Lecture 7
(Part 2)

The Discrete Fourier Transform
Properties of the DFT

How the finite length assumption and the implicit periodicity affect the properties of the DFT

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Fourier Transform Properties

How a change in one domain affects the other domain

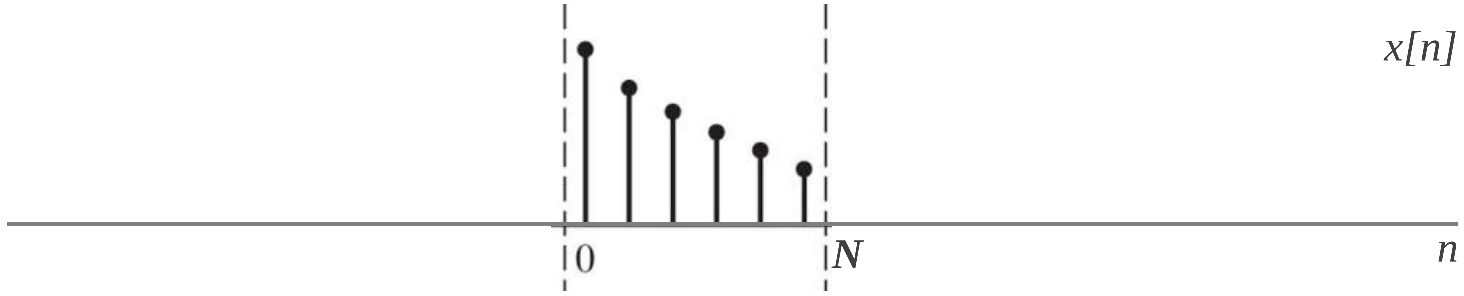
Property	Signal (Time Domain)	Transform (Frequency Domain)
	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$	$X(\omega)=\int_{-\infty}^{\infty}x(t).e^{-j\omega t}.dt$
Symmetry (Duality)	$x(t)$ $X(t)$	$X(\omega)$ $2\pi x(-\omega)$
Linearity	$\alpha x(t)+\beta y(t)$	$\alpha X(\omega)+\beta Y(\omega)$
Time-shift	$x(t-\tau)$	$e^{-j\omega\tau}X(\omega)$
Frequency-shift	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Impulse	$\delta(t)$ $\delta(t-\tau)$	1 $e^{-j\omega\tau}$
Complex exponential	1 $e^{j\omega_0 t}$	$2\pi\delta(\omega)$ $2\pi\delta(\omega-\omega_0)$
Cosine	$\cos(\omega_0 t)=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
Sine	$\sin(\omega_0 t)=\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}$	$j\pi\delta(\omega+\omega_0)-j\pi\delta(\omega-\omega_0)$
Impulse train	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)\quad \omega_0=2\pi/T$
Time Convolution	$x(t)*y(t)$	$X(\omega)Y(\omega)$
Frequency convolution	$x(t)y(t)$	$\frac{1}{2\pi}X(\omega)*Y(\omega)$
Symmetric signals	$x(t)=x^*(-t)$	$X(\omega)=X^*(\omega)$ real
Real signals	$x(t)=x^*(t)$	$X(\omega)=X^*(-\omega)$ symmetric

Discrete Fourier Transform

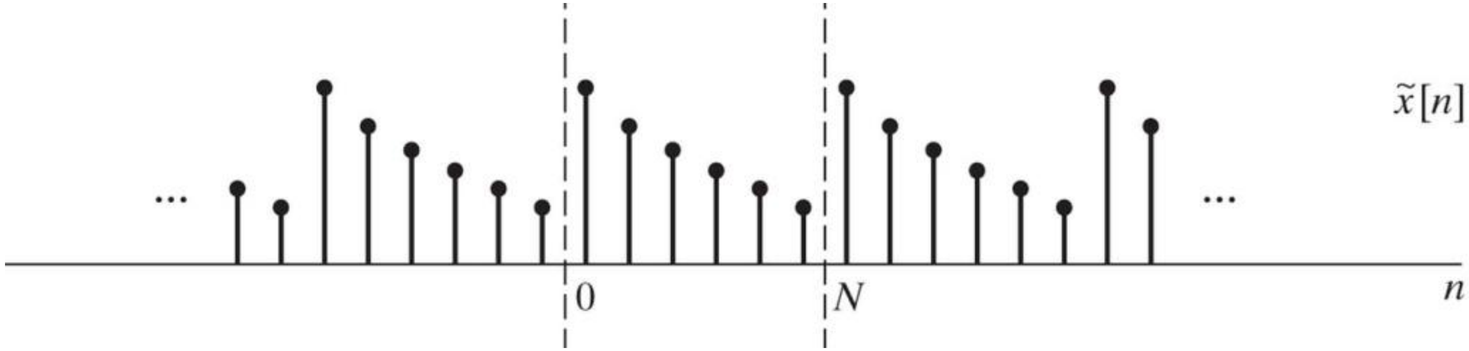
Windowing and Frequency Sampling

Finite length assumption and implicit periodicity

Signal is undefined outside 0 ≤ n ≤ N-1 (finite length)



Signal is implicitly periodic when using the DFT to represent it



Discrete Fourier Transform

Discrete Fourier Transform Properties

Property	Signal	Transform	
	$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{jkn\frac{2\pi}{N}}$	$X[k]=\sum_{n=0}^{N-1}x[n]e^{-jkn\frac{2\pi}{N}}$	
Linearity	$\alpha x[n]+\beta y[t]$	$\alpha X[k]+\beta Y[k]$	Slide 05
Symmetry	$x[n]$ $X[n]$	$X[k]$ $Nx[-k]$	
Cyclic Time-shift	$x[(n-n_0)\bmod N]$	$e^{-jkn_0\frac{2\pi}{N}}X[k]$	Slide 06
Cyclic Frequency-shift	$e^{jk_0n\frac{2\pi}{N}}x[n]$	$X[(k-k_0)\bmod N]$	
Impulse	$\delta[n]$ $\delta[(n-n_0)\bmod N]$	1 $e^{-jkn_0\frac{2\pi}{N}}$	
Complex exponential	1 $e^{jk_0n\frac{2\pi}{N}}$	$N\delta[k]$ $N\delta[(k-k_0)\bmod N]$	
Cosine	$\cos(nk_0\frac{2\pi}{N})=\frac{e^{jnk_0\frac{2\pi}{N}}+e^{-jnk_0\frac{2\pi}{N}}}{2}$	$\frac{N}{2}\delta[(k-k_0\frac{2\pi}{N})\bmod N]$ $+\frac{N}{2}\delta[(k+k_0\frac{2\pi}{N})\bmod N]$	
Sine	$\sin(nk_0\frac{2\pi}{N})=\frac{e^{jnk_0\frac{2\pi}{N}}-e^{-jnk_0\frac{2\pi}{N}}}{2j}$	$\frac{jN}{2}\delta[(k+k_0\frac{2\pi}{N})\bmod N]$ $-\frac{jN}{2}\delta[(k-k_0\frac{2\pi}{N})\bmod N]$	
Time Convolution	$x[n]\otimes y[n]$	$X[k]Y[k]$	Slide 12
Frequency convolution	$x[n]y[n]$	$\frac{1}{N}X(\omega)\otimes Y(\omega)$	
Symmetric signals	$x[n]=x^*[-n]$	$X[k]=X^*[k]$ real	
Real signals	$x[n]=x^*[n]$	$X[k]=X^*[N-k]$ symmetric	
Complex conjugate	$x^*[n]$	$X^*[N-k]$	

Discrete Fourier Transform

Linearity

Finite length issues

$$y[n] = ax_1[n] + bx_2[n] \overset{DFT}{\longleftrightarrow} = Y[k] = aX_1[k] + bX_2[k],$$

$$\text{where } x_1[n] \overset{DFT}{\longleftrightarrow} X_1[k] \,\&\, x_2[n] \overset{DFT}{\longleftrightarrow} X_2[k]$$

For this to be meaningful both DFTs $X_1[k]$ and $X_2[k]$ should be computed with same number of points N

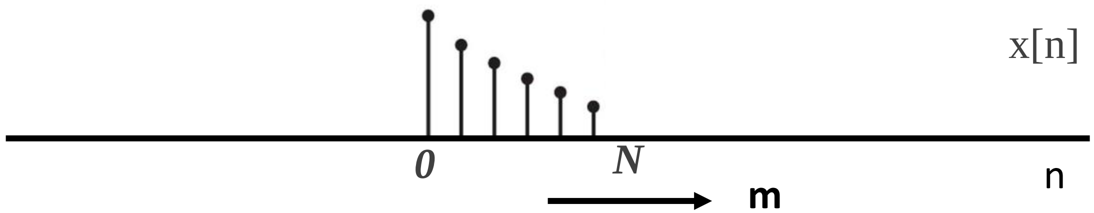
Discrete Fourier Transform

Circular Shift

Linear shift : problem due to finite length assumption of DFT

DTFT Time Shift: $x[n-m] \overset{DTFT}{\longleftrightarrow} e^{-j\Omega m} X(\Omega)$

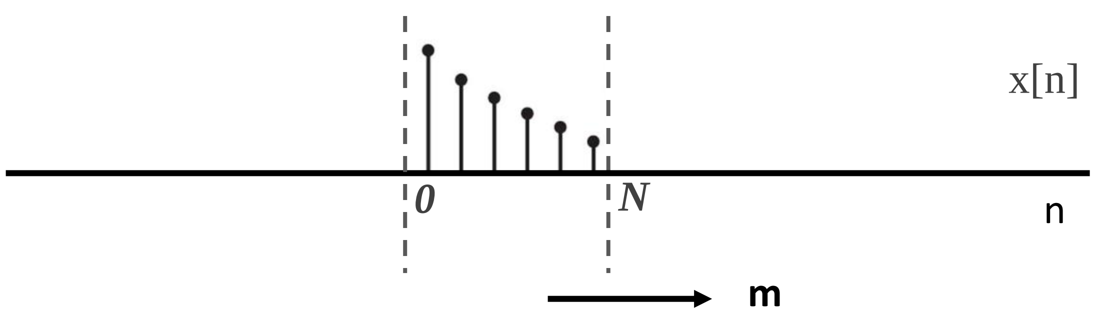
$$X(\Omega) = \boxed{\sum_{n=-\infty}^{\infty}} x[n]\, e^{-j\Omega n}$$



x[n] defined from −∞ to ∞

DFT Time Shift: $? \overset{DFT}{\longleftrightarrow} e^{-j(2\pi k/N)m} X[k]$

$$X[k] = \boxed{\sum_{n=0}^{N-1}} x[n]\, e^{-j(2\pi k/N)n}$$



x[n] undefined outside

0 ≤ n ≤ N-1

Cannot shift sequence linearly

Circular Shift

Circular shift as linear shift of periodic signal

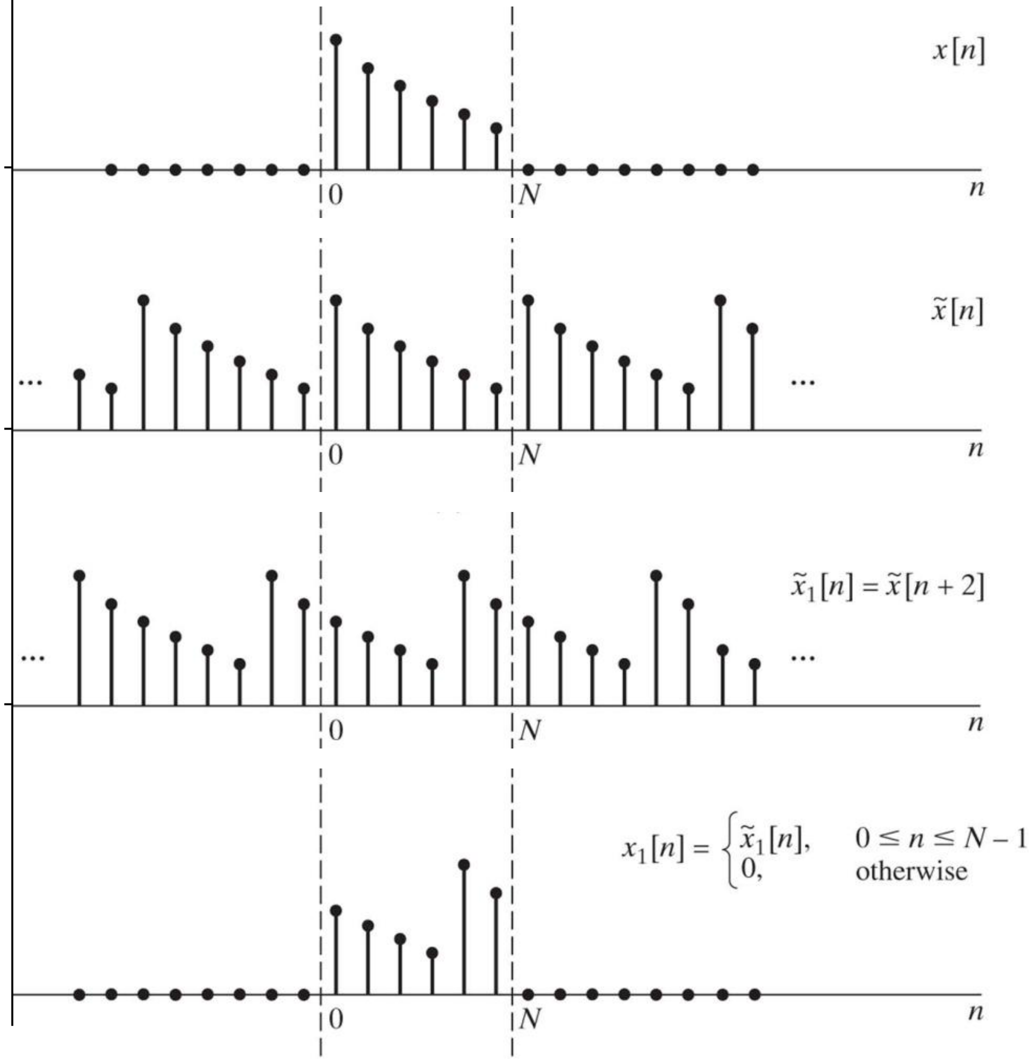
a) $x[n]$ **- Input signal**

b) $\tilde{x}[n]$ **- Periodic version of x[n]**

c) $\tilde{x}[n-m]$ **- Linear shift**

m=-2 (m<0 => time advance)

d) $x_1[n]$ **- Circularly shifted x[n]**

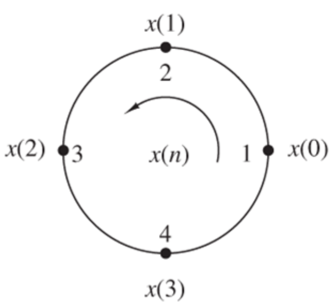


Discrete Fourier Transform

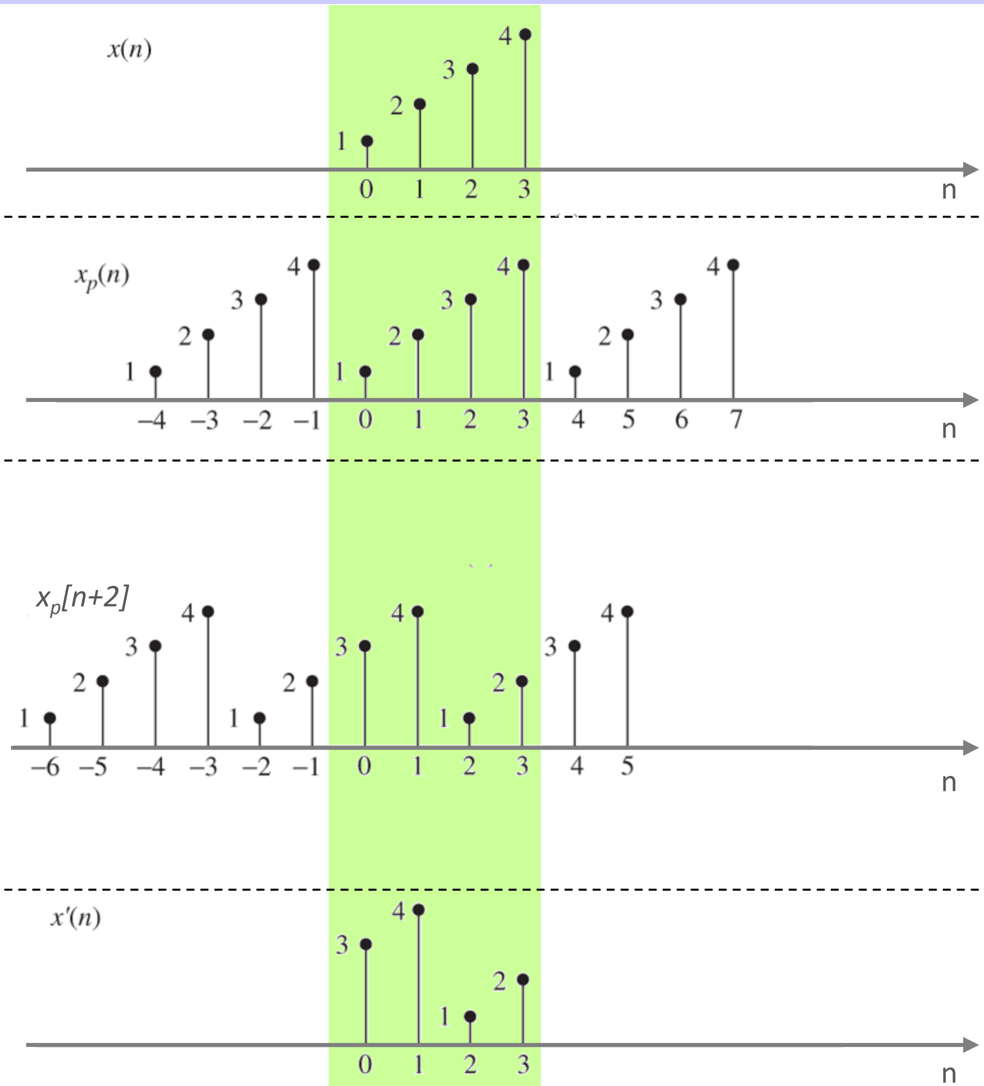
Circular Shift

Circular shift as cylinder rotation

- Think of the periodic extension of a signal as wrapping the signal around a cylinder with an N-point circumference
- As we traverse the cylinder repeatedly what we see is the periodic extension of x[n] (denoted as $x_p[n]$)
- A linear shift of the signal $x_p[n]$ corresponds to a rotation of the cylinder



- Periodic signals can be characterised by a single period - keep the *0* to *N-1* part only



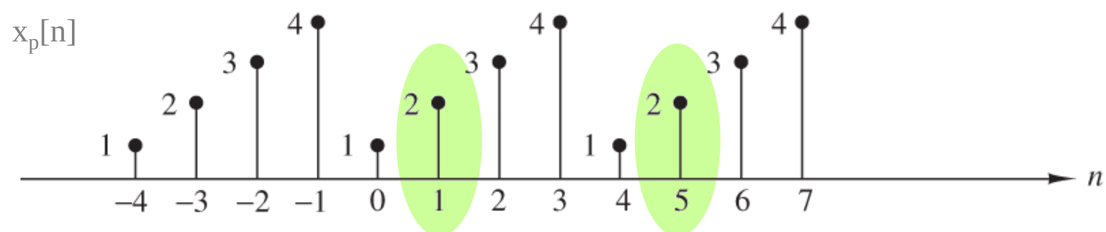
Circular Shift

Circular shift using modulo arithmetic

1. Periodic signals can be characterised by a single period

$$x_p[n] \quad 0 \le n < N,$$

$$x_p[n] = x_p[n - qN]$$



m_1: remainder of division m/N   ***N***: DFT length

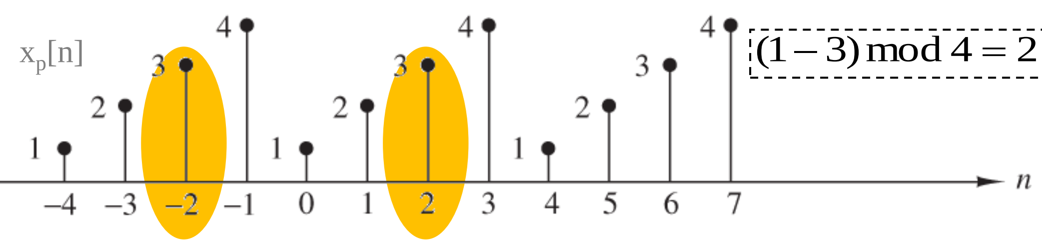
2. Any shift $m \ge N$ cannot be distinguished from a shorter shift m_1 where $m = m_1 + (q \times N)$

q: quotient from division m/N

3. If n-m lies outside the 1st period we can find a corresponding value within the 1st period

$$(n - m) \bmod N = (n - m) - qN$$

$$0 \le m \bmod N < N$$



4. $y[n] = x[(n - m) \bmod N]$

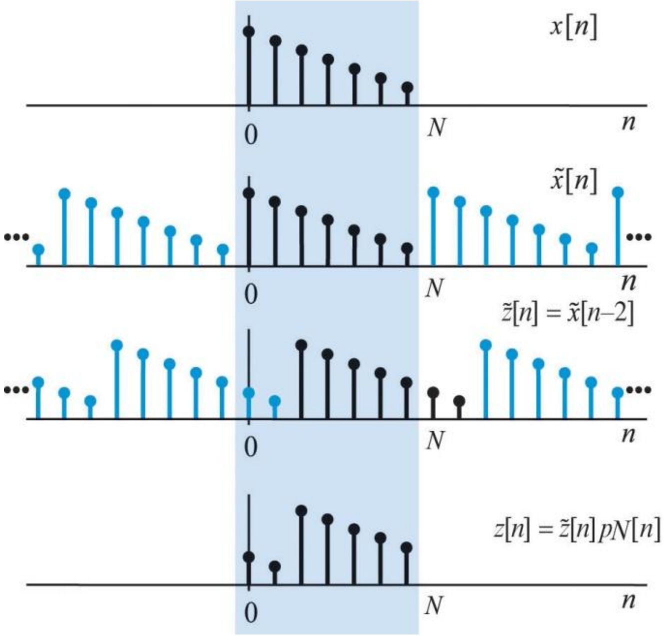
The mod function is defined as the amount by which a number exceeds the largest integer multiple of the divisor that is not greater than that number

Discrete Fourier Transform

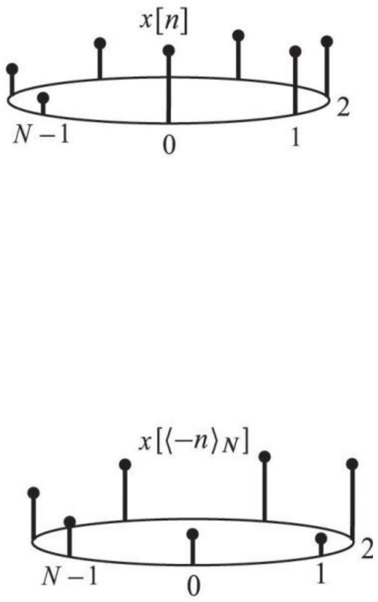
Circular Shift

Yet another interpretation...

Linear shift of periodic signal



Circular buffer rotation



Circular shift Matlab implementation

Shift sequence *x* by amount *k* assuming an *N*-point DFT

The input sequence is padded with zeros if its length is less than N (the length of the DFT)

```
function y=cirshift0(x,k,N)
% Circular shift of a sequence
if length(x) > N; error('N < length(x)'); end
x=[x zeros(1,N-length(x))];
n=(0:1:N-1); y=x(mod(n-k,N)+1);
```

Circular Shift

DFT of circularly shifted signal

What is the DFT of $y[n] = x[(n-m) \bmod N]$, where $x[n] \overset{DFT}{\leftrightarrow} X[k]$

$$Y[k] = \sum_{n=0}^{N-1} y[n] e^{-jkn\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[(n-m) \bmod N] e^{-jkn\frac{2\pi}{N}}$$

Substitute

$$n' = (n-m) \bmod N$$

$$(n-m) \bmod N = (n-m) - qN \Rightarrow n' = n - m - qN \Rightarrow n = n' + m + qN$$

$$Y[k] = \sum_{n'=0}^{N-1} x[n'] e^{-jk(n'+m+qN)\frac{2\pi}{N}} = e^{-jkm\frac{2\pi}{N}} \underbrace{\boxed{e^{-jkqN\frac{2\pi}{N}}}}_{1} \underbrace{\boxed{\sum_{n'=0}^{N-1} x[n'] e^{-jkn'\frac{2\pi}{N}}}}_{X[k]}$$

$$Y[k] = e^{-jkm\frac{2\pi}{N}} X[k]$$

Discrete Fourier Transform

Circular Convolution

Differences with linear convolution

Linear Convolution: $$y[n] = x_1[n] * x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m] \times x_2[n-m]$$

1. Time reversal	$x_2[-m]$	Change to circular reversal
2. Linear shift of one signal relative to the other	$x_2[-m+n]$	Change to circular shift
3. Multiplication of the two sequences	$x[m] \times x_2[n-m]$	
4. Summation of the product	$\sum_{m=0}^{N-1} x[m] \times x_2[n-m]$	

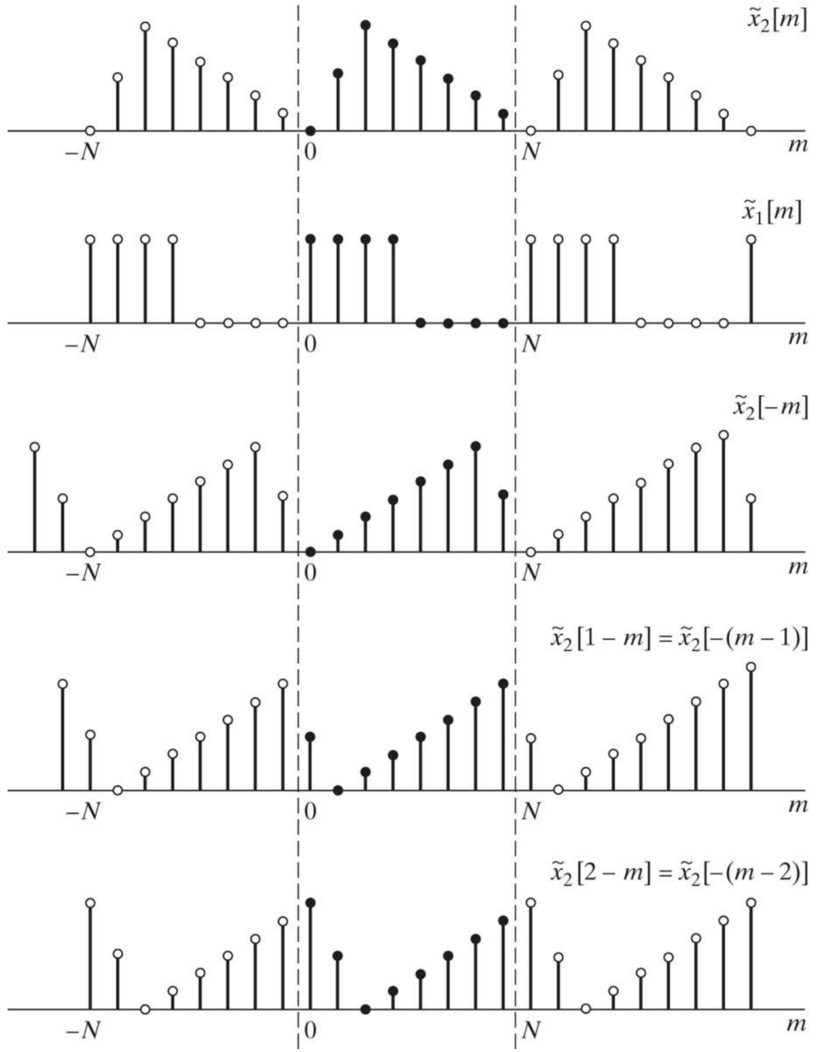
Circular Convolution: $$y[n] = x_1[n] \otimes x_2[n] = \sum_{m=0}^{N-1} x_1[m] \times x_2[(n-m) \bmod N]$$

Discrete Fourier Transform

Circular Convolution

Circular convolution as periodic reversal followed by periodic time shifting

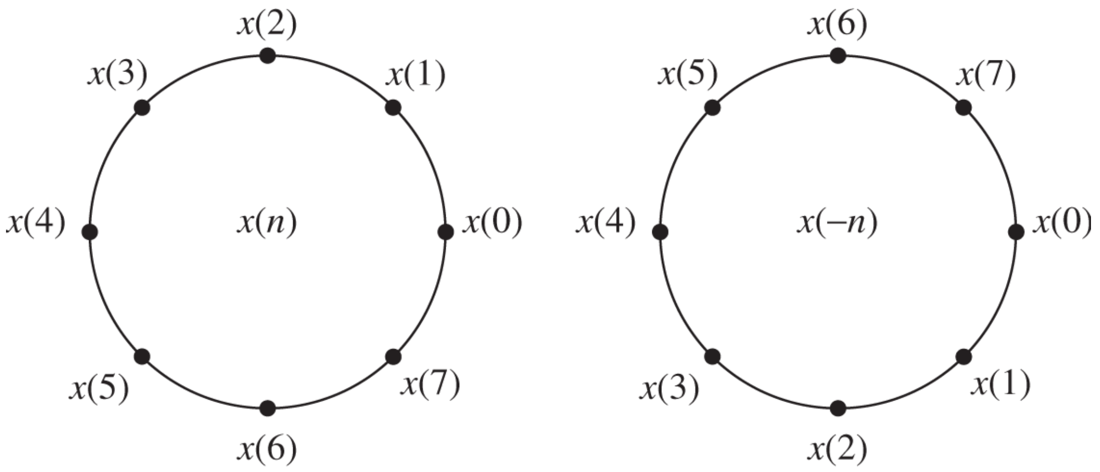
- Periodic extension of signal $x_2[m]$
- Periodic extension of signal $x_1[m]$
- Reversed periodic extension of signal $x_2[m]$
- Reversed periodic extension of signal $x_2[m]$ shifted by one sample
- Reversed periodic extension of signal $x_2[m]$ shifted by two samples



Circular Convolution

Circular convolution as circular reversal followed by circular time shifting

Circular Reversal:

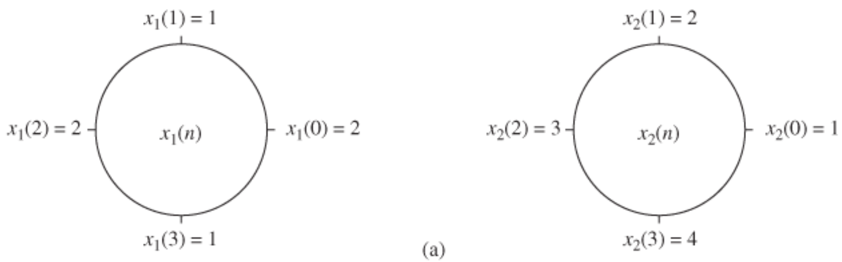


Turn cylinder over

$$x[-n] = x[(-n \bmod N)]$$

Circular Convolution

Circular convolution as cylinder reversal followed by cylinder rotation

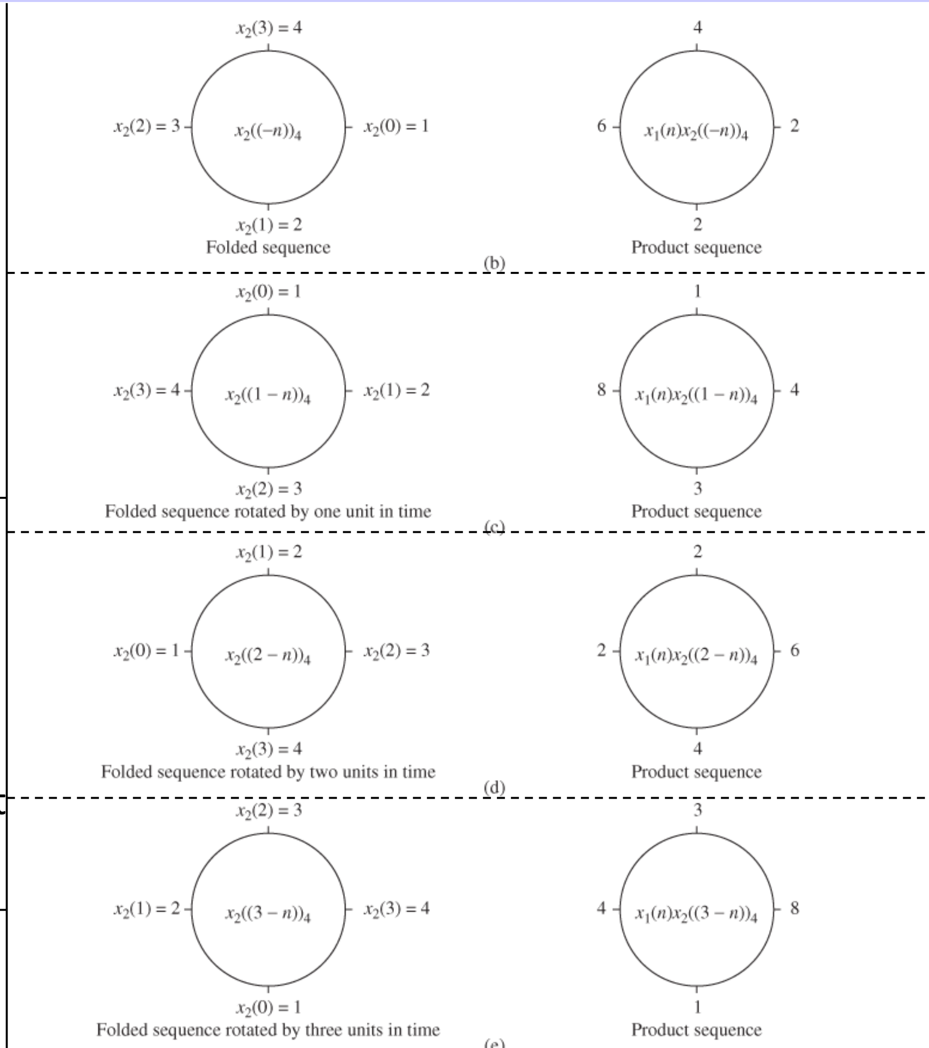


Signals to be convolved

$$x_1[n] \otimes x_2[n] = \sum_{m=0}^{N-1} x_1[m] \times x_2[(n-m) \bmod N]$$

- Circular time reversal (turn cylinder over)
- Circular shifting (rotate cylinder by one sample)
- Multiply terms
- Add each product sequence to get the output samples

$$x_1[n] \otimes x_2[n] = \{14, 16, 14, 16\}$$

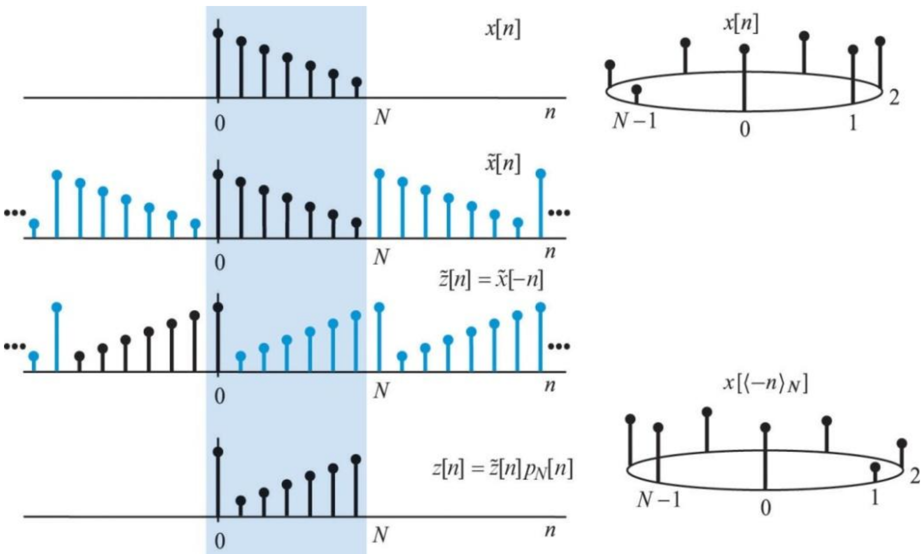


Discrete Fourier Transform

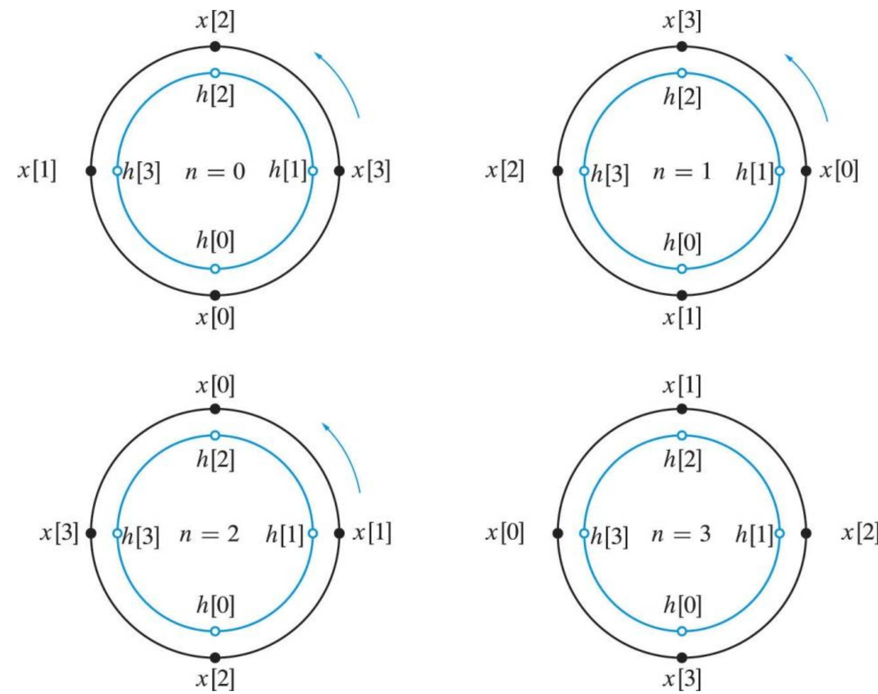
Circular Convolution

Yet another interpretation...

Time Reversal



Circular shift and multiplication



Filter (h[n]) : anticlockwise arrangement

Signal x[n] : clockwise arrangement (time reversal)

Discrete Fourier Transform

Circular Convolution

DFT of circular convolution of two signals

What is the DFT of $y[n] = x_1[n] \otimes x_2[n] = \sum_{m=0}^{N-1} x[m] y[(n-m) \bmod N],$

$$\text{where } x_1[n] \overset{DFT}{\leftrightarrow} X_1[k] \ \& \ x_2[n] \overset{DFT}{\leftrightarrow} X_2[k]$$

$$Y[k] = \sum_{n=0}^{N-1} y[n] e^{-jkn\frac{2\pi}{N}} = \sum_{n=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x_1[m] x_2[(n-m) \bmod N] \right\} e^{-jkn\frac{2\pi}{N}}$$

Substitute

$$n' = (n-m) \bmod N$$

$$(n-m) \bmod N = (n-m) - qN \quad \Rightarrow n' = n - m - qN \quad \Rightarrow n = n' + m + qN$$

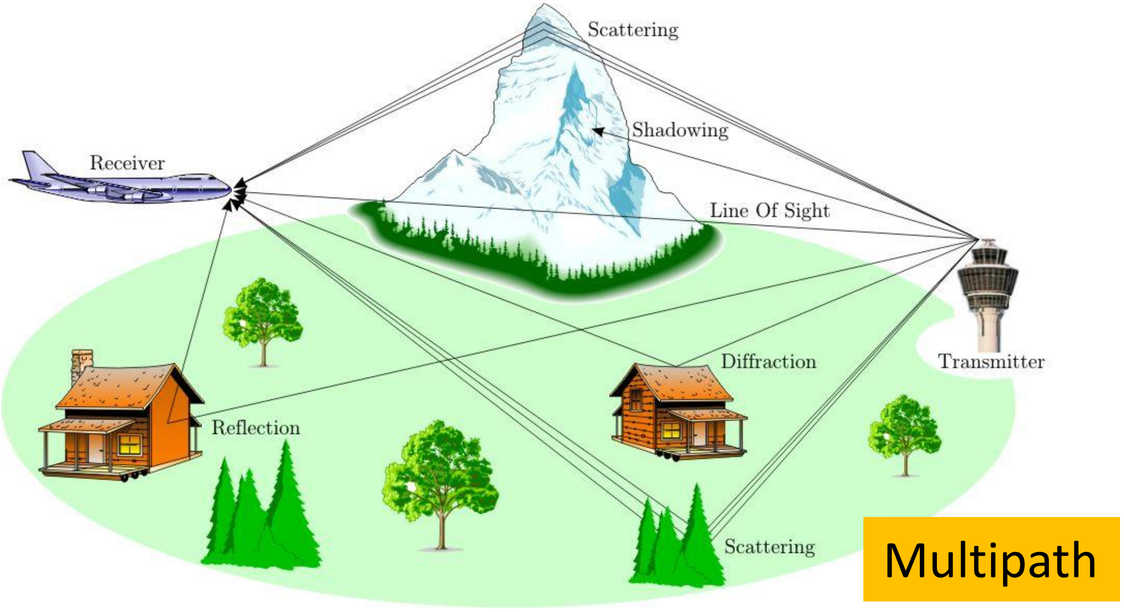
$$Y[k] = \sum_{n'=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] x_2[n'] e^{-jk(m+n'+qN)\frac{2\pi}{N}} = \sum_{n'=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] x_2[n'] e^{-jkm\frac{2\pi}{N}} e^{-jkn'\frac{2\pi}{N}} e^{-jkqN\frac{2\pi}{N}}$$

$$Y[k] = \sum_{m=0}^{N-1} x_1[m] e^{-jkm\frac{2\pi}{N}} \sum_{n'=0}^{N-1} x_2[n'] e^{-jkn'\frac{2\pi}{N}} = X_1[k] X_2[k]$$

Discrete Fourier Transform

Circular Convolution

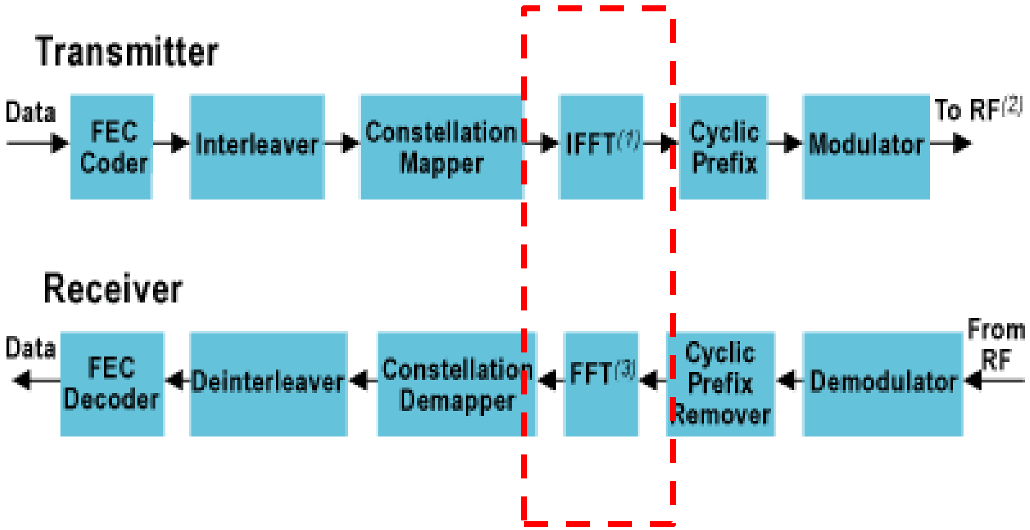
OFDM example



- Transmitting data is equivalent to convolving the data with a filter, the impulse response of which represents the multipath effects of the channel
- In order to remove the effect of the channel a filter with the inverse impulse response has to be created at the receiver to cancel out multipath **(equalisation)**
- Complex operation, many taps needed, not always successful

Circular Convolution

OFDM example



- Perform an IFFT on the time domain data (i.e. assume data are DFT coefficients)
- At the receiver perform an FFT to get the original values - FFT takes us to the frequency domain
- Divide above spectrum with channel spectrum to remove effect of multipath - No need for filter design
- For the above to be valid the convolution with the channel has to be circular – extend original data periodically after the IFFT (cyclic prefix)