Lecture 15

Design of FIR Filters using Optimisation & Variable Transformation

Design through the use of approximation criteria
Design of mirror filters

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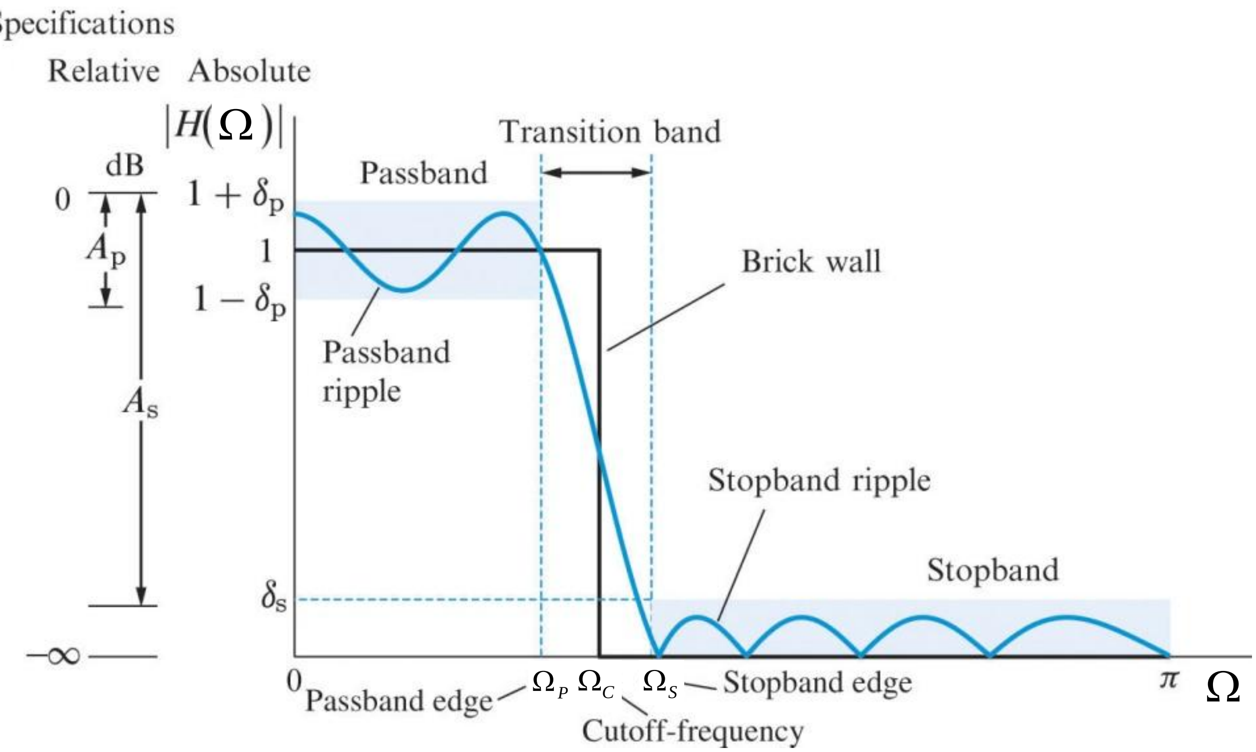
Filter Specification

Absolute and Relative Specifications

Absolute Specifications		Relative Specifications
$1-\delta_p \le \lvert H(\Omega)\rvert \le 1+\delta_p$, $\delta_p << 1$	$0 \le \Omega \le \Omega_p$	$-A_p \le \lvert H(\Omega)\rvert \le 0$ (in dB) , $A_p = 20\log_{10}\left[(1+\delta_p)/(1-\delta_p)\right]$
$\lvert H(\Omega)\rvert \le \delta_s$, $\delta_s << 1$	$\Omega_s \le \Omega \le \pi$	$\lvert H(\Omega)\rvert \le -A_s$ (in dB) , $A_s = -20\log_{10}(\delta_s)$



Optimum Approximation of Ideal Frequency Response

Why / What

Aim of Filter Design Method

Design a filter that "best" approximates the shape of some ideal response given a filter length constraint

Need an approximation criterion to define what "best" means

Aim of Filter Design Method (Restated)

Design a filter that minimises a measure of deviation from the ideal response given a filter length constraint

Deviation measured in terms of ripple in pass-band and stop-band & width of transition band

$$\boxed{\delta_p}\ ,\ \boxed{\Omega_p - \Omega_s}\ ,\ \boxed{\delta_s}\ ,\ \boxed{L}$$

Pass-band ripple | Transition band width | Stop-band ripple | Filter length

Optimum Approximation of Ideal Frequency Response

Approximation (Optimality) Criteria

Design filter that minimises one of the following measures of error relative to ideal response

Mean-Squared Error Approximation Minimise the mean squared error

$$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d(\Omega) - H(\Omega)\right|^2 d\Omega$$

Minimax Approximation Minimise the maximum error

$$\varepsilon = \max_{\Omega\in B}\left|H_d(\Omega) - H(\Omega)\right|$$

Maximally-flat Approximation Minimise the squared error at a certain frequency

$$\varepsilon\left(\Omega_0\right) = \left|H_d(\Omega_0)\right|^2 - \left|H(\Omega_0)\right|^2$$

FIR Filter Design using Windowing

Mean-Squared Error Approximation and Windowing

FIR Filter design using a rectangular window is optimal in terms of the mean-squared error

Mean-Squared Error Approximation Minimise the mean squared error

$$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_d(\Omega) - H(\Omega) \right|^2 d\Omega$$

Parseval's Theorem

$$Energy = \sum_{n=-\infty}^{\infty} \left| h_d[n] \right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_d(\Omega) \right|^2 d\Omega$$

$$\varepsilon^2 = \sum_{n=0}^{M} \left| h_d[n] - h[n] \right|^2 + \sum_{n=-\infty}^{-1} \left| h_d[n] \right|^2 + \sum_{n=M+1}^{\infty} \left| h_d[n] \right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_d(\Omega) - H(\Omega) \right|^2 d\Omega$$

Mean squared error minimised if

$$\sum_{n=0}^{M} \left| h_d[n] - h[n] \right|^2 = 0$$

Mean squared error minimised if the ideal impulse response (infinite) is truncated

Truncation means multiplication with a rectangular window

Use of other windows does not minimise the mean squared error

FIR Filter Design using Frequency Sampling

Mean-Squared Error Approximation and Frequency Sampling

FIR Filter design using frequency sampling and least-squares error minimisation

Mean-Squared Error Approximation

Minimise the mean squared error

$$\varepsilon^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_d(\Omega) - H(\Omega) \right|^2 d\Omega$$

In Matlab

```
B=firls(N,F,A)
```

Returns a length N+1 linear phase (real, symmetric coefficients) FIR filter which has the best approximation to the desired frequency response described by F and A in the least squares sense. F is a vector of frequency band edges in pairs, in ascending order between 0 and 1. 1 corresponds to the Nyquist frequency or half the sampling frequency. A is a real vector the same size as F which specifies the desired amplitude of the frequency response of the resultant filter B.

FIR Filter Design using Windowing / Frequency Sampling

Design & performance limitations

1. Individual control over ripples in the pass-band or stop-band is not possible – designed filters typically result in approximately equal errors in the pass-band and stop-band
   - The resulting pass-band and stop-band parameters are equal even though often the specification is more strict in the stop band than in the pass band

2. The ripple of the window is not necessarily uniform (decays as we move away from discontinuity points according to side-lobe pattern of the window)
   - The error is greatest on either side of a discontinuity of the ideal frequency response and the error becomes smaller for frequencies away from the discontinuity

FIR Filter Design using Optimisation

Minimax Approximation Optimisation (Equiripple Optimisation)

Aim of Filter Design Method:

To design a filter that minimises the maximum approximation error given a desired filter order

$$\min_{\{h[n]:0\le n\le M\}}\left[\max_{\Omega}\left(\left|E(\Omega)\right|\right)\right]$$

- An algorithm exists for solving this optimisation problem:

  The Remez exchange algorithm also known as Parks & McClellan algorithm

- It can be shown that this leads to an equiripple filter – a filter with amplitude response that oscillates uniformly between the tolerance bounds of each band

- Given pass-band and stop-band edge frequencies and a filter order the method will calculate coefficients that provide the same ripple in pass-band & stop-band

FIR Filter Design using Optimisation

Minimax Approximation Optimisation (Equiripple Optimisation)

Aim of Filter Design Method:

To design a filter that minimises the maximum approximation error given a desired filter order

$$\min_{\{h[n]:0\leq n\leq M\}}\left[\max_{\Omega}\left(\left|E(\Omega)\right|\right)\right]$$

In Matlab

```
B=firpm(N,F,A)
```

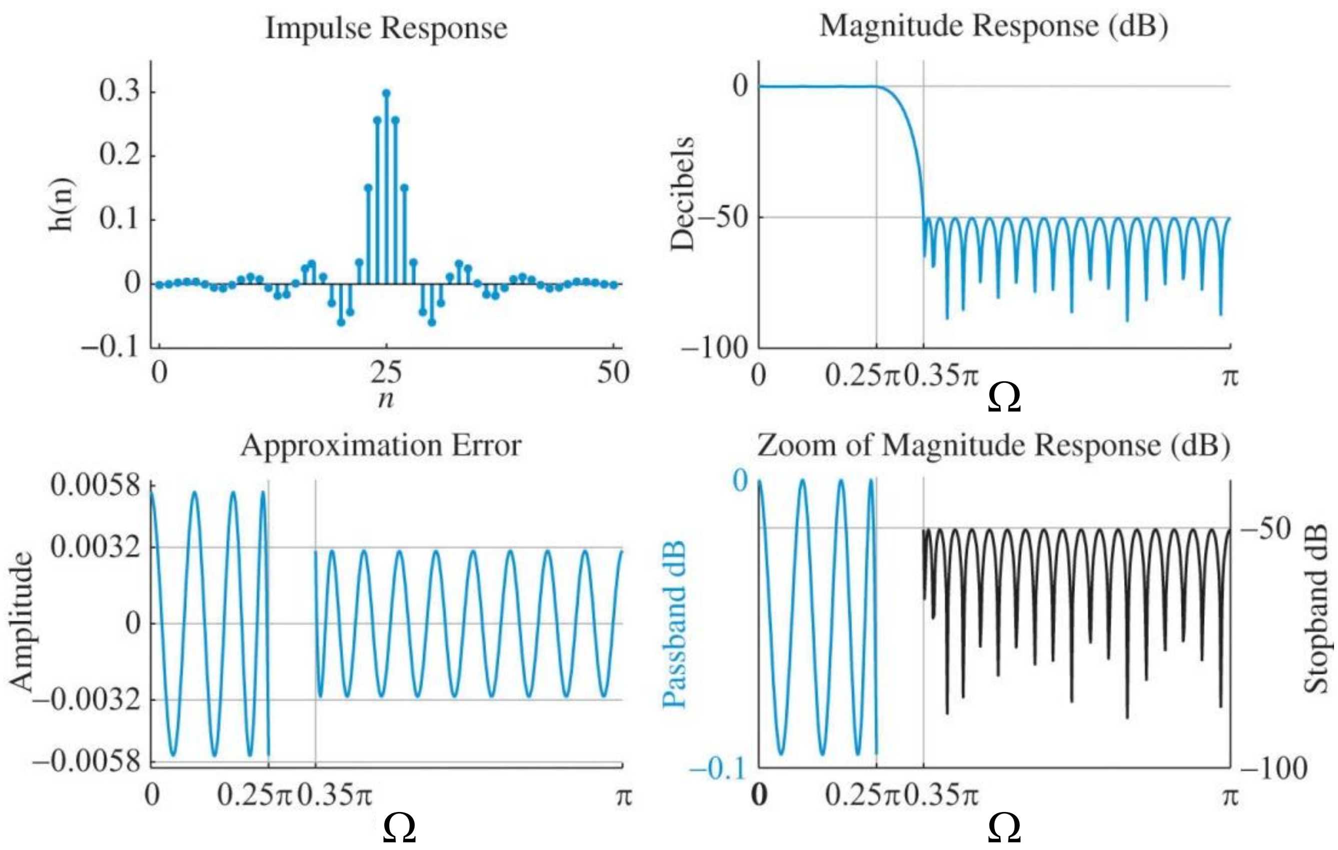
Designs a linear-phase FIR filter using the Parks-McClellan algorithm. The Parks-McClellan algorithm uses the Remez exchange algorithm and Chebyshev approximation theory to design filters with an optimal fit between the desired and actual frequency responses. The filters are optimal in the sense that the maximum error between the desired frequency response and the actual frequency response is minimized. Filters designed this way exhibit an equiripple behavior in their frequency responses and are sometimes called equiripple filters. B = firpm(N,F,A) returns row vector B containing the N+1 coefficients of the order N FIR filter whose frequency-amplitude characteristics match those given by vectors F and A.

FIR Filter Design using Optimisation

Minimax Approximation Optimisation (Equiripple Optimisation)

Aim of Filter Design Method:

To design a filter that minimises the maximum approximation error given a desired filter order



FIR Filter Design using Optimisation

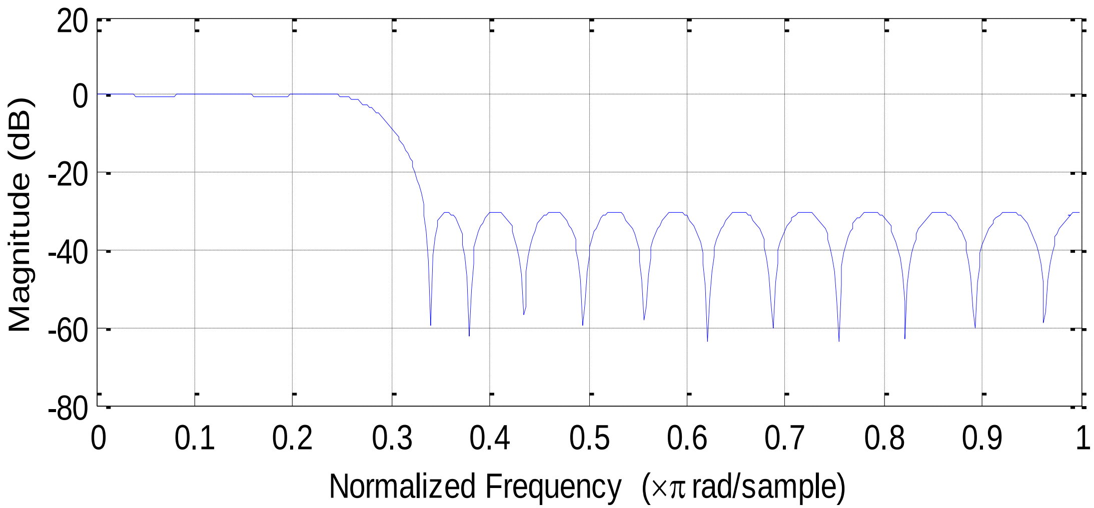
Equiripple Optimisation Example - Equiripple design of even length filter in Matlab

Design a low-pass 32nd order FIR filter with the following spec: $|H(\Omega)| = \begin{cases} 1 & |\Omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{3} < |\Omega| \le \pi \end{cases}$

Command ‘firpm’ in Matlab implements the Parks-McClellan algorithm

Order | Frequency points of interest | Desired response

b = firpm (32, [0,0.25,0.3333,1], [1,1,0,0]);

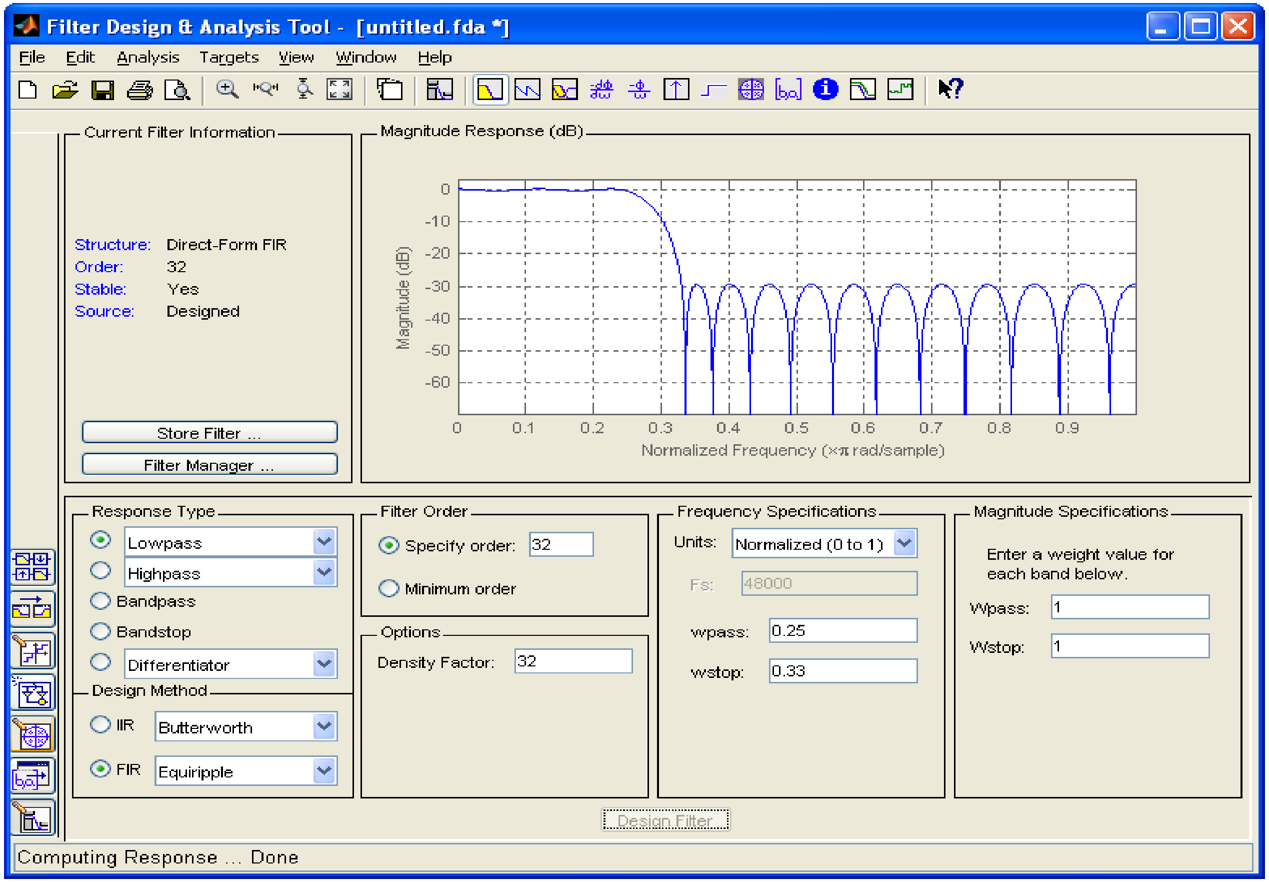


32nd order low pass filter designed using the Remez exchange (Parks-McClellan) algorithm

FIR Filter Design using Optimisation

Equiripple Optimisation Example - Equiripple design of even length filter in Matlab

Design a low-pass 32nd order FIR filter with the following spec: $|H(\Omega)| = \begin{cases} 1 & |\Omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{3} < |\Omega| \le \pi \end{cases}$



FIR Filter Design using Variable Transformations

Mirror Filters – Transforming low pass to high pass and vice versa

- Substitute z with $-z$ => Reflection about the origin of the z-plane

$H_M(z) = H(-z)$ (filters with real coefficients =>reflection about imaginary axis)

- Frequency response reflected about π/2
- Low-pass transformed to high-pass - High-pass transformed to low-pass

- In time domain this becomes equivalent to changing the sign of odd coefficients

$$H_M(z) = H(-z) \Leftrightarrow h_M = (-1)^n h(n)$$

In Matlab

```
g = firlp2hp(b)
```

Transforms the lowpass FIR filter b into a Type I highpass FIR filter g with zero-phase response. Filter b can be any FIR filter, including a nonlinear-phase filter. The passband and stopband ripples of g will be equal to the passband and stopband ripples of b.

FIR Filter Design using Variable Transformations

Mirror Filters – Example

Use a Rectangular window and variable transformation to design a 6th order (7 tap) high pass FIR filter with a cut-off of 100Hz and sampling frequency 1200Hz.

$$\Omega_{HP} = 2\pi . f_{HP} / f_s = 2\pi .100/1200 = \pi / 6 \ , \qquad \left|H_{HP}(\Omega)\right| = \begin{cases} 0 & \left|\Omega\right| \le \Omega_{HP} \\ 1 & \Omega_{HP} < \left|\Omega\right| \le \pi \end{cases}$$

$$\Omega_{LP} = \pi - \Omega_{HP} = 5\pi / 6 \ , \qquad \left|H_{LP}(\Omega)\right| = \begin{cases} 1 & \left|\Omega\right| \le \Omega_{LP} \\ 0 & \Omega_{LP} < \left|\Omega\right| \le \pi \end{cases}$$

For a 6th order linear phase filter we have a group delay of 6/2 =3 giving a phase term of $e^{-j3\Omega}$

$$H_{LP}(\Omega) = \begin{cases} e^{-j3\Omega} & \left|\Omega\right| \le \Omega_{LP} \\ 0 & \Omega_{LP} < \left|\Omega\right| \le \pi \end{cases}$$

FIR Filter Design using Variable Transformations

Mirror Filters – Example

- Take the inverse DTFT to get ideal impulse response

$$h_{LP}[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\Omega_{LP}}^{\Omega_{LP}} e^{-jT_d\Omega}e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\Omega_{LP}}^{\Omega_{LP}} e^{j(n-T_d)\Omega}d\Omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\Omega(n-T_d)}}{j(n-T_d)}\right]_{-\Omega_{LP}}^{\Omega_{LP}} = \frac{e^{j(n-T_d)\Omega_{LP}} - e^{-j(n-T_d)\Omega_{LP}}}{2\pi j(n-T_d)} = \frac{\sin\left((n-T_d)\Omega_{LP}\right)}{\pi(n-T_d)} = \frac{\Omega_{LP}}{\pi}\operatorname{sinc}\left((n-T_d)\Omega_{LP}\right)$$

- Multiply by rectangular window to get finite impulse response

$$h_{LP}[n] = \{0.1061 \quad -0.1378 \quad 0.1592 \quad 0.8333 \quad 0.1592 \quad -0.1378 \quad 0.1061\}$$

$$H_{LP}(z) = 0.1061 - 0.1378z^{-1} + 0.1592z^{-2} + 0.8333z^{-3} + 0.1592z^{-4} - 0.1378z^{-5} + 0.1061z^{-6}$$

- Variable transformation

$$H_{HP}(z) = H_{LP}(-z) = 0.1061 + 0.1378z^{-1} + 0.1592z^{-2} - 0.8333z^{-3} + 0.1592z^{-4} + 0.1378z^{-5} + 0.1061z^{-6}$$

$$h_{HP}[n] = \{0.1061 \quad 0.1378 \quad 0.1592 \quad -0.8333 \quad 0.1592 \quad 0.1378 \quad 0.1061\}$$

FIR Filter Design using Variable Transformations

Mirror Filters – Example