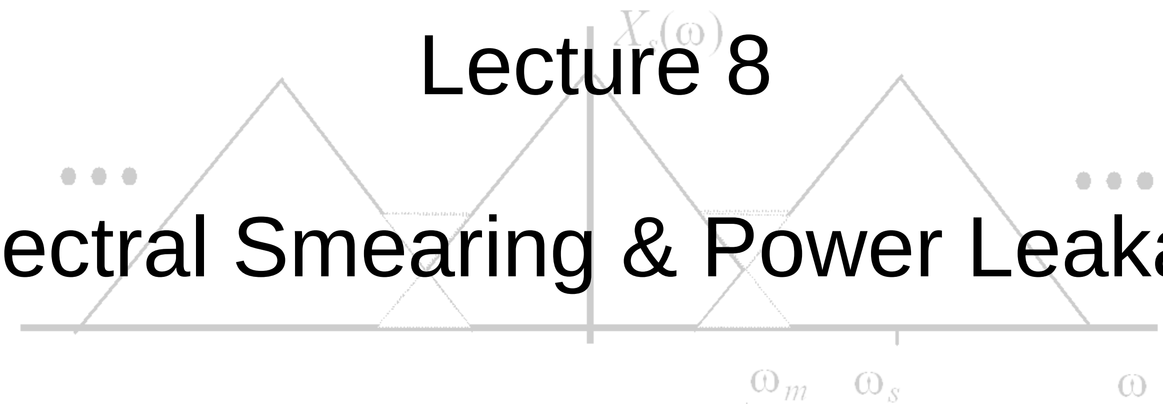
Lecture 8

Spectral Smearing & Power Leakage



Spectral Analysis using the DFT
Effects of Windowing and Spectral Sampling

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Spectral Smearing & Power Leakage

Spectral Analysis using the DFT

DFT Analysis of Sinusoidal Signals

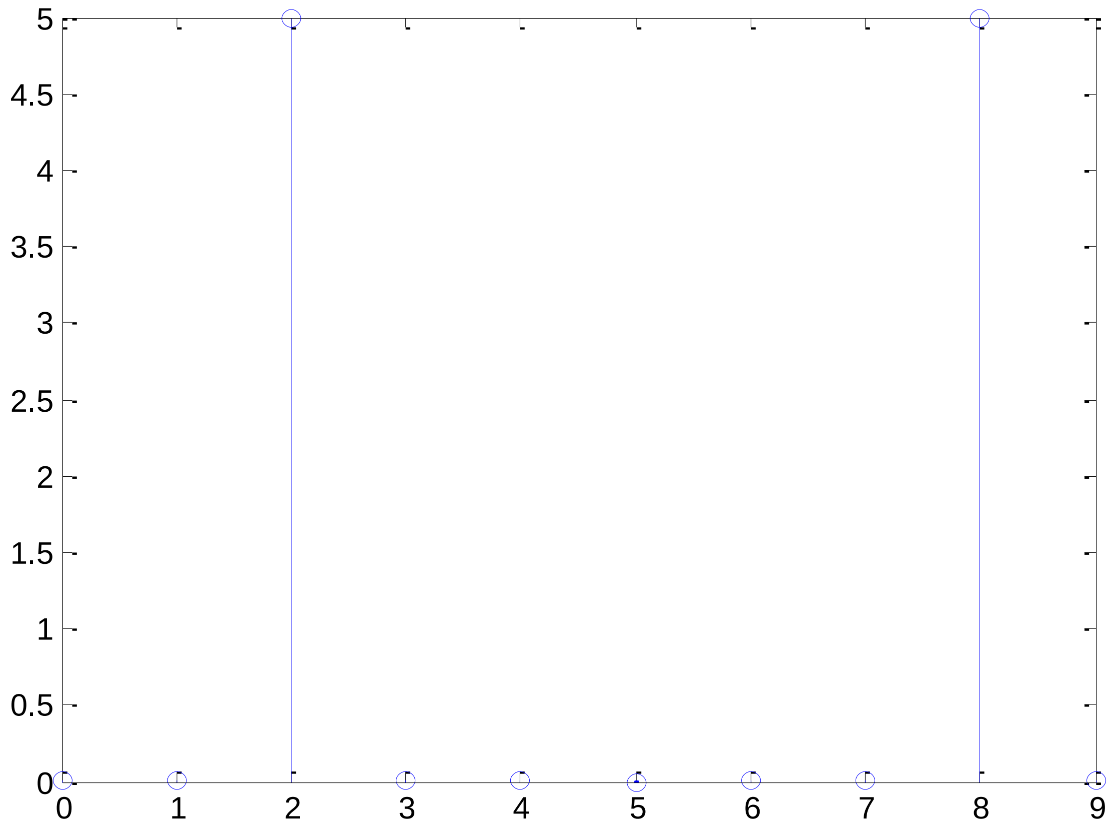
Matlab code

```
% DFT Length
N=10;
n=0:N-1;

% Signal cos(2πF0n)
Omega0=2*pi*0.2;
xn=cos(Omega0*n);

% DFT spectrum
Xk=fft(xn);
k=0:N-1;
stem(k,abs(Xk))
```

Plotted spectrum (DFT)



Expected Spectrum (DTFT)

Two impulses at Ω_0 and 2π-Ω_0

Spectrum looks as expected

Spectral Smearing & Power Leakage

Spectral Analysis using the DFT

DFT Analysis of Sinusoidal Signals

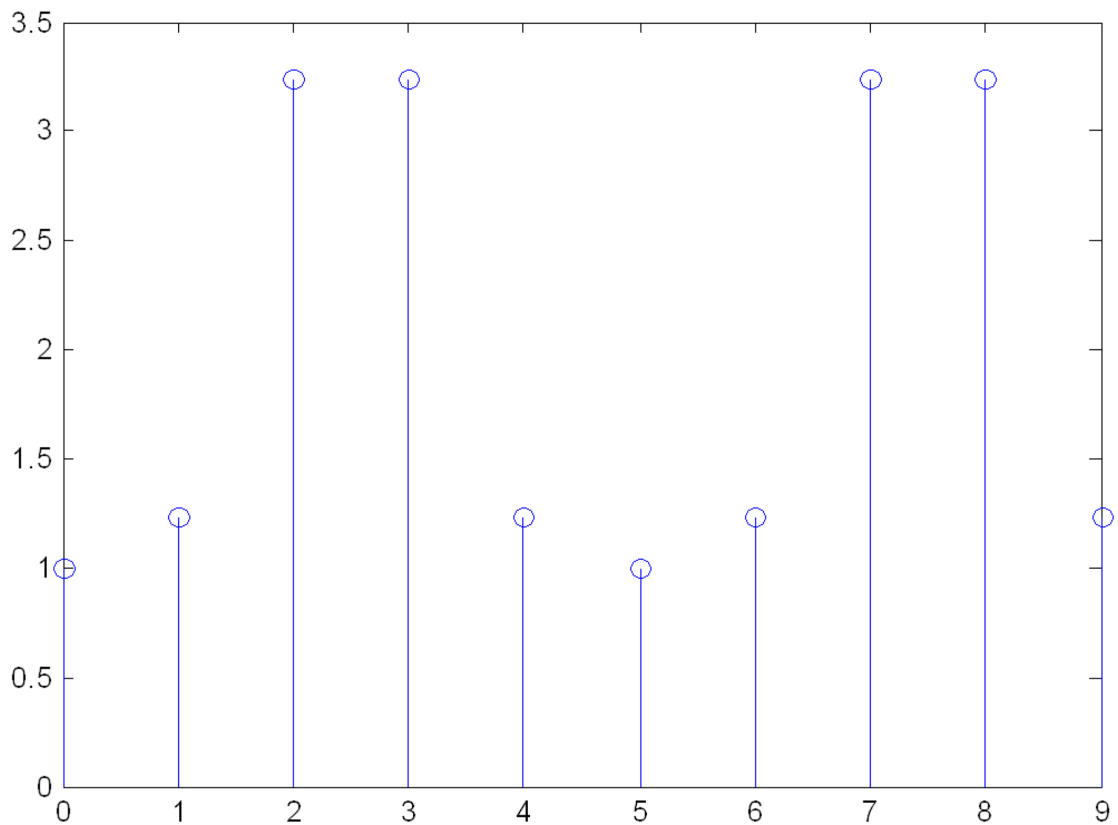
Matlab code

```
% DFT Length
N=10;
n=0:N-1;

% Signal cos(2πF0n)
Omega0=2*pi*0.25;
xn=cos(Omega0*n);

% DFT spectrum
Xk=fft(xn);
k=0:N-1;
stem(k,abs(Xk))
```

Plotted spectrum (DFT)



Expected Spectrum (DTFT)

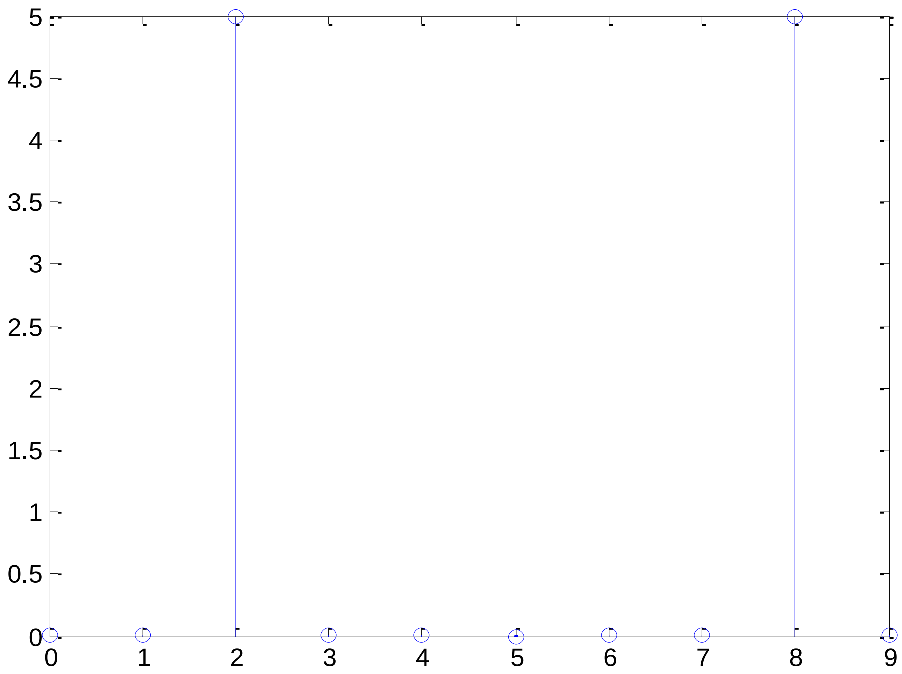
Two impulses at Ω_0 and $2\pi\text{-}\Omega_0$

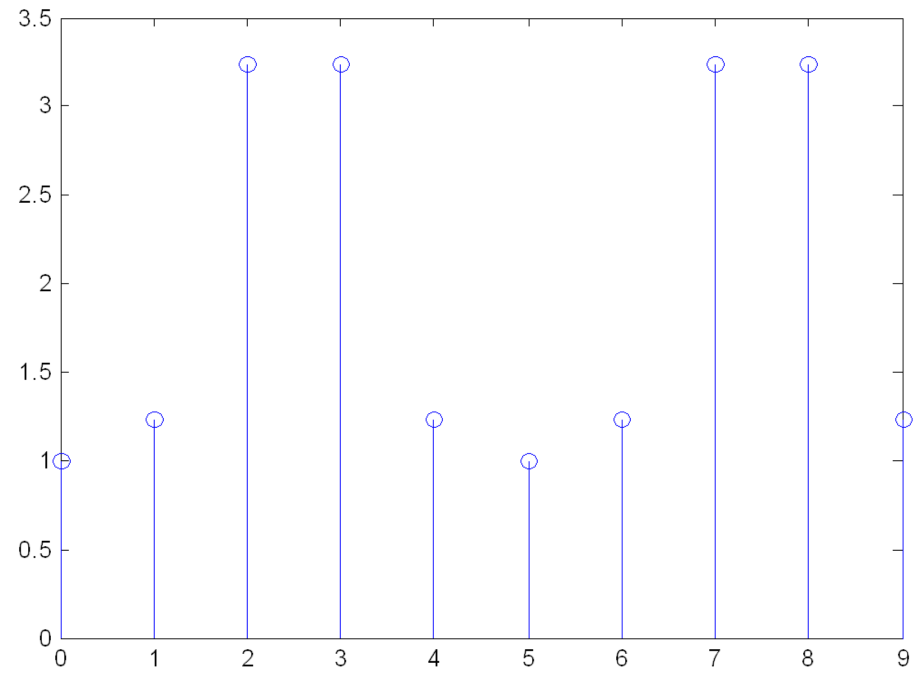
Ooops.....

Spectral Smearing & Power Leakage

Spectral Analysis using the DFT

DFT Analysis of Sinusoidal Signals





What is going on ?

Spectral Analysis using the DFT

From the DTFT to the DFT

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\Omega n} \qquad \Bigg| \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$

Use a finite portion of the signal with N samples

Windowing

Sample the frequency domain taking N samples

Spectral Sampling

DFT

Discrete Fourier Transform

Spectral Analysis using the DFT

From the DTFT to the DFT

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

- **Effect of Windowing on the Spectrum**
  - **Properties of different Windows**

Spectral Smearing & Power Leakage

- **Effect of Spectral Sampling**
  - **Frequency resolution**

Misleading picture of the true spectrum

DFT

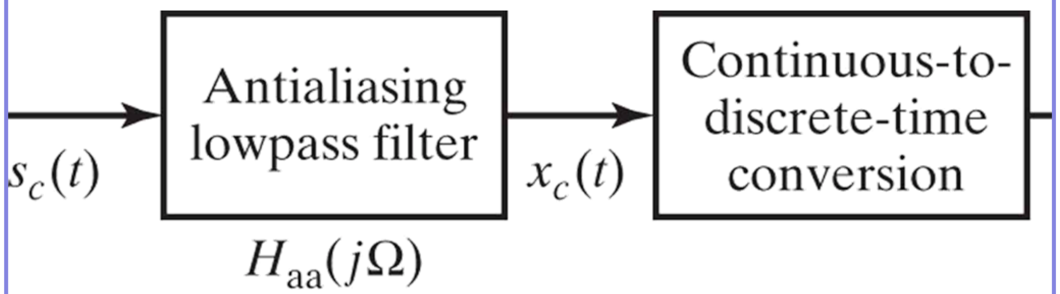
$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$$

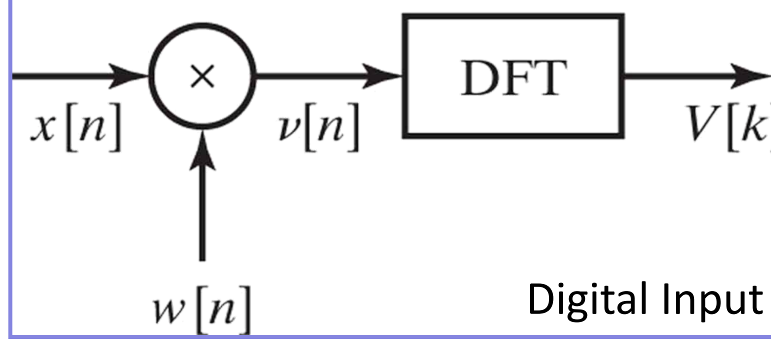
Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

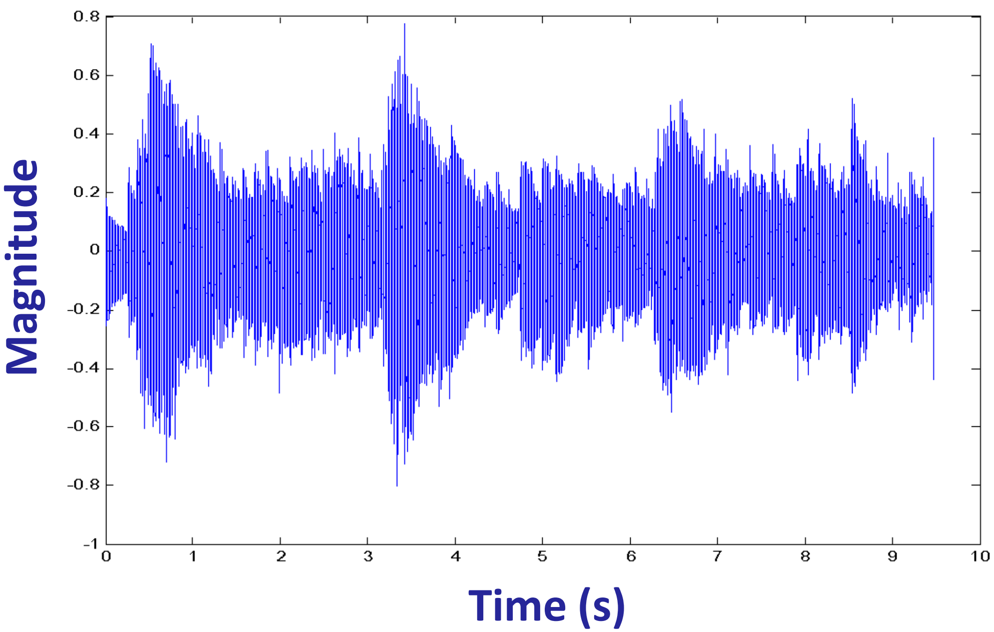
Why windowing



Analogue Input – A/D Conversion



Digital Input



Audio signal duration : 10 sec

Sampling rate : 44 KHz

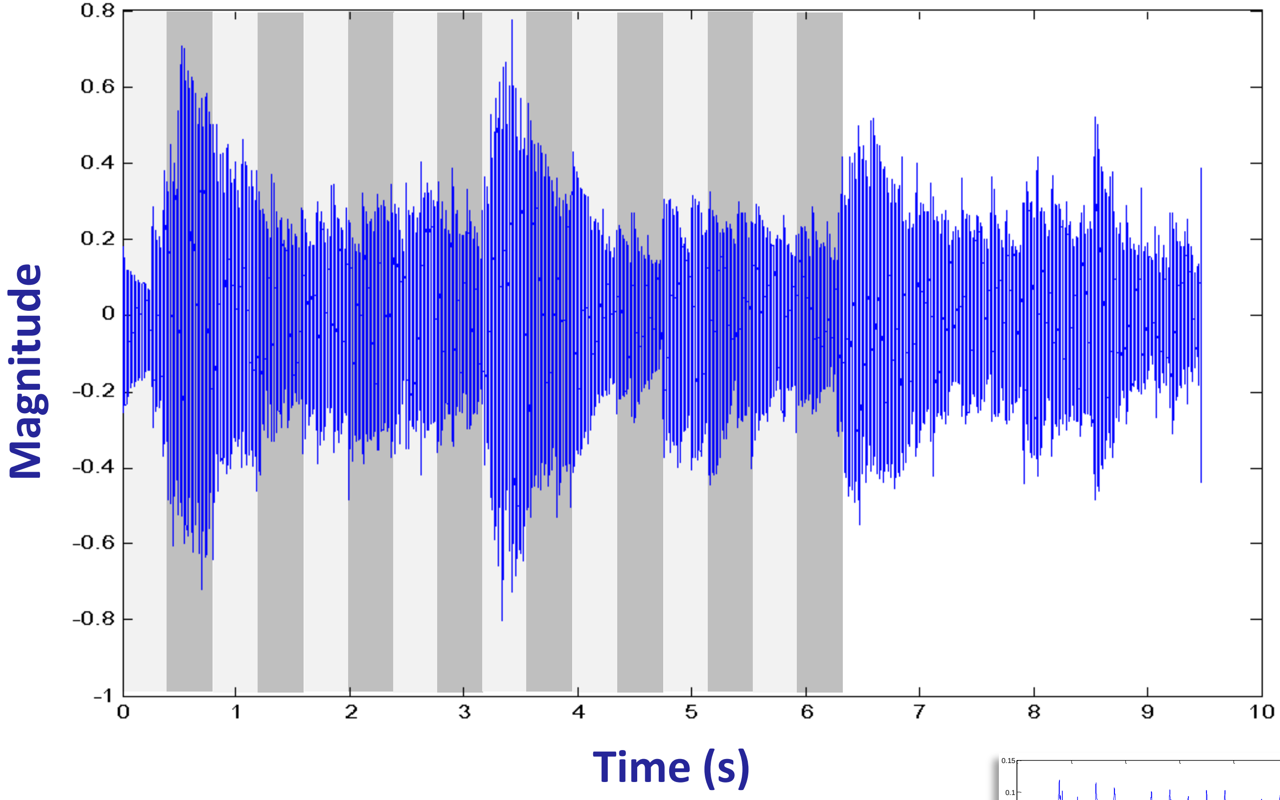
Total number of samples : 440,000

Minimum DFT Length : 440,000 !

Spectral Smearing & Power Leakage

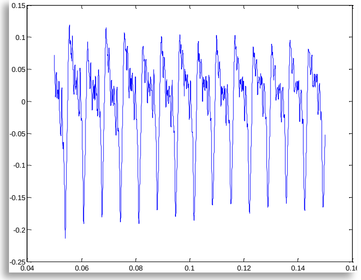
The Effect of Windowing on the DFT Spectrum

Windowing for very long or indefinitely long signals



Rectangular window

Smaller segments analysed separately



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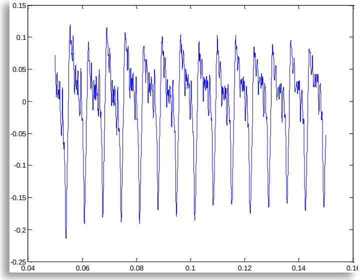

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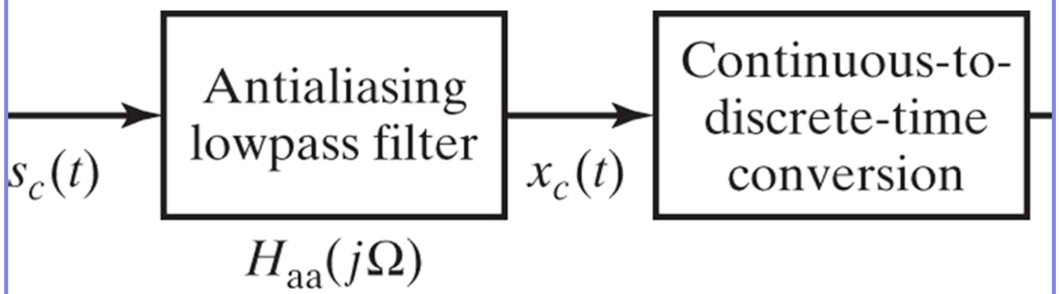
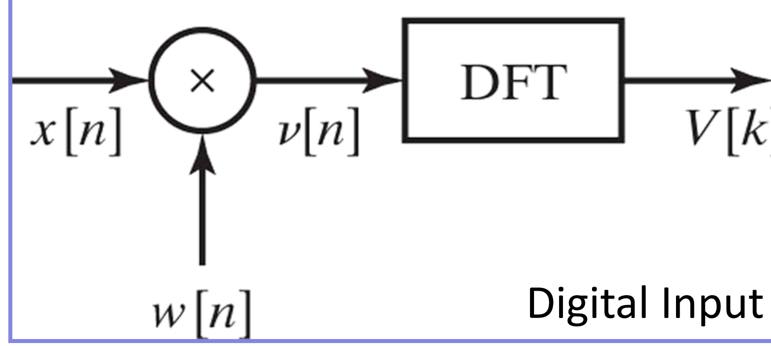
Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

DFT Analysis of Sinusoidal Signals



Analogue Input – A/D Conversion



Digital Input

1. Continuous time signal

$$s_c(t) = A_0 \cos(\omega_0 t) + A_1 \cos(\omega_1 t)\ , \quad -\infty < t < \infty$$

2. Sampling

$$x[n] = A_0 \cos(\Omega_0 n) + A_1 \cos(\Omega_1 n), \quad -\infty < n < \infty$$

3. Multiply signal with window

$$v[n] = A_0 w[n] \cos(\Omega_0 n) + \\ + A_1 w[n] \cos(\Omega_1 n)$$

4. DTFT of windowed signal v[n]

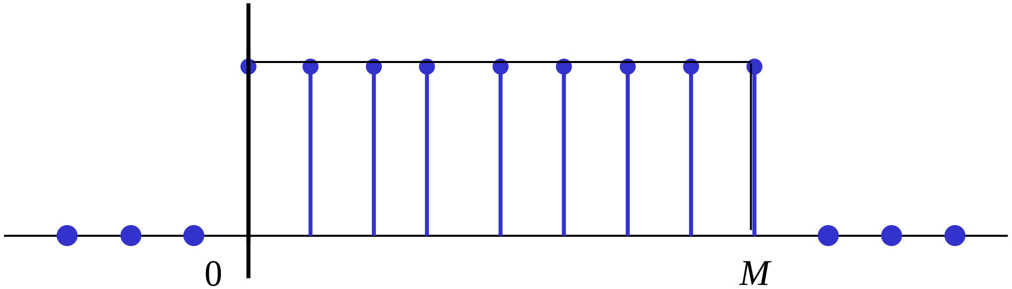
$$v[n] = x[n] w[n] \overset{DTFT}{\Leftrightarrow} V[\Omega] = \frac{1}{2\pi} X[\Omega] \otimes w[\Omega] = V[\Omega] = \frac{A_0}{2} W(\Omega - \Omega_0) + \frac{A_0}{2} W(\Omega + \Omega_0) + \frac{A_1}{2} W(\Omega - \Omega_1) + \frac{A_1}{2} W(\Omega + \Omega_1)$$

Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

DTFT of rectangular window

$$w[n]=\begin{cases}1 & 0\le n\le M\\ 0 & \text{elsewhere}\end{cases}$$



$$W(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=\sum_{n=0}^{M}e^{-j\Omega n}\quad =1+e^{-j\Omega}+e^{-j2\Omega}+\ldots+e^{-j(M-1)\Omega}+e^{-jM\Omega} \qquad (1)$$

Geometric series : $1+\alpha+\alpha^2+\ldots+\alpha^M=\dfrac{1-\alpha^{M+1}}{1-\alpha}$ with $\alpha=e^{-j\Omega}$ (2)

$$\stackrel{(1,2)}{\Rightarrow} W(\Omega)=\frac{1-e^{-j\Omega(M+1)}}{1-e^{-j\Omega}}=\frac{e^{j\Omega/2}}{e^{j\Omega/2}}\,\frac{1-e^{-j\Omega(M+1)}}{1-e^{-j\Omega}}\stackrel{(3,4)}{=}e^{-j\Omega(M/2)}\,\frac{e^{j\Omega(M+1)/2}-e^{-j\Omega(M+1)/2}}{e^{j\Omega/2}-e^{-j\Omega/2}}$$

$$1-e^{-j\Omega(M+1)}=e^{-j\Omega\frac{(M+1)}{2}}\left(e^{j\Omega\frac{(M+1)}{2}}-e^{-j\Omega\frac{(M+1)}{2}}\right) \qquad (3)$$

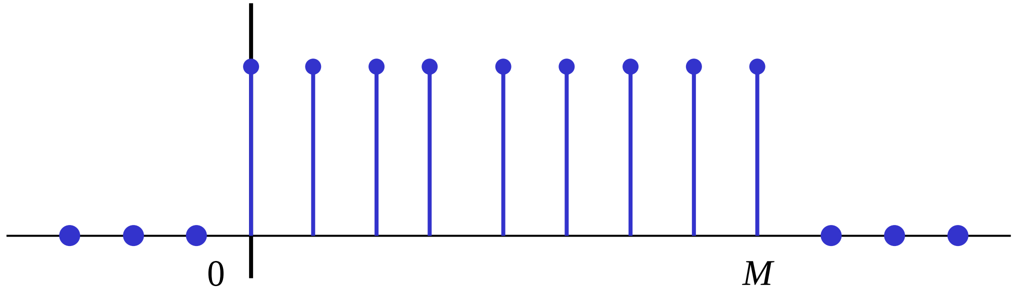
$$e^{-j\Omega\frac{(M+1)}{2}}e^{j\frac{\Omega}{2}}=e^{-j\Omega(M/2)} \qquad (4)$$

Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

DTFT of rectangular window

$$w[n]=\begin{cases}1 & 0\le n\le M\\ 0 & \text{elsewhere}\end{cases}$$



$$W(\Omega)=e^{-j\Omega(M/2)}\frac{e^{j\Omega(M+1)/2}-e^{-j\Omega(M+1)/2}}{e^{j\Omega/2}-e^{-j\Omega/2}} \overset{5}{\Rightarrow} W(\Omega)=e^{-j\Omega(M/2)}\frac{\sin(\Omega(M+1)/2)}{\sin(\Omega/2)}$$

$$e^{j\Omega(M+1)/2}-e^{-j\Omega(M+1)/2}=2j\sin(\Omega(M+1)/2),\quad e^{j\Omega/2}-e^{-j\Omega/2}=2j\sin(\Omega/2)$$

(5)

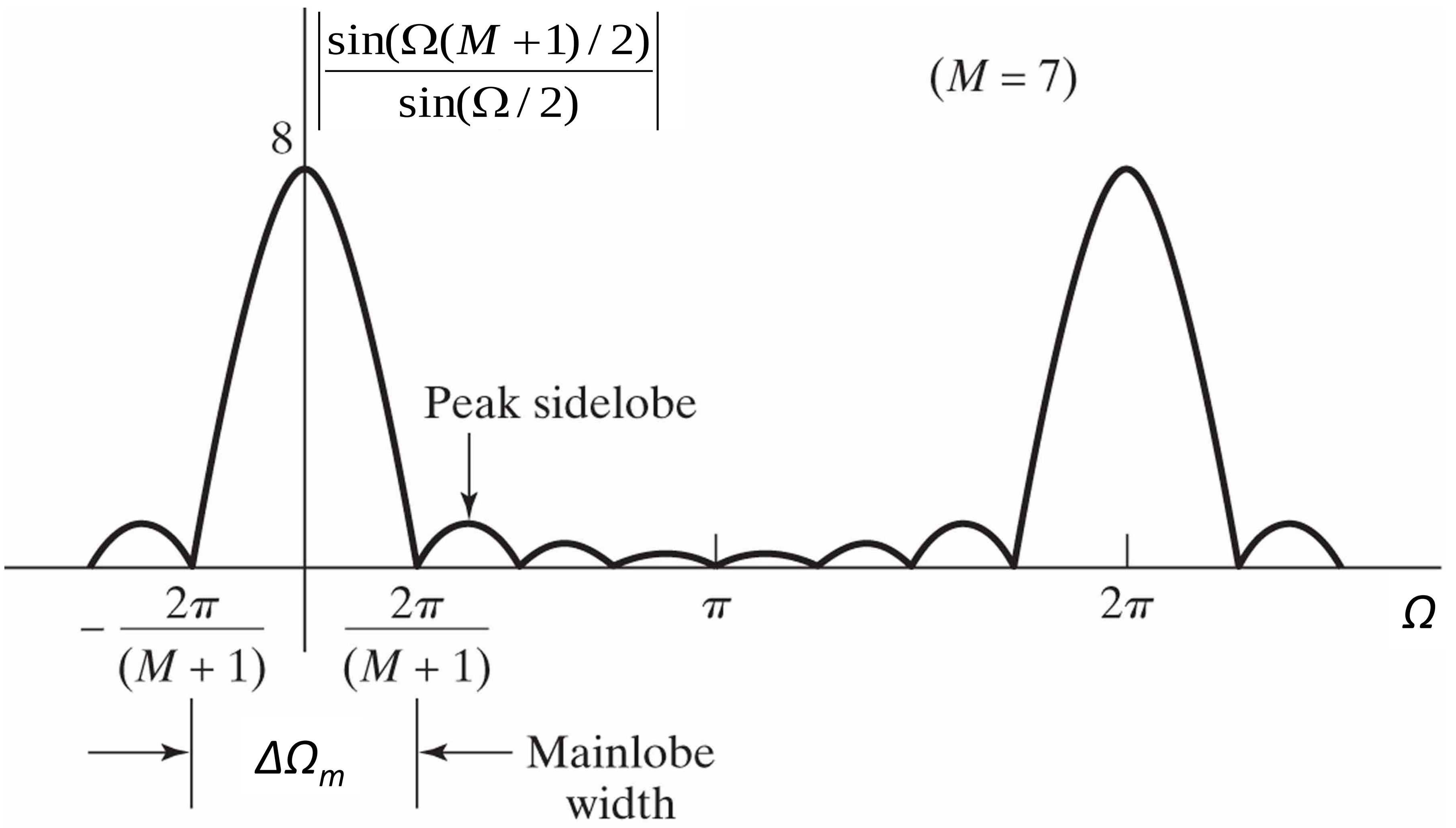
$$\left.\begin{matrix}W(\Omega)=0\\ -\pi<\Omega<\pi\end{matrix}\right\}\Rightarrow \sin(\Omega(M+1)/2)=0 \quad\Rightarrow \left.\begin{matrix}\Omega(M+1)/2=k\pi\\ k=0,1,2\ldots\end{matrix}\right\}\Rightarrow\quad \Omega=\frac{k2\pi}{M+1}$$

$$M=7:\ W(\Omega)=0\ \ at\quad 2\pi/8,\quad 4\pi/8,\quad 6\pi/8$$

Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

DTFT of rectangular window



Spectral Smearing & Power Leakage

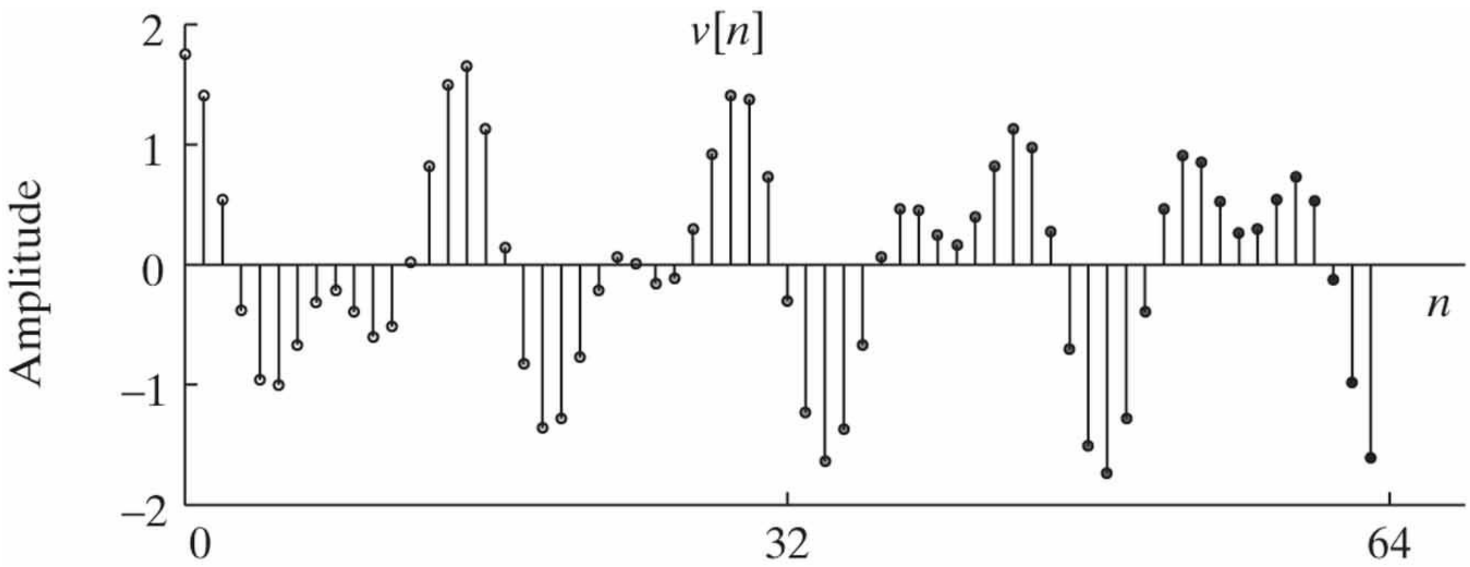
The Effect of Windowing on the DFT Spectrum

DTFT of Windowed Signal

Windowed signal in the time domain

$$v[n] = A_0 w[n]\cos(\Omega_0 n) + A_1 w[n]\cos(\Omega_1 n), \;\; 0 \le n < 64$$

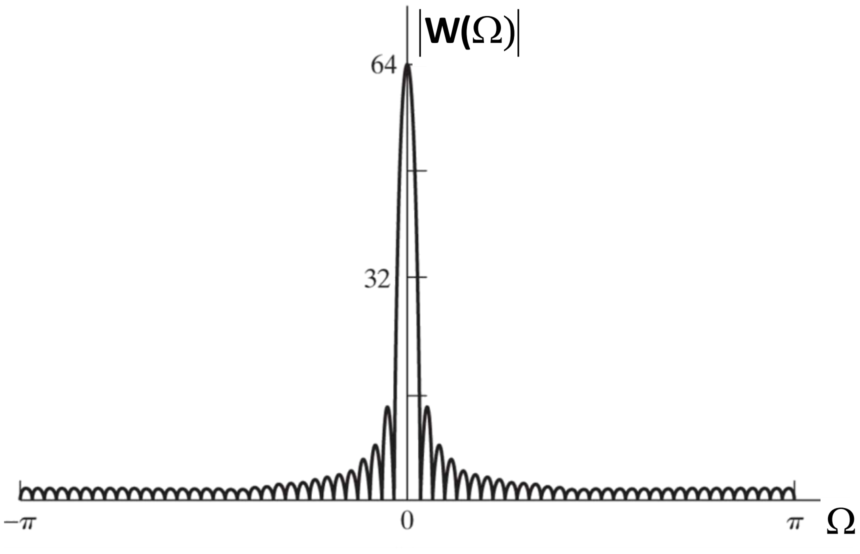
$A_0 = 1$, $A_1 = 0.75$, $\Omega_0 = 2\pi/14$, $\Omega_1 = 4\pi/15$



Spectral Smearing & Power Leakage

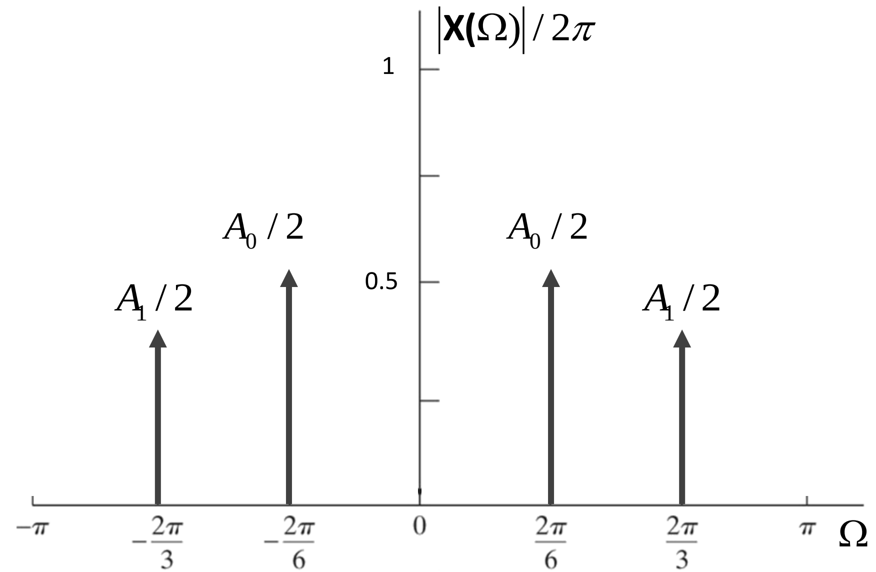
The Effect of Windowing on the DFT Spectrum

DTFT of Windowed Signal



DTFT of length 64 rectangular window

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{elsewhere} \end{cases}$$



DTFT of signal

$$x[n] = A_0 \cos(\Omega_0 n) + A_1 \cos(\Omega_1 n)$$

with $A_0 = 1$, $A_1 = 0.75$, $\Omega_0 = 2\pi/6$, $\Omega_1 = 2\pi/3$

$$V[\Omega] = \frac{A_0}{2} W(e^{j(\Omega-\Omega_0)}) + \frac{A_0}{2} W(e^{j(\Omega+\Omega_0)}) + \frac{A_1}{2} W(e^{j(\Omega-\Omega_1)}) + \frac{A_1}{2} W(e^{j(\Omega+\Omega_1)})$$

Spectral Smearing & Power Leakage

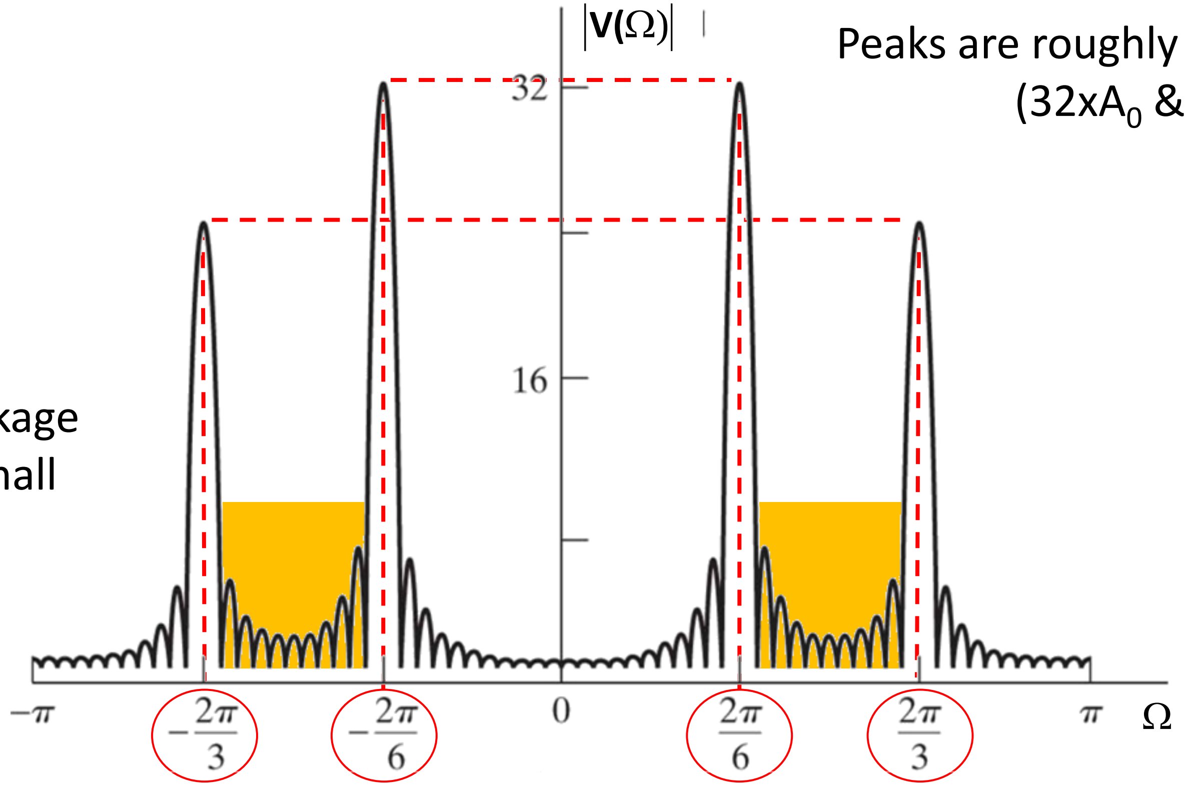
The Effect of Windowing on the DFT Spectrum

DTFT of Windowed Signal - Power leakage

$f_s = 10kHz$

$\Delta f \approx 1.6kHz$

Power leakage is very small



Peaks are roughly where expected ($32xA_0$ & $32xA_1$)

The two frequencies can be resolved

$$\Delta\Omega = \Omega_1 - \Omega_0 = \frac{\pi}{3} = 2\pi\Delta F \Rightarrow \Delta F = \frac{1}{6}, \ \Delta f = \Delta F \times f_s = \frac{1}{6} \times 10kHz = 1.66kHz$$

Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

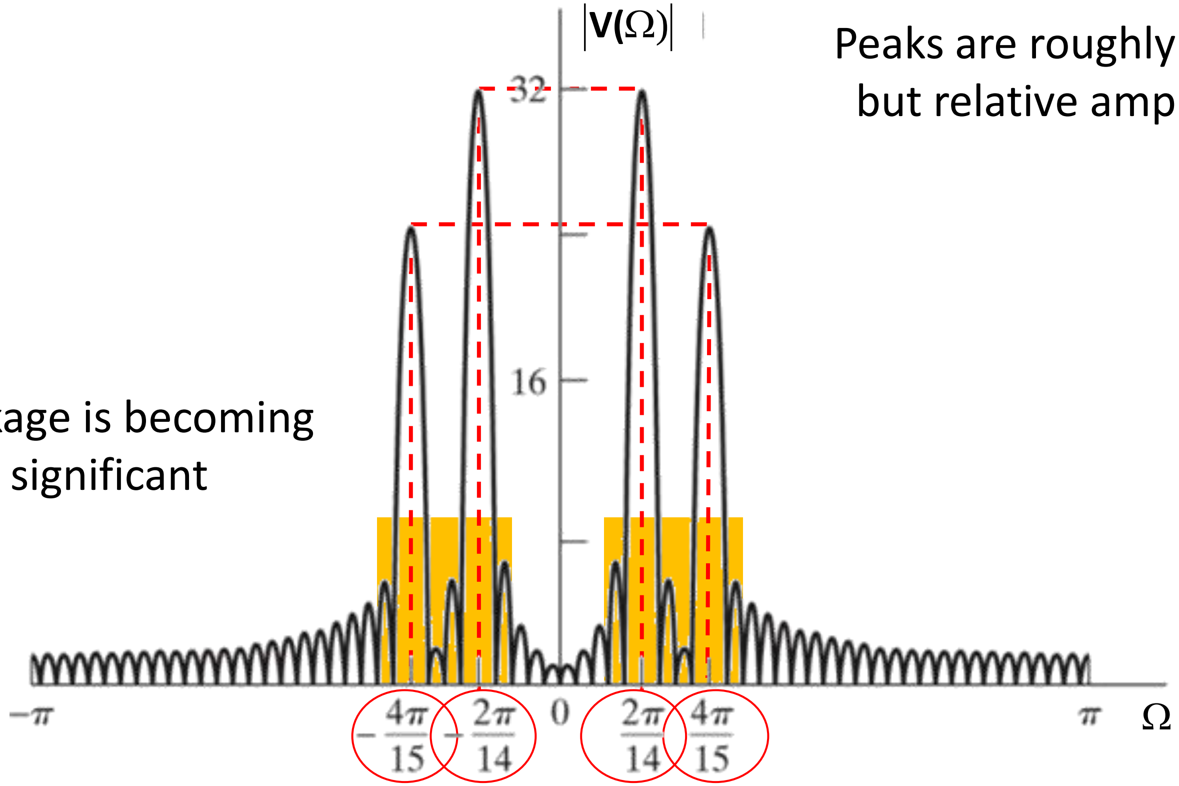
DTFT of Windowed Signal - Power leakage

$f_s = 10kHz$

$\Delta f \approx 620Hz$

Peaks are roughly where expected but relative amplitudes affected

Power leakage is becoming more significant



The two frequencies can still be resolved

Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

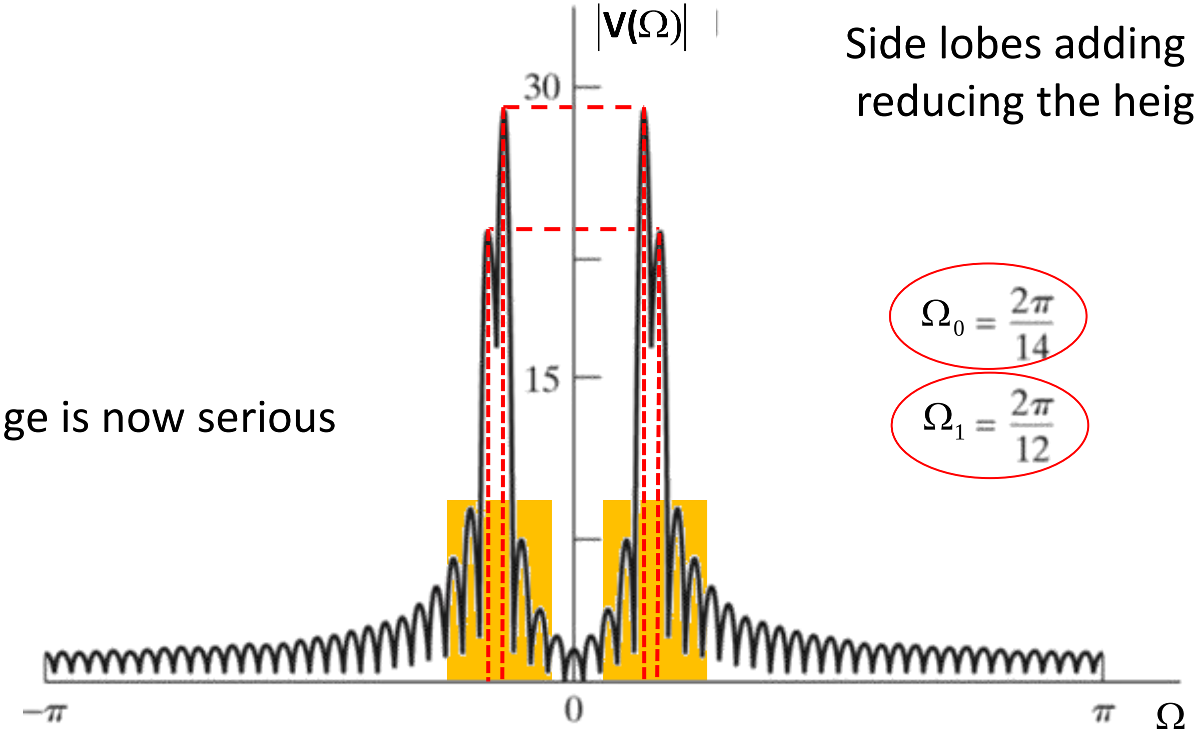
DTFT of Windowed Signal - Power leakage

$f_s = 10kHz$

$\Delta f \approx 120Hz$

Side lobes adding out of phase are reducing the heights of the peaks

Power leakage is now serious



$\Omega_0 = \frac{2\pi}{14}$

$\Omega_1 = \frac{2\pi}{12}$

The two frequencies can barely be resolved

Spectral Smearing & Power Leakage

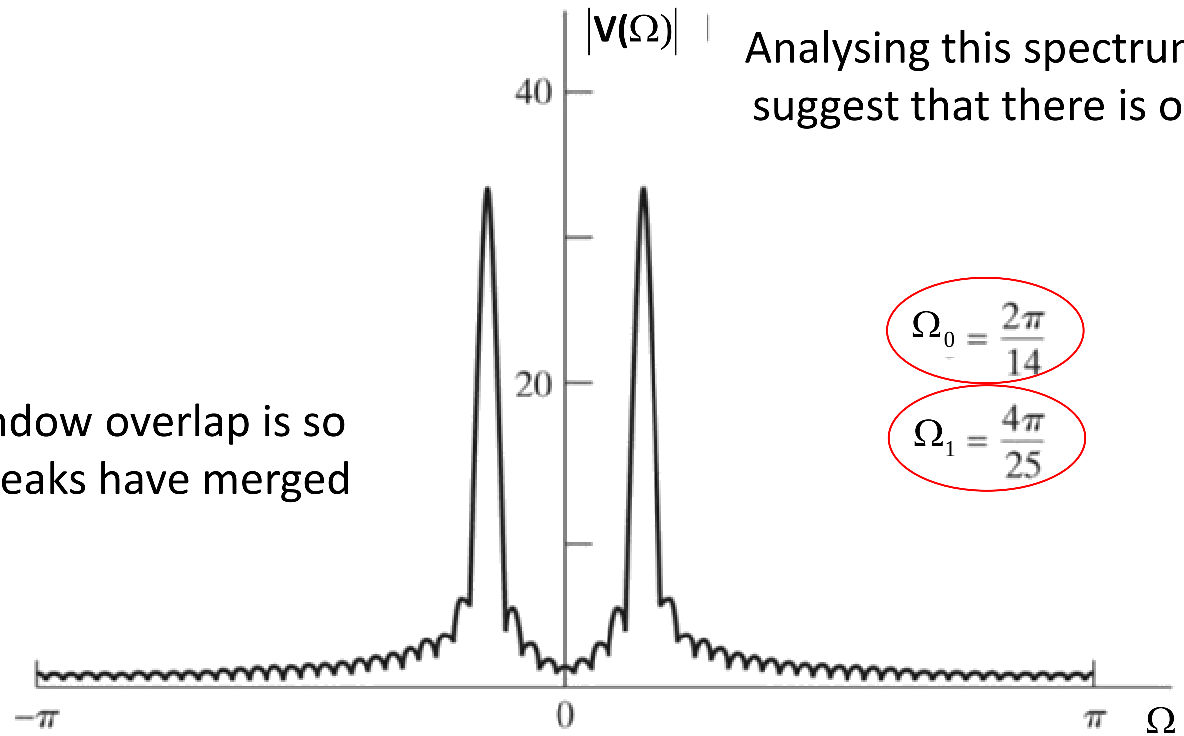
The Effect of Windowing on the DFT Spectrum

DTFT of Windowed Signal - Power leakage

$f_s = 10kHz$

$\Delta f \approx 85Hz$

Analysing this spectrum would wrongly suggest that there is only one sinusoid



$\Omega_0 = \frac{2\pi}{14}$

$\Omega_1 = \frac{4\pi}{25}$

With this window overlap is so severe that peaks have merged

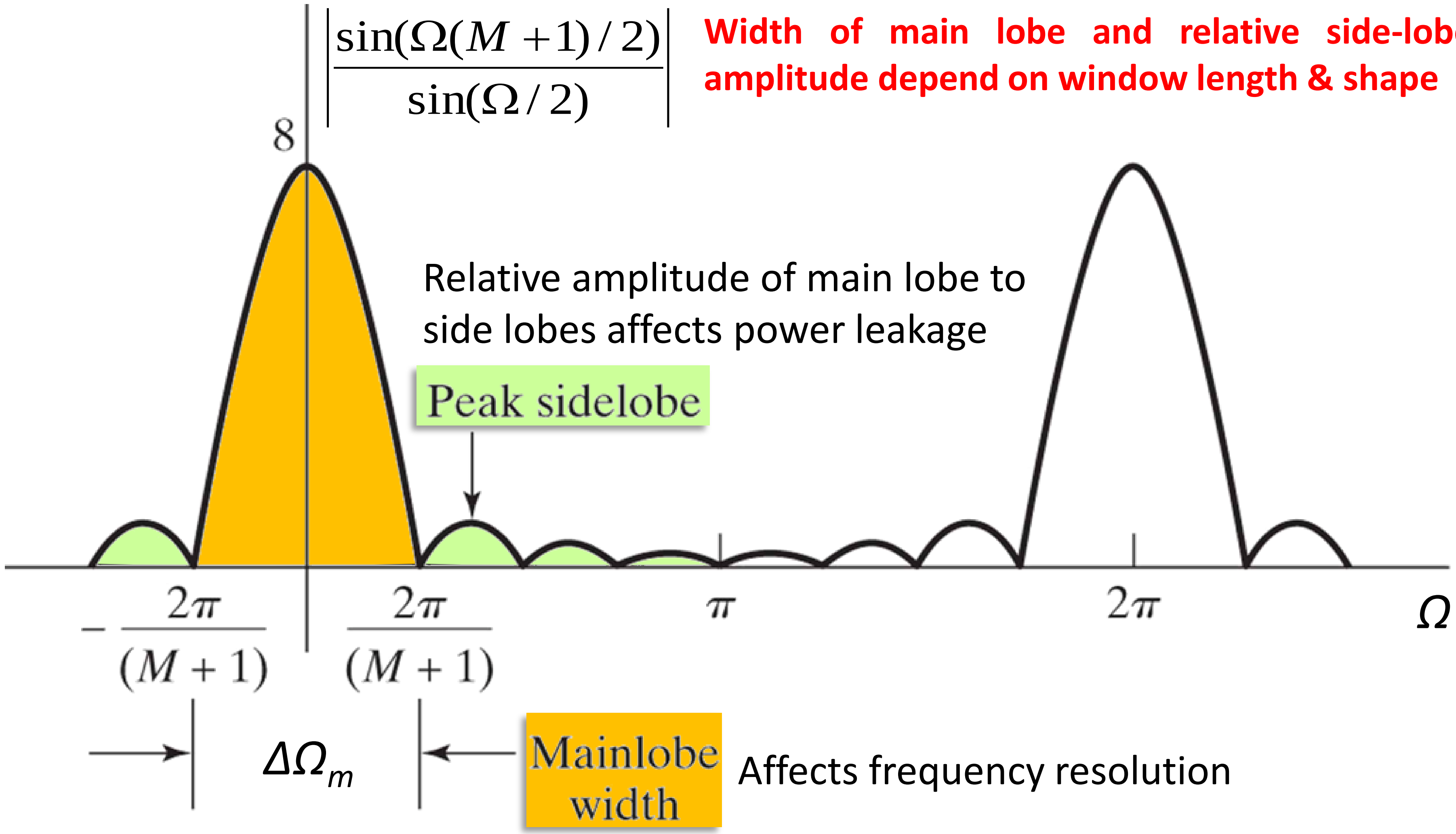
The two frequencies cannot be resolved any more

Spectral Smearing & Power Leakage

The Effect of Windowing on the DFT Spectrum

Spectral smearing, power leakage and reduced resolution

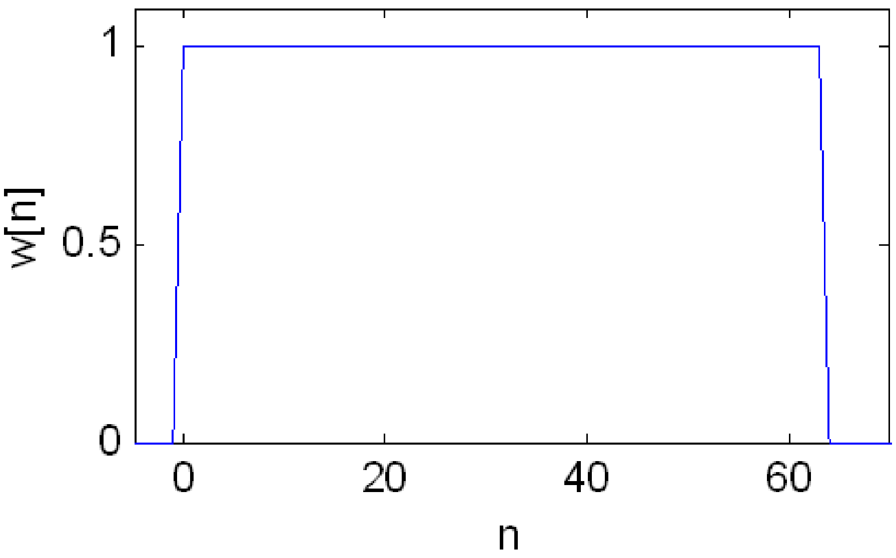
Width of main lobe and relative side-lobe amplitude depend on window length & shape

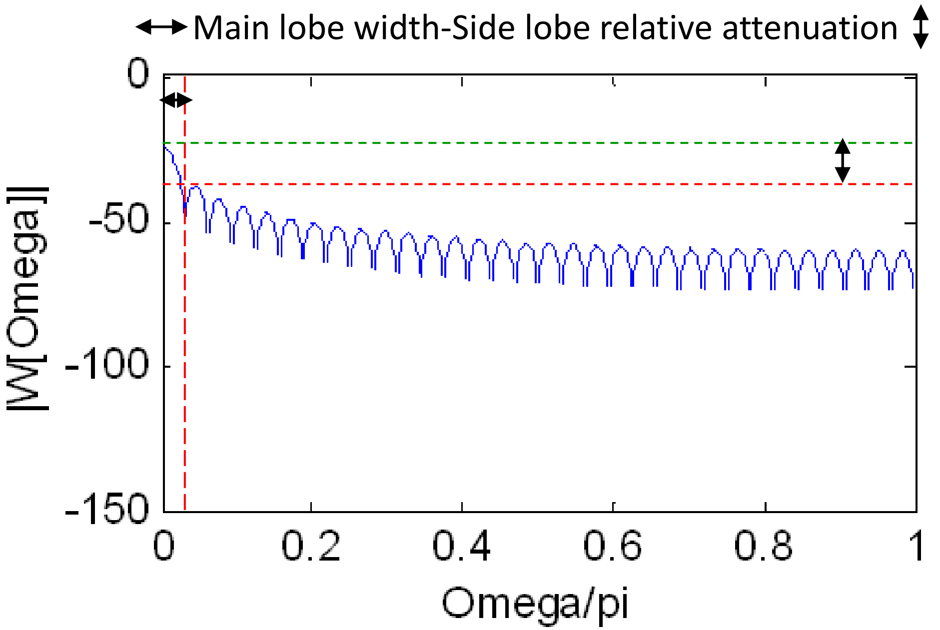


Types of Window

Rectangular Window

Largest side-lobe relative amplitude of all commonly used windows





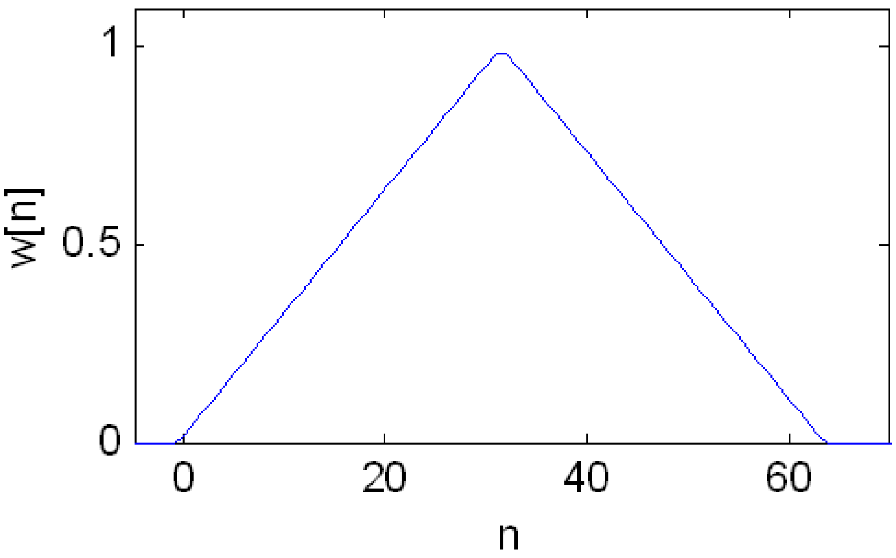
Narrowest main lobe ($\Delta_{ml}=4\pi/L$)

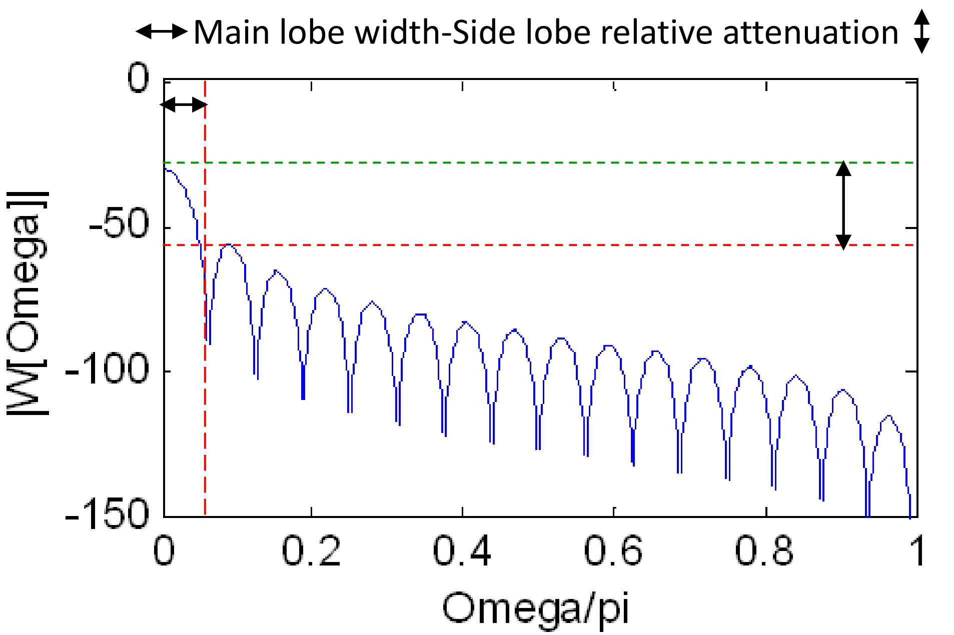
Spectral Smearing & Power Leakage

Types of Window

Triangular Window

Wider main lobe width





Lower side-lobe relative amplitude

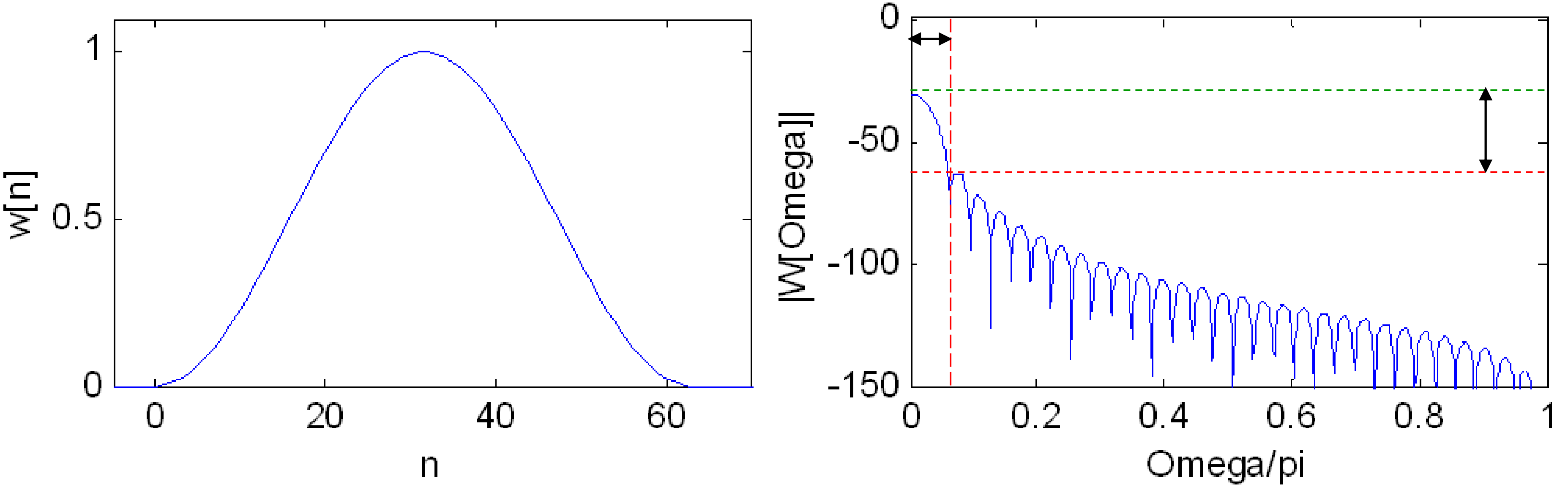
Spectral Smearing & Power Leakage

Types of Window

Hanning Window (raised cosine)

Main lobe width $\Delta_{ml}=8\pi/L$

↔ Main lobe width-Side lobe relative attenuation ↕



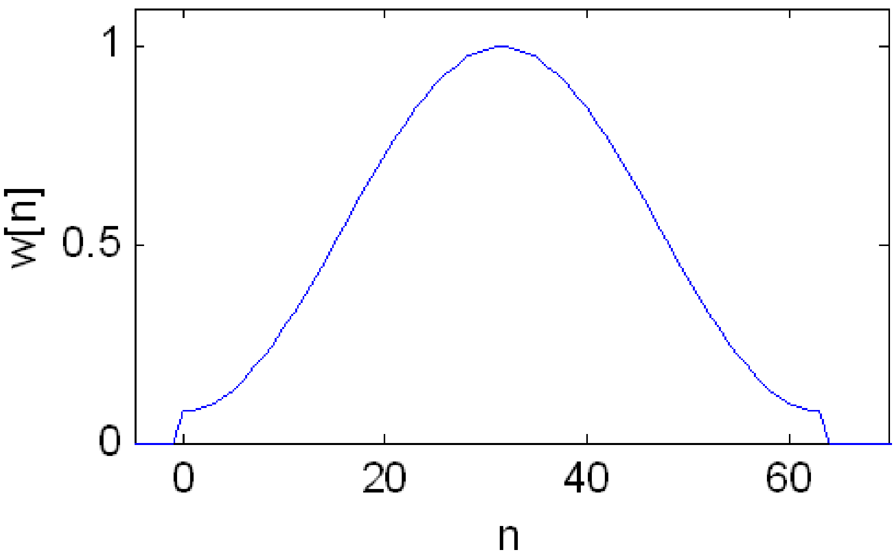
Lower side-lobe relative amplitude

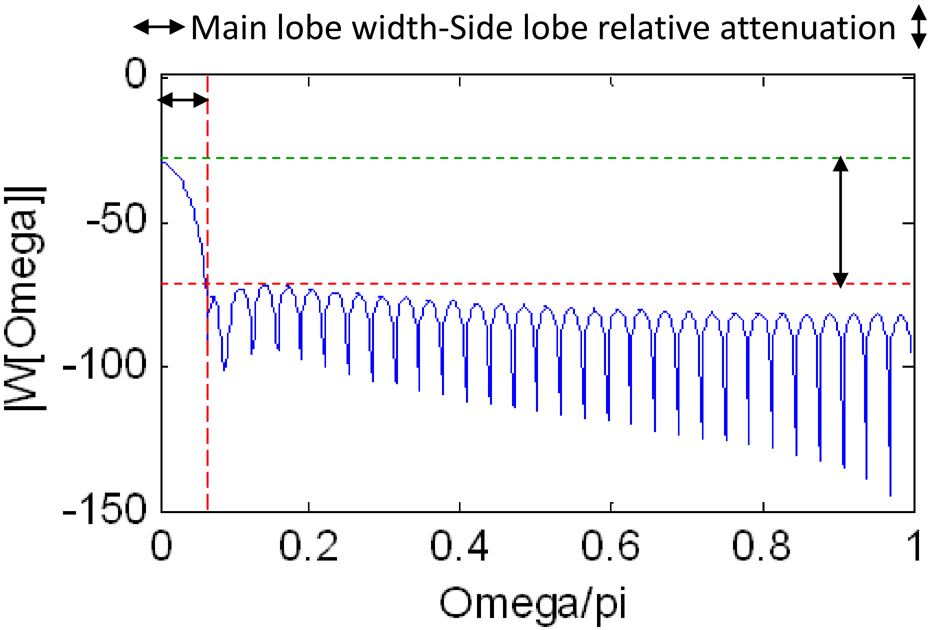
Spectral Smearing & Power Leakage

Types of Window

Hamming Window

Main lobe width $\Delta_{ml}=8\pi/L$





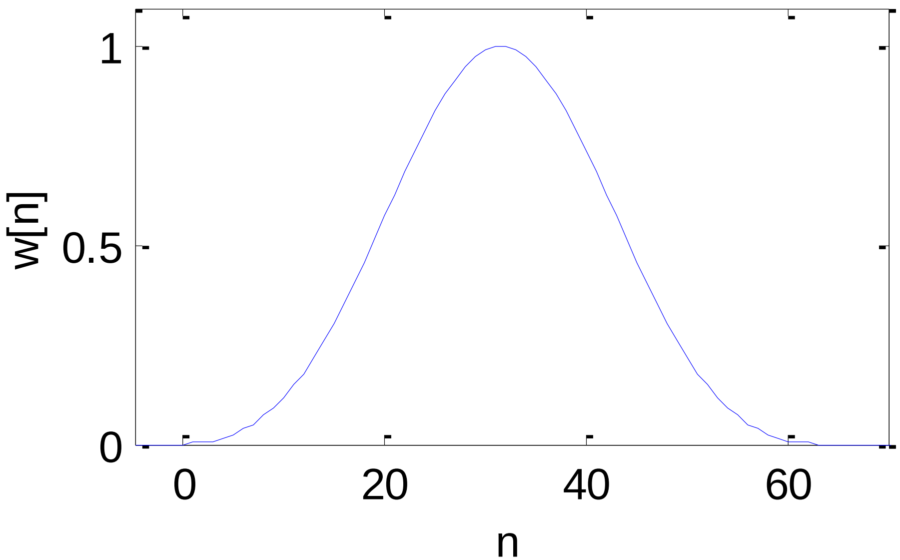
Lower side-lobe relative amplitude

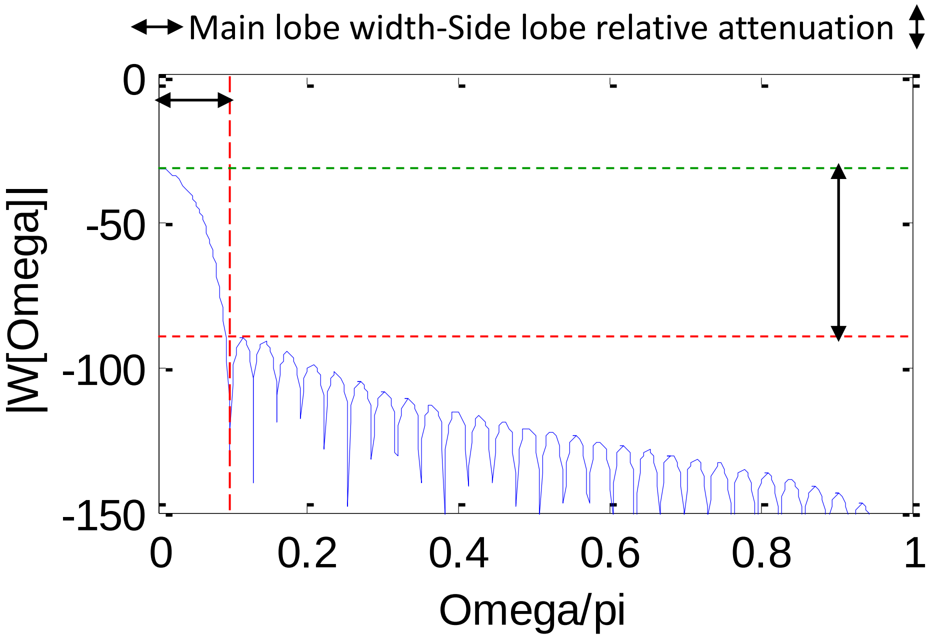
Spectral Smearing & Power Leakage

Types of Window

Blackman Window

Main lobe width $\Delta_{ml}=12\pi/L$





Lower side-lobe relative amplitude

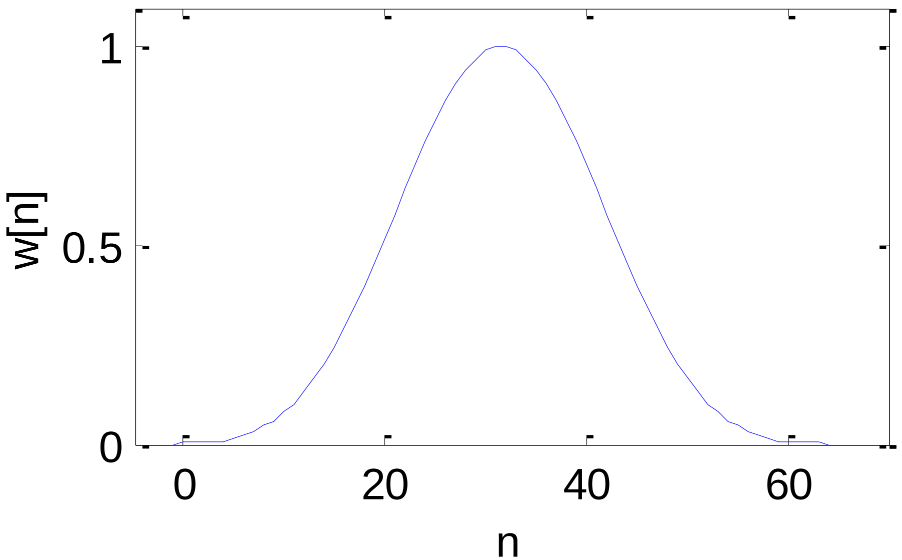
Spectral Smearing & Power Leakage

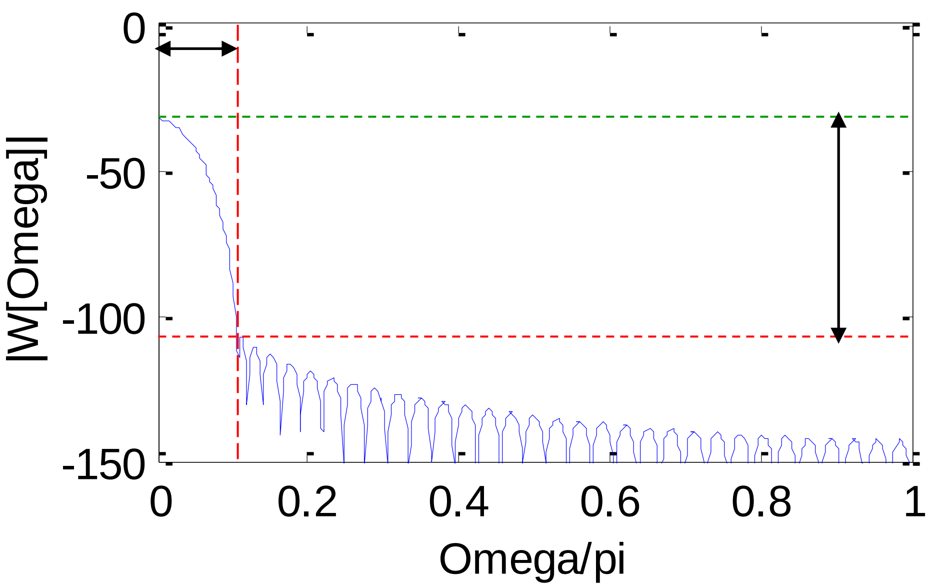
Types of Window

Kaiser-Bessel Window

Wider main lobe width

↔ Main lobe width-Side lobe relative attenuation ↕





Lower side-lobe relative amplitude

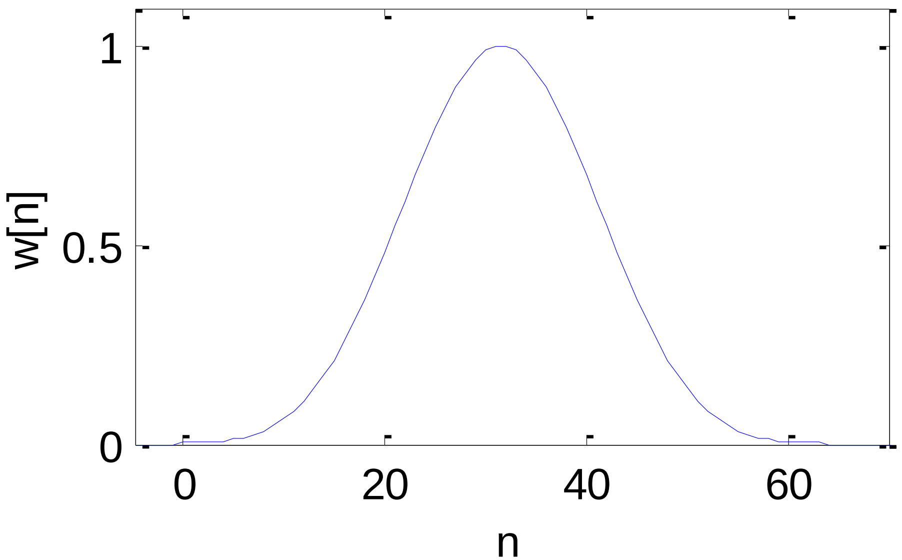
Trade-off between main lobe width and side-lobe relative amplitude possible

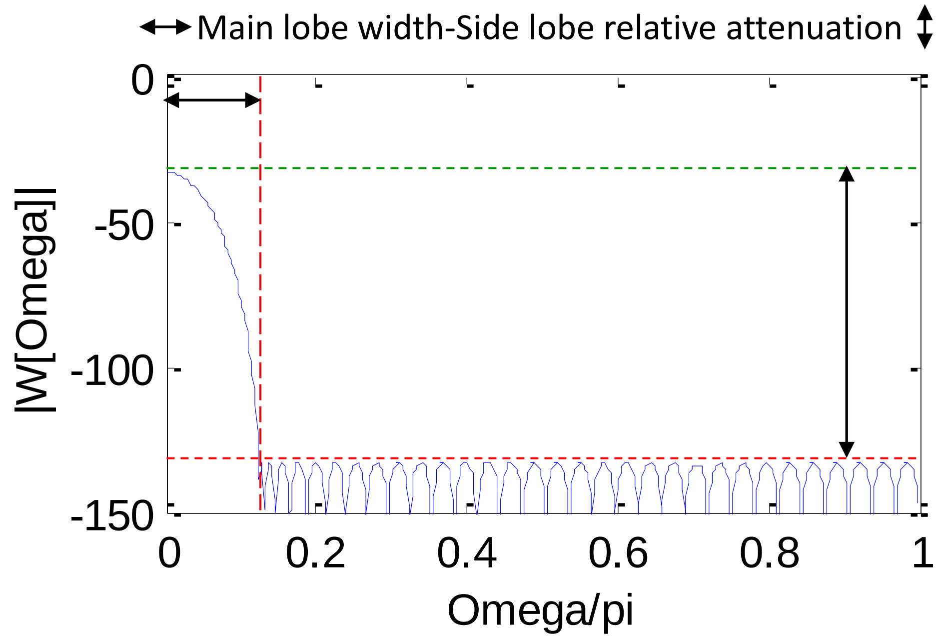
Spectral Smearing & Power Leakage

Types of Window

Dolph-Chebyshev Window

Wider main lobe width





Lower side-lobe relative amplitude

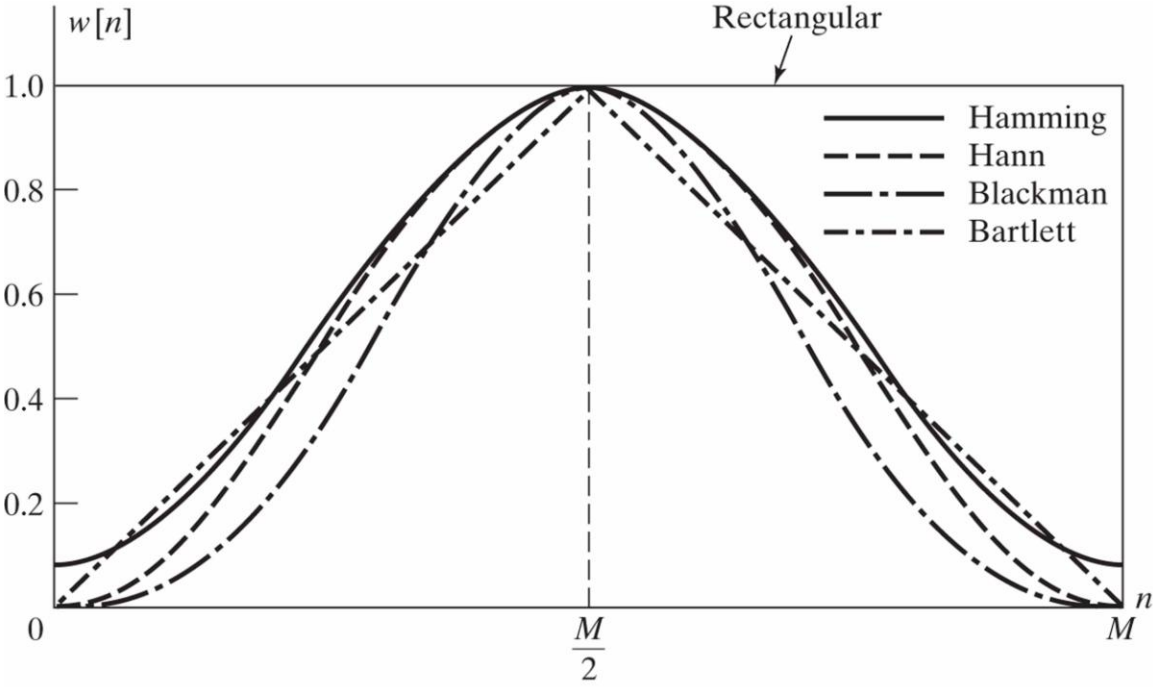
Trade-off between main lobe width and side-lobe relative amplitude possible

Spectral Smearing & Power Leakage

Comparison of commonly used windows

Tapering the window smoothly to zero reduces the side-lobe amplitude

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe
Rectangular	-13	$4\pi/(M+1)$
Bartlett	-25	$8\pi/M$
Hann	-31	$8\pi/M$
Hamming	-41	$8\pi/M$
Blackman	-57	$12\pi/M$



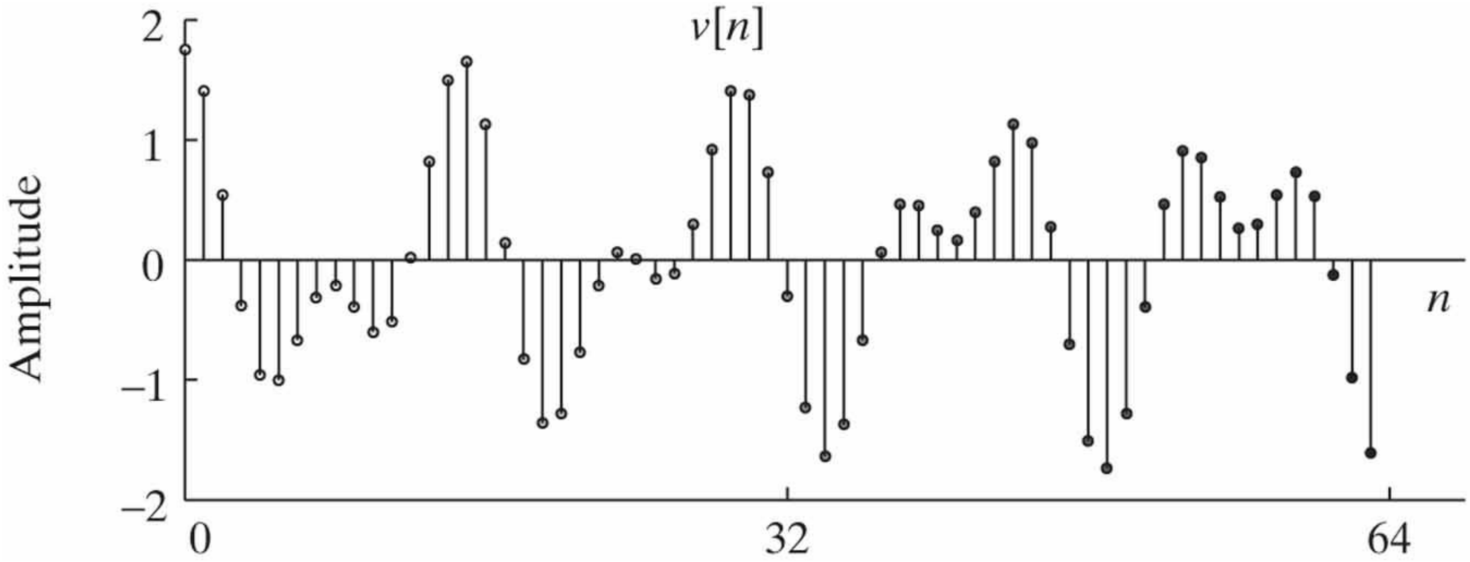
The Effect of Windowing on the DFT Spectrum

DTFT of Windowed Signal

Windowed signal in the time domain

$$v[n] = A_0 w[n]\cos(\Omega_0 n) + A_1 w[n]\cos(\Omega_1 n),\ \ 0 \le n < 64$$

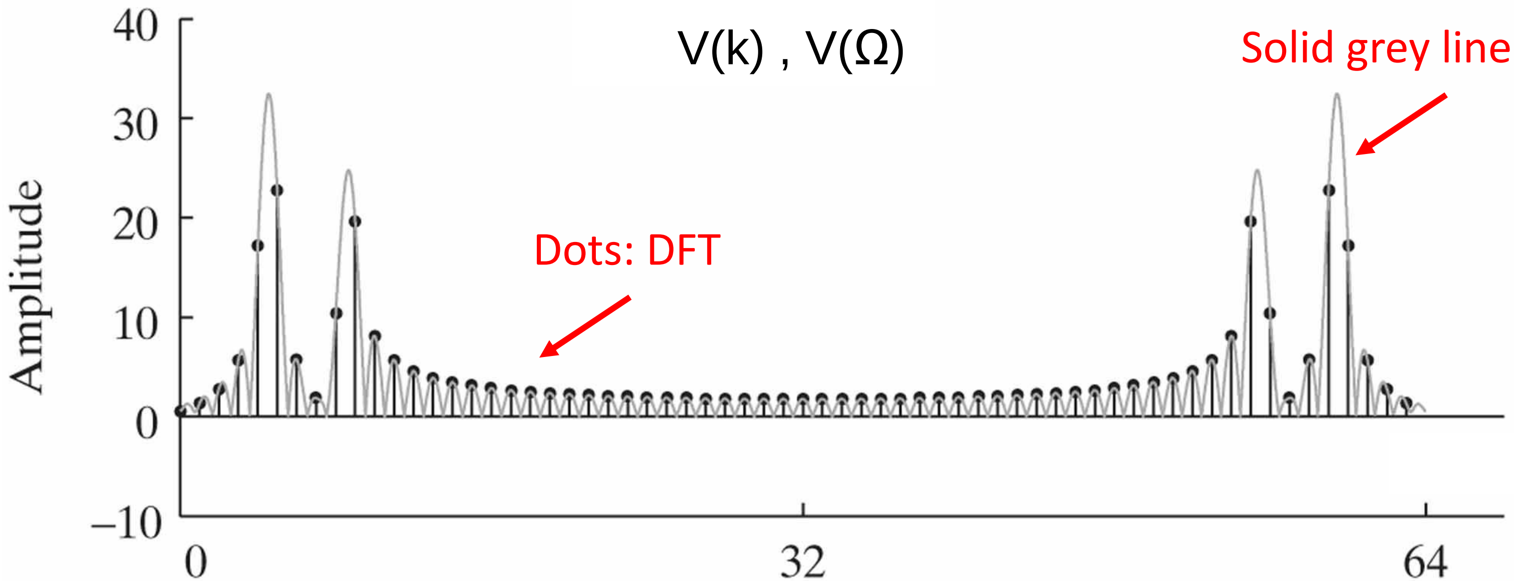
$A_0 = 1$, $A_1 = 0.75$, $\Omega_0 = 2\pi/14$, $\Omega_1 = 4\pi/15$



Spectral Smearing & Power Leakage

The Effect of Spectral Sampling

Spectral smearing & spectral sampling: misleading picture of true spectrum



$$\left.\begin{array}{l}\Omega_0 = \dfrac{2\pi}{N} k_0 \Rightarrow k_0 = \dfrac{N\Omega_0}{2\pi} \\ \Omega_0 = 2\pi/14 \ , \ N = 64\end{array}\right\} \Rightarrow \begin{array}{l} k_0 = 4.571.. \\ k = 4 < k_0 < k = 5 \end{array}$$

$$\left.\begin{array}{l}\Omega_1 = \dfrac{2\pi}{N} k_1 \Rightarrow k_1 = \dfrac{N\Omega_1}{2\pi} \\ \Omega_1 = 2\pi/7.5 \ , \ N = 64\end{array}\right\} \Rightarrow \begin{array}{l} k_1 = 8.533.. \\ k = 8 < k_1 < k = 9 \end{array}$$

The locations of peaks in DFT do not necessarily coincide with the exact frequency locations of the peaks in the DTFT since the true spectrum peaks can lie between spectrum samples

$v[n] = A_0 w[n]\cos(\Omega_0 n) + A_1 w[n]\cos(\Omega_1 n), \ 0 \le n < 64$ A_0 =1, A_1 =0.75, Ω_0=2π/14, Ω_1=4π/15

Spectral Smearing & Power Leakage

Spectral Smearing & Periodicity

Spectral smearing as a result of a lack of periodicity

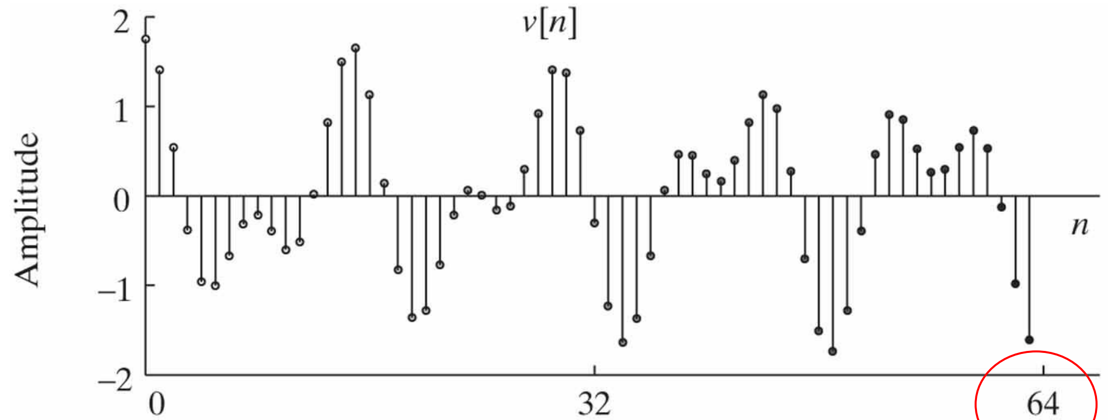
For the signal to be periodic in 64

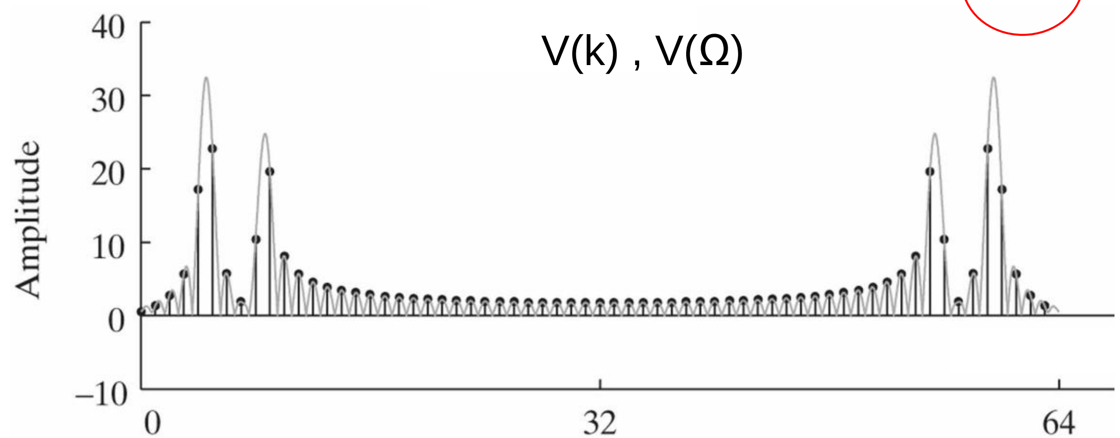
$$\Omega_0 N = 2\pi k \Rightarrow \Omega_0 = 2\pi \frac{k}{N}$$

$\Omega_0 = 2\pi/14$, $\Omega_1 = 4\pi/15$

$$\frac{2\pi}{14} = \frac{2\pi}{64} k \Rightarrow k = \frac{64}{14} = 4.571..$$

$$\frac{2\pi}{7.5} = \frac{2\pi}{64} k \Rightarrow k = \frac{64}{7.5} = 8.533..$$





DT periodicity (Lecture 5): $x[n] = x[n+N]$, N integer

$$\cos(\Omega_0 n) = \cos[\Omega_0 (n+N)] = \cos(\Omega_0 n + \Omega_0 N)$$

$$\Omega_0 N = 2\pi k \Rightarrow \Omega_0 = 2\pi \frac{k}{N}$$

same applies to $e^{j\Omega_0 n} = e^{j\Omega_0 (n+N)}$

Spectral Smearing & Power Leakage

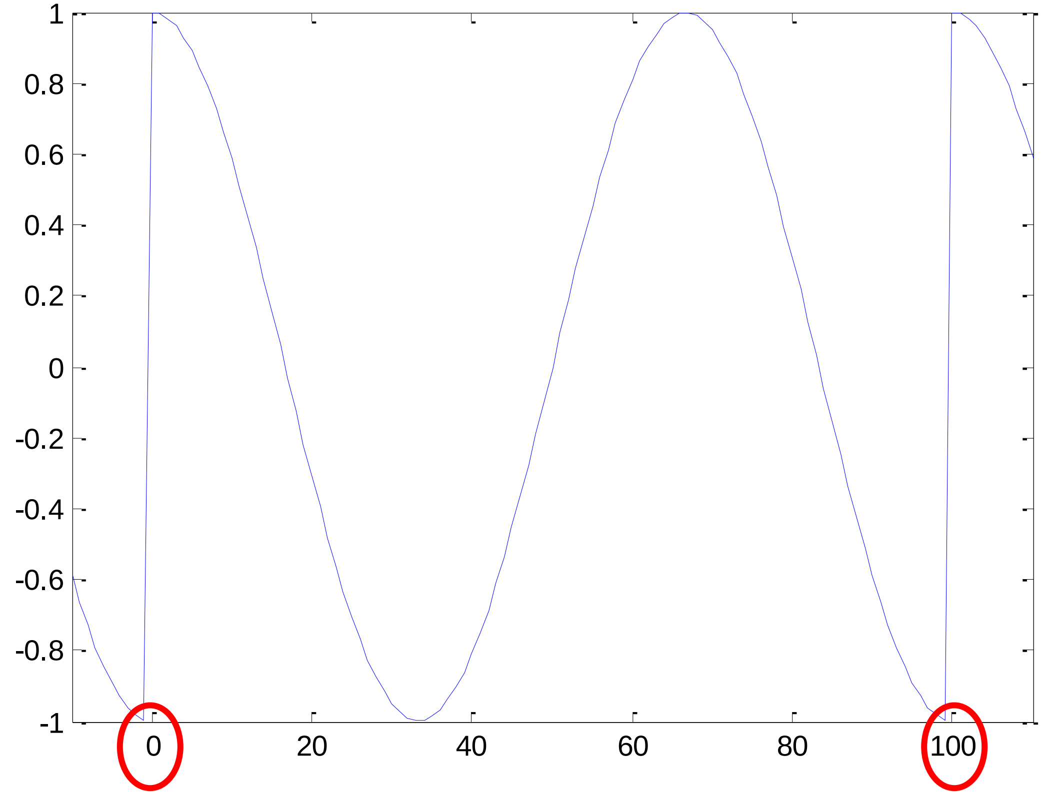
Spectral Smearing & Periodicity

Spectral smearing as a result of discontinuities at the borders

Windowing the signal => Potential discontinuities between $x_p[mN-1]$ and $x_p[mN]$

Discontinuities ⇔ multiple frequencies

```
N=100;
Omega0=0.015*2*pi;
n=0:N-1;
xn=cos(Omega0*n);
xp = [xn,xn,xn];
n2 = -N:2*N-1;
plot(n2,xp)
axis([-10,110,-1,1]);
```



Spectral Smearing & Power Leakage

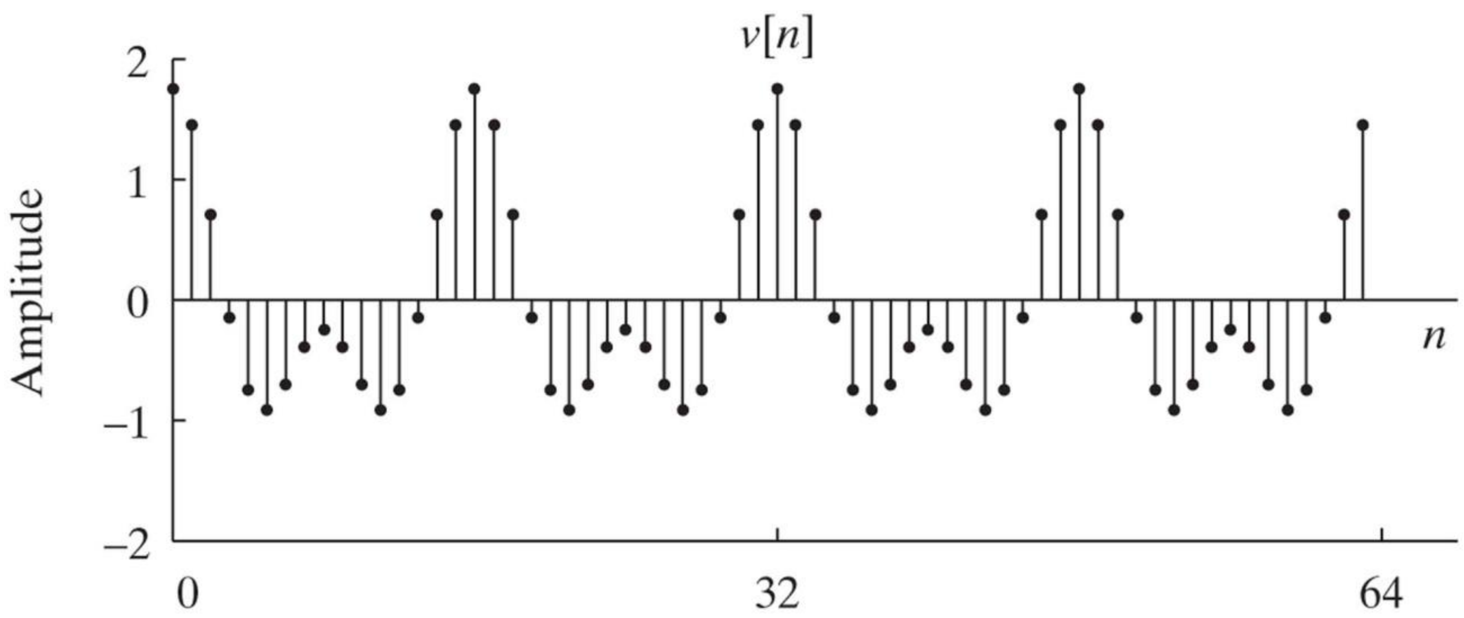
The Effect of Windowing on the DFT Spectrum

DTFT of Windowed Signal

Windowed signal in the time domain

$$v[n] = A_0 w[n]\cos(\Omega_0 n) + A_1 w[n]\cos(\Omega_1 n),\ \ 0 \le n < 64$$

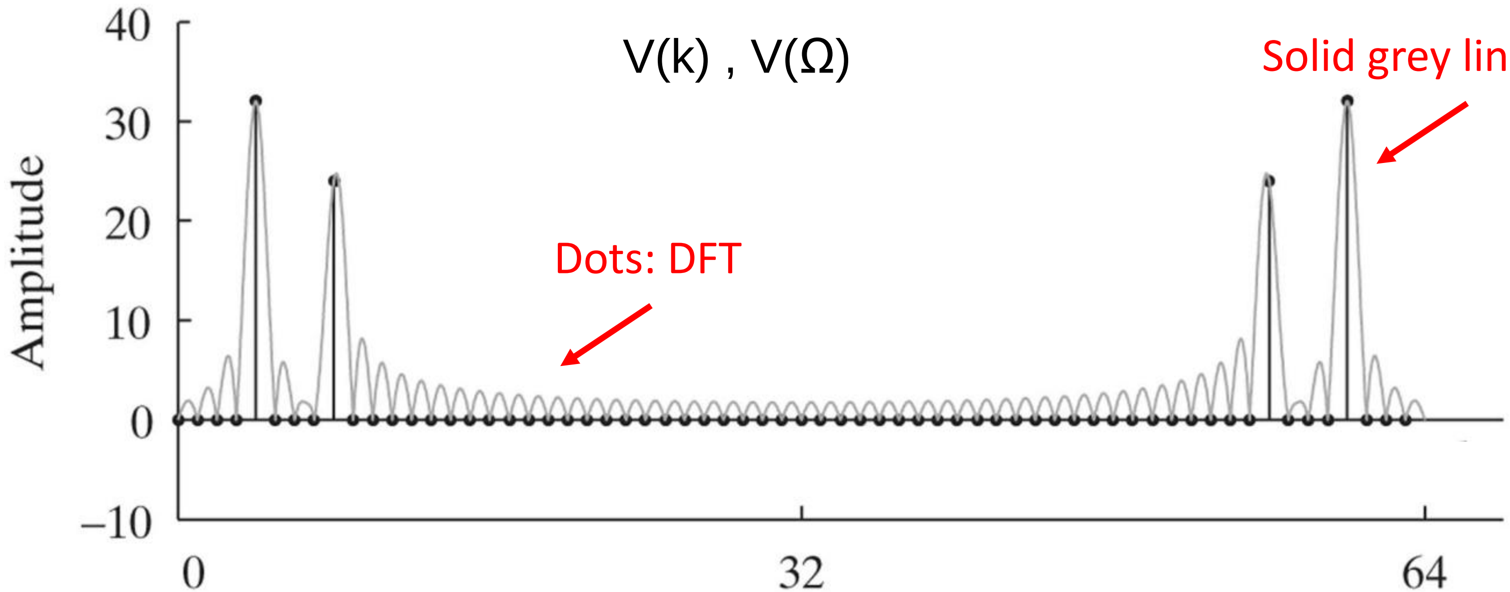
$A_0 = 1$, $A_1 = 0.75$, $\Omega_0 = 2\pi/16$, $\Omega_1 = 2\pi/8$



Spectral Smearing & Power Leakage

The Effect of Spectral Sampling

Spectral smearing & spectral sampling: misleading picture of true spectrum



$$\left.\begin{array}{l}\Omega_0 = \dfrac{2\pi}{N}k_0 \Rightarrow k_0 = \dfrac{N\Omega_0}{2\pi} \\ \Omega_0 = 2\pi/16\ ,\ N = 64\end{array}\right\} \Rightarrow k_0 = 4$$

$$\left.\begin{array}{l}\Omega_1 = \dfrac{2\pi}{N}k_1 \Rightarrow k_1 = \dfrac{N\Omega_1}{2\pi} \\ \Omega_1 = 2\pi/8\ ,\ N = 64\end{array}\right\} \Rightarrow k_1 = 8$$

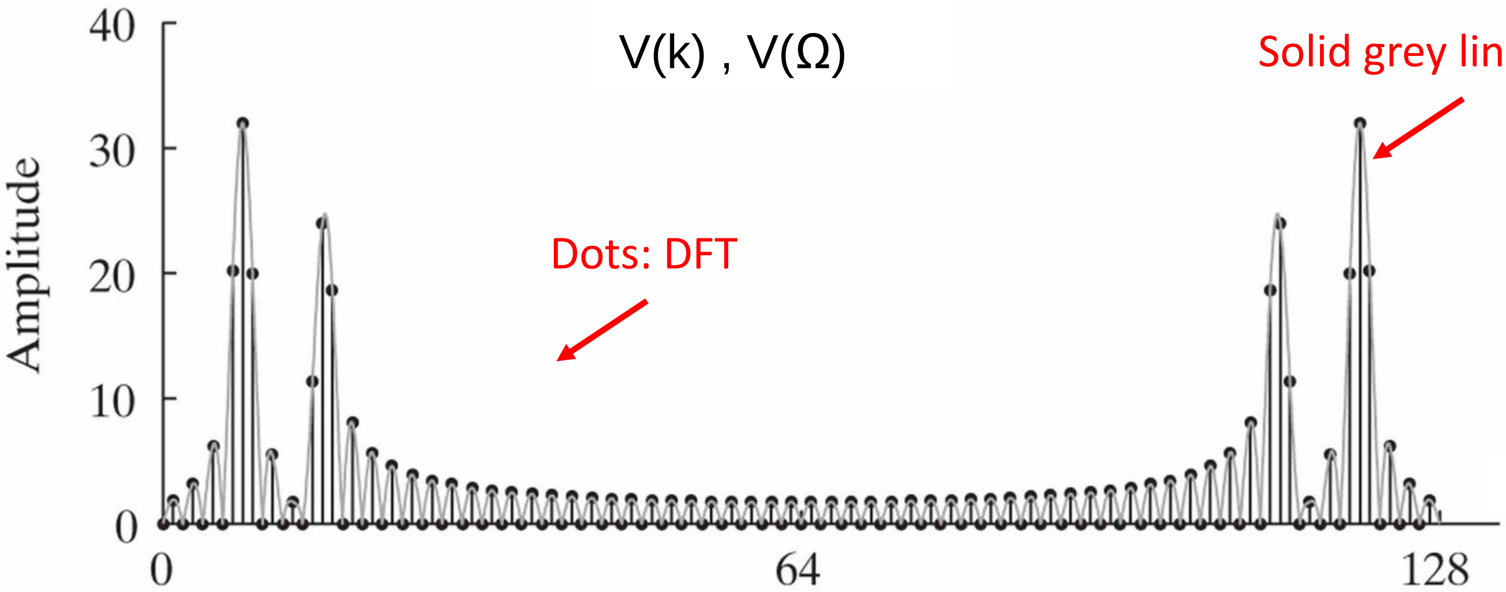
Although the signal has significant frequency content at almost all frequencies, the DFT does not show that because of the sampling of the spectrum (Ω_0 and Ω_1 are multiples of $2\pi/N$)

$v[n] = A_0 w[n]\cos(\Omega_0 n) + A_1 w[n]\cos(\Omega_1 n),\ 0 \le n < 64$ $A_0 = 1$, $A_1 = 0.75$, $\Omega_0 = 2\pi/16$, $\Omega_1 = 2\pi/8$

Spectral Smearing & Power Leakage

The Effect of Spectral Sampling

What happens if we increase the spectral sampling rate (DFT length - zero padding)



V(k) , V(Ω)

Solid grey line : DTFT

Dots: DFT

$$\left.\begin{array}{l}\Omega_0 = \dfrac{2\pi}{N} k_0 \Rightarrow k_0 = \dfrac{N\Omega_0}{2\pi} \\ \Omega_0 = 2\pi/16, \quad N = 128\end{array}\right\} \Rightarrow k_0 = \frac{128}{16} = 8$$

$$\left.\begin{array}{l}\Omega_1 = \dfrac{2\pi}{N} k_1 \Rightarrow k_1 = \dfrac{N\Omega_1}{2\pi} \\ \Omega_1 = 2\pi/8, \quad N = 128\end{array}\right\} \Rightarrow k_1 = \frac{128}{8} = 16$$

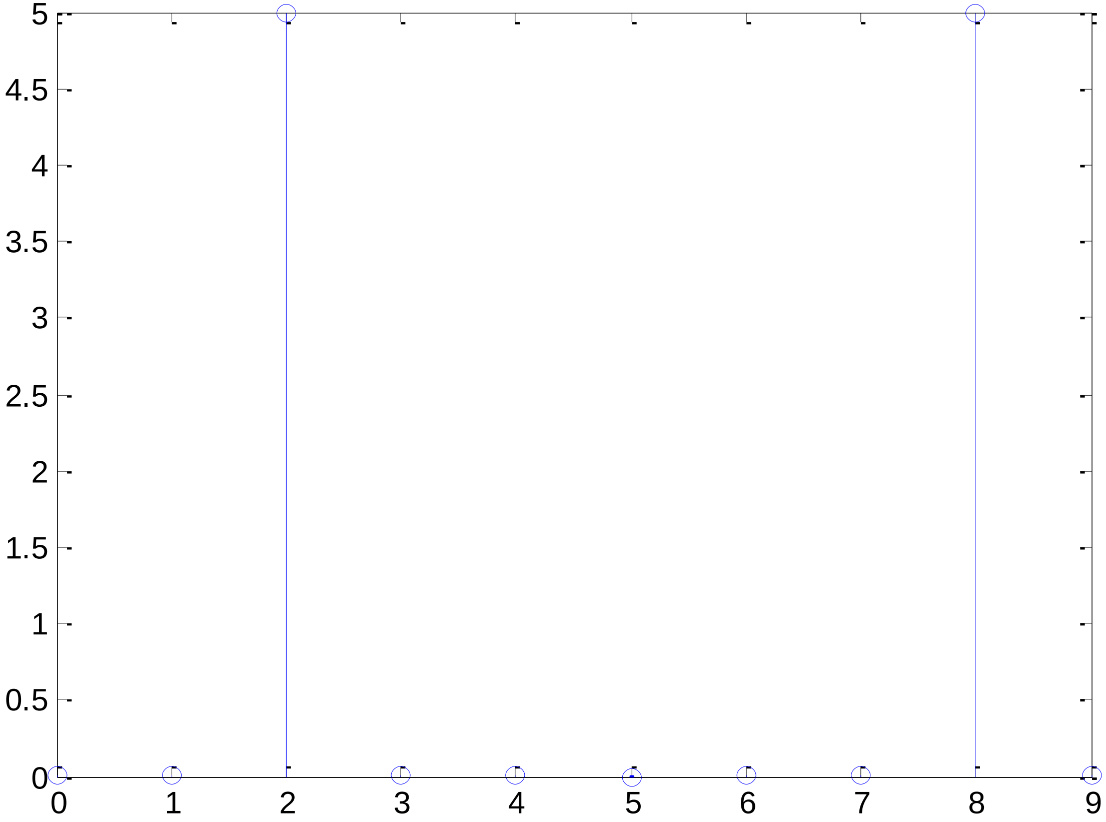
With finer sampling of the spectrum the presence of significant spectral content at other frequencies becomes apparent

$v[n] = A_0 w[n]\cos(\Omega_0 n) + A_1 w[n]\cos(\Omega_1 n),\ 0 \le n < 64$ A_0 =1, A_1 =0.75, Ω_0=2π/16, Ω_1=2π/8

Spectral Smearing & Power Leakage

Problem with DFT : Spectrum not always what we expect

Matlab




```
N=10;
n=0:N-1;
Omega0=0.2*2*pi;
xn=cos(n*Omega0)
Xk=fft(xn);
k=0:N-1;
stem(k,abs(Xk))
```

Spectrum looks as expected

$k_1 = 10 \times 2 \times 0.2\pi / 2\pi = 2$

$k_2 = 10 - 2 = 8$

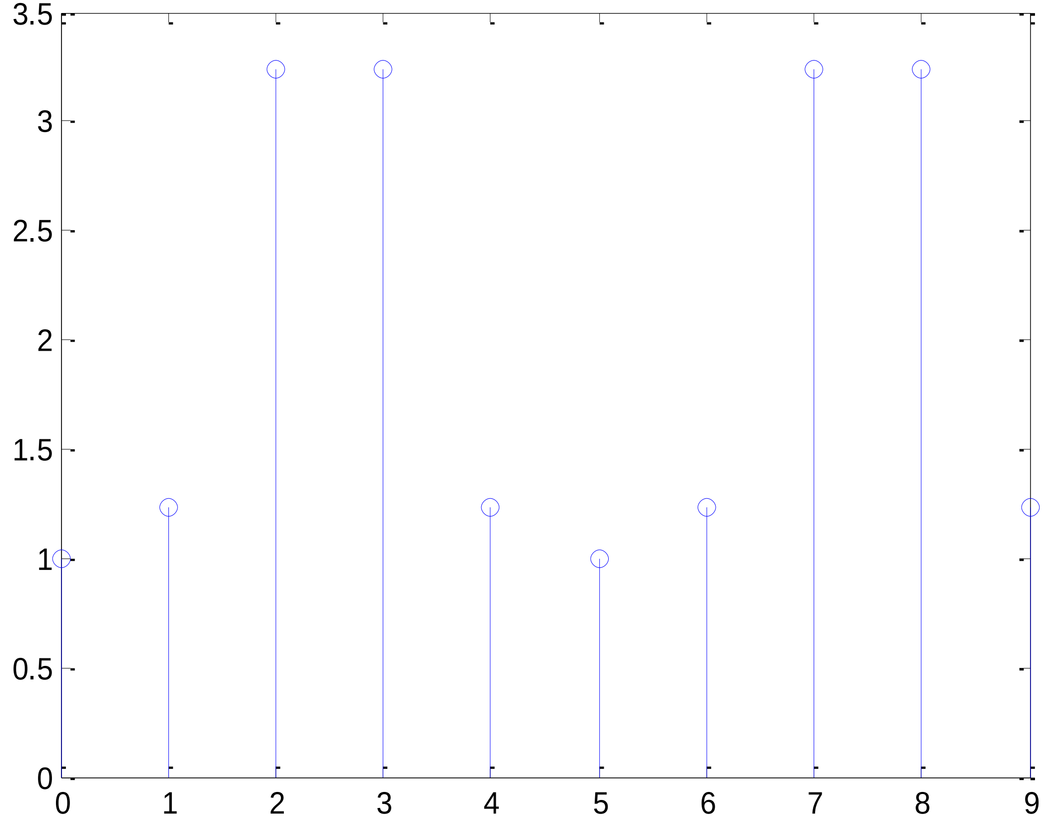
$k_1 = N\Omega_0 / 2\pi$,

$k_2 = N - k_1$

Spectral Smearing & Power Leakage

Problem with DFT : Spectrum not always what we expect

Matlab




```
N=10;
n=0:N-1;
Omega0=0.25*2*pi;
xn=cos(n*Omega0);
Xk=fft(xn);
k=0:N-1;
stem(k,abs(Xk))
```

Ooops….

$k_1 = 10 \times 0.25 \times 2\pi / 2\pi = 2.5$

$k_2 = 10 - 2.5 = 7.5$

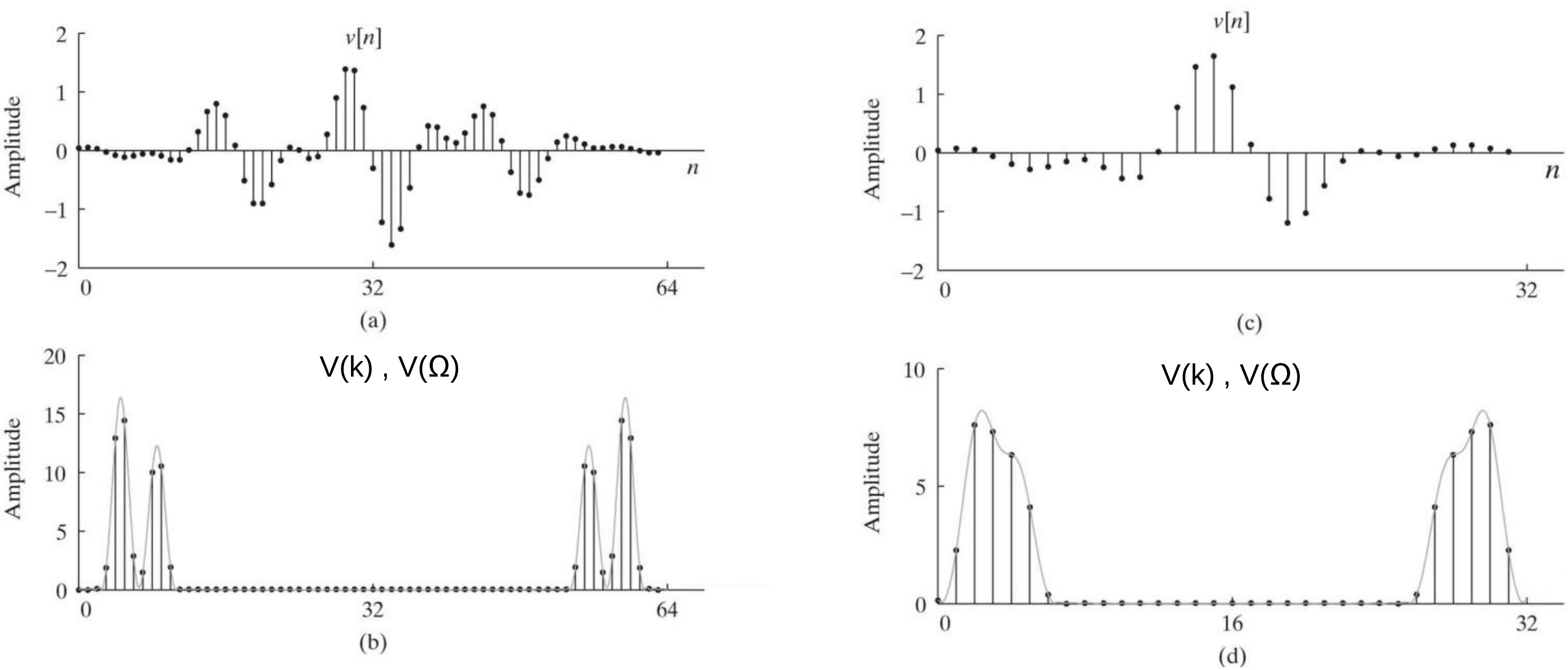
$k_1 = N\Omega_0 / 2\pi$,

$k_2 = N - k_1$

Spectral Smearing & Power Leakage

Windowing & Zero Padding

Keiser Window – Different window lengths

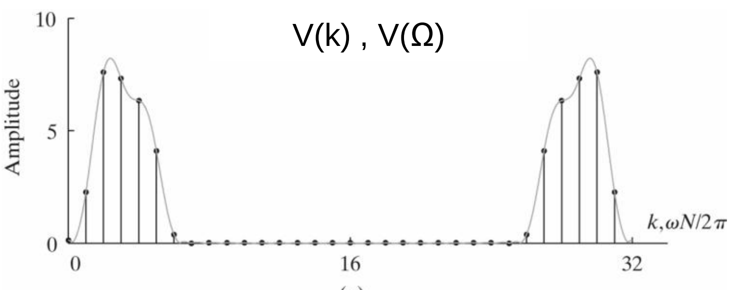


DFT analysis with Kaiser window. (a) Windowed sequence for L = 64. (b) Magnitude of DFT for L = 64. (c) Windowed sequence for L = 32. (d) Magnitude of DFT for L = 32.

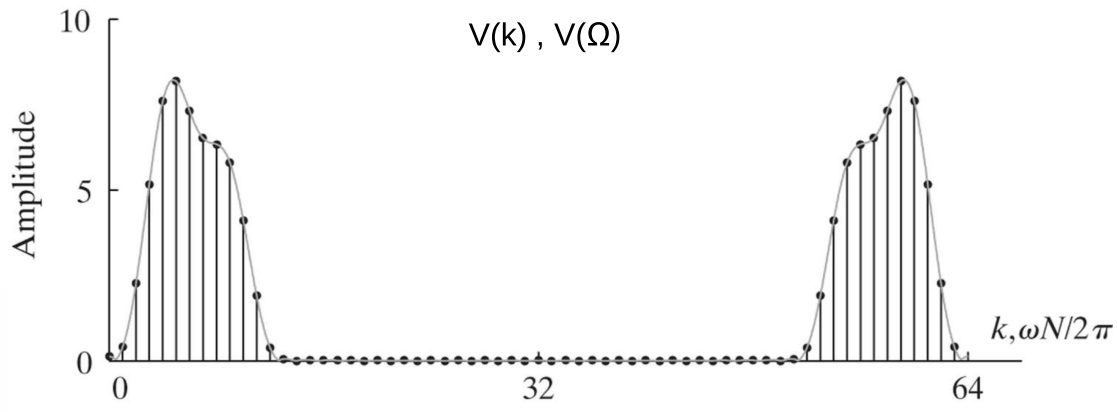
Spectral Smearing & Power Leakage

Windowing & Zero Padding

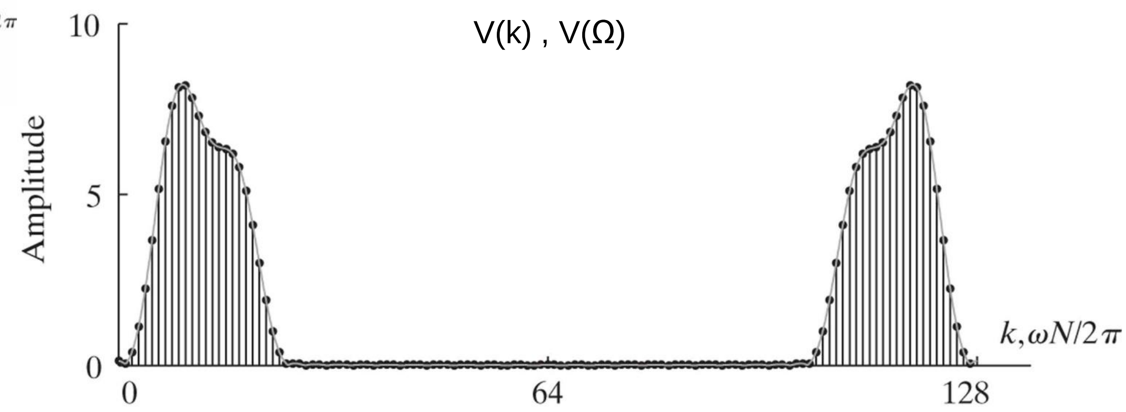
Keiser Window – Fixed window length with zero padding



(a)



(b)



(c)

Spectral Smearing & Power Leakage

Time Frequency Tradeoffs

Large Window

\+ Good resolution in frequency

\+ Less spectral smearing / power leakage

– Poor resolution in time

– Higher complexity

Small Window

\+ Good resolution in time

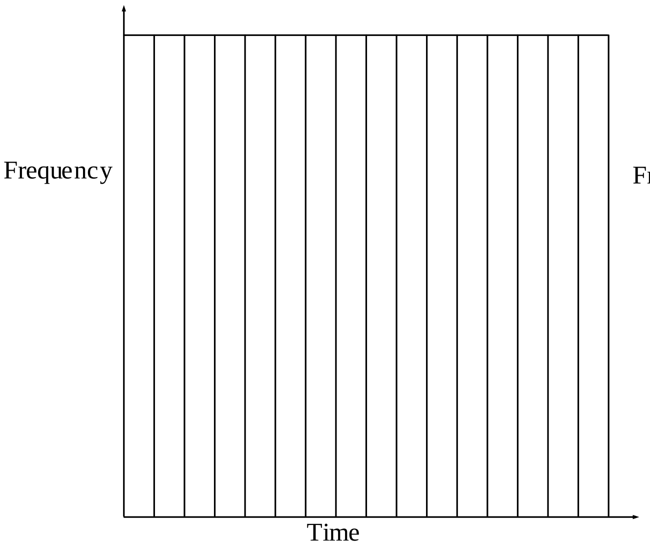
\+ Lower complexity

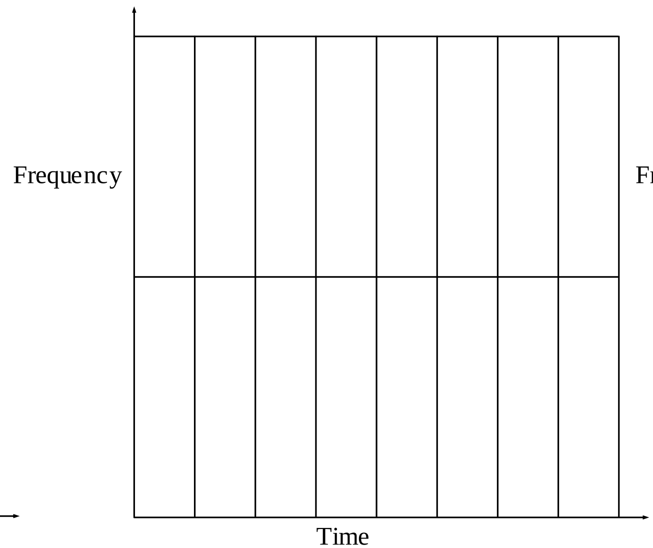
– Poor resolution in frequency

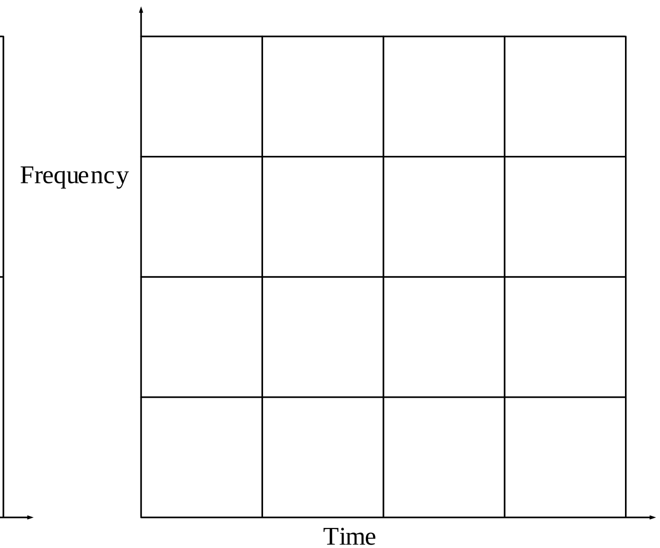
– More spectral smearing / power leakage

Spectral Smearing & Power Leakage

Uncertainty Principle: Resolution in time x Resolution in frequency = constant







- Can't know time and frequency accurately
- An impulse function is localised in time but unbounded in frequency
- A complex exponential is localised in frequency but infinite in time