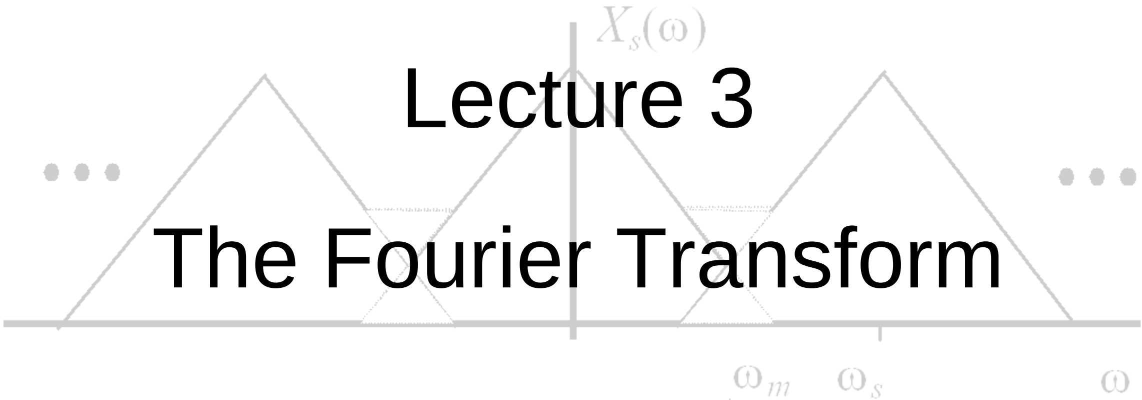


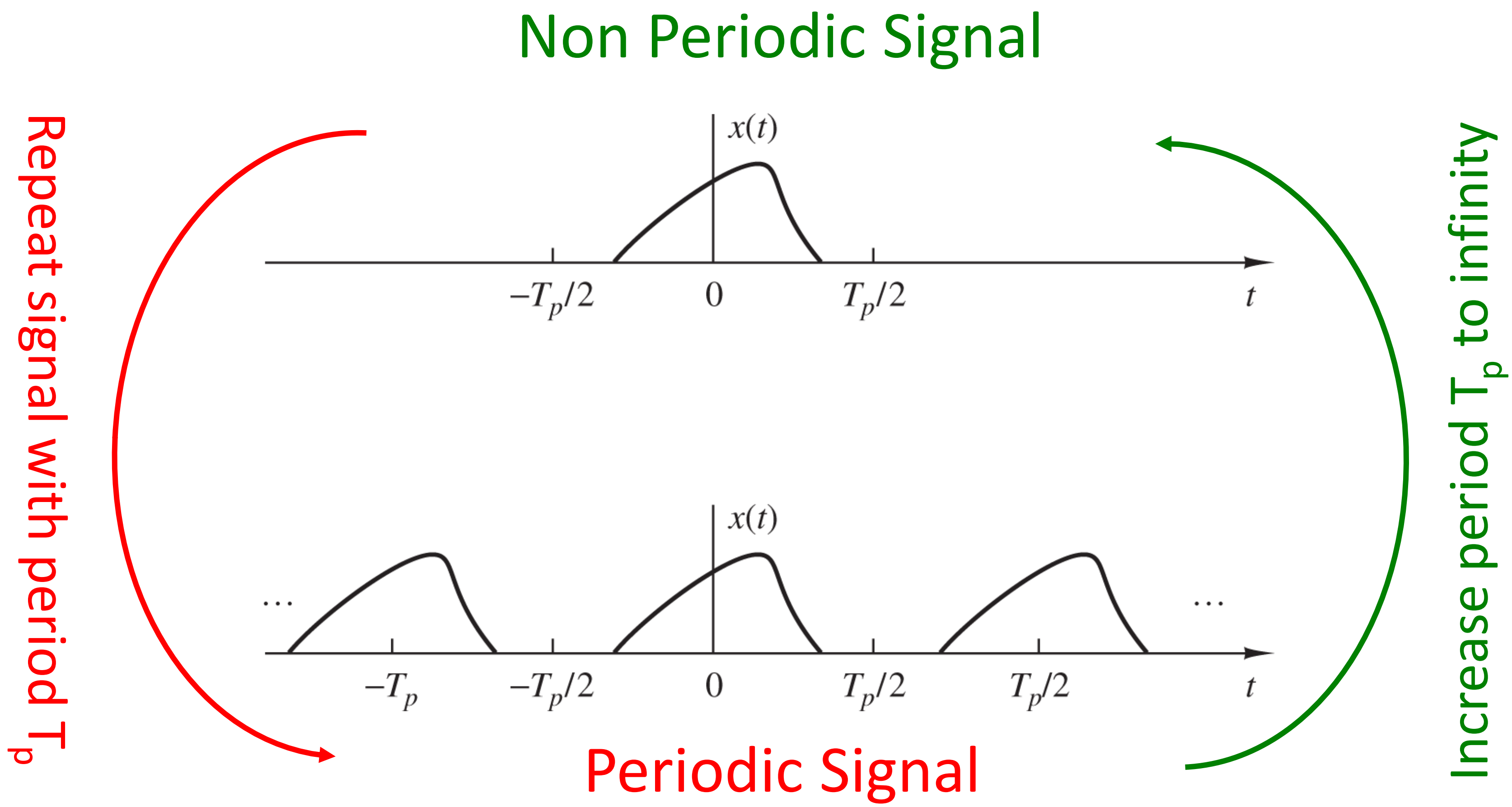
Lecture 3

The Fourier Transform

Representing continuous time non periodic signals with complex exponentials

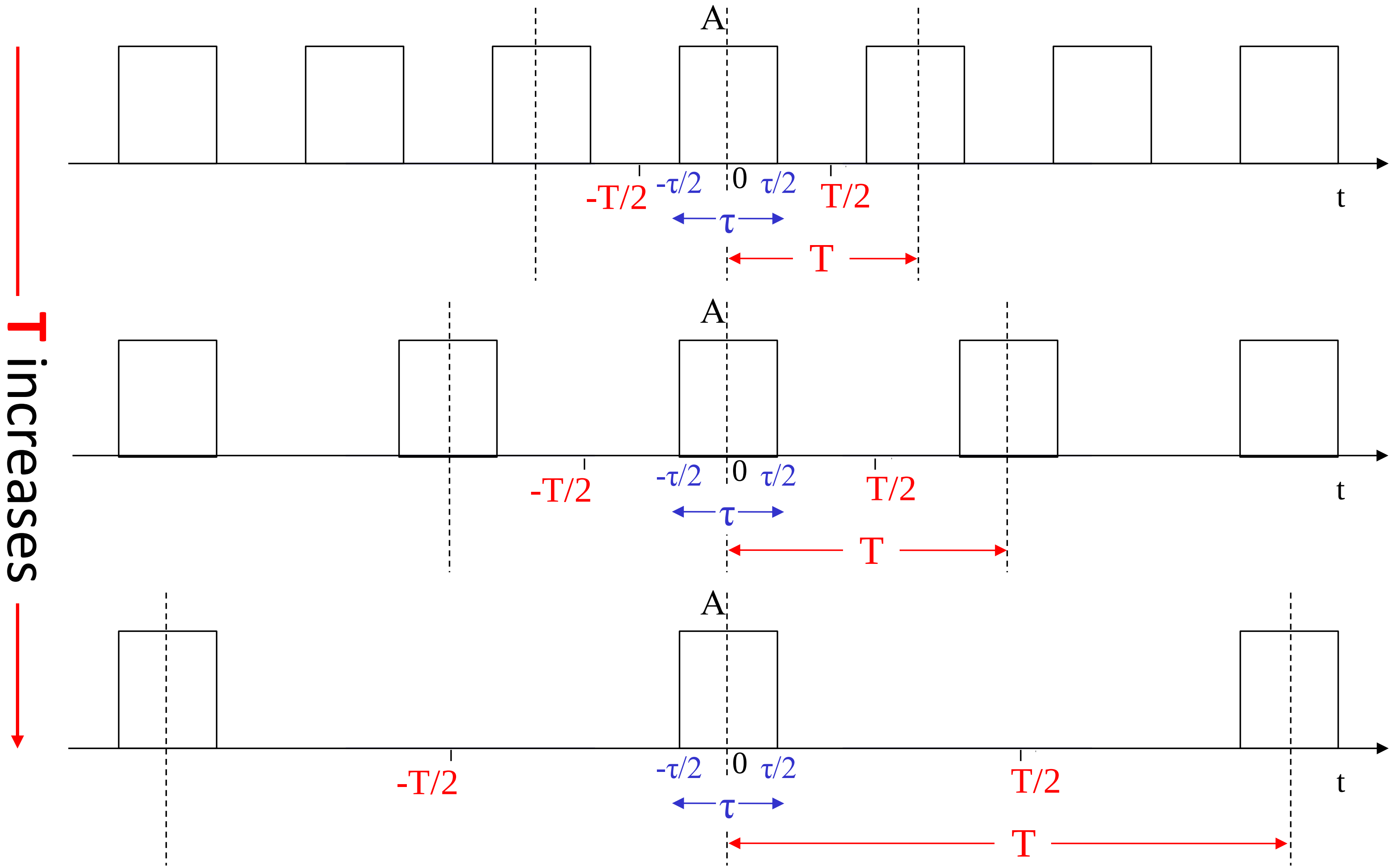
Dr. Dimitris Agrafiotis, Dept. Of Electrical & Electronic Engineering

From the Fourier Series to the Fourier Transform



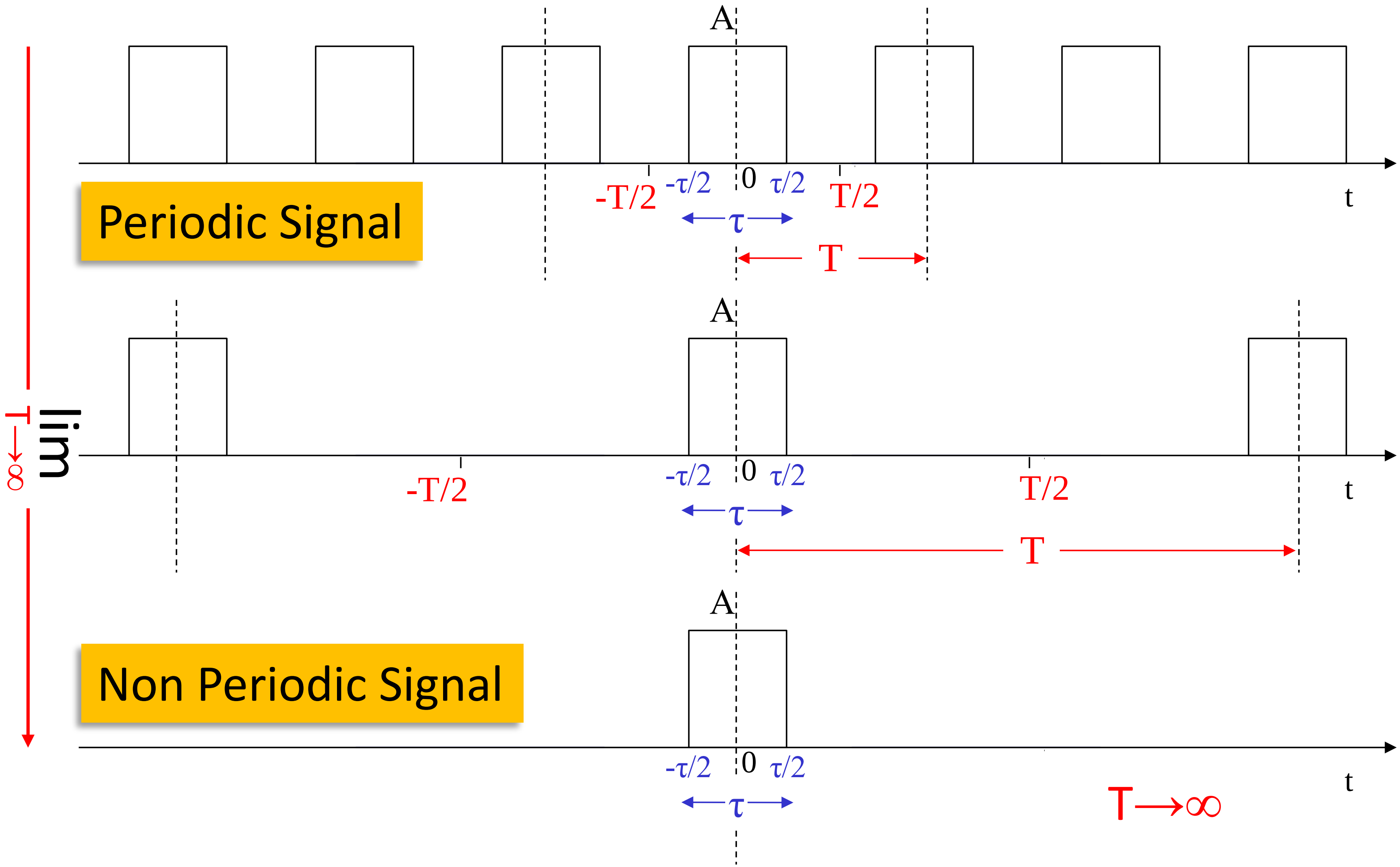
From the Fourier Series to the Fourier Transform – Q1

What happens to the Fourier series as the period T of the signal increases?



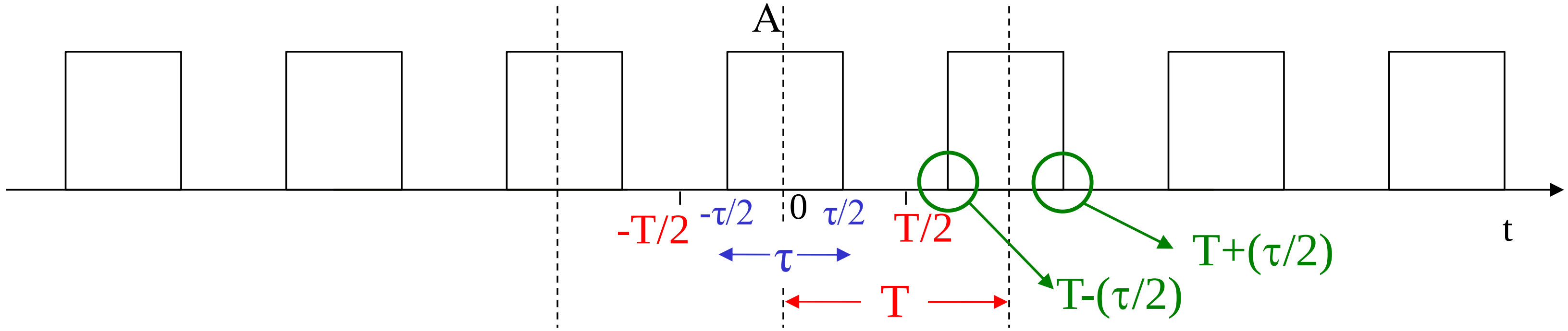
From the Fourier Series to the Fourier Transform – Q2

What happens to the Fourier series in the limit as period T goes to infinity ?



From the Fourier Series to the Fourier Transform

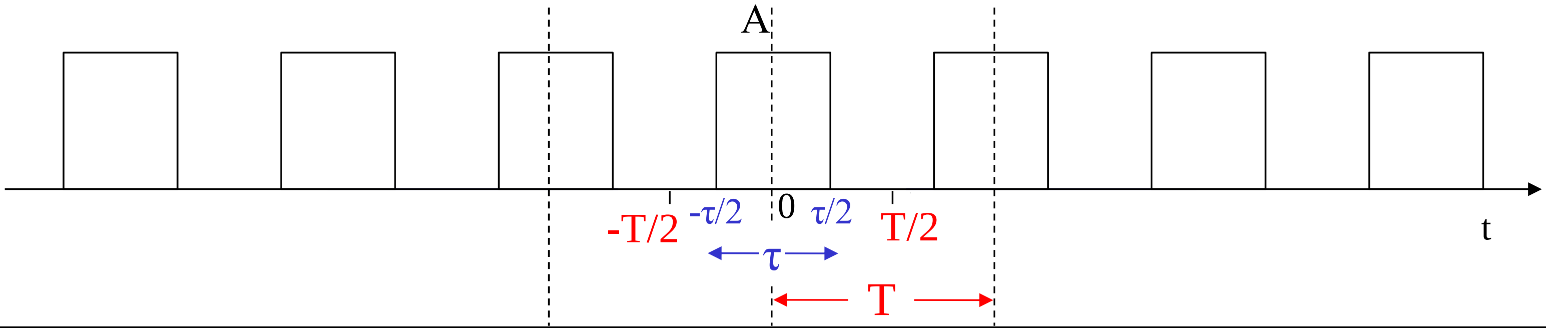
Find the Fourier Series coefficients of a pulse wave



1. Define x(t) :
$$x(t)=\begin{cases} A & -\dfrac{\tau}{2}\le t\le\dfrac{\tau}{2} \\ 0 & -\dfrac{T}{2}<t<-\dfrac{\tau}{2},\quad \dfrac{\tau}{2}<t\le\dfrac{T}{2} \end{cases}$$

2. Check signal is periodic : $x(t+T)=x(t)$

3. Apply forward transform : $c_k=\dfrac{1}{T}\displaystyle\int_T x(t)e^{-jk\omega_0 t}dt$



$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt$$

$$x(t) = \begin{cases} A & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & -\frac{T}{2} < t < -\frac{\tau}{2}, \quad \frac{\tau}{2} < t \le \frac{T}{2} \end{cases}$$

$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jk\omega_0 t}dt \qquad = \frac{A}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-A}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-\tau/2}^{\tau/2} = \frac{-A}{jk\omega_0 T}\left[e^{\frac{-jk\omega_0\tau}{2}} - e^{\frac{jk\omega_0\tau}{2}}\right]$$

Euler's formula:

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$-\quad e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} - e^{j\theta} = -2j\sin(\theta)$$

$$= \frac{-A}{jk\omega_0 T}\left[-2j\sin(k\omega_0\tau/2)\right] \qquad = \frac{2A}{T}\,\frac{\sin(k\omega_0\tau/2)}{k\omega_0}$$

Multiply with $\frac{\tau/2}{\tau/2}$

$$= \frac{A\tau}{T}\,\frac{\sin(k\omega_0\tau/2)}{k\omega_0\tau/2} \qquad \Rightarrow c_k = \frac{A\tau}{T}\operatorname{sinc}(k\omega_0\tau/2)$$

$\frac{\sin(x)}{x}$ *is the sinc function*

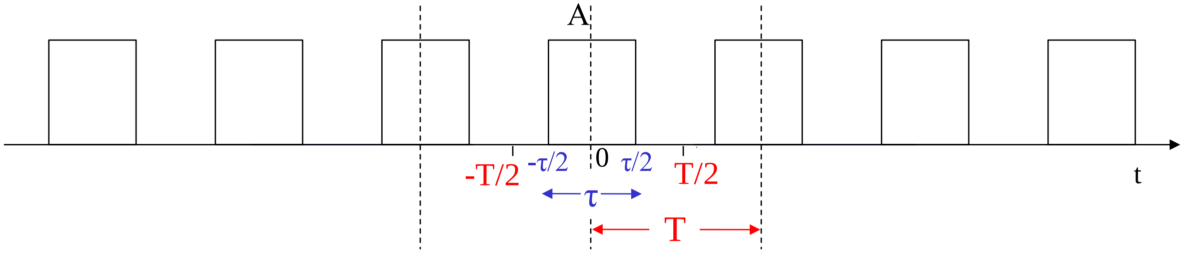
To maintain continuity sinc(x) is defined as equal to 1 for x =0 as sin(x) ≈x for very small x

From the Fourier Series to the Fourier Transform

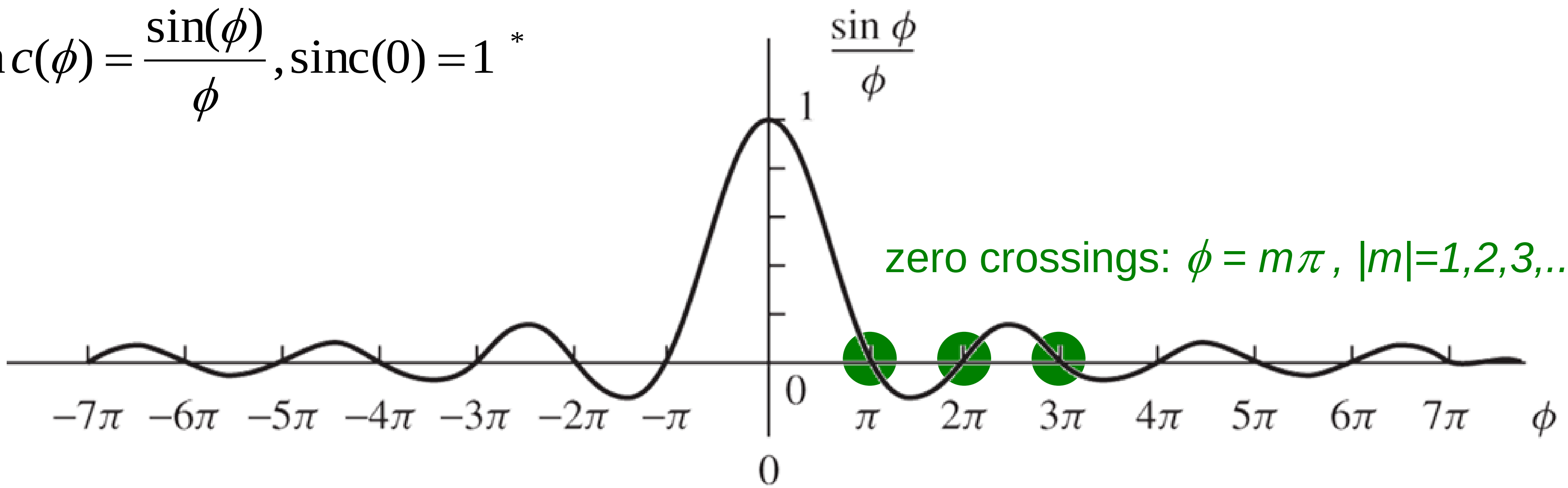
Find the Fourier Series coefficients of a pulse wave

$$c_k = \frac{A\tau}{T}\,\mathrm{sinc}(k\omega_0\tau/2)$$

Spectrum has shape of sinc(φ)



$$\mathrm{sinc}(\phi) = \frac{\sin(\phi)}{\phi},\ \mathrm{sinc}(0) = 1\ ^*$$



zero crossings: $\phi = m\pi$, $|m|$=1,2,3,..

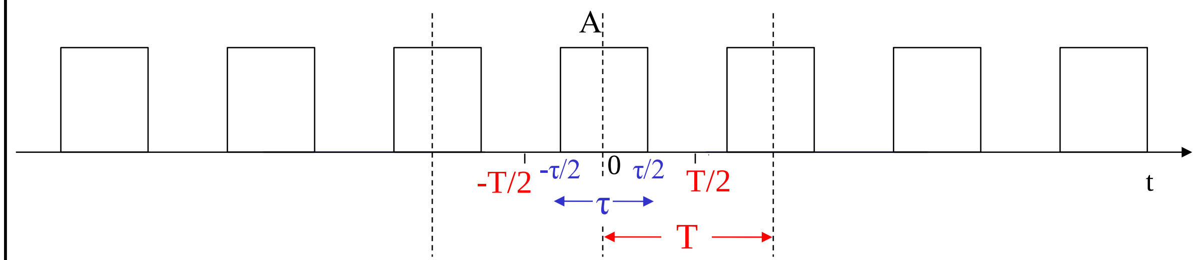
$$^*\ when\ \phi \approx 0,\ \mathrm{sinc}(\phi) = \frac{\sin(\phi)}{\phi} \approx \frac{\phi}{\phi} = 1$$

From the Father Series to the Fourier Transform

Find the Fourier Series coefficients of a pulse wave

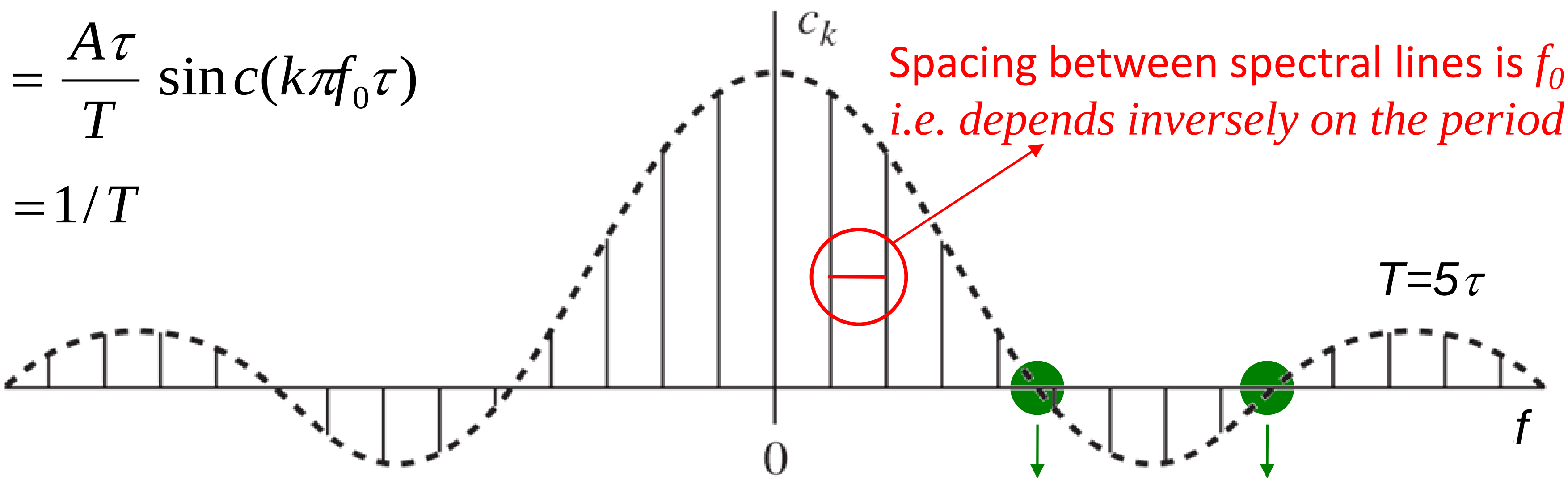
$$c_k = \frac{A\tau}{T}\,\mathrm{sinc}(k\omega_0\tau/2)$$

Scaled samples of sinc(φ)



$$c_k = \frac{A\tau}{T}\,\mathrm{sinc}(k\pi f_0\tau)$$

$$f_0 = 1/T$$



zero crossings: $\pi(kf_0)\tau = m\pi \Leftrightarrow kf_0 = m/\tau \Leftrightarrow f = m/\tau$

i.e. zero crossings don't depend on period

Fourier Transform

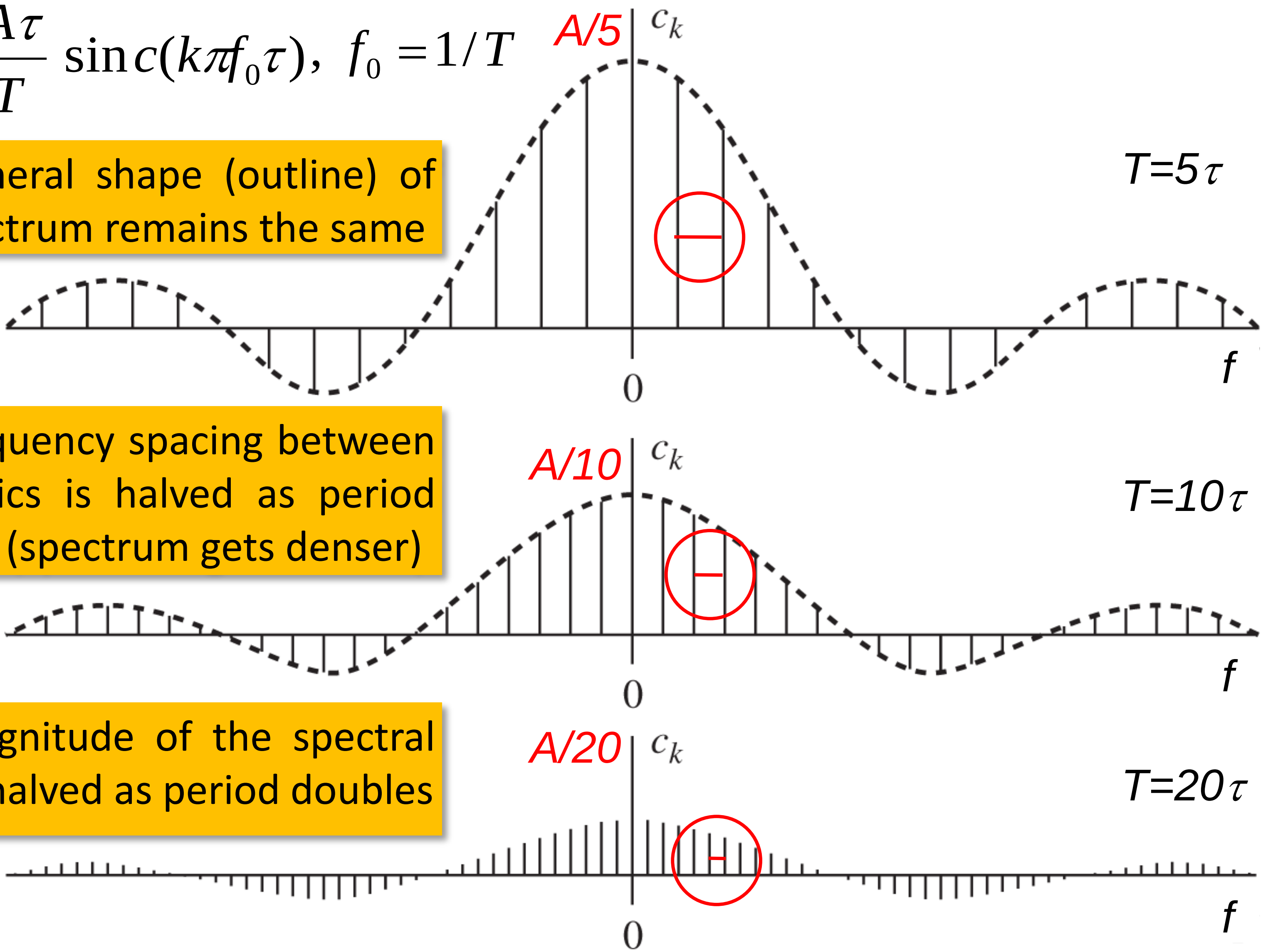
What happens to the Fourier series as period T increases?

$$c_k = \frac{A\tau}{T}\,\mathrm{sinc}(k\pi f_0 \tau),\ f_0 = 1/T$$

The general shape (outline) of the spectrum remains the same

The frequency spacing between harmonics is halved as period doubles (spectrum gets denser)

The magnitude of the spectral lines is halved as period doubles



What happens in the limit as period T goes to infinity ?

$$X_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt$$

$$as \;\; T \to \infty \, , \frac{1}{T} \to 0$$

$$\lim_{T\to\infty} X_k = 0$$

Define $X_k^{'} = T\, X_k$

$$X_k^{'} = \int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \qquad (1)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{X_k^{'}}{T} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} X_k^{'}\, e^{jk\omega_0 t}\,\frac{\omega_0}{2\pi}$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$x(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X_k^{'}\, e^{jk\omega_0 t}\omega_0 \qquad (2)$$

What happens in the limit as period T goes to infinity ?

$$X_k' = \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt \quad (1)$$

$$x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X_k' e^{jk\omega_0 t} \omega_0 \quad (2)$$

$$\omega_0 \to d\omega \ , \ k\omega_0 \to \omega \ , \ X_k' \to X(\omega)$$

$$\int_{-T/2}^{T/2} \to \int_{-\infty}^{\infty} \ , \ \sum_{k=-\infty}^{\infty} \to \int_{-\infty}^{\infty} \quad (3)$$

(1) (3)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Forward Fourier Transform

Analysis function

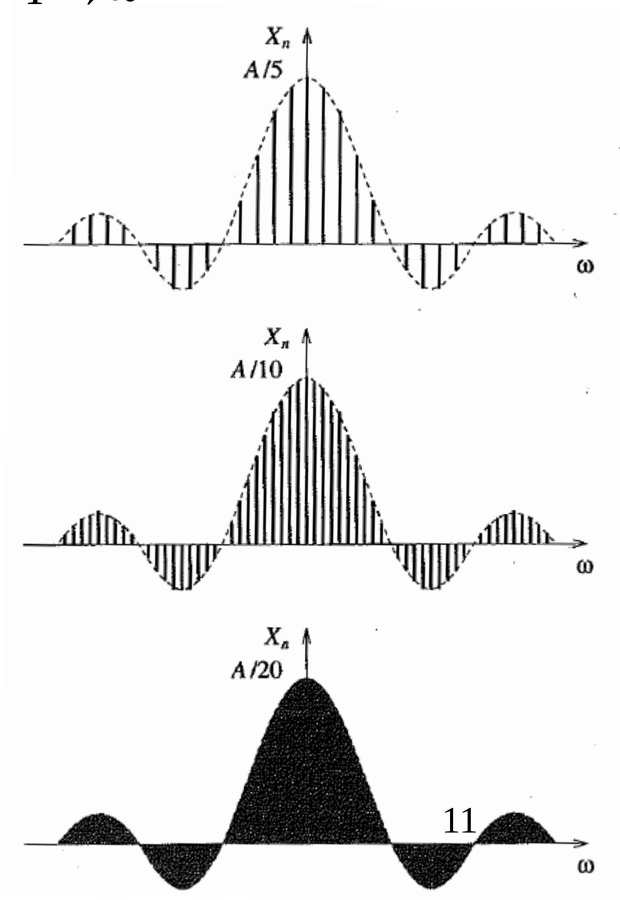
(2) (3)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Inverse Fourier Transform

Synthesis function

$$\lim_{T\to\infty}$$



Fourier Transform

From the Fourier Series to the Fourier Transform

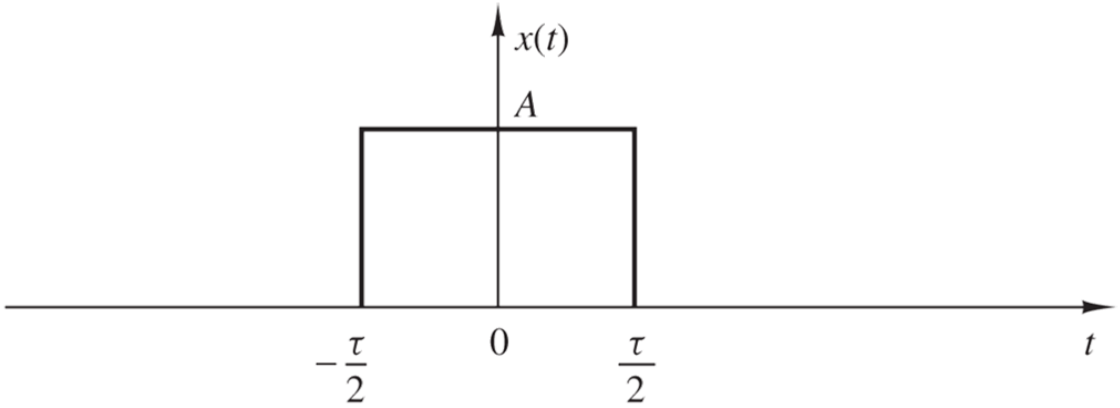
Transform	Synthesis Equation (inverse transform)	Analysis Equation (forward transform)
Fourier Series	$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$	$X_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$
Fourier Transform *Radial Frequency*	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
Normal Frequency	$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$	$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

Why is there no $1/2\pi$ factor in the ordinary frequency version?

Fourier Transform

Determine the Fourier Transform of a rectangular pulse

$$x(t) = \begin{cases} A & |t| \le \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} dt \; = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2} = A\frac{\left(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right)}{-j\omega}$$

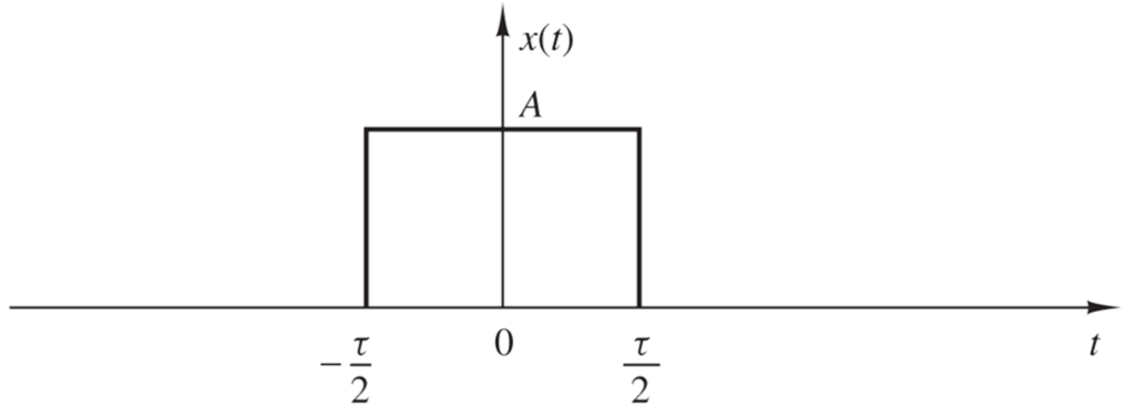
$$= A\frac{\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right)}{j\omega} \quad = \frac{2Aj\sin(\omega\tau/2)}{j\omega} = \frac{2A\sin(\pi f\tau)}{2\pi f} = A\tau\frac{\sin(\pi f\tau)}{\pi f\tau}$$

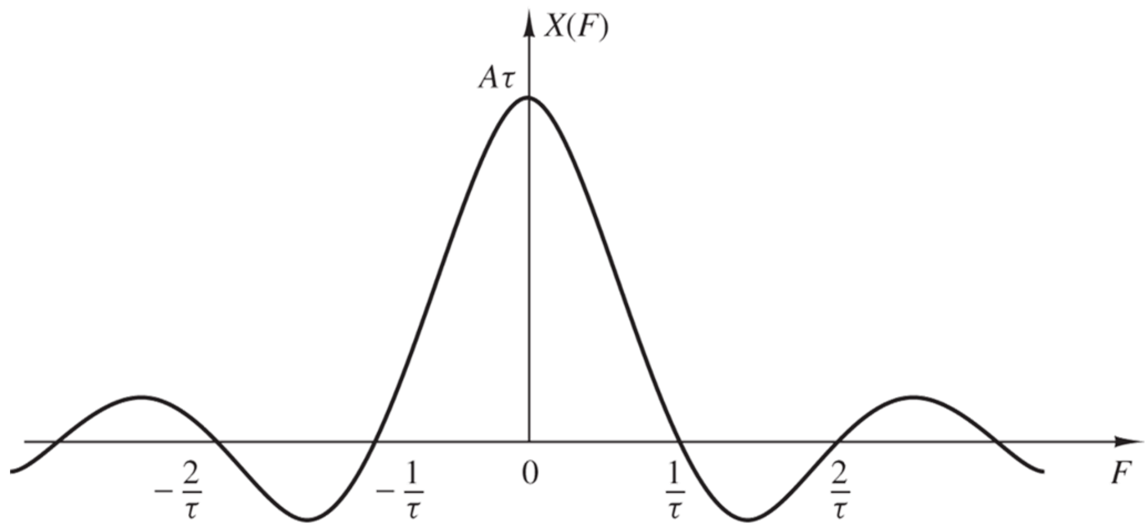
$$X(f) = A\tau\,\mathrm{sinc}(\pi f\tau) \quad \Leftrightarrow \quad X(\omega) = A\tau\,\mathrm{sinc}(\omega\tau/2)$$

Fourier Transform

Determine the Fourier Transform of a rectangular pulse

$$x(t) = \begin{cases} A & |t| \le \tau / 2 \\ 0 & \text{elsewhere} \end{cases}$$



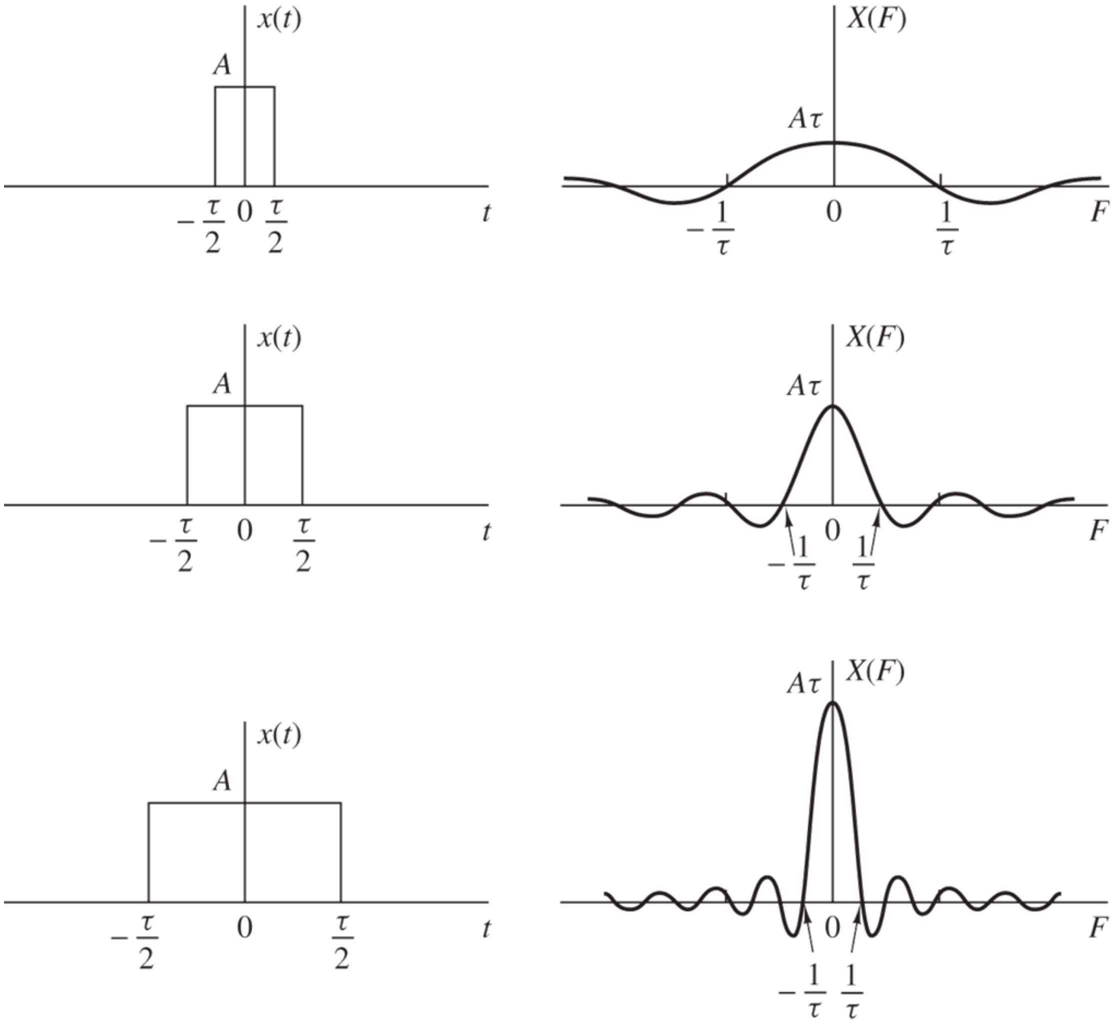


$$X(f) = A\tau\,\mathrm{sinc}(\pi f \tau)$$

Fourier Transform

Determine the Fourier Transform of a rectangular pulse

$$x(t) = \begin{cases} A & |t| \le \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$



Effect of pulse width

$$X(f) = A\tau \,\mathrm{sinc}(\pi f \tau)$$

Fourier Transform Properties

How a change in one domain affects the other domain

Property	Signal (Time Domain)	Transform (Frequency Domain)	
	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$	$X(\omega)=\int_{-\infty}^{\infty}x(t).e^{-j\omega t}.dt$	
Symmetry (Duality)	$x(t)$ $X(t)$	$X(\omega)$ $2\pi x(-\omega)$	Slide 27
Linearity	$\alpha x(t)+\beta y(t)$	$\alpha X(\omega)+\beta Y(\omega)$	Slide 17
Time-shift	$x(t-\tau)$	$e^{-j\omega\tau}X(\omega)$	Slide 20
Frequency-shift	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$	Slide 23
Impulse	$\delta(t)$ $\delta(t-\tau)$	1 $e^{-j\omega\tau}$	
Complex exponential	1 $e^{j\omega_0 t}$	$2\pi\delta(\omega)$ $2\pi\delta(\omega-\omega_0)$	
Cosine	$\cos(\omega_0 t)=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$	
Sine	$\sin(\omega_0 t)=\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}$	$j\pi\delta(\omega+\omega_0)-j\pi\delta(\omega-\omega_0)$	
Impulse train	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0) \quad \omega_0=2\pi/T$	Slide 28
Time Convolution	$x(t)*y(t)$	$X(\omega)Y(\omega)$	Slide 22
Frequency convolution	$x(t)y(t)$	$\frac{1}{2\pi}X(\omega)*Y(\omega)$	Slide 23
Symmetric signals	$x(t)=x^*(-t)$	$X(\omega)=X^*(\omega)$ real	
Real signals	$x(t)=x^*(t)$	$X(\omega)=X^*(-\omega)$ symmetric	Slide 25

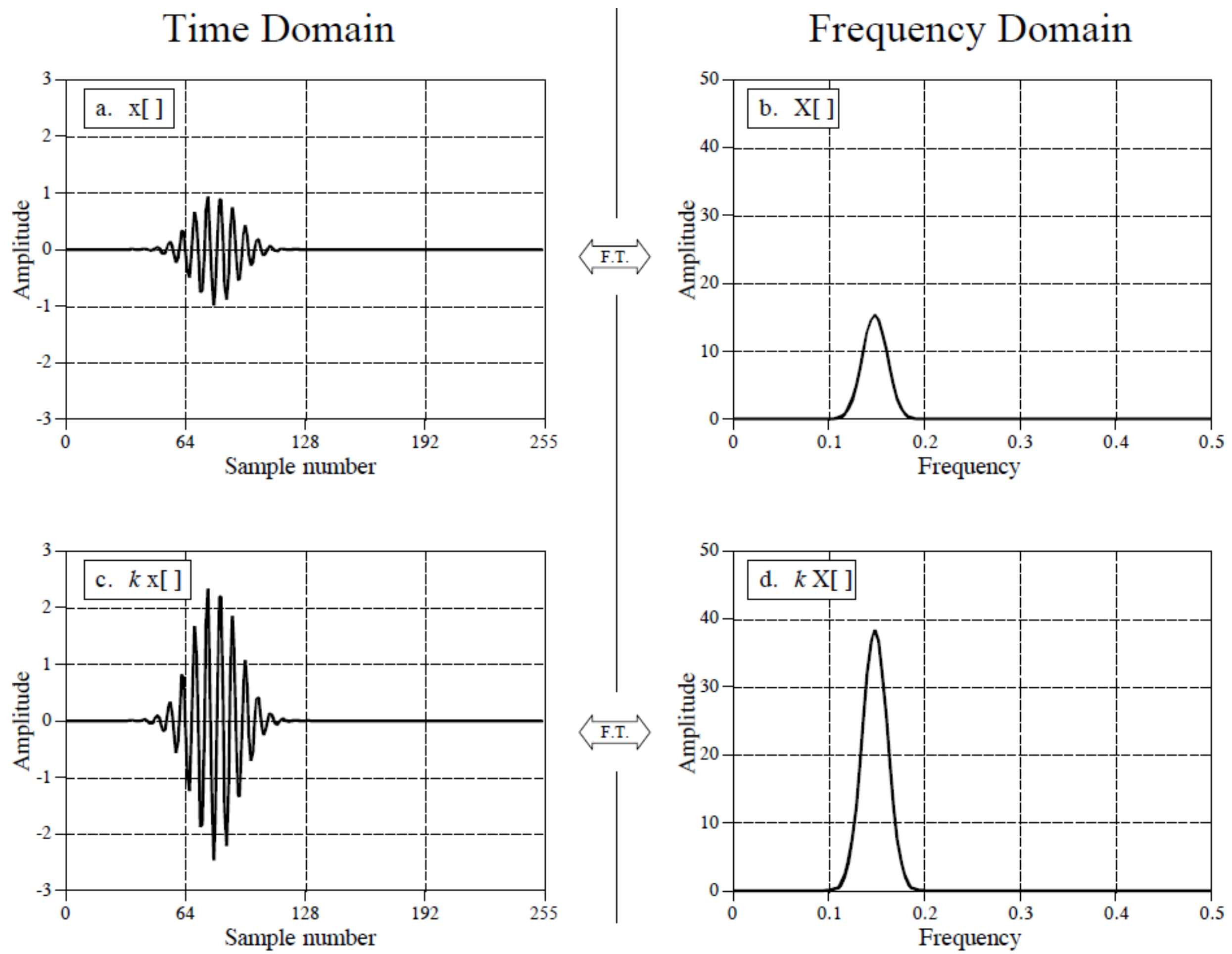
Fourier Transform Properties

Linearity

$$\alpha\, x(t) + \beta\, y(t) \overset{F}{\leftrightarrow} \alpha\, X(\omega) + \beta\, Y(\omega)$$

Homogeneity (scaling property)

- If the amplitude is changed in one domain, it is changed by the same amount in the other domain.
- In other words, *scaling* in one domain corresponds to *scaling* in the other domain



Note that these graphs have been created using the Discrete Fourier Transform

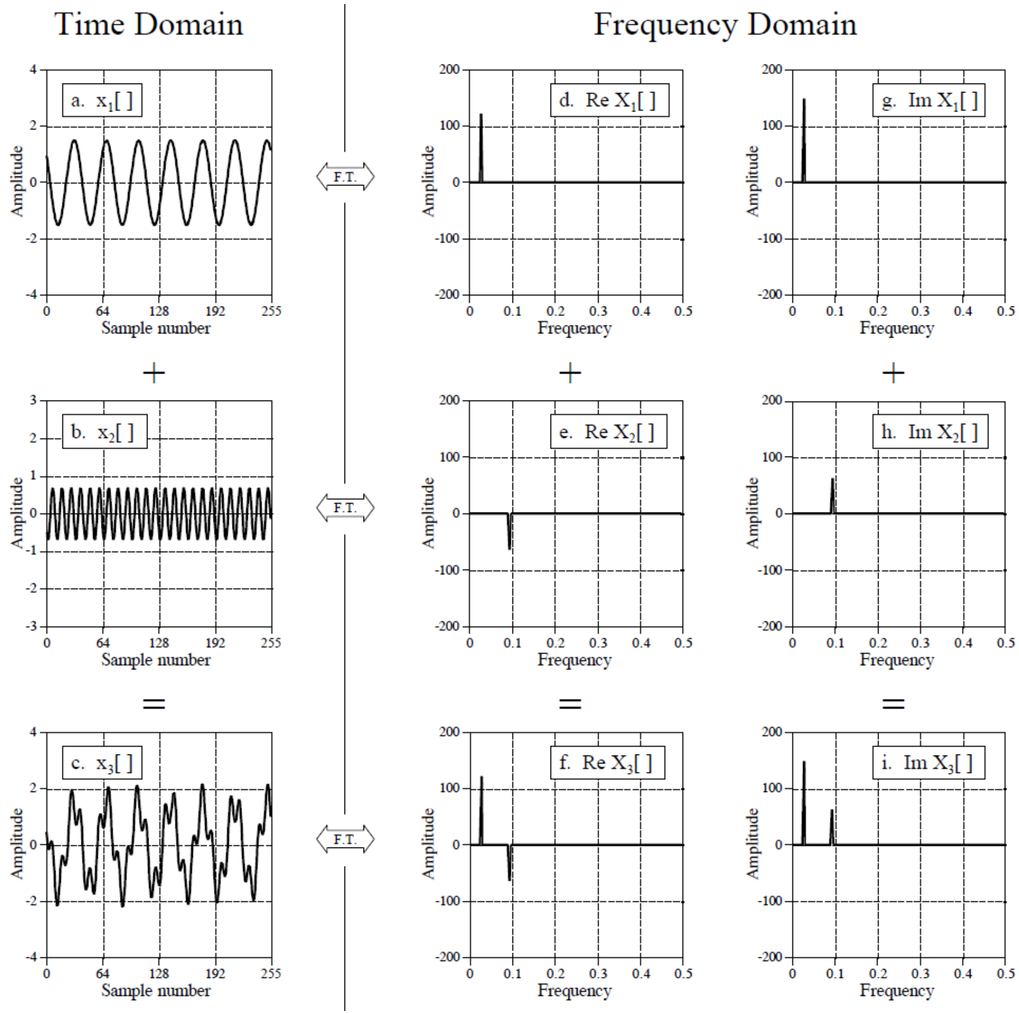
Fourier Transform Properties

Linearity

$$\alpha\, x(t) + \beta\, y(t) \overset{F}{\leftrightarrow} \alpha\, X(\omega) + \beta\, Y(\omega)$$

Additivity

- Adding two or more signals in one domain results in the corresponding signals being added in the other domain.
- In this illustration, the time domain signals in (a) and (b) are added to produce the signal in (c). This results in the corresponding real and imaginary parts of the frequency spectra being added.



Note that these graphs have been created using the Discrete Fourier Transform

Fourier Transform Properties

Linearity $\alpha\, x(t) + \beta\, y(t) \overset{F}{\leftrightarrow} \alpha\, X(\omega) + \beta Y(\omega)$

Proof (obvious…)

$$\mathcal{F}\left[c_1 x_1(t) + c_2 x_2(t)\right] = \int_{-\infty}^{\infty} \left[c_1 x_1(t) + c_2 x_2(t)\right] e^{-j\omega t} dt$$

$$= c_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + c_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$= c_1 X_1(\omega) + c_2 X_2(\omega)$$

Fourier Transform Properties

Time Shift

$$x(t-\tau) \overset{F}{\leftrightarrow} e^{-j\omega\tau} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad , \quad Y(\omega) = ?$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t-\tau) e^{-j\omega t} dt$$

Substitute $t' = t - \tau$

$$Y(\omega) = \int_{-\infty}^{\infty} x(t') e^{-j\omega(t'+\tau)} dt'$$

$$e^{-j\omega(t'+\tau)} = e^{-j\omega t'} e^{-j\omega\tau}$$

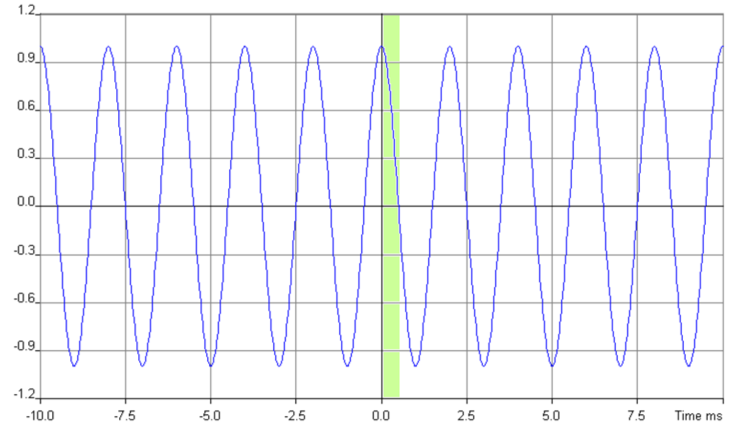
$$= e^{-j\omega\tau} \int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt' \Rightarrow Y(\omega) = e^{-j\omega\tau} X(\omega)$$

Shifting a signal in time changes linearly the phase of its spectrum ($\omega\tau$). The magnitude of the spectrum is not affected

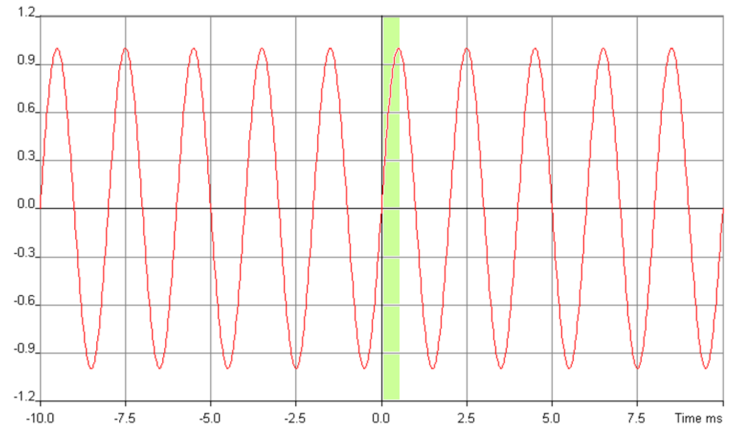
τ = 0.5ms
T = 2ms(f = 500Hz)

$$Y(\omega) = e^{-j\frac{\pi}{2}} X(\omega)$$

$$x(t) = \cos\omega t, \; T = 2ms$$



Delay of 0.5 ms (shift to the right)

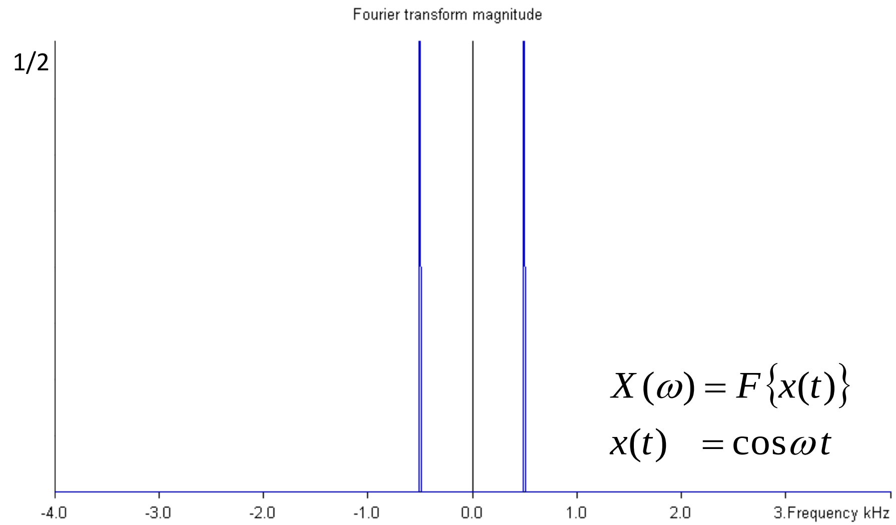


$$y(t) = x(t-\tau) = \cos\omega(t-\tau), \; \tau = 0.5ms$$

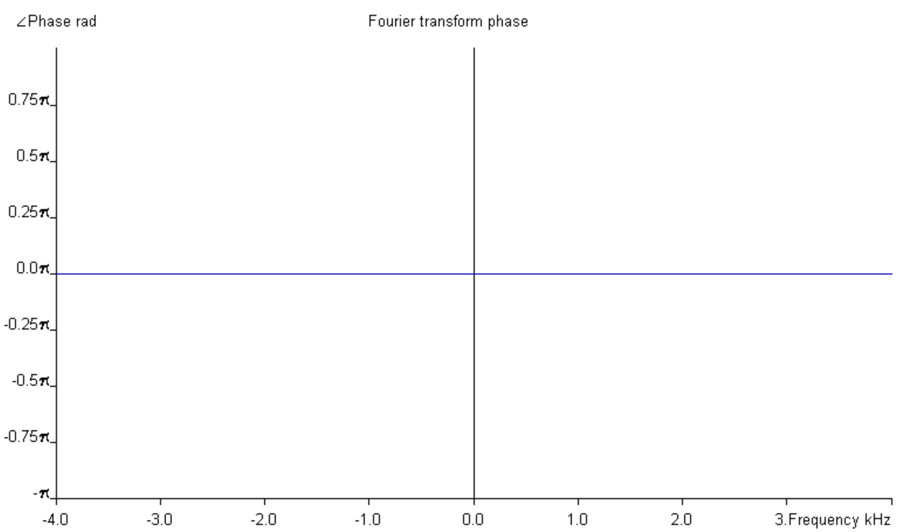
Fourier Transform Properties

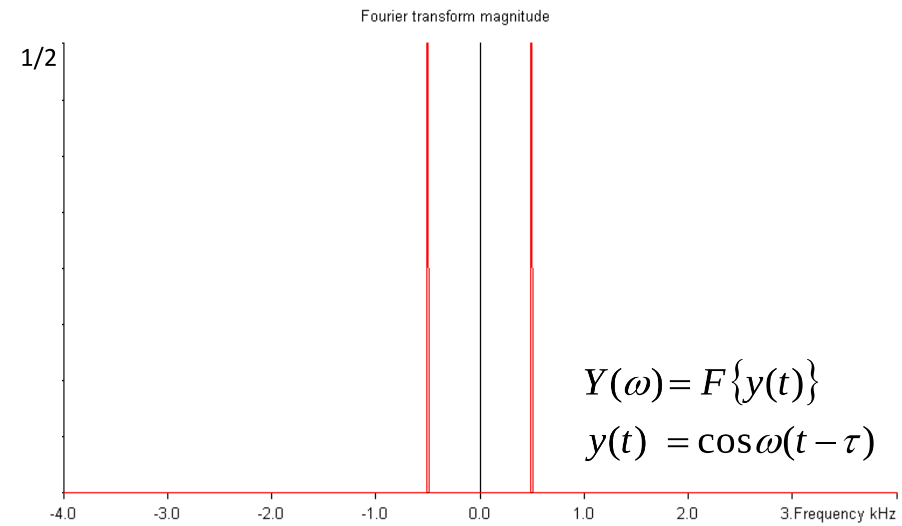
Time Shift

$$x(t-\tau) \overset{F}{\leftrightarrow} e^{-j\omega\tau} X(\omega)$$

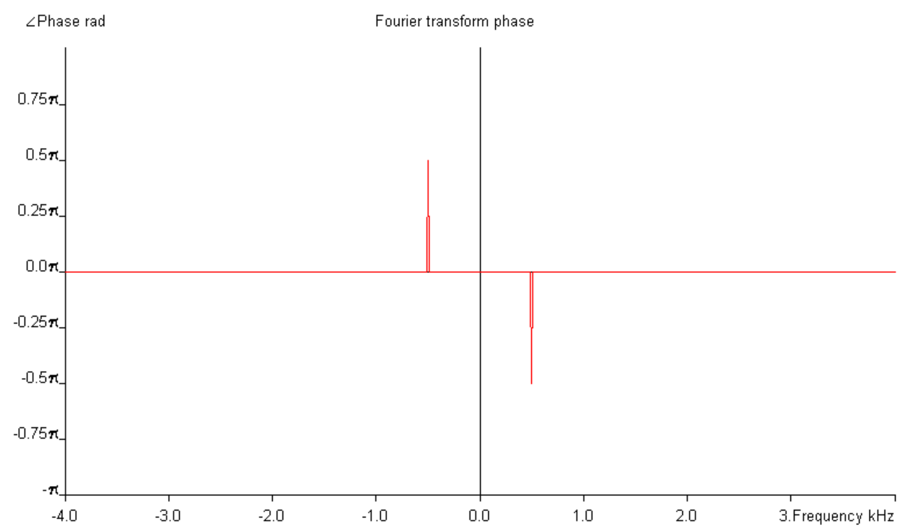


$$X(\omega) = F\{x(t)\}$$
$$x(t) = \cos\omega t$$





$$Y(\omega) = F\{y(t)\}$$
$$y(t) = \cos\omega(t-\tau)$$



Fourier Transform Properties

Time Convolution

$$h(t)*x(t)\overset{F}{\leftrightarrow}H(\omega)X(\omega)$$

$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$	The convolution integral
$Y(\omega)=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau\right)e^{-j\omega t}dt$	Take the Fourier transform of the convolution integral
$Y(\omega)=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x(\tau)h(t-\tau)e^{-j\omega t}dt\right)d\tau$	Interchange the order of integration
$Y(\omega)=\int_{-\infty}^{\infty}x(\tau)\left(\int_{-\infty}^{\infty}h(t-\tau)e^{-j\omega t}dt\right)d\tau$	Move *x(τ)* (constant) outside inner integral
$Y(\omega)=\int_{-\infty}^{\infty}x(\tau)e^{-j\omega\tau}H(\omega)d\tau$	Use the time shift property of the FT to evaluate the inner integral
$Y(\omega)=H(\omega)\int_{-\infty}^{\infty}x(\tau)e^{-j\omega\tau}d\tau=H(\omega)X(\omega)$	Move *H(ω)* (constant) outside integral

Convolution in time becomes multiplication in frequency

Fourier Transform Properties

Frequency Convolution

$$x(t)y(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

- Convolution in frequency is equivalent to multiplication in time

This leads to the Frequency Shift property...

Frequency Shift

$$e^{j\omega_0 t} x(t) \overset{F}{\leftrightarrow} X(\omega - \omega_0)$$

- Multiplication with a complex exponential in time results in frequency convolution with a delta function
- The frequency of the baseband signal is shifted around that of the delta function

...which is the basis of Amplitude Modulation

Amplitude Modulation

$$\cos(\omega_0 t) x(t) \overset{F}{\leftrightarrow} \tfrac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$$

- Multiplication with a sinusoid

$$F[x(t)\cos(\omega_0 t)] = F\left[\frac{1}{2}x(t)\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right] \overset{\text{linearity}}{=} \frac{1}{2}F\left[x(t)e^{j\omega_0 t}\right] + \frac{1}{2}F\left[x(t)e^{-j\omega_0 t}\right] \Leftrightarrow$$

$$\overset{\text{frequency shift}}{=} \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$$

Fourier Transform Properties

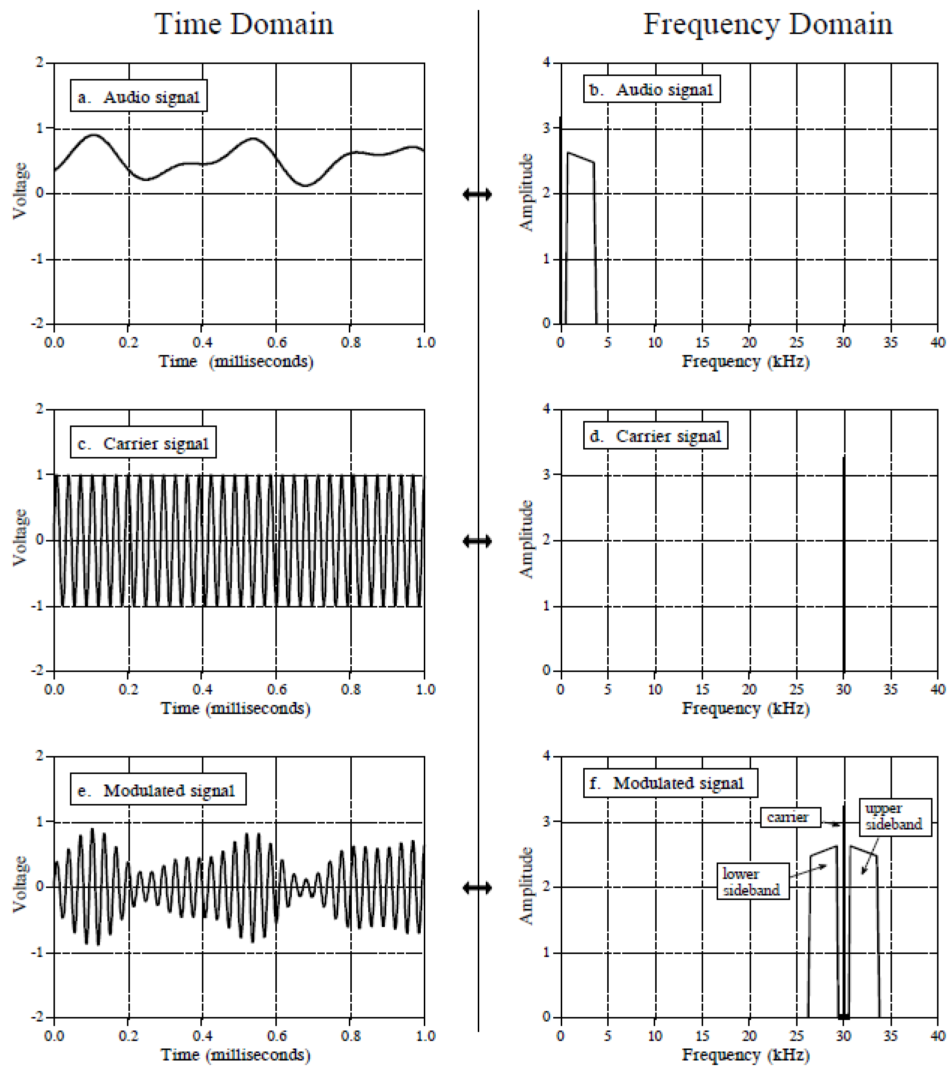
Amplitude modulation

$$\cos(\omega_0 t)x(t) \overset{F}{\leftrightarrow} \tfrac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$$

$x(t)$ $\leftrightarrow$ $X(\omega)$

$\cos(\omega_0 t)$ $\leftrightarrow$ $\dfrac{1}{2}[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$

$\cos(\omega_0 t)x(t)$ $\leftrightarrow$ $\dfrac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$



Note that only the positive frequencies are shown

Fourier Transform Properties

Real Signals $\quad x(t)=x^*(t) \overset{F}{\leftrightarrow} X(\omega)=X^*(-\omega)$

The spectrum of real valued signals displays Hermitian symmetry

$$\boxed{x(t)=x^*(t)}^{(1)} \Leftrightarrow x_R(t)+jx_I(t)=x_R(t)-jx_I(t) \Leftrightarrow 2jx_I(t)=0 \Leftrightarrow x_I(t)=0$$

Mathematical Proof

$$X^*(-\omega)=\left(\int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt\right)^* = \int_{-\infty}^{\infty} x(t)^* e^{j(-\omega)t}dt \overset{(1)}{=} \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(\omega)$$

What does it mean?

$$\left\{\begin{matrix} X(\omega) & = X_R(\omega) & + & jX_I(\omega) \\ X^*(-\omega) & = X_R(-\omega) & - & jX_I(-\omega) \end{matrix}\right\} \Rightarrow \begin{matrix} X_R(-\omega)=X_R(\omega) \\ X_I(-\omega)=-X_I(\omega) \end{matrix}$$

$$|X(-\omega)|=\sqrt{X_R^2(\omega)+X_I^2(\omega)}=|X(\omega)| \quad , \quad \angle X(-\omega)=\tan^{-1}\frac{-X_I(\omega)}{X_R(\omega)}=-\angle X(\omega)$$

Spectral plots of real signals are normally displayed only for positive frequencies (esp. with sampled signals)

Fourier Transform Properties

Real Signals

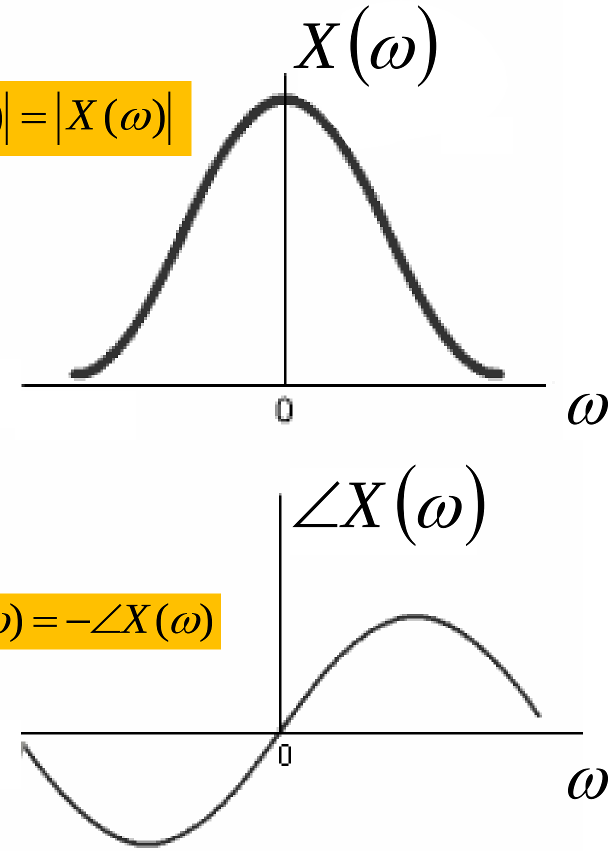
$$x(t) = x^*(t) \overset{F}{\leftrightarrow} X(\omega) = X^*(-\omega)$$

- In the frequency domain, real-valued signals/systems have always even symmetric amplitude spectrum/response and odd-symmetric phase spectrum/response with respect to the zero frequency (origin, two-sided spectra)

- Complex signals don’t (need to) have any symmetry properties in general e.g., the spectral support (region of non-zero amplitude spectrum) can basically be anything

$|X(-\omega)| = |X(\omega)|$

$\angle X(-\omega) = -\angle X(\omega)$

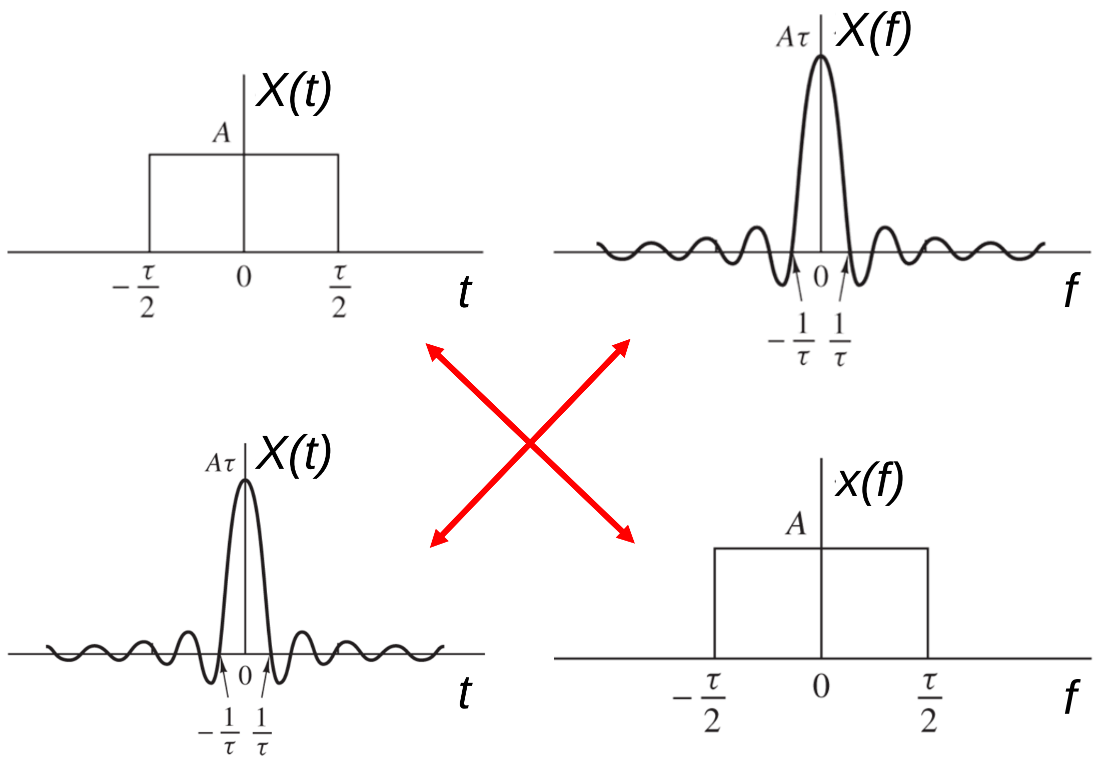


Hermitian symmetry

Fourier Transform Properties

Duality (Symmetry) $if\ \ F[x(t)] = X(f),\ then\ F[X(t)] = x(-f)$

- If $x(t)$ has a Fourier Transform $X(\omega)$, then if we form a new function of time that has the functional form of the transform, $X(t)$, it will have a Fourier Transform $x(\omega)$ that has the functional form of the original time function (but is a function of frequency).



$$if\ \ F[x(t)] = X(\omega),\ then\ F[X(t)] = 2\pi x(-\omega)$$

Fourier Transform

Fourier Transform of Periodic Signals

The Fourier transform of periodic signals is the Fourier series

$$g(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)\overset{F}{\leftrightarrow}G(\omega)=\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$$

FT of impulse train is a scaled impulse train with impulses at $k\omega_0$ where $\omega_0 = 2\pi/T$

$x(t)$ periodic: $x(t+T)=x(t)$ can be expressed as

$$x(t)=g(t)*x'(t)\ ,\quad x'(t)=\begin{cases}x(t) & 0\le t<T\\ 0 & \text{elsewhere}\end{cases}$$

Convolution in time becomes multiplication in frequency…

$$X(\omega)=G(\omega)X'(\omega)\ =\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)\int_0^T x(t)e^{-j\omega t}dt$$

Rearrange to get …

$$X(\omega)=\left(\omega_0\int_0^T x(t)e^{-j\omega t}.dt\right)\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$$

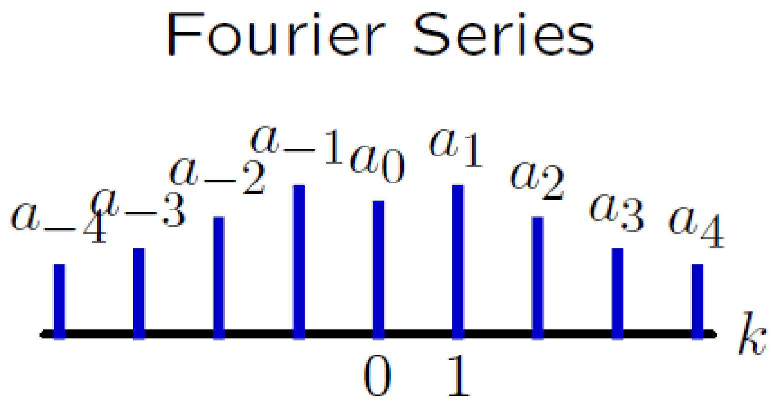
A weighted set of impulses at multiples of ω_0. Zero elsewhere…

$$\omega_0\int_0^T x(t)e^{-jk\omega_0 t}dt\ =\frac{2\pi}{\boxed{T}}\boxed{\int_0^T x(t)e^{-jk\omega_0 t}dt}=2\pi\boxed{X_k}$$

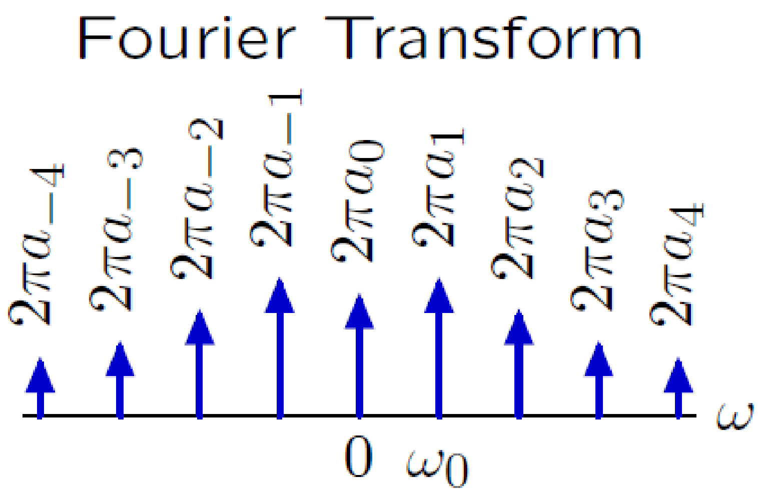
Where the weights are … the Fourier Series coefficients

Fourier Transform of Periodic Signals

The Fourier transform of periodic signals is the Fourier series



$$X_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$



$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k \, \delta(\omega - k\omega_0)$$