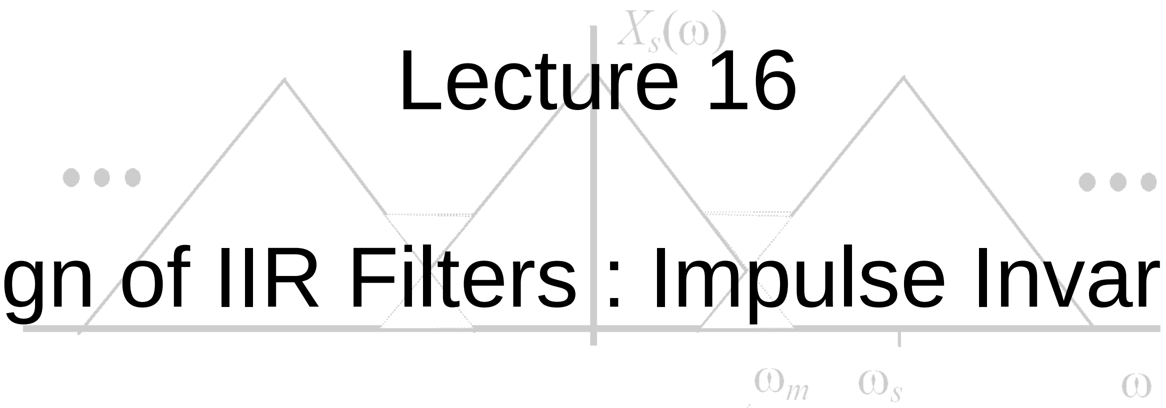
Lecture 16

Design of IIR Filters : Impulse Invariance



Design by sampling the impulse response of analogue filters

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Design of IIR Filters: Impulse Invariance

Infinite Impulse Response (IIR) Filters

Filter definition

Difference equation

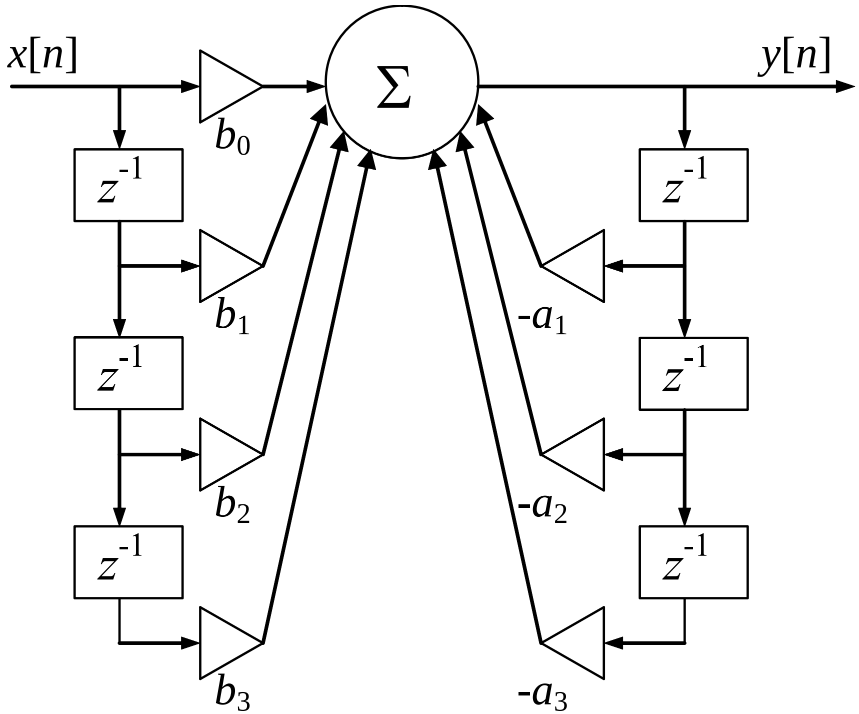
$$y[n]=\sum_{k=0}^{N} b_k x[n-k]-\sum_{k=1}^{N} a_k y[n-k]$$

Transfer function

$$H(z)=\sum_{k=0}^{N} b_k z^{-k} \Big/ \sum_{k=0}^{N} a_k z^{-k}$$

Fraction of products of 1st order terms

$$H(z)=\prod_{k=1}^{N}(z-z_k) \Big/ \prod_{k=1}^{N}(z-p_k)$$



Design of IIR Filters: Impulse Invariance

From Continuous Time to Discrete Time Filters

First there were continuous time filters only …..

When digital signal processing came along, design of digital filters relied on mapping/transformation of well-known continuous time filter designs to discrete time ones

Impulse Invariance and **Bilinear Transformation** are two methods providing such a transformation/mapping

Impulse Invariance

Impulse Invariance means that the impulse response of the digital filter will be similar to that of the continuous time filter – achieved through sampling of the impulse response of the continuous time filter

Bilinear Transformation

Bilinear Transformation applies a non-linear mapping of the analogue frequency axis to the digital frequency one

Design of IIR Filters: Impulse Invariance

Impulse Invariance

Design Procedure

1. Start with an analogue prototype/transfer function $H_a(s)$
2. Use the inverse Laplace transform to obtain the analogue prototype impulse response $h_a(t)$
3. Sample the impulse response $h_a(t)$ to obtain $h[n] = h_a(nT)$
4. Find the desired transfer function by taking the **z transform of h[n]**

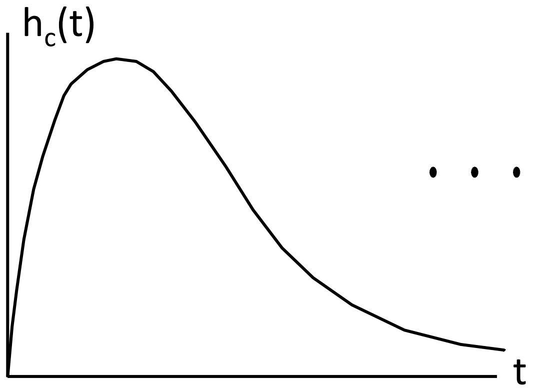
Results of Sampling

- The impulse response of the discrete filter h[n] is identical to that of the analogue filter at t =nT (hence the name impulse invariance)
- Sampling in the time domain will cause frequency aliasing

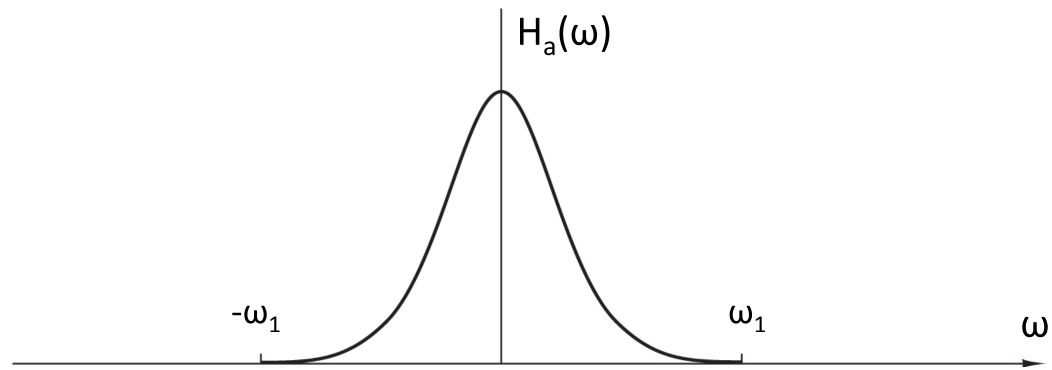
Impulse Invariance

IIR Filter design through sampling of the continuous time impulse response

Analogue Filter - Impulse Response



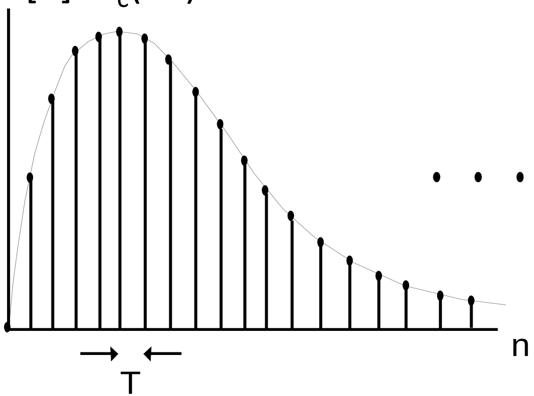
Frequency Response



Discrete Time Filter - Impulse Response

Impulse Response Transformation $(t \rightarrow nT)$

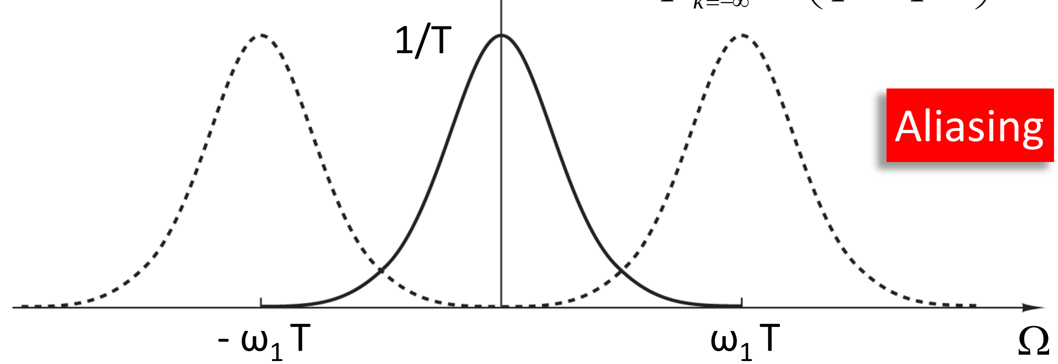
$h[n]=h_c(nT)$



Frequency Response

Frequency response transformation $(\omega \rightarrow \Omega=\omega T)$ *linear transformation*

$$H(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(\frac{\Omega}{T}-\frac{2\pi}{T}k\right)$$

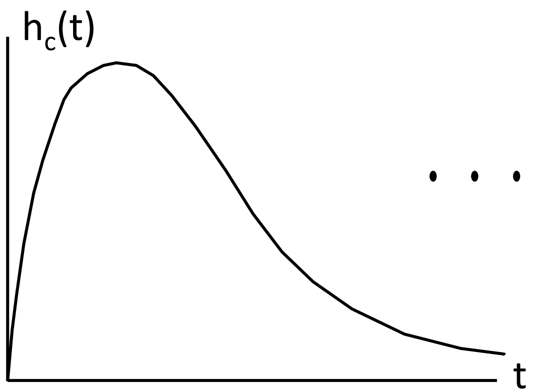


Design of IIR Filters: Impulse Invariance

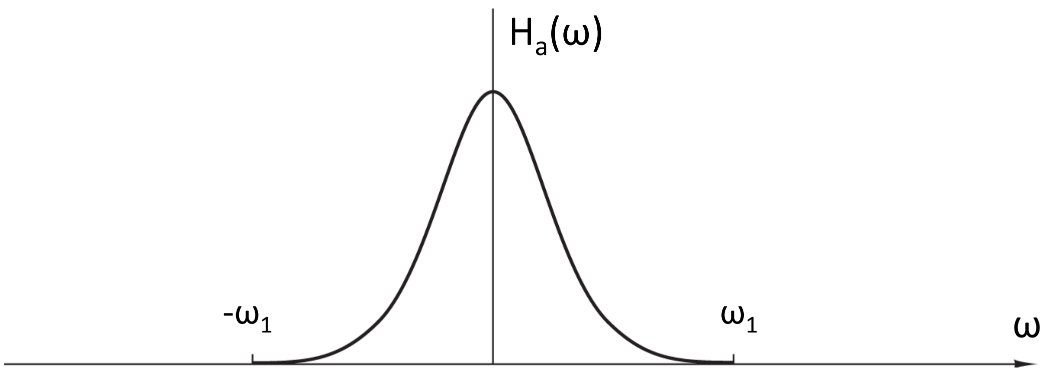
Impulse Invariance

IIR Filter design through sampling of the continuous time impulse response

Analogue Filter - Impulse Response

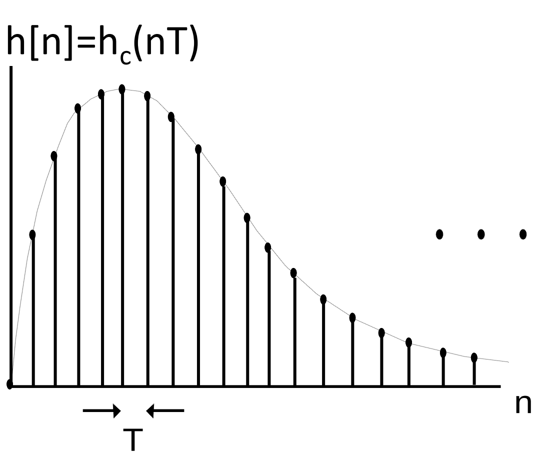


Frequency Response



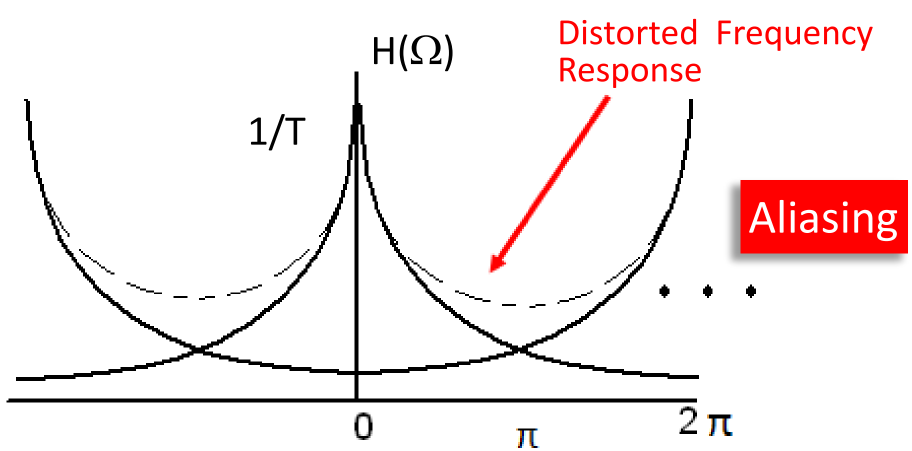
Discrete Time Filter - Impulse Response

Impulse Response Transformation ($t \rightarrow nT$)



Frequency Response

Frequency response transformation ($\omega \rightarrow \Omega=\omega T$) linear transformation



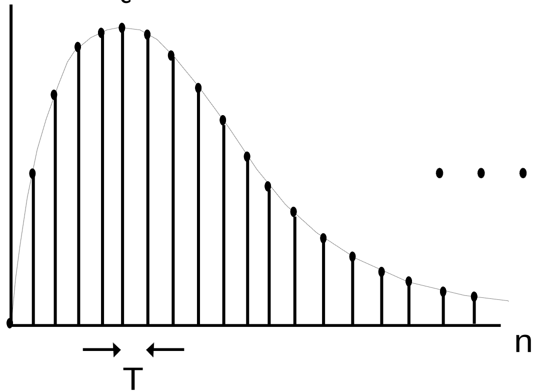
Impulse Invariance

Design limitations

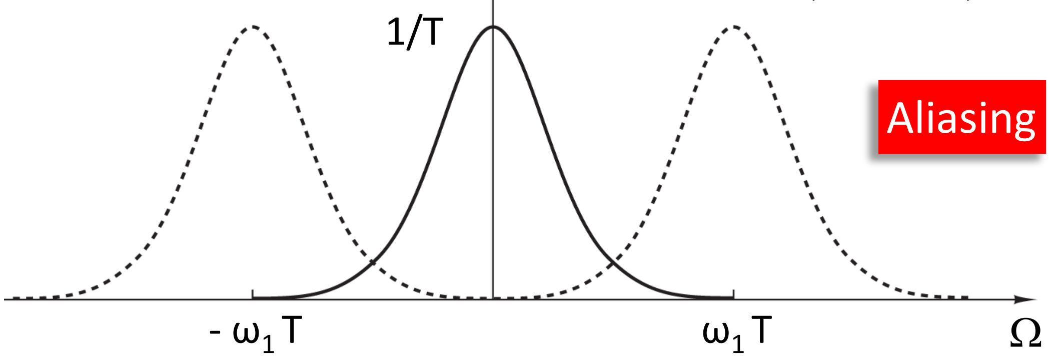
- The frequency response of the continuous-time filter has to be band-limited for aliasing not to occur
- High pass and band-stop filter design not possible with Impulse invariance
- Practical continuous time filters are not exactly band-limited and some aliasing occurs
- In practice to compensate for aliasing the continuous-time filter may be somewhat overdesigned (lower cut-off, higher attenuation in stop-band)

Discrete Time Filter - Impulse Response	Frequency Response
Impulse Response Transformation ($t \rightarrow nT$)	*Frequency response transformation ($\omega \rightarrow \Omega = \omega T$)* *linear transformation*

$h[n]=h_c(nT)$



$$H(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(\frac{\Omega}{T} - \frac{2\pi}{T}k\right)$$



Design of IIR Filters: Impulse Invariance

From Continuous Time to Discrete Time Filters

From the s-plane to the z-plane

$$H(s)=\sum_{k=0}^{N} b_k s^k \Big/ \sum_{k=0}^{N} a_k s^k \xrightarrow{\text{Mapping of S-plane to Z-plane}} H(z)=\sum_{k=0}^{N} b_k z^{-k} \Big/ \sum_{k=0}^{N} a_k z^{-k}$$

Mapping of the s-plane to the z-plane with Impulse Invariance

$$h(t)=\int_{-\infty}^{\infty} H(s)e^{st}dt$$

$$H(z)=\sum_{n=-\infty}^{\infty} h[n].z^{-n}$$

$$y(t)=h(t)*x(t) \xrightarrow{\text{Sampling of Impulse response}} y[n]=h[n]*x[n]$$

Indirect mapping of the s-plane to the z-plane through sampling of the impulse response

From Continuous Time to Discrete Time Filters

From the s-plane to the z-plane – Mapping requirements

1. Essential properties of the continuous-time frequency response should be preserved in the frequency response of the resulting discrete-time filter

   - The imaginary axis of the s-plane should map onto the unit circle of the z-plane

2. A stable continuous-time filter should be transformed to a stable discrete-time filter

   - If the continuous-time filter has poles only in the left half of the s-plane the discrete-time filter should have poles only inside the unit circle in the z-plane

Impulse Invariance

From the s-plane to the z-plane - Transformation of the system function

- The system function of the causal continuous-time (analogue) filter can be expressed in terms of a partial fraction expansion:

$$H_a(s) = \sum_{k=0}^{N} b_k s^k \Big/ \sum_{k=0}^{N} a_k s^k \Leftrightarrow \boxed{H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}}$$

- The impulse response of the analogue filter is the inverse Laplace transform of $H_\alpha(s)$:

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t}$$

- The impulse response of the discrete-time filter is obtained by sampling the impulse response of the analogue filter:

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{s_k nT} = \sum_{k=1}^{N} A_k e^{(s_k T)^n}$$

- The system function of the discrete-time filter is the Z transform of this impulse response:

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}}$$

Design of IIR Filters: Impulse Invariance

Impulse Invariance

Transformation of the system function

Compare system functions: $H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$ and $H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$

A pole $s=s_k$ on the s-plane maps onto a pole $z = e^{s_k T}$ on the z-plane

More generally …

$$\left.\begin{aligned} z &= e^{sT} \\ s &= \sigma + j\omega \end{aligned}\right\} \quad \left.\begin{aligned} z &= e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} \\ z &= re^{j\Omega} \end{aligned}\right\} \quad re^{j\Omega} = e^{\sigma T} e^{j\omega T} \Rightarrow \begin{aligned} r &= e^{\sigma T} \\ \Omega &= \omega T \end{aligned}$$

$$\left.\begin{aligned} r &= e^{\sigma T} \\ \sigma &< 0 \end{aligned}\right\} \quad 0 < r < 1$$

The left half side of the s-plane is mapped inside the unit circle on the z-plane => Stable continuous time filters are transformed into stable discrete-time filters

$$\left.\begin{aligned} r &= e^{\sigma T} \\ \sigma &= 0 \end{aligned}\right\} \quad r = 1$$

The $j\omega$ axis is mapped on the unit circle

Design of IIR Filters: Impulse Invariance

Impulse Invariance

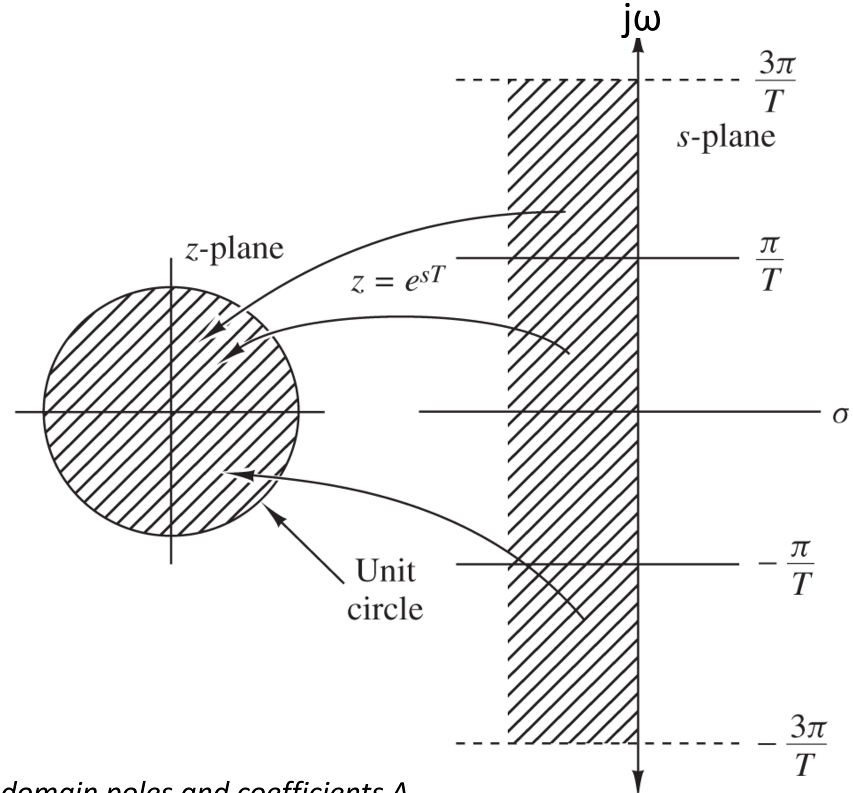
Frequency mapping – many to one

Compare system functions: $H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$ and $H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$

$\Omega = \omega T$, $-\pi < \Omega < \pi$: $-\pi / T \leq \omega \leq \pi / T$ maps to $-\pi < \Omega < \pi$

$\Omega = \omega T$, $-\pi < \Omega < \pi$: $\pi / T \leq \omega \leq 3\pi / T$ also maps to $-\pi < \Omega < \pi$

$\Omega = \omega T$, $-\pi < \Omega < \pi$: $(2k - 1)\pi / T \leq \omega \leq (2k + 1)\pi / T$ maps to $-\pi < \Omega < \pi$



Zeros are not mapped the same way as the poles ; they are a function of the z-domain poles and coefficients A_k

Design of IIR Filters: Impulse Invariance

Design Procedure

1. Design an analogue prototype filter e.g. Butterworth, Eliptic, Chebychev

   Start with analogue prototype filter with a normalized low-pass frequency response (i.e. with a cut-off frequency of 1 rad/s) and convert normalized low-pass to general low-pass

   $$s \rightarrow s/\omega_{ca}$$

2. Factorise and use partial fractions to split into a sum of 1st order terms

   $$H_a(s) = \frac{1}{s - s_p}$$

3. Map each 1st order analogue term to 1st order digital term

   $$H_d(z) = \frac{1}{1 - z_p z^{-1}}, \quad z_p = e^{s_p T}$$

4. Rearrange to standard form

Design of IIR Filters: Impulse Invariance

Example 1

Design a low-pass digital filter with f_c=100Hz and T=1.6ms using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

1. Scale frequency of analogue filter to get desired cut off frequency:

   replace s by $s/\omega_c = s/2\pi f_c$ which gives $$H(s) = \frac{\omega_c^2}{s^2 + \omega_c\sqrt{2}s + \omega_c^2}$$

2. Factorize H(s) and use partial fractions to express as sum of 2 single pole filters

   $$H(s) = \frac{\omega_c^2}{s^2 + \omega_c\sqrt{2}s + \omega_c^2} = \frac{C}{s-p} + \frac{C^*}{s-p^*}$$

   where $$p = \frac{-\omega_c\sqrt{2}(1-j)}{2} \qquad C = \frac{-j\omega_c}{\sqrt{2}}$$

3. Find the poles in z using substitution $z_p = e^{pT}$: $z_p = e^{\frac{-\omega_c\sqrt{2}(1-j)}{2}T}, z_p^*$ (2 complex conjugate poles)

   resulting in transfer function: $$H(z) = \frac{C}{1-e^{pT}z^{-1}} + \frac{C^*}{1-e^{p^*T}z^{-1}} = \frac{2C_r - 2e^{p_rT}\left(C_r\cos(p_iT) + C_i\sin(p_iT)\right)z^{-1}}{1 - 2e^{p_rT}\cos(p_iT)z^{-1} + e^{2p_rT}z^{-2}}$$

   where C_r, C_i, p_r and p_i are the real and imaginary parts of C and p respectively.

Design of IIR Filters: Impulse Invariance

Example 1

Design a low-pass digital filter with f_c=100Hz and T=1.6ms using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

1. Scale frequency of analogue filter to get desired cut off frequency:

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

replace s with s/ω_c

$$H(s)=\frac{1}{\frac{s^2}{\omega_c^2}+\sqrt{2}\frac{s}{\omega_c}+1}=\frac{1}{\frac{s^2+\sqrt{2}s+\omega_c^2}{\omega_c^2}}=\boxed{\frac{\omega_c^2}{s^2+\sqrt{2}s+\omega_c^2}}$$

Example 1

Design a low-pass digital filter with f_c=100Hz and T=1.6ms using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

2. Factorize H(s) and use partial fractions to express as sum of two 1st order transfer functions $H(s)=\frac{\omega_c^2}{s^2+\sqrt{2}s+\omega_c^2}$

I. Find poles (roots of denominator polynomial): $as^2+bs+c \Rightarrow roots=\frac{-b\pm\sqrt{b^2-4ac}}{2a},\ a=1,\ b=\omega_c\sqrt{2},\ c=\omega_c{}^2$

$$p_{1,2}=\frac{-\omega_c\sqrt{2}\pm\sqrt{\left(\omega_c\sqrt{2}\right)^2-4\omega_c^2}}{2}=\frac{-\omega_c\sqrt{2}\pm\sqrt{2\omega_c^2-4\omega_c^2}}{2}=\frac{-\omega_c\sqrt{2}\pm\sqrt{-2\omega_c^2}}{2}\Leftrightarrow \boxed{p_{1,2}=\frac{-\omega_c\sqrt{2}(1\pm j)}{2}}$$

and factorise transfer function: $H(s)=\frac{\omega_c^2}{s^2+\omega_c\sqrt{2}s+\omega_c^2}=\frac{\omega_c^2}{(s-p)(s-p^*)}$

II. Apply partial fraction expansion: $H(s)=\frac{C}{s-p}+\frac{C^*}{s-p^*}\Leftrightarrow(s-p)H(s)=(s-p)\left(\frac{C}{(s-p)}+\frac{C^*}{(s-p^*)}\right)$

$$\Leftrightarrow(s-p)H(s)=C+\frac{C^*(s-p)}{(s-p^*)}\Leftrightarrow(s-p)\frac{\omega_c^2}{(s-p)(s-p^*)}=C+\frac{C^*(s-p)}{(s-p^*)}\Leftrightarrow\frac{\omega_c^2}{(s-p^*)}=C+\frac{C^*(s-p)}{(s-p^*)},$$

evaluate at $s=p\Leftrightarrow\frac{\omega_c^2}{(p-p^*)}=C=\frac{\omega_c^2}{\left(\frac{-\omega_c\sqrt{2}(1-j)-\left(-\omega_c\sqrt{2}(1+j)\right)}{2}\right)}=\frac{\omega_c^2}{\left(\frac{2j\omega_c\sqrt{2}}{2}\right)}=\frac{\omega_c^2}{j\omega_c\sqrt{2}}=\frac{\omega_c}{j\sqrt{2}}\frac{j}{j}\Leftrightarrow \boxed{C=\frac{-j\omega_c}{\sqrt{2}}}$

Design of IIR Filters: Impulse Invariance

Example 1

Design a low-pass digital filter with f_c=100Hz and T=1.6ms using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

3. Find the poles in z using substitution $z_p=e^{pT}$:

$z_p=e^{\frac{-\omega_c\sqrt{2}(1-j)}{2}T},\ z_p^*$ and formulate the resulting transfer function H(z)

$$H(z)=\frac{C}{1-e^{pT}z^{-1}}+\frac{C^*}{1-e^{p^*T}z^{-1}}=\frac{\left(1-e^{p^*T}z^{-1}\right)C+\left(1-e^{pT}z^{-1}\right)C^*}{\left(1-e^{pT}z^{-1}\right)\left(1-e^{p^*T}z^{-1}\right)}=\frac{\boxed{C+C^*}-\boxed{(e^{p^*T}C+e^{pT}z^{-1}C^*)}z^{-1}}{1-\boxed{(e^{pT}+e^{p^*T})}z^{-1}+\boxed{e^{(p+p^*)T}}z^{-2}}$$

C_r, C_i, p_r and p_i : real & imaginary parts of C and p respectively

Simplify / expand numerator

$$\left.\begin{array}{l}C=C_r+C_i\\ C^*=C_r-C_i\end{array}\right\}\Rightarrow C+C^*=2C_r\quad,\quad \left.\begin{array}{l}p=p_r+p_i\\ p^*=p_r-p_i\end{array}\right\}\Rightarrow \begin{array}{l}e^{pT}=e^{(p_r+p_i)T}=e^{p_rT}e^{p_iT}\\ e^{p^*T}=e^{(p_r-p_i)T}=e^{p_rT}e^{-p_iT}\end{array}\quad,\quad \begin{array}{l}e^{p_iT}=\cos(p_iT)+j\sin(p_iT)\\ e^{-p_iT}=\cos(p_iT)-j\sin(p_iT)\end{array}$$

Simplify / expand denominator

$$(e^{pT}+e^{p^*T})=e^{p_rT}(e^{p_iT}+e^{-p_iT})=e^{p_rT}(\cos(p_iT)+j\sin(p_iT)+\cos(p_iT)-j\sin(p_iT))=2e^{p_rT}\cos(p_iT)$$

$$e^{(p+p^*)T}=e^{(p_r+p_i+p_r-p_i)T}=e^{2p_rT}$$

Example 1

Design a low-pass digital filter with f_c=100Hz and T=1.6ms using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(z)=\frac{2C_r-2e^{p_rT}\left(C_r\cos\left(p_iT\right)+C_i\sin\left(p_iT\right)\right)z^{-1}}{1-2e^{p_rT}\cos(p_iT)z^{-1}+e^{2p_rT}z^{-2}}$$

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

$$H(z)=\frac{2C_r-2e^{p_rT}\left(C_r\cos\left(p_iT\right)+C_i\sin\left(p_iT\right)\right)z^{-1}}{1-2e^{p_rT}\cos(p_iT)z^{-1}+e^{2p_rT}z^{-2}}$$

$$C=\frac{-j\omega_c}{\sqrt{2}} \qquad C=C_r+C_i$$

$$C_r=0 \qquad C_i=-j(\omega_c/\sqrt{2})$$

$$H(z)=\frac{-2e^{p_rT}C_i\sin\left(p_iT\right)z^{-1}}{1-2e^{p_rT}\cos(p_iT)z^{-1}+e^{2p_rT}z^{-2}}$$

$$p=-\omega_c\sqrt{2}(1-j)/2 \qquad p=p_r+p_i$$

$$p_r=-\omega_c\sqrt{2}/2 \qquad p_i=j\omega_c\sqrt{2}/2$$

$$\omega_c=2\pi f_c \qquad f_c=100$$

$$\omega_c=628.32\,rad\,s^{-1}$$

$$C_i=-444.29j \qquad p_r=-444.29 \qquad p_i=j444.29$$

$$H(z)=\frac{284.80z^{-1}}{1-0.7445z^{-1}+0.2413z^{-2}}$$

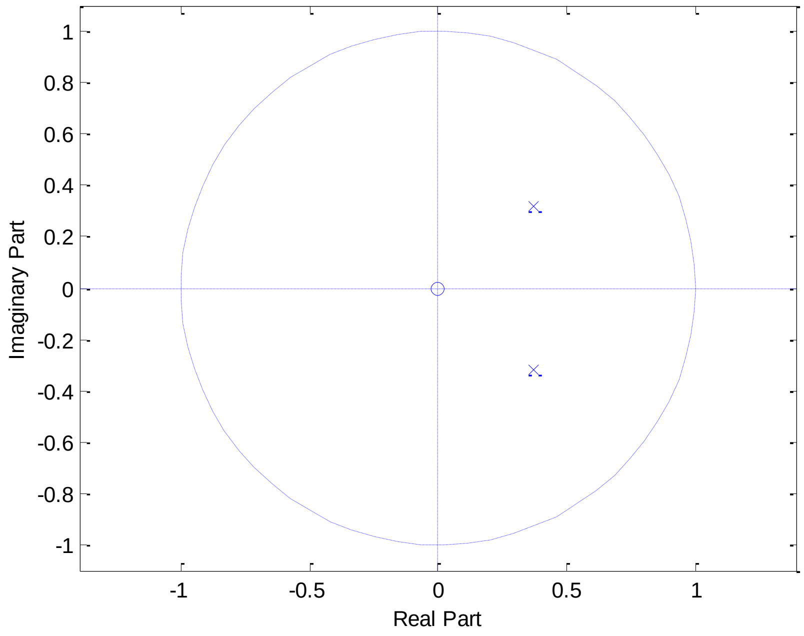
Design of IIR Filters: Impulse Invariance

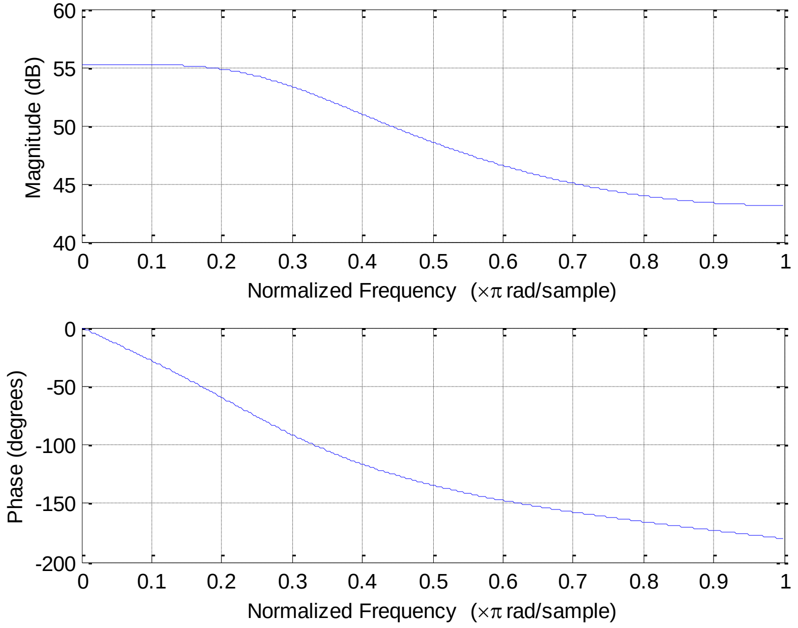
Example 1

Design a low-pass digital filter with f_c=100Hz and T=1.6ms using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

$$H(z)=\frac{284.80z^{-1}}{1-0.7445z^{-1}+0.2413z^{-2}}$$





Example 2

Design a low-pass digital filter with f_c=100Hz and f_s=1200Hz using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

1. Scale the frequency : replace s by s/ω_c $\quad H(s)=\dfrac{\omega_c^2}{s^2+\omega_c\sqrt{2}s+\omega_c^2}$

2. Factorize H(s) and use partial fractions

$$H(s)=\frac{\omega_c^2}{s^2+\omega_c\sqrt{2}s+\omega_c^2}=\frac{A}{s-p}+\frac{B}{s-p^*}=\frac{(A+B)s-pB-p^*A}{s^2-(p+p^*)s+pp^*}\Rightarrow$$

$$A+B=0\Rightarrow B=-A \quad (1) \qquad , \qquad \omega_c^2=-pB-p^*A \quad (2)$$

$$p,p^*=\frac{-\omega_c\sqrt{2}\pm\sqrt{2\omega_c^{\ 2}-4\omega_c^{\ 2}}}{2}=\frac{-\omega_c\sqrt{2}(1\pm j)}{2} \quad (3)$$

$$\omega_c^{\ 2}\overset{[2]}{=}-pB-p^*A\overset{[3]}{=}\frac{\omega_c\sqrt{2}}{2}(1-j)B+\frac{\omega_c\sqrt{2}}{2}(1+j)A=\frac{\omega_c\sqrt{2}}{2}(B-jB+A+jA)=$$

$$\overset{[1]}{=}\frac{\omega_c\sqrt{2}}{2}(-A+jA+A+jA)=\frac{\omega_c\sqrt{2}}{2}2jA\Rightarrow A=\frac{\omega_c^{\ 2}}{j\omega_c\sqrt{2}}\overset{\times\frac{j\sqrt{2}}{j\sqrt{2}}}{\Rightarrow} A=\frac{-j\omega_c\sqrt{2}}{2} \quad B=\frac{j\omega_c\sqrt{2}}{2} \quad (4)$$

Design of IIR Filters: Impulse Invariance

Example 2

Design a low-pass digital filter with f_c=100Hz and f_s=1200Hz using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

3. Substitution to avoid carrying all terms in intermediate steps:

$$\alpha=(\omega_c\sqrt{2})/2 \Rightarrow \begin{Bmatrix} p &= -\alpha+j\alpha \\ p^* &= -\alpha-j\alpha \end{Bmatrix}, \begin{Bmatrix} A = -j\alpha \\ B = j\alpha \end{Bmatrix} \qquad (5)$$

4. Map each 1st order analogue term to a 1st order digital term

$$H(s)=\frac{A}{s-p}+\frac{B}{s-p^*}=\frac{(A+B)s-pB-p^*A}{s^2-(p+p^*)s+pp^*} \overset{z_p=e^{pT}}{\Leftrightarrow} H(z)=\frac{A}{1-e^{pT}z^{-1}}+\frac{B}{1-e^{p^*T}z^{-1}}$$

$$=\frac{A+B-(Ae^{p^*T}+Be^{pT})z^{-1}}{1-(e^{pT}+e^{p^*T})z^{-1}+e^{pT}e^{p^*T}z^{-2}} \overset{[5]}{=} \frac{-(-j\alpha e^{-\alpha T-j\alpha T}+j\alpha e^{-\alpha T+j\alpha T})z^{-1}}{1-(e^{-\alpha T+j\alpha T}+e^{-\alpha T-j\alpha T})z^{-1}+e^{-\alpha T+j\alpha T}e^{-\alpha T-j\alpha T}z^{-2}}$$

$$=\frac{j\alpha e^{-\alpha T}(e^{-j\alpha T}-e^{j\alpha T})z^{-1}}{1-e^{-\alpha T}(e^{j\alpha T}+e^{-j\alpha T})z^{-1}+e^{-2\alpha T}z^{-2}}=\frac{-j\alpha e^{-\alpha T}\,2j\sin(\alpha T)z^{-1}}{1-2e^{-\alpha T}\cos(\alpha T)z^{-1}+e^{-2\alpha T}z^{-2}} \Leftrightarrow$$

$$H(z)=\frac{2\alpha e^{-\alpha T}\sin(\alpha T)z^{-1}}{1-2e^{-\alpha T}\cos(\alpha T)z^{-1}+e^{-2\alpha T}z^{-2}}$$

Example 2

Design a low-pass digital filter with f_c=100Hz and f_s=1200Hz using impulse invariance, based on a 2nd order Butterworth response (with a normalised cut-off frequency):

$$H(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

$$H(z)=\frac{2\alpha e^{-\alpha T}\sin(\alpha T)z^{-1}}{1-2e^{-\alpha T}\cos(\alpha T)z^{-1}+e^{-2\alpha T}z^{-2}} \quad , \quad \alpha=(\omega_c\sqrt{2})/2$$

$$\left.\begin{aligned}\alpha &= \frac{\omega_c\sqrt{2}}{2}\\ \omega_c &= 2\pi f_c = 2\pi 100\end{aligned}\right\} \Rightarrow a=\frac{2\pi f_c\sqrt{2}}{2}=100\pi\sqrt{2}\Rightarrow a=444.2883 \quad (6)$$

$$T=1/f_s \Rightarrow aT=0.3702 \quad (7)$$

[6,7]

$$\Rightarrow H(z)=\frac{222.033\,z^{-1}}{1-1.2876\,z^{-1}+0.4769\,z^{-2}}$$

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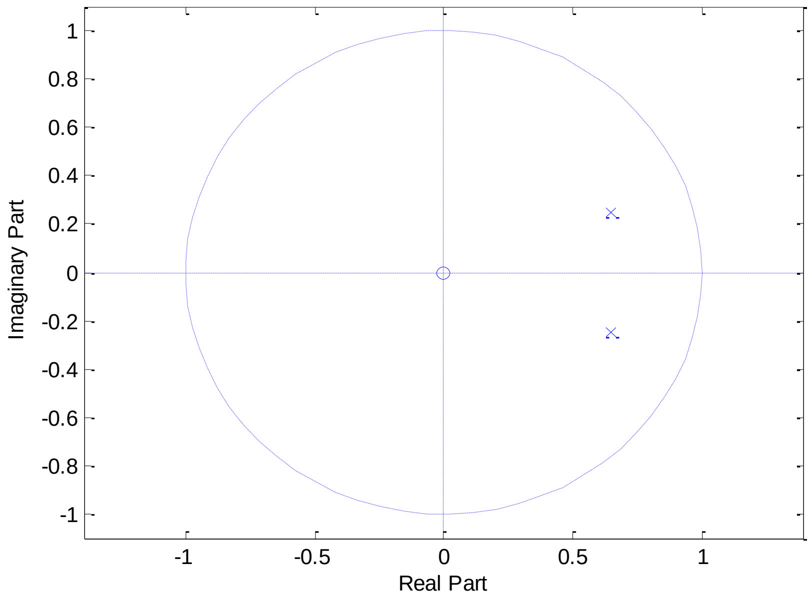

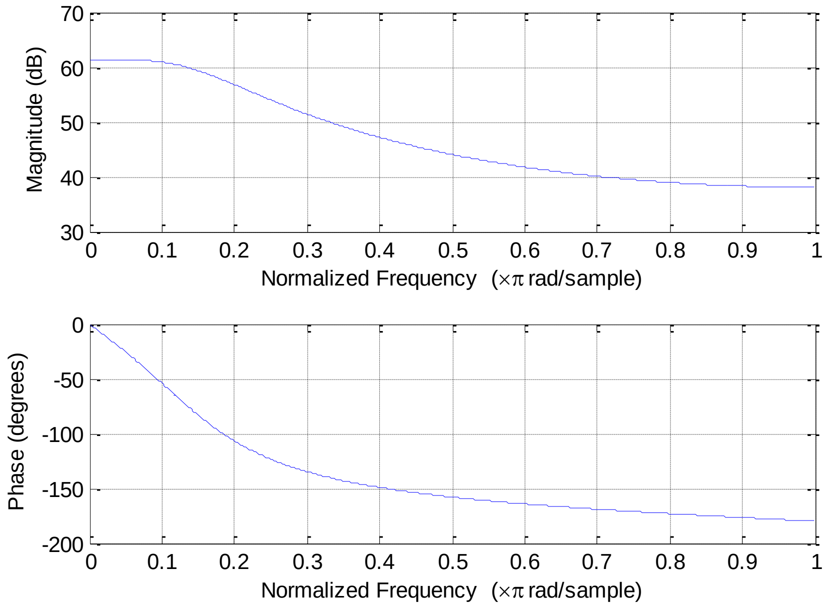

Example 3

Use the impulse invariance method to design an IIR digital filter based on the analogue prototype (with normalised cut-off freq. of 1 rads/sec): $H(s)=\dfrac{1}{s^2+\sqrt{2}s+1}$

Assume a cut-off frequency of 150Hz and a sampling frequency of 1.28KHz

1. Move the cut-off frequency from 1Hz to 150Hz.

$s \rightarrow s/\omega_c$ where $\omega_c = 2\pi 150 = 942.48$

$$H_a'(s)=\frac{\omega_c^2}{s^2+\sqrt{2}\omega_c s+\omega_c^2}$$

Express as partial fractions:

$$H_a'(s)=\frac{C_1}{s-p_1}+\frac{C_2}{s-p_2}$$

$$p_1=\frac{-\omega_c(1-j)}{\sqrt{2}}=-666.43(1-j)$$

$$p_2=p_1^*$$

$$C_1=\frac{\omega_c j}{\sqrt{2}}=-666.43j$$

$$C_2=C_1^*$$

Using

$$\alpha=\frac{\omega_c}{\sqrt{2}}=666.643$$

$$H(z)=\frac{C_1}{1-e^{p_1T}z^{-1}}+\frac{C_2}{1-e^{p_2T}z^{-1}}$$

$$=\frac{C_1+C_2-\left(C_1e^{p_2T}+C_2e^{p_1T}\right)z^{-1}}{1-(e^{p_1T}+e^{p_2T})z^{-1}+e^{p_1T}e^{p_2T}z^{-2}}$$

$$=\frac{-(-\alpha je^{-j\alpha T}e^{-\alpha T}+\alpha je^{+j\alpha T}e^{-\alpha T})}{1-e^{-\alpha T}2\cos(\alpha T)z^{-1}+e^{-2\alpha T}z^{-2}}$$

$$=\frac{\alpha e^{-\alpha T}2\sin(\alpha T)}{1-e^{-\alpha T}2\cos(\alpha T)z^{-1}+e^{-2\alpha T}z^{-2}}$$

$$=\frac{393.93z^{-1}}{1-1.031z^{-1}+0.353z^{-2}}$$