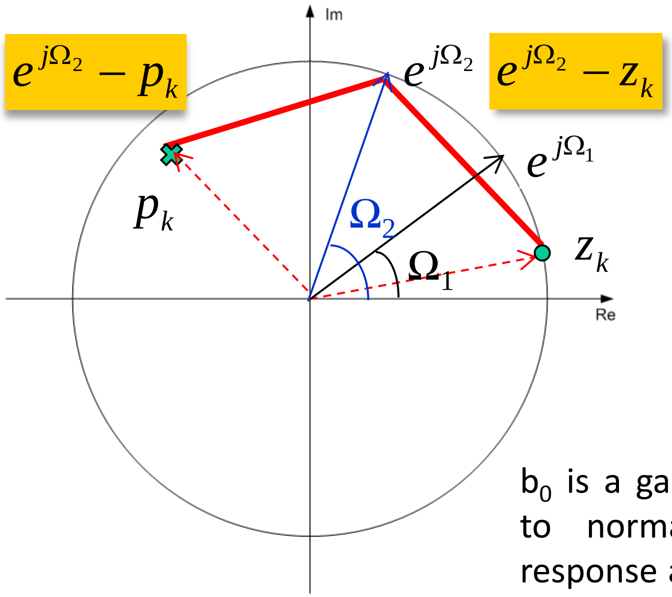
Suitable for designing simple filters where filter parameters need not be specified precisely

Frequency Response from Pole Zero Map

Geometric Interpretation

$$H(e^{j\Omega}) = b_0 \frac{\prod_{k=1}^{N}(1 - z_k e^{-j\Omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\Omega})} = b_0 \frac{\prod_{k=1}^{N}(e^{j\Omega} - z_k)}{\prod_{k=1}^{N}(e^{j\Omega} - p_k)} \overset{r=1}{\Leftrightarrow} H(z) = b_0 \frac{\prod_{k=1}^{N}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} = b_0 \frac{\prod_{k=1}^{N}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$





b_0 is a gain term selected to normalise the freq. response at some selected frequency $|H(\Omega_0)|=1$

Frequency Response from Pole Zero Map

Geometric Interpretation

$$H(e^{j\Omega}) = b_0 \frac{\prod_{k=1}^{N}(1 - z_k e^{-j\Omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\Omega})} = b_0 \frac{\prod_{k=1}^{N}(e^{j\Omega} - z_k)}{\prod_{k=1}^{N}(e^{j\Omega} - p_k)} \overset{r=1}{\Longleftrightarrow} H(z) = b_0 \frac{\prod_{k=1}^{N}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} = b_0 \frac{\prod_{k=1}^{N}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

Expressed in polar form

$$\left.\begin{aligned} (e^{j\Omega} - z_k) &= V_k(\Omega) e^{j\Theta_k(\Omega)} \\ (e^{j\Omega} - p_k) &= U_k(\Omega) e^{j\Phi_k(\Omega)} \end{aligned}\right\} \quad \begin{aligned} V_k(\Omega) &\equiv \left|e^{j\Omega} - z_k\right|, \quad \Theta_k(\Omega) \equiv \angle(e^{j\Omega} - z_k) \\ U_k(\Omega) &\equiv \left|e^{j\Omega} - p_k\right|, \quad \Phi_k(\Omega) \equiv \angle(e^{j\Omega} - p_k) \end{aligned}$$

Magnitude of Frequency Response

$$\left|H(\Omega)\right| = \left|b_0\right| \frac{V_1(\Omega).....V_N(\Omega)}{U_1(\Omega)U_2(\Omega).....U_N(\Omega)}$$

Phase of Frequency Response

$$\angle H(\Omega) = \Theta_1(\Omega) + \Theta_2(\Omega)..... + \Theta_N(\Omega) - \left[\Phi_1(\Omega) + \Phi_2(\Omega)..... + \Phi_N(\Omega)\right]$$

Filter Design by Pole-Zero Placement

Frequency Response from Pole Zero Map

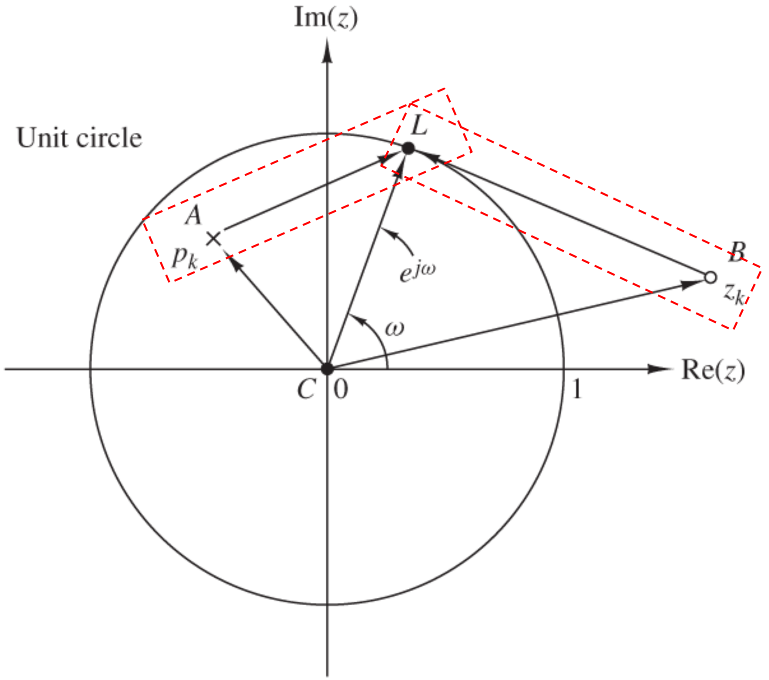
Geometric Interpretation

Magnitude of Frequency Response

CL = CA+AL, AL = CL-CA → $(e^{j\Omega} - p_k)$

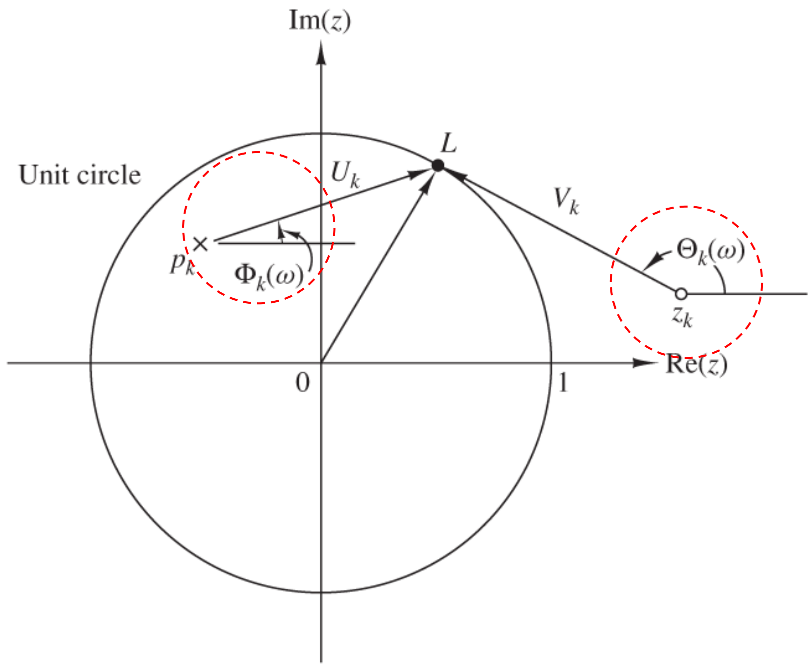
CL = CB+BL, BL = CL-CB → $(e^{j\Omega} - z_k)$

CL = $e^{j\Omega}$, CA = p_k , CB = z_k



Phase of Frequency Response

The phases $\Phi_k(\Omega)$ and $\Theta_k(\Omega)$ are the angles of the vectors AL and BL



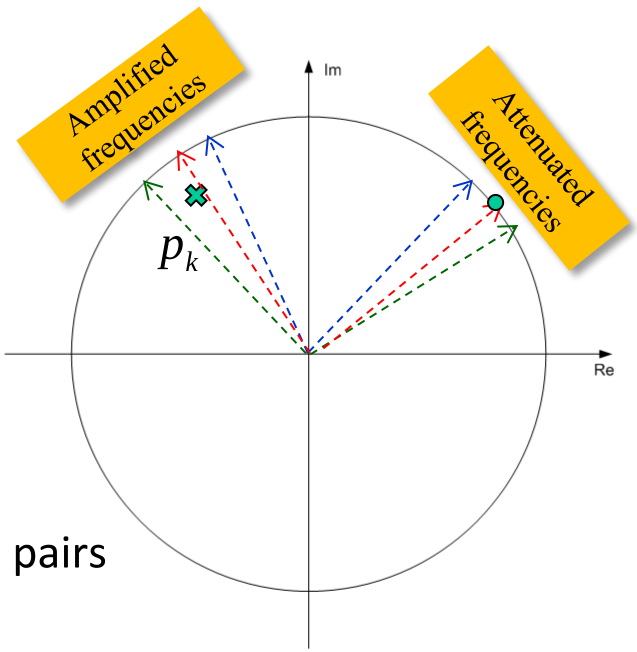
Design by Pole Zero Placement

Things to remember

- Presence of zero close-to/on unit circle => magnitude at frequencies that correspond to points close to that zero will be small
- Presence of pole close to the unit circle => magnitude at frequencies that correspond to points close to that point will be large

Poles have the opposite effect of zeros

- Place poles close to frequencies that are to be amplified
- Place zeros close to frequencies that are to be attenuated
- Use of both poles and zeros offers greater flexibility in terms of the frequency responses that are possible
- All poles should lie inside the unit circle for stability
  Zeros can be placed anywhere
- All complex zeros and poles must occur in complex-conjugate pairs in order for the filter coefficients to be real
- Poles and zeros on the origin do not influence the magnitude response



Filter Design by Pole-Zero Placement

Simple FIR Filters

First order filters

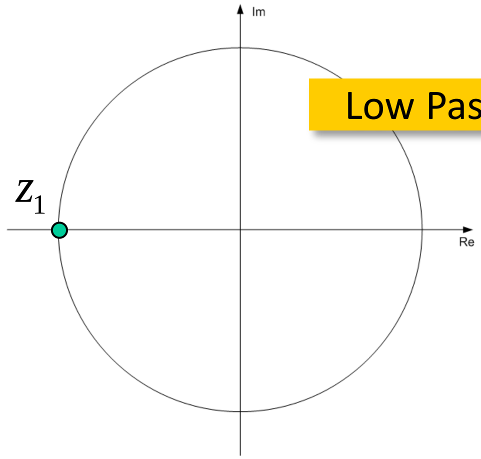
- $y[n] = x[n] - z_1 x[n-1]$
- $H(z) = 1 - z_1 z^{-1}$
- The zero is often placed on the unit circle...

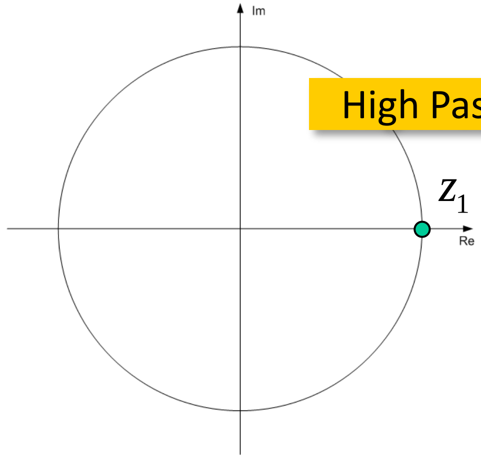
- At *-1* to give a 1st order low pass filter

$$H(z) = 1 + z^{-1}$$

- At *1* to give a 1st order high pass filter

$$H(z) = 1 - z^{-1}$$





Simple FIR Filters

First order filters

- $y[n] = x[n] - z_1 x[n-1]$
- $H(z) = 1 - z_1 z^{-1} \Leftrightarrow \boxed{\dfrac{z - z_1}{z}}$

  Multiply numerator & denominator with z

- The zero is often placed on the unit circle...

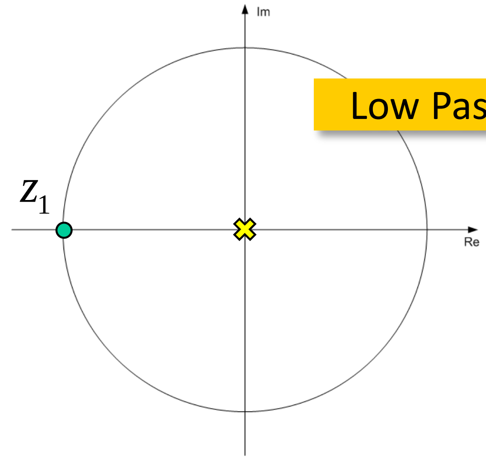
- At *-1* to give a 1st order low pass filter

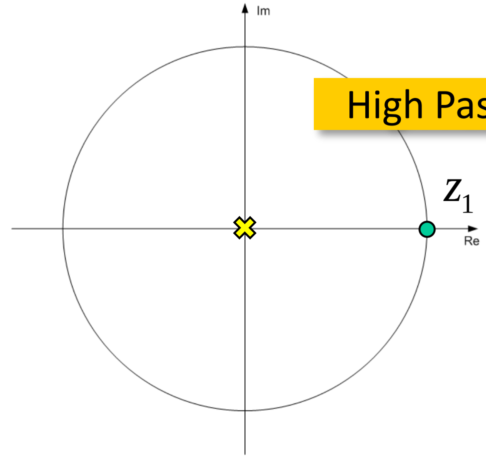
$$H(z) = 1 + z^{-1} \Leftrightarrow \boxed{\frac{z+1}{z}}$$

- At *1* to give a 1st order high pass filter

$$H(z) = 1 - z^{-1} \Leftrightarrow \boxed{\frac{z-1}{z}}$$

- Filters are still FIR





Filter Design by Pole-Zero Placement

Simple FIR Filters

Second order filters

- If the coefficients are real the filter can be factorised to give two zeros z_1 and z_2

- z_1 and z_2 are either real …
- in which case the filter is a cascade of two 1^{st} - order FIR filters
- H(z) = H1(z) H2(z) = $(1 \pm z_1 z^{-1})(1 \pm z_2 z^{-1})$
- Band pass filter if zeros are at -1,1

- … or they form a complex conjugate pair

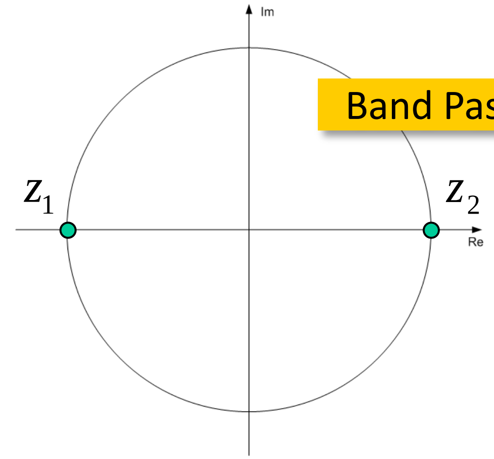
$$z_1, z_2 = re^{\pm j\theta}$$

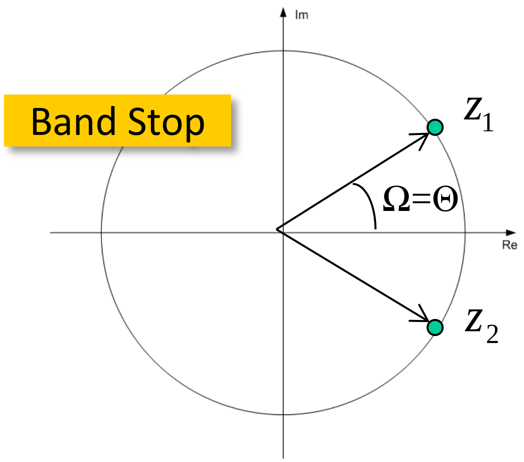
- with transfer function

$$H(z) = (1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1}) = 1 - 2r\cos(\theta) z^{-1} + r^2 z^{-2}$$

$$\Omega = \theta$$

- Band stop filter for r=1 (blocks frequencies)





Filter Design by Pole-Zero Placement

Resonators

First and second order IIR filters

- 1^{st} order **low pass** (resonates at/close to 0)

$$H(z) = \frac{1}{1 - rz^{-1}} \Leftrightarrow \frac{z}{z - r}$$

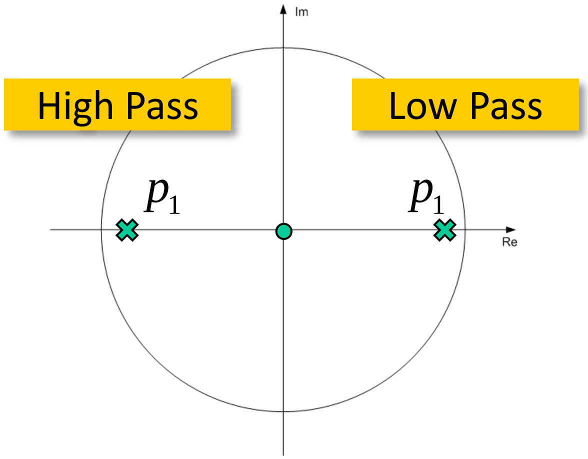
- 1^{st} order **high pass** (resonates at/close to π)

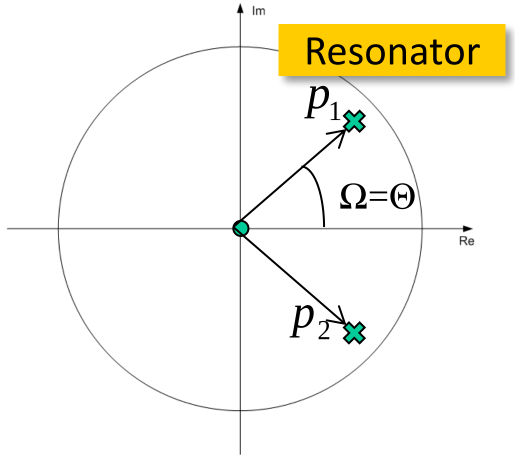
$$H(z) = \frac{1}{1 + rz^{-1}} \Leftrightarrow \frac{z}{z + r}$$

- 2^{nd} order **resonator** (resonates at /close to θ)

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}}$$

- If r is chosen to be just less than 1 (poles just inside the unit circle), then these filters will resonate (give high gain) for the resonant frequency θ.



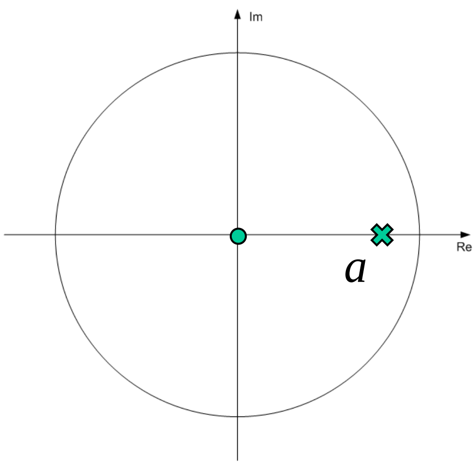


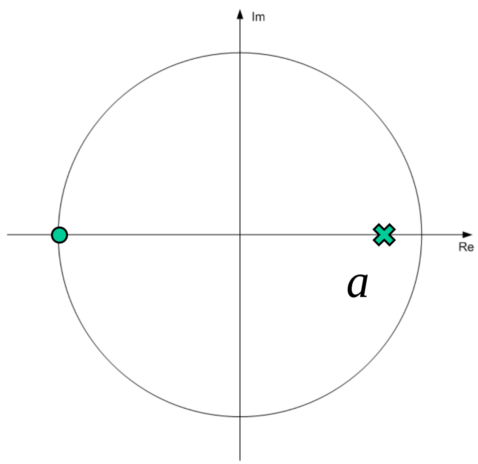
Examples

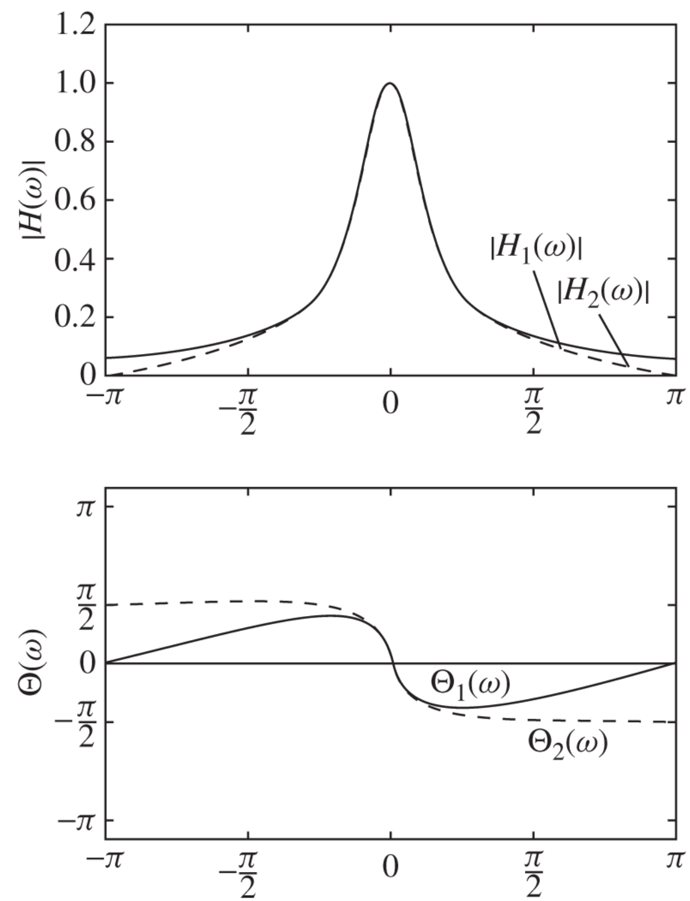
Low pass, High pass, Band pass filters

- Low pass IIR

$$H_1(z) = (1-a)\frac{1}{1-az^{-1}} \quad \text{and} \quad H_2(z) = \frac{(1-a)}{2}\frac{1+z^{-1}}{1-az^{-1}}$$







- Both filters have unity gain at H(Ω=0)

$$for\ z = e^{j\Omega}\ with\ \Omega = 0 \quad H_2(\Omega = 0) = \frac{(1-a)}{2}\frac{1+1}{1-a} = 1$$

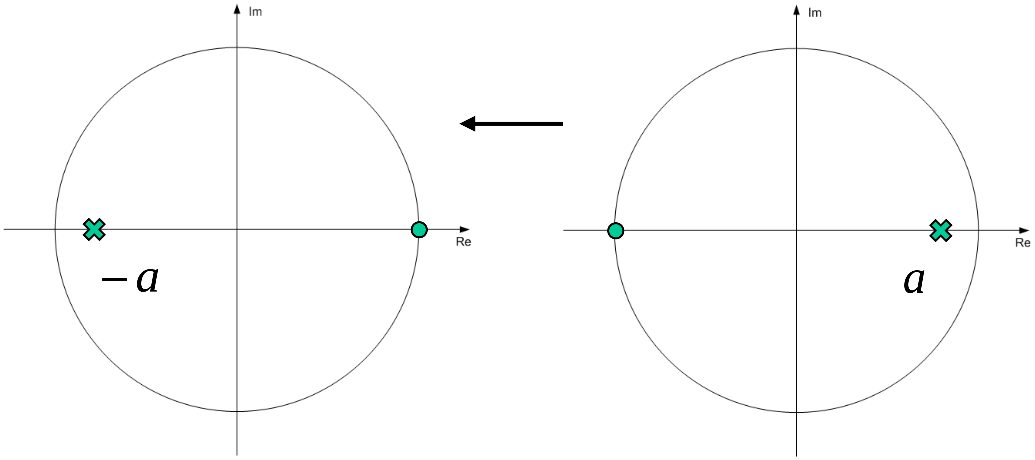
Filter Design by Pole-Zero Placement

Examples

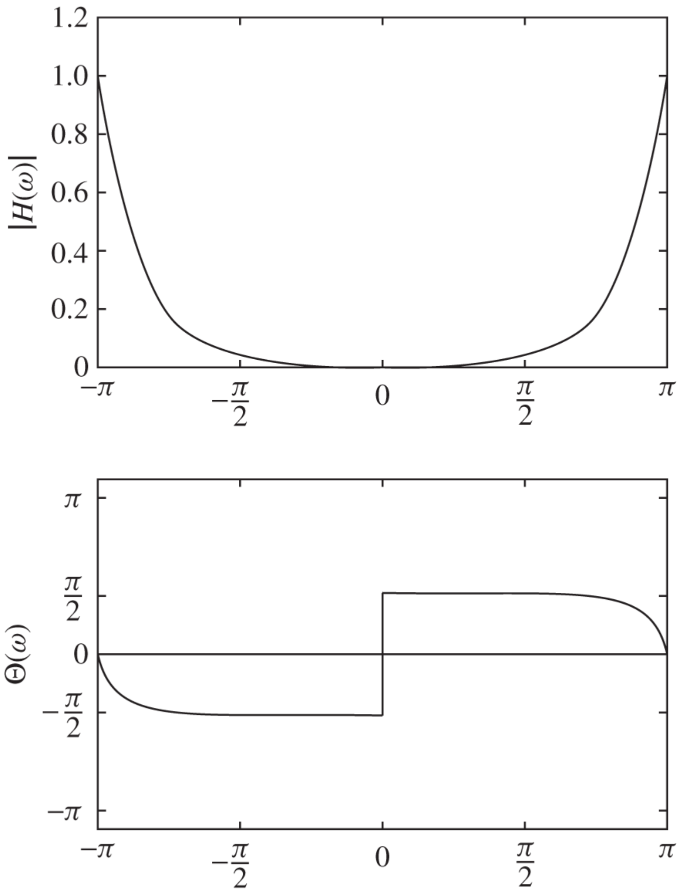
Low pass, High pass, Band pass filters

Obtain a high pass filter from the low pass

$$H(z)=\frac{(1-a)}{2}\frac{1-z^{-1}}{1+az^{-1}} \longleftarrow H_2(z)=\frac{(1-a)}{2}\frac{1+z^{-1}}{1-az^{-1}}$$



Reflect the pole zero locations about the imaginary axis



Examples

Low pass, High pass, Band pass filters

A two-pole low pass filter has the transfer function: $H(z)=\dfrac{b_0}{(1-pz^{-1})^2}$ .Determine the values of b_0 and p such that the frequency response $H(\Omega)$ satisfies the conditions :

$$H(0)=1 \;\; and \;\; \left|H(\frac{\pi}{4})\right|^2=\frac{1}{2}$$

$$at\ \Omega=0\ we\ have\ H(0)=\frac{b_0}{(1-p)^2}=1 \Rightarrow b_0=(1-p)^2 \qquad (1)$$

$$at\ \Omega=\frac{\pi}{4}\ we\ have\ H(\frac{\pi}{4})=\frac{b_0}{(1-pe^{-j\pi/4})^2}\overset{(1)}{=}\frac{(1-p)^2}{(1-pe^{-j\pi/4})^2}=\frac{(1-p)^2}{(1-p\cos(\pi/4)+jp\sin(\pi/4))^2}=$$

$$=\frac{(1-p)^2}{(1-p/\sqrt{2}+jp/\sqrt{2})^2} \Rightarrow \boxed{|H(\Omega)|}=\frac{(1-p)^2}{\left(\sqrt{(1-p/\sqrt{2})^2+(p/\sqrt{2})^2}\right)^2}=\frac{(1-p)^2}{(1-p/\sqrt{2})^2+(p/\sqrt{2})^2}$$

$$=\frac{(1-p)^2}{(1-p/\sqrt{2})^2+(p^2/2)}=\frac{1}{\sqrt{2}} \Rightarrow p=0.32 \qquad (2)$$

$$\overset{1,2}{\Longrightarrow} H(z)=\frac{0.46}{(1-0.32z^{-1})^2}$$

Filter Design by Pole-Zero Placement

Examples

Low pass, High pass, Band pass filters

Design a two pole band pass filter that has the centre of its passband at $\Omega = \pi / 2$ and zero in its frequency response at $\Omega = 0$ and $\Omega = \pi$

The filter must have poles at $p_{1,2} = re^{\pm j\pi/2}$

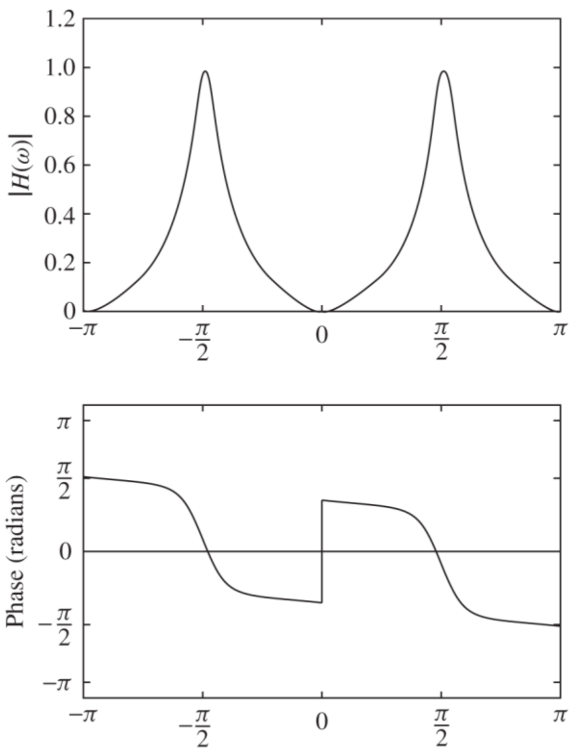
$$\Leftrightarrow p_{1,2} = r(\cos(\pi/2) \pm j\sin(\pi/2)) = \pm jr \quad (1)$$

The filter must have zeros at $z = 1$ and $z = -1$ (2)

$$\overset{1,2}{\Rightarrow} H(z) = G\frac{(z-1)(z+1)}{(z-jr)(z+jr)}$$

For a magnitude response of $\frac{1}{\sqrt{2}}$ $at\ \Omega = 4\pi/9$

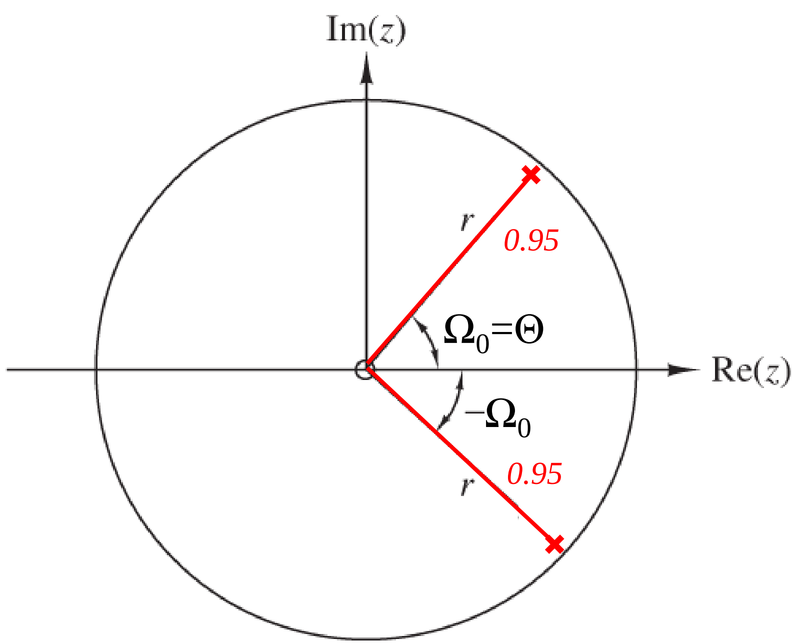
$$G = 0.15\ and\ r^2 = 0.7$$

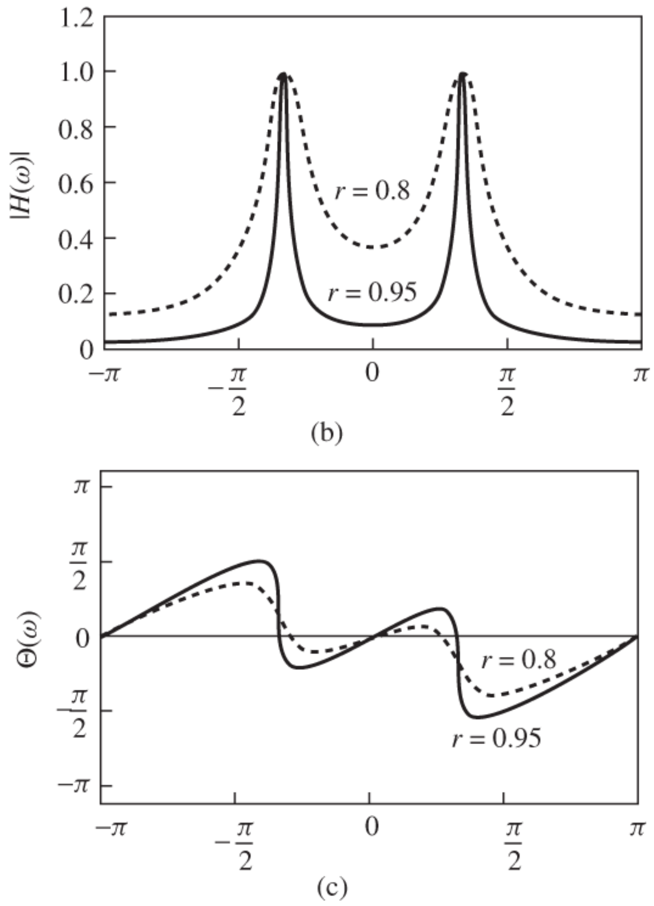


Examples

Resonator – Effect of *r*

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}}$$





Filter Design by Pole-Zero Placement

Notch Filters

Removing an isolated frequency or narrowband interference

- The 1st or 2nd order FIR and IIR sections can be easily combined to give notch filters, where the zero(s) are placed on the unit circle and the pole(s) are placed at corresponding angle(s) but just inside the unit circle

$$H(z) = \frac{1 - z^{-1}}{1 - rz^{-1}}$$

$$H(z) = \frac{1 + z^{-1}}{1 + rz^{-1}}$$

$$H(z) = \frac{1 - 2\cos(\theta) z^{-1} + z^{-2}}{1 - 2r\cos(\theta) z^{-1} + r^2 z^{-2}}$$

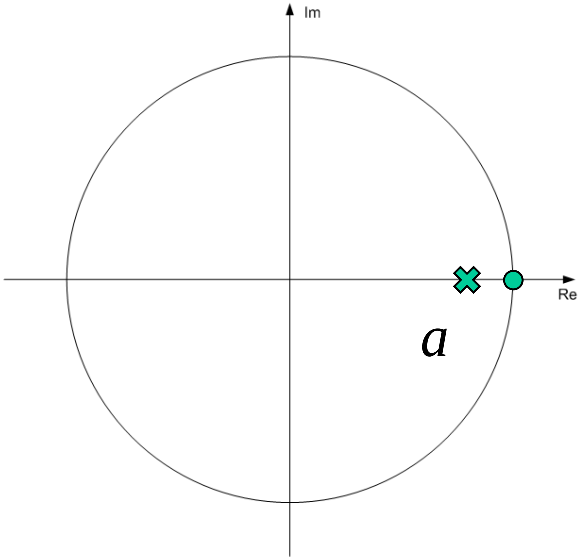
- The closeness of the pole-zero combination gives a frequency response which is close to unity for all frequencies except those particularly close to the zero.

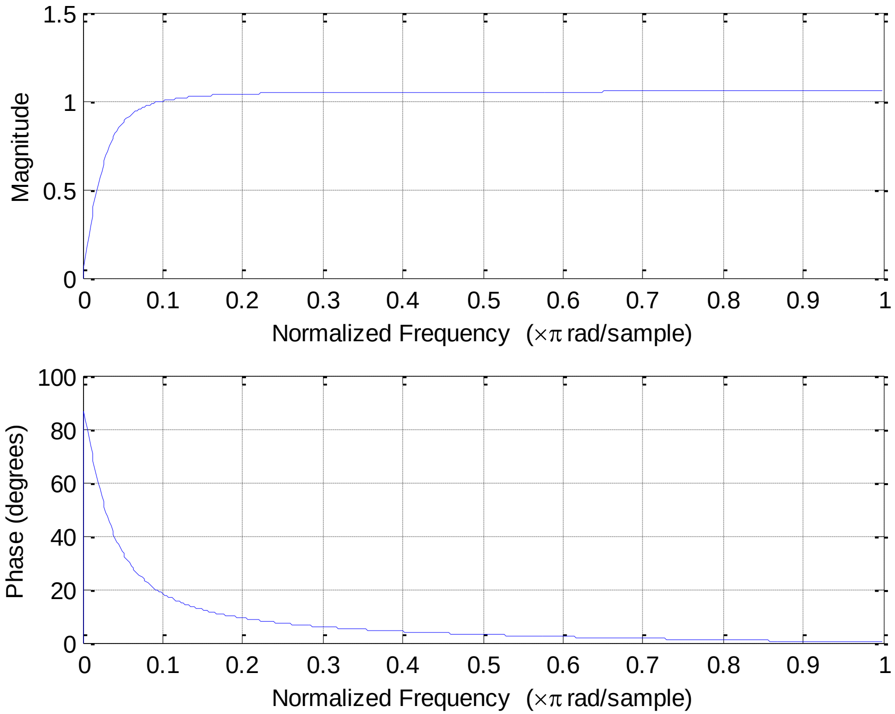
Examples

Notch Filters

Design a notch filter for removing DC signal components

$$H(z) = \frac{1 - z^{-1}}{1 - az^{-1}}$$





Filter Design by Pole-Zero Placement

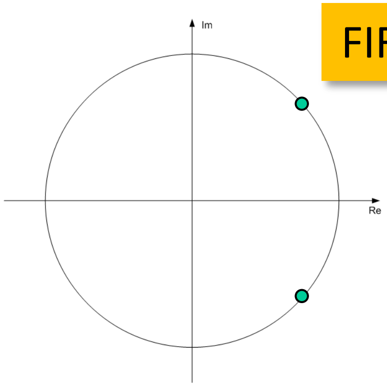
Examples

Notch Filters

A signal sampled at 400Hz suffers from narrow-band noise centred a 50 Hz. Design both 2nd order FIR and IIR notch filters to remove this noise.

- $\Omega = \pm 2\pi \dfrac{f}{f_s} = 2\pi \dfrac{50}{400} = \pm \dfrac{\pi}{4}$
- Design a 2nd order FIR filter with zeros at $e^{\pm j\pi/4}$
- $H(z) = (1 - e^{j\pi/4} z^{-1})(1 - e^{-j\pi/4} z^{-1}) = 1 - 2\cos(\pi/4) + z^{-2}$

FIR

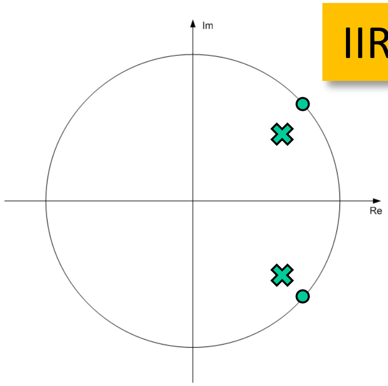


- 2nd order IIR filter with zeros at $e^{\pm j\pi/4}$ and poles at $re^{\pm j\pi/4}$

  where r<1 for stability

$$H(z) = \frac{(1 - e^{j\pi/4} z^{-1})(1 - e^{-j\pi/4} z^{-1})}{(1 - re^{j\pi/4} z^{-1})(1 - re^{-j\pi/4} z^{-1})} = \frac{1 - 2\cos(\pi/4) + z^{-2}}{1 - 2r\cos(\pi/4) + r^2 z^{-2}}$$

IIR

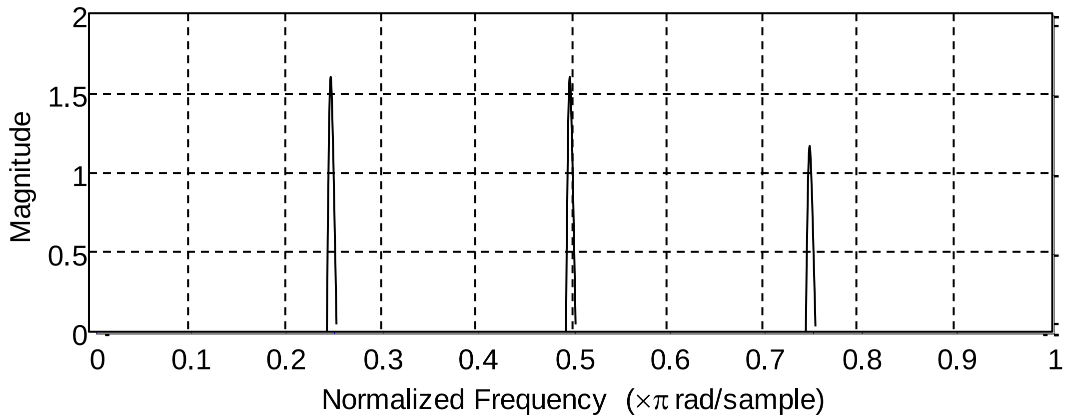


Filter Design by Pole-Zero Placement

Comb Filters

Removing harmonically related interference

- Comb filters are based on the factorization of $\boxed{1-r^N z^{-N}}$ which has *N* roots at $\boxed{re^{jk2\pi/N}}$ for $0 \le k < N$ i.e. the roots are evenly spaced around a circle of radius *r*
- FIR forms often use a radius of 1 (giving zero gain at the zeros).
- IIR forms often place poles just inside the unit circle with zeros on the unit circle. (thus they behave like a series of notch filters)



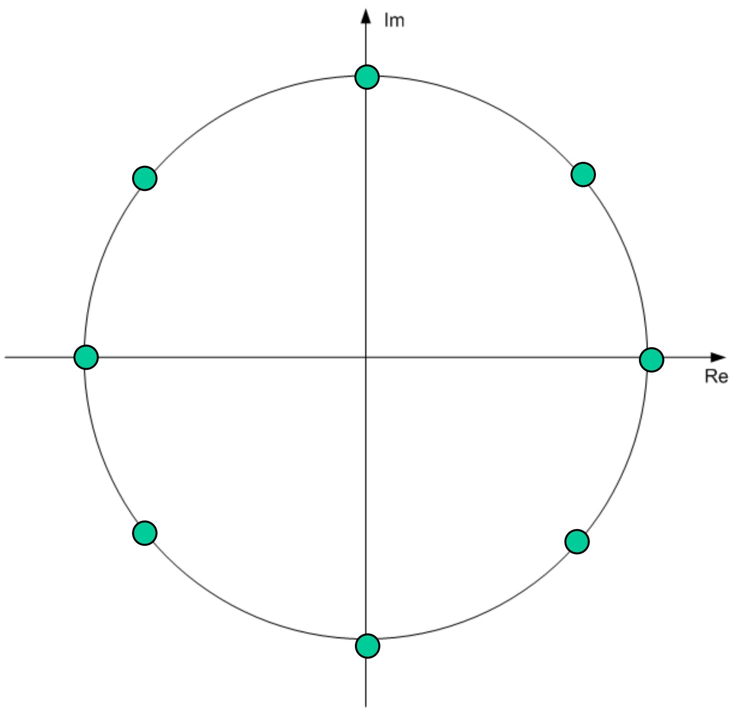
Filter Design by Pole-Zero Placement

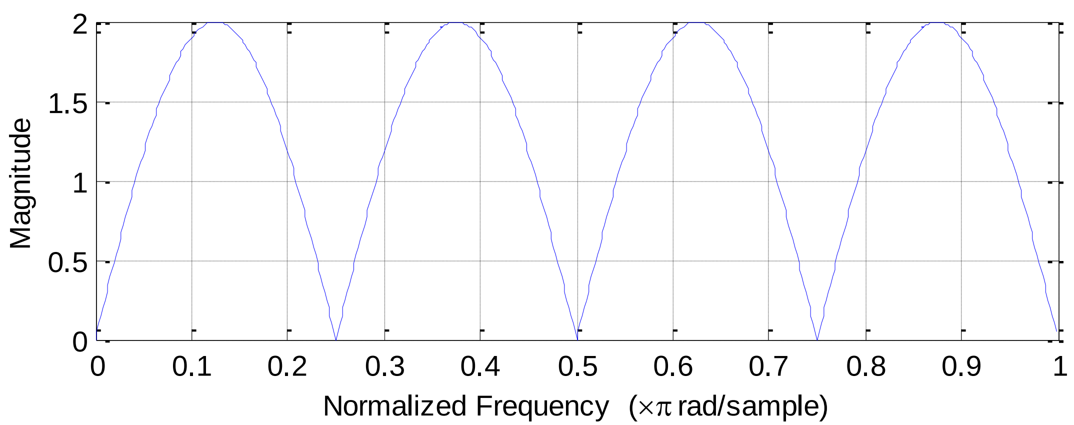
Examples

Comb Filters (8th Order FIR)

$$H(z) = 1 - z^{-8} = \prod_{k=0}^{7} (1 - e^{j2k\pi/8} z^{-1})$$

Removes frequencies at multiples of π/4





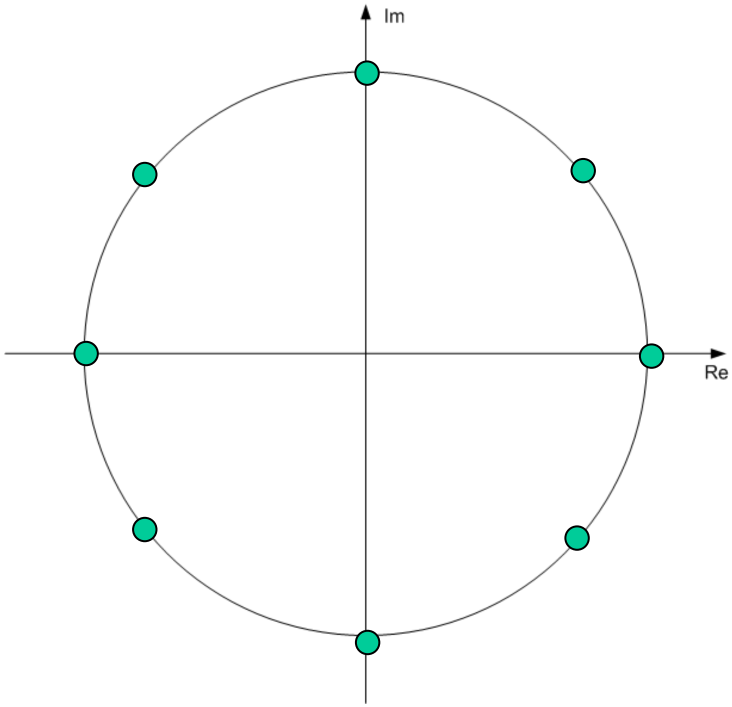
Filter Design by Pole-Zero Placement

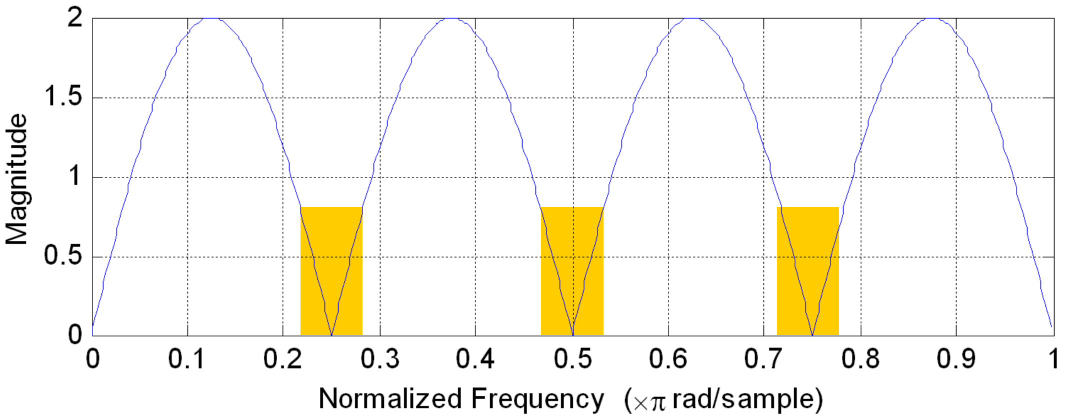
Examples

Comb Filters (8th Order FIR)

$$H(z) = 1 - z^{-8} = \prod_{k=0}^{7} \left(1 - e^{j2k\pi/8} z^{-1}\right)$$

Removes frequencies at multiples of π/4





Wide transition => many frequencies attenuated

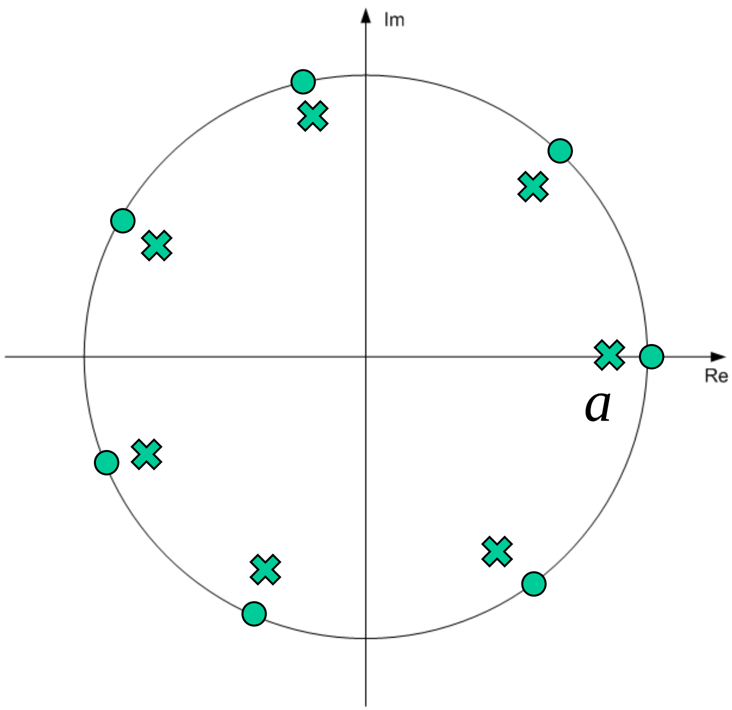
Filter Design by Pole-Zero Placement

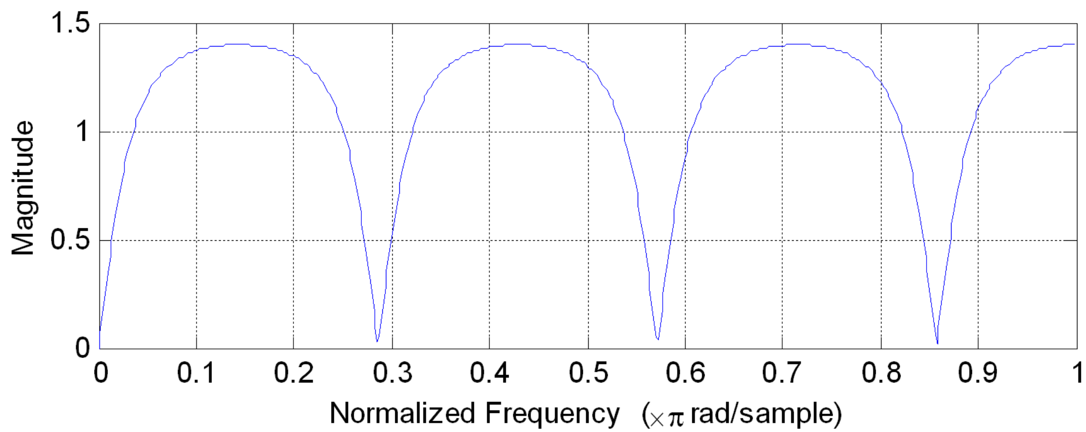
Examples

Comb Filters (7th Order IIR)

$$H(z) = \frac{1 - z^{-7}}{1 - \alpha^7 z^{-7}} = \prod_{k=0}^{6} \frac{(1 - e^{j2k\pi/7} z^{-1})}{(1 - \alpha e^{j2k\pi/7} z^{-1})}$$

Removes frequencies at multiples of 2π/7





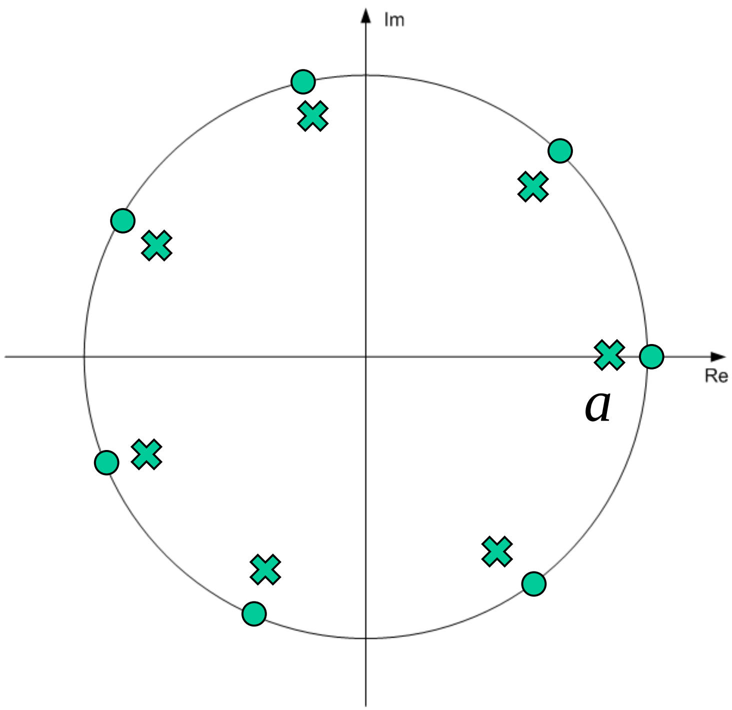
Filter Design by Pole-Zero Placement

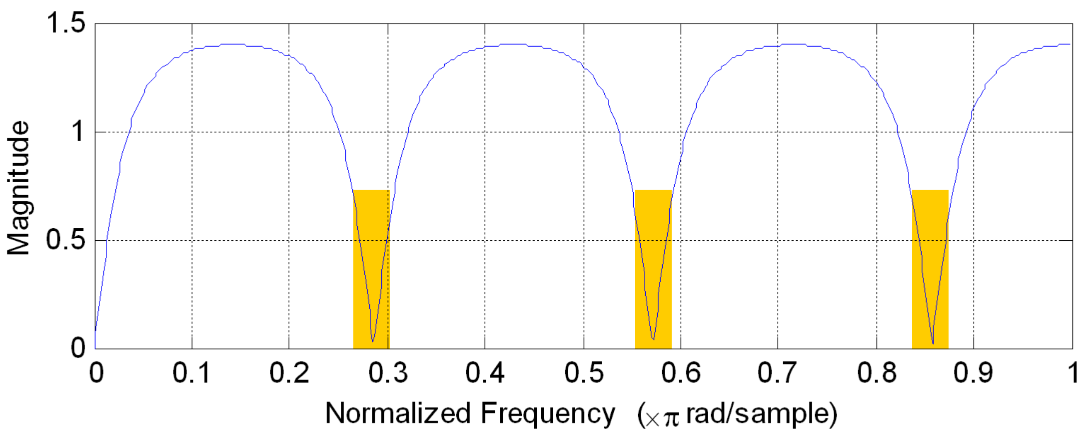
Examples

Comb Filters (7th Order IIR)

$$H(z) = \frac{1 - z^{-7}}{1 - \alpha^7 z^{-7}} = \prod_{k=0}^{6} \frac{(1 - e^{j2k\pi/7} z^{-1})}{(1 - \alpha e^{j2k\pi/7} z^{-1})}$$

Removes frequencies at multiples of 2π/7





Sharper transition due to the presence of poles

Filter Design by Pole-Zero Placement

Examples

Cascading Simple Filters to Create a Comb Filter

Design a filter to remove 2 isolated frequencies at 100Hz and 150Hz assuming a sampling rate of 600Hz. Use a cascade of 2 notch filters

$$\Omega_1 = 2\pi\, f_1 / f_s = 2\pi 100/600 = \pi/3$$
$$\Omega_2 = 2\pi\, f_2 / f_s = 2\pi 150/600 = \pi/2$$

$$H_1(z) = \frac{1 - 2\cos(\Omega_1) z^{-1} + z^{-2}}{1 - 2r\cos(\Omega_1) z^{-1} + r^2 z^{-2}} = \frac{1 - z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$$H_2(z) = \frac{1 - 2\cos(\Omega_2) z^{-1} + z^{-2}}{1 - 2r\cos(\Omega_2) z^{-1} + r^2 z^{-2}} = \frac{1 + z^{-2}}{1 + 0.81z^{-2}}$$

$$H(z) = H_1(z)H_2(z) = \frac{1 - z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}} \frac{1 + z^{-2}}{1 + 0.81z^{-2}} = \frac{1 - z^{-1} + 2z^{-2} - z^{-3} + z^{-4}}{1 - 0.9z^{-1} + 1.62z^{-2} - 0.729z^{-3} + 0.656z^{-4}}$$