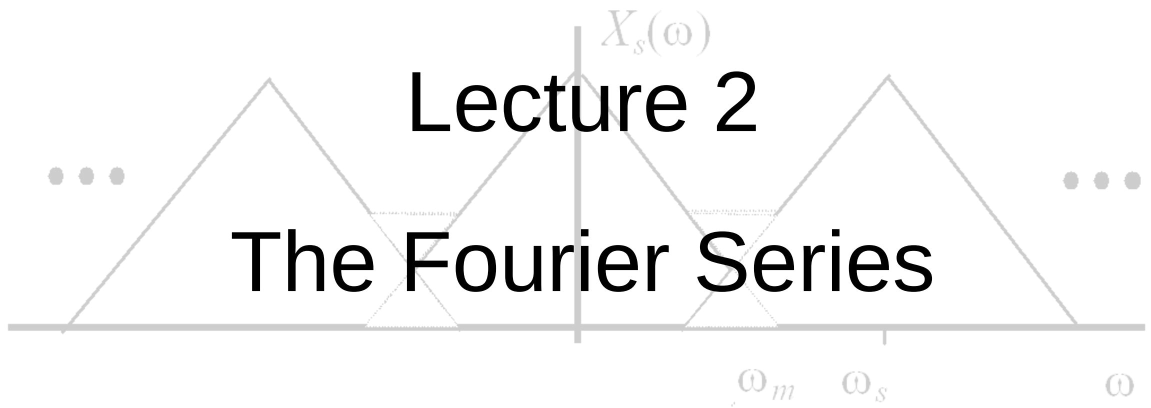



# Lecture 2

# The Fourier Series

Representing continuous time periodic signals
with harmonically related complex exponentials

# Fourier Series



Harmonically Related

Complex Exponential

Periodic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Fourier Series Coefficients

Representing continuous time periodic signals with harmonically related complex exponentials

# Fourier Series



Periodic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

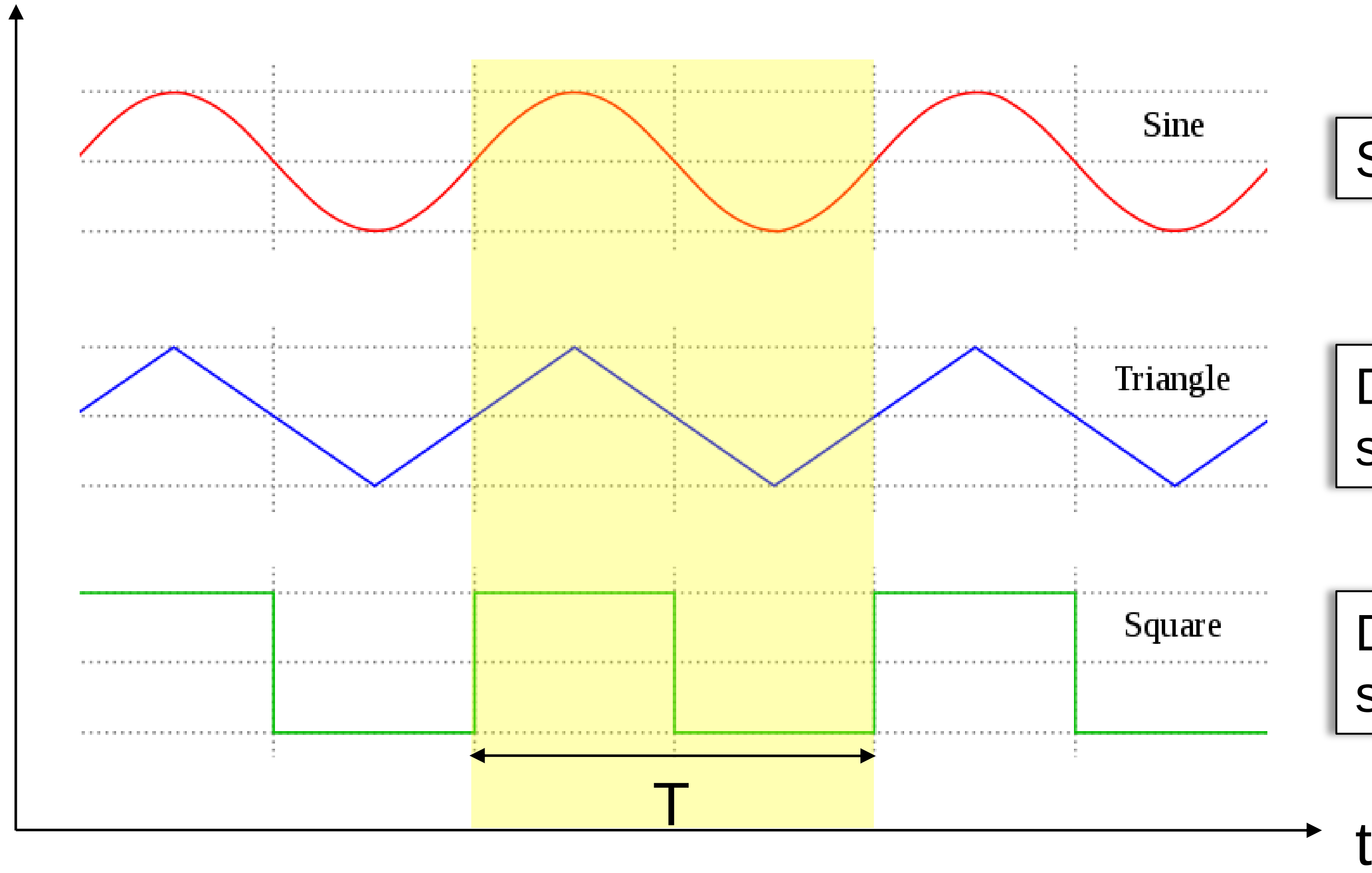
Representing continuous time periodic signals with harmonically related complex exponentials

# Fourier Series



## Periodic Signals

$$x(t) = x(t+T)$$ T : fundamental period, $\omega_0 = 2\pi/T$ : fundamental frequency



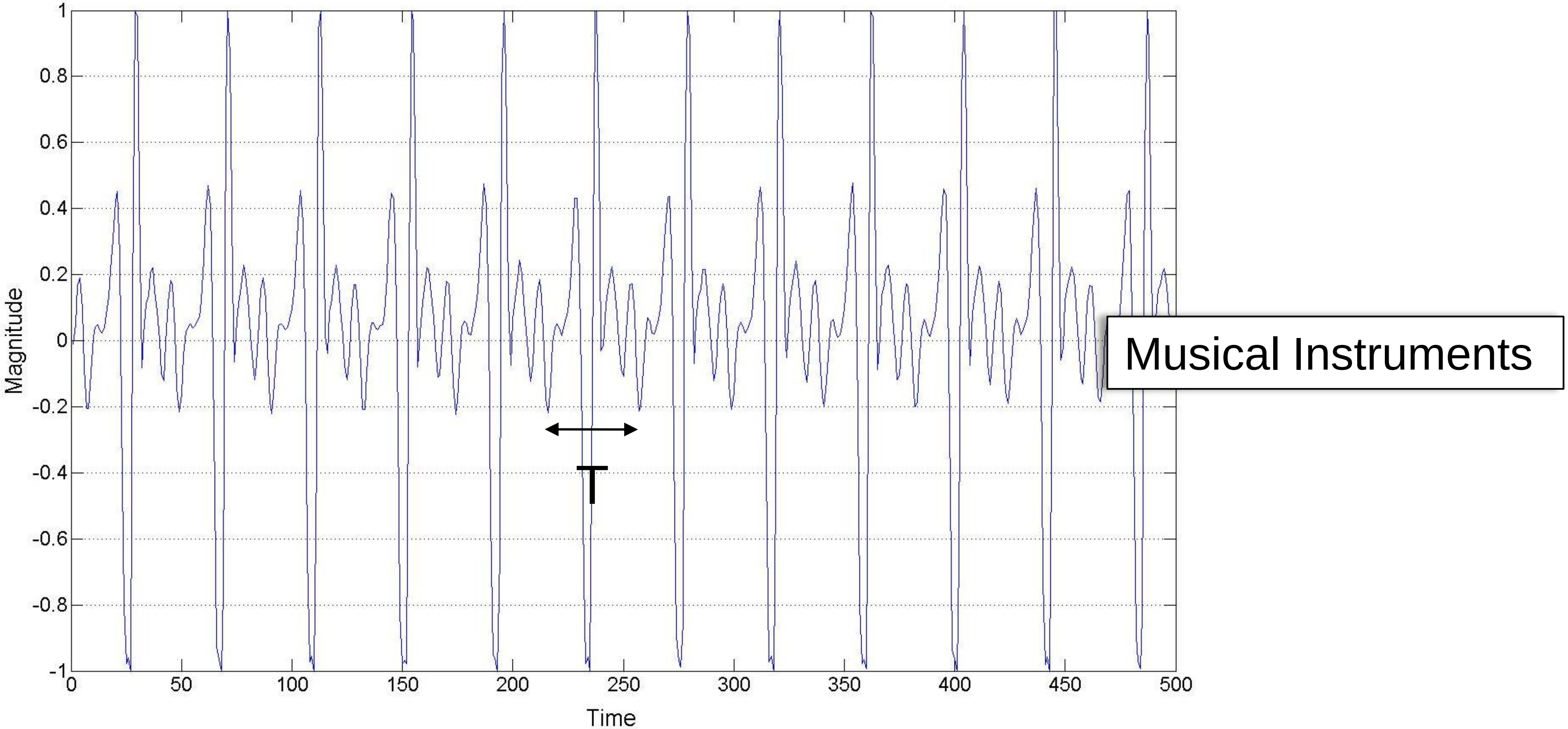

Smooth signal

Discontinuous slopes

Discontinuous signal

## Periodic Signals

$$x(t) = x(t+T)$$ T : fundamental period, $\omega_0 = 2\pi/T$ : fundamental frequency

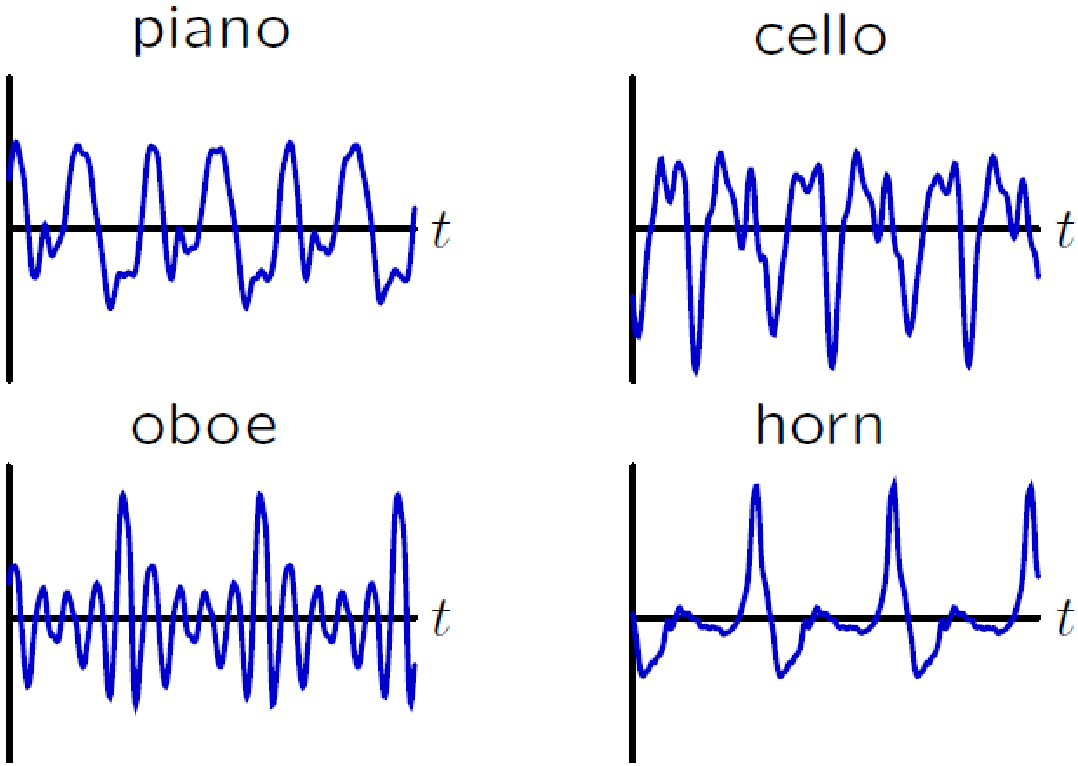


# Fourier Series



## Periodic Signals

$$x(t) = x(t+T)$$ T : fundamental period, $\omega_0 = 2\pi/T$ : fundamental frequency




Musical Instruments

# Fourier Series



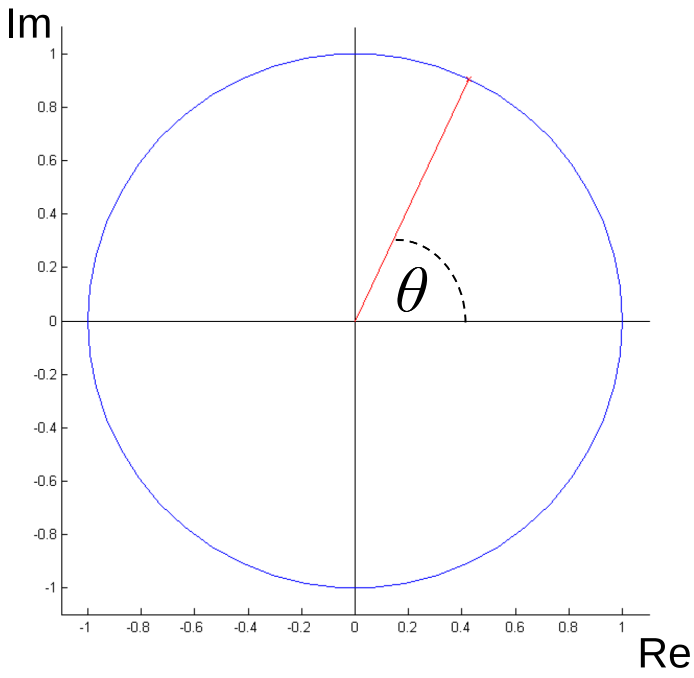
Complex Exponential

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

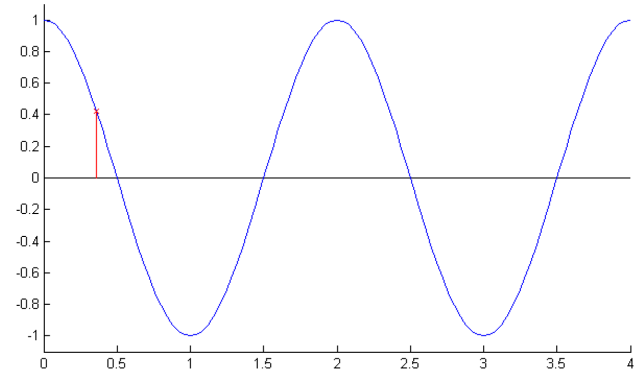
Representing continuous time periodic signals with harmonically related complex exponentials

## Complex Exponentials

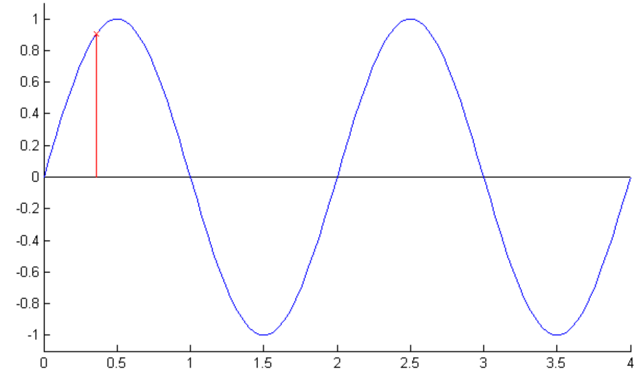
$$e^{\pm j\omega_0 t} = \cos(\omega_0 t) \pm j\sin(\omega_0 t) \quad \text{Euler's Formula}$$




$$e^{\pm j\omega_0 t}$$



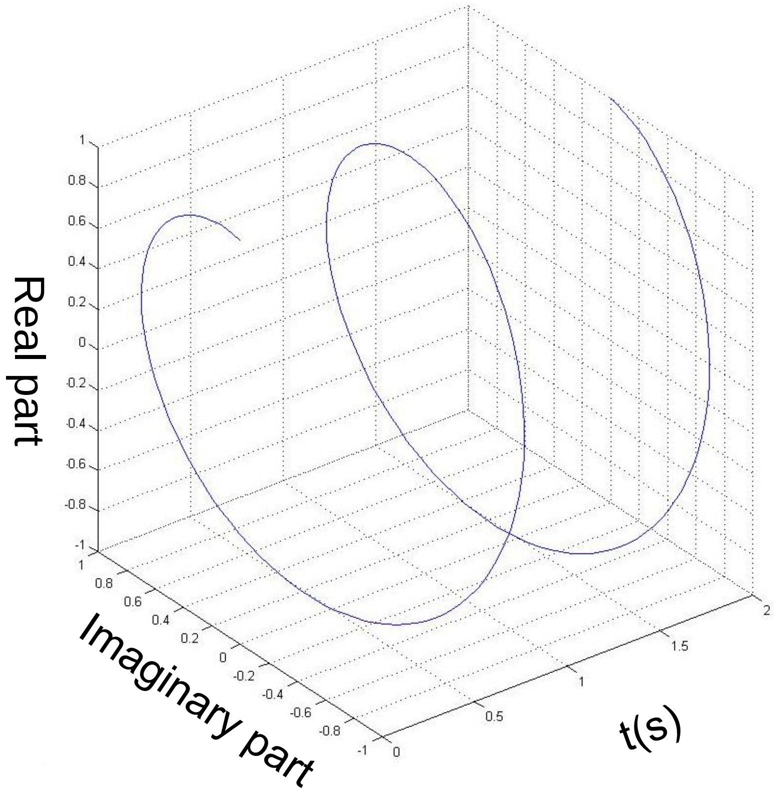
$$\cos(\omega_0 t)$$



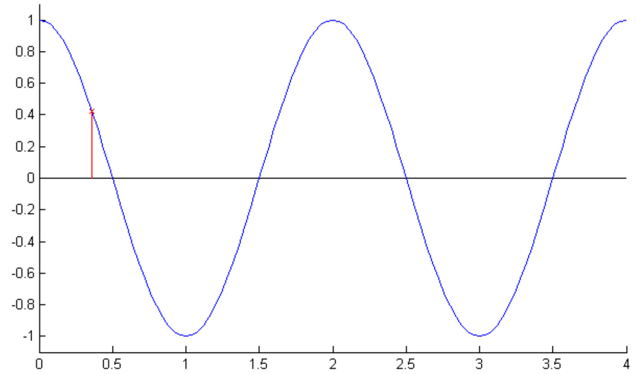
$$\sin(\omega_0 t)$$

## Complex Exponentials

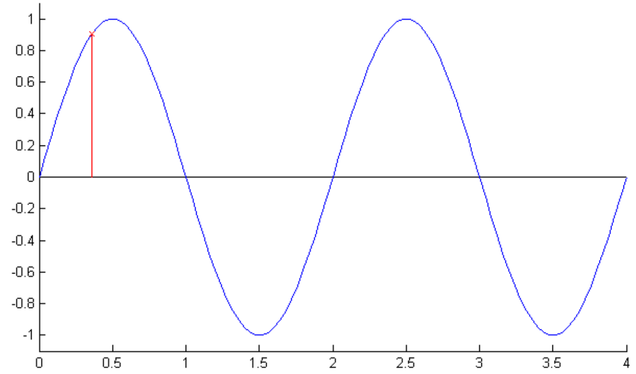
$$e^{\pm j\omega_0 t} = \cos(\omega_0 t) \pm j\sin(\omega_0 t)$$ Euler's Formula




$$e^{\pm j\omega_0 t}$$



$$\cos(\omega_0 t)$$



$$\sin(\omega_0 t)$$

## Complex Exponentials

### Why not cosines & sines ?

$$x(t) = \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

**Trigonometric** form of the Fourier Series

$$\stackrel{(1)}{\Rightarrow}$$

$$x(t) = \tfrac{1}{2} a_0 e^{0.j\omega_0 t} + \sum_{k=1}^{\infty} a_k \frac{(e^{jk\omega_0 t} + e^{-jk\omega_0 t})}{2} + b_k \frac{(e^{jk\omega_0 t} - e^{-jk\omega_0 t})}{2j}$$

$$\Leftrightarrow$$

$$x(t) = \tfrac{1}{2} a_0 e^{0.j\omega_0 t} + \sum_{k=1}^{\infty} (\frac{1}{2} a_k + \frac{1}{2j} b_k) e^{jk\omega_0 t} + (\frac{1}{2} a_k - \frac{1}{2j} b_k) e^{-jk\omega_0 t}$$

$$\stackrel{(2)}{\Rightarrow}$$

$$x(t) = \tfrac{1}{2} a_0 e^{0.j\omega_0 t} + \sum_{k=1}^{\infty} (\frac{a_k - jb_k}{2}) e^{jk\omega_0 t} + (\frac{a_k + jb_k}{2}) e^{-jk\omega_0 t}$$

$$\stackrel{(3)}{\Rightarrow}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad (1)$$

$$\frac{1}{2j} b_k \times \frac{j}{j} = -\frac{j}{2} b_k \qquad (2)$$

$$\sum_{k=1}^{\infty} e^{-jk\omega_0 t} = \sum_{k=-\infty}^{-1} e^{jk\omega_0 t} \qquad (3)$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad , \quad c_k = \begin{cases} a_0 / 2 & k = 0 \\ (a_k - jb_k)/2 & k > 0 \\ (a_k + jb_k)/2 & k < 0 \end{cases}$$

**Complex** form of the Fourier Series

## Complex Exponentials

### Why not cosines & sines ?

$$x(t) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

**Trigonometric** form of the Fourier Series

\+ Complex form has 1 set of coefficients (two for the trigonometric form)

– Complex form coefficients $c_k$ are in general complex ($a_k$ , $b_k$ are real)

$\Rightarrow c_k = re^{j\theta}, magnitude\, r, phase\, \theta$

– Negative and positive frequencies are present in complex form (positive only in trigonometric)

\+ Complex Fourier series of real signal exhibits Hermitian (complex conjugate) symmetry $|c_{-k}| = |c_k|$ , $\angle c_{-k} = -\angle c_k$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad , \quad c_k = \begin{cases} a_0/2 & k=0 \\ (a_k - jb_k)/2 & k>0 \\ (a_k + jb_k)/2 & k<0 \end{cases}$$

**Complex** form of the Fourier Series

# Fourier Series



Harmonically Related Complex Exponential

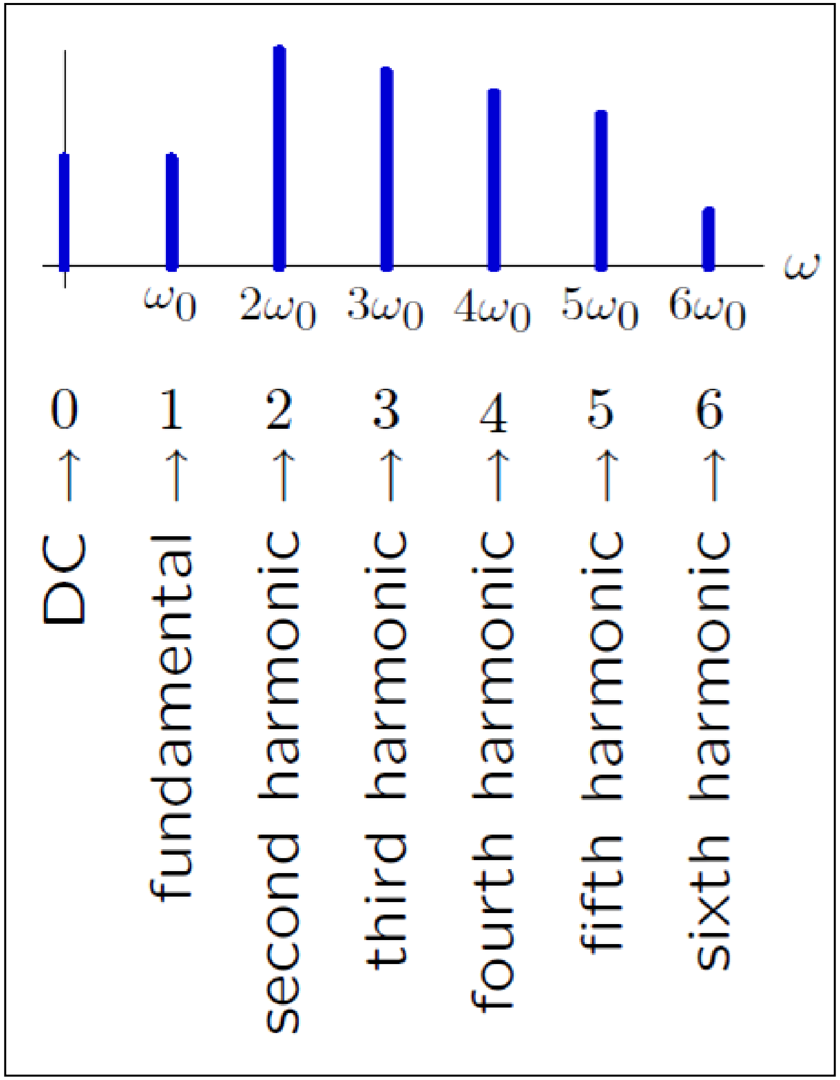
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Representing continuous time periodic signals with harmonically related complex exponentials

# Fourier Series

## Harmonically Related Complex Exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$



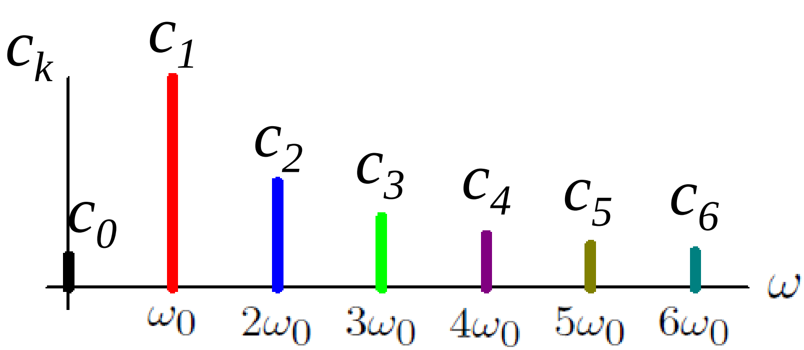

- The Frequency of each complex exponential (i.e. of the cosines and sines taking part in the synthesis) is a multiple of the fundamental frequency
- Fundamental frequency = 1/T , where T is the period of x(t)
- $\omega_0 = 2\pi/T$

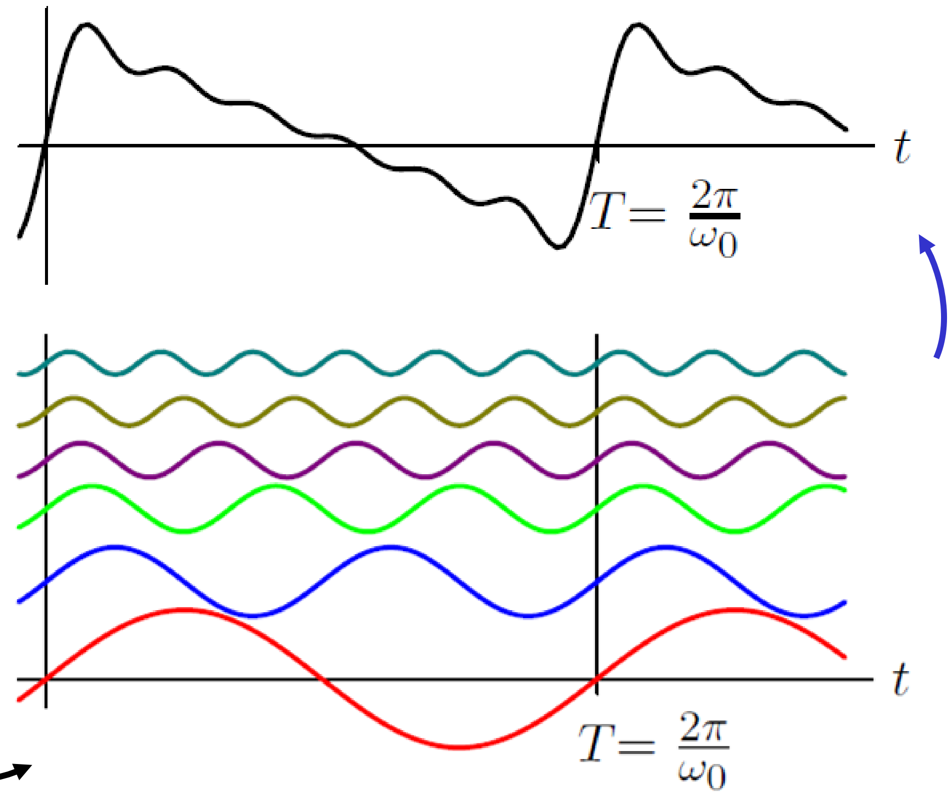
# Fourier Series

## Harmonically Related Complex Exponentials

### Synthesis Function (Inverse Transform)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$




$c_k = ?$ Analysis Function (Forward Transform)

# Fourier Series



$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Fourier Series Coefficients

Representing continuous time periodic signals with harmonically related complex exponentials

# Fourier Series

## Harmonically Related Complex Exponentials

### Analysis Function (Forward Transform)

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Inverse Fourier Series (Synthesis Function)

$$x(t)e^{-jl\omega_0 t}=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jl\omega_0 t} \quad (1)$$

$$\int_T x(t)e^{-jl\omega_0 t}dt=\int_T \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jl\omega_0 t}dt \quad (2)$$

$$\int_T x(t)e^{-jl\omega_0 t}dt=\sum_{k=-\infty}^{\infty} c_k \int_T e^{j(k-l)\omega_0 t}dt \quad (3)$$

Multiply both parts with: $e^{-jl\omega_0 t}$ (1)

where *l* is an integer

Integrate over T (2)

Interchange the order of summation & integration & combine exponentials (3)

$$c_k =$$

Forward Fourier Series (Analysis Function)

# Fourier Series



## Harmonically Related Complex Exponentials

### Analysis Function (Forward Transform)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Inverse Fourier Series (Synthesis Function)

$$\int_T x(t)e^{-jl\omega_0 t}dt = \sum_{k=-\infty}^{\infty} c_k \int_T e^{j(k-l)\omega_0 t}dt$$

$$\int_T e^{j(k-l)\omega_0 t}dt = 0, for\, k \neq l, \int_T e^{j(k-l)\omega_0 t}dt = T, for\, k = l$$

$$\int_T x(t)e^{-jl\omega_0 t}dt = c_l T \Rightarrow c_l = \frac{1}{T}\int_T x(t)e^{-jl\omega_0 t}dt$$

Handout Page 6 : Integrating over $k$ complete periods of a cos, sin or complex exponential gives zero

$$\int_0^T \cos(k\omega t).dt = \begin{matrix} 0 & k \neq 0 \\ T & k = 0 \end{matrix},$$

$$\int_0^T \sin(k\omega t).dt = 0$$

$$\int_0^T e^{jk\omega t}.dt = \begin{matrix} 0 & k \neq 0 \\ T & k = 0 \end{matrix} \qquad (4)$$

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Forward Fourier Series (Analysis Function)

# Fourier Series



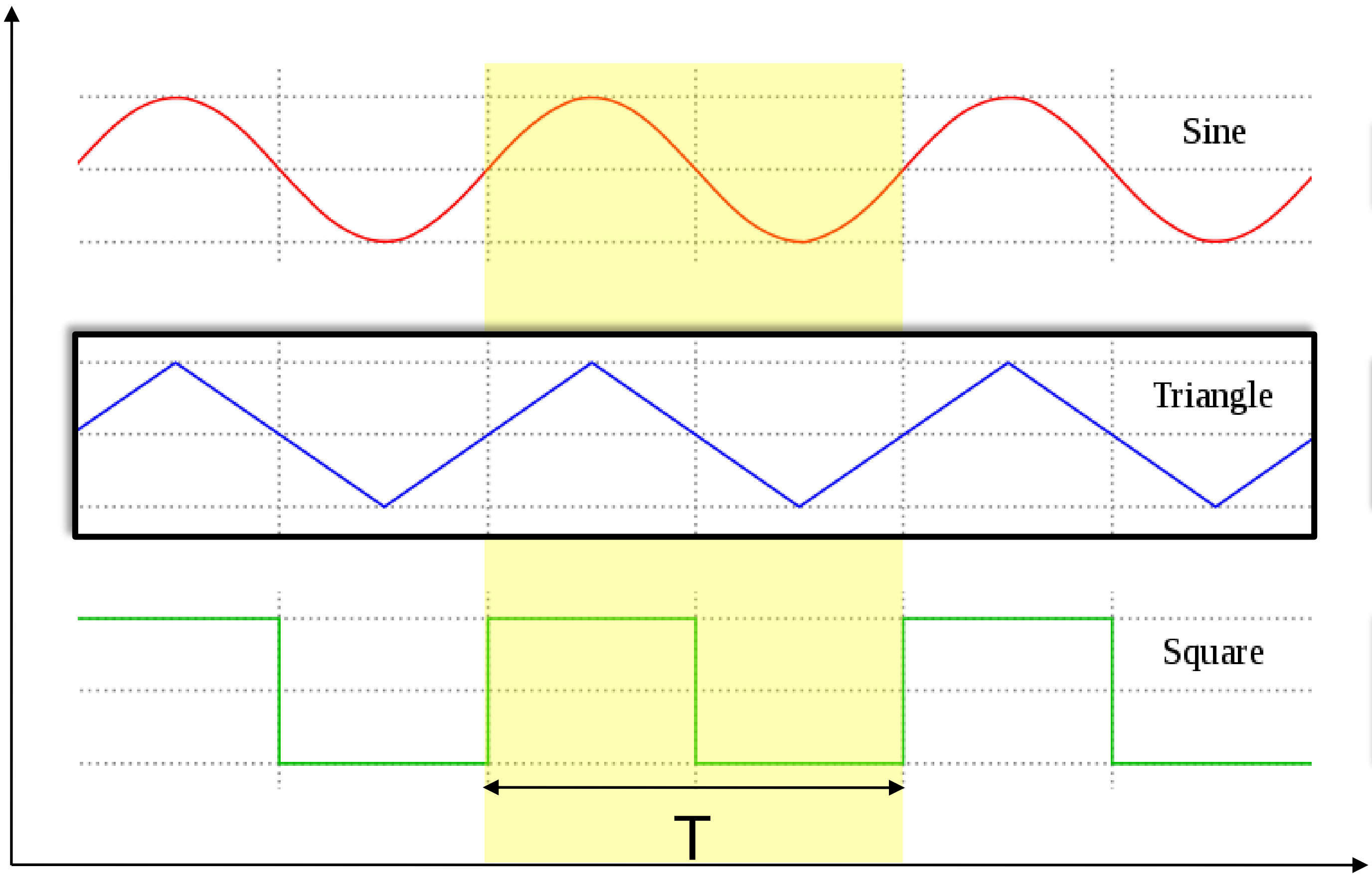
Can we represent all periodic signals with harmonics?
What about discontinuous signals?

| Analysis | Synthesis |
| --- | --- |
| $c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$ | $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ |

Representing continuous time periodic signals with harmonically related complex exponentials

## Can we represent all periodic signals with harmonics?

### Triangle Wave



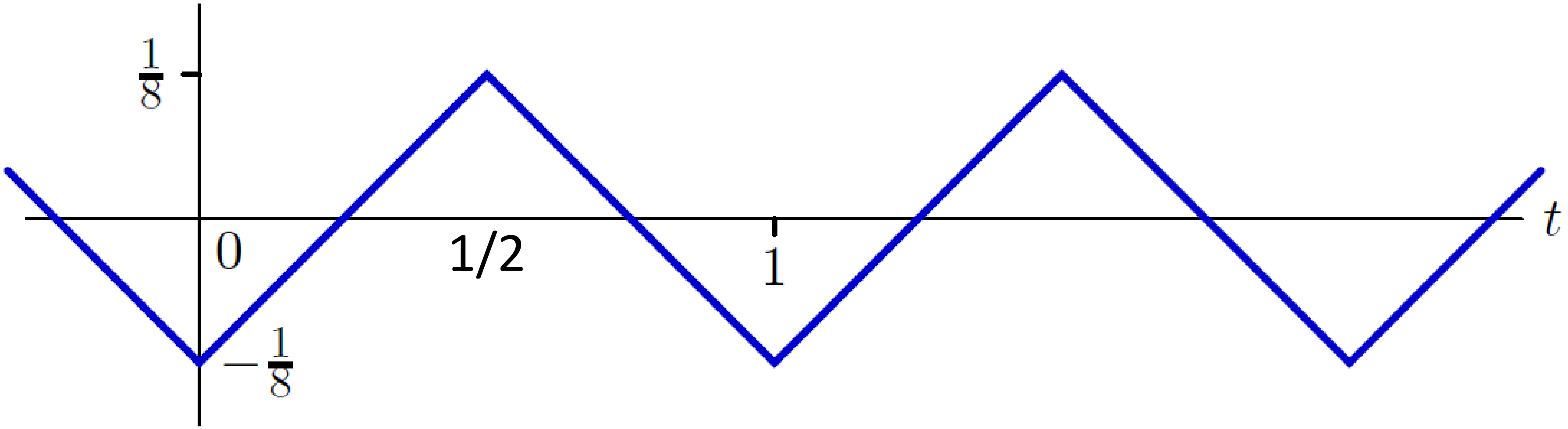

Smooth signal

Discontinuous slopes

Discontinuous signal

# Can we represent all periodic signals with harmonics?

## Triangle Wave




$$x(t) = \begin{cases} -\frac{1}{2}t + \frac{3}{8}, & \frac{1}{2} \le t < 1 \\ \frac{1}{2}t - \frac{1}{8}, & 0 \le t < \frac{1}{2} \end{cases}$$

1. Apply the forward transform

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

2. Synthesise triangular wave by adding together harmonics weighted by $c_k$

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Triangle Wave

1. Integration by parts $\int_a^b f(t)g(t)dt = \left[f(t)\int g(t)dt\right]_a^b - \int_a^b f'(t)\left(\int g(t)dt\right)dt$

   where $f(t) = t$, $g(t) = e^{-jk\omega_0 t}$, $a=0$, $b=1/2$ ($a=1/2$, $b=1$ for the second part)

2. Time-shift property of Fourier series (page 19)

3. Linearity property of Fourier series (page 19)

to solve the integral

$$x(t) = \begin{cases} -\frac{1}{2}t + \frac{3}{8}, & \frac{1}{2} \le t < 1 \\ \frac{1}{2}t - \frac{1}{8}, & 0 \le t < \frac{1}{2} \end{cases}$$

1. Apply the forward transform

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

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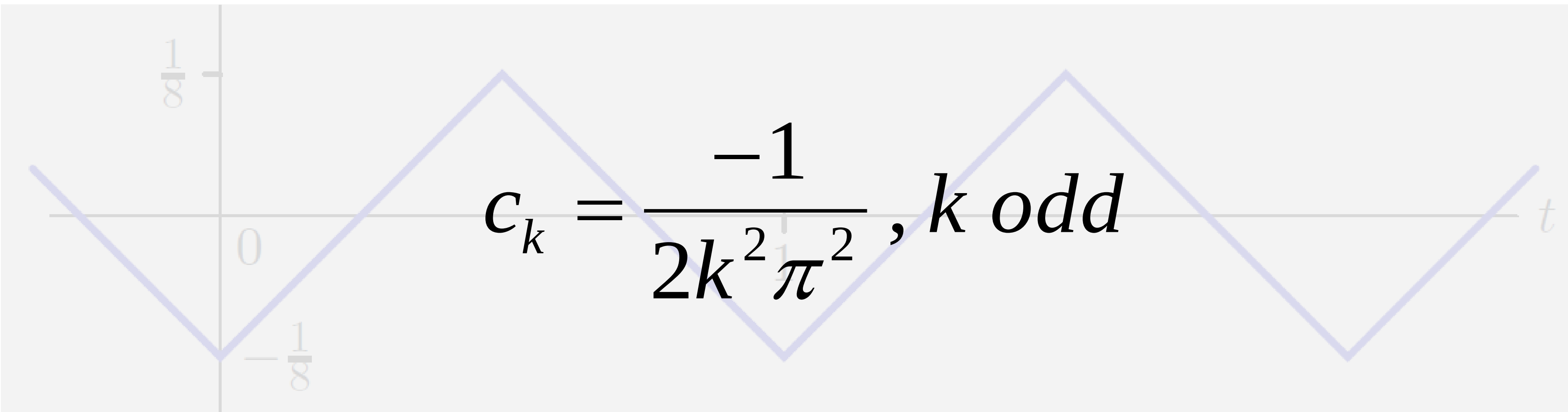
# Fourier Series



## Can we represent all periodic signals with harmonics?

### Triangle Wave




$$c_k = \frac{-1}{2k^2\pi^2}, k\ odd$$

$$x(t) = \begin{cases} -\frac{1}{2}t + \frac{3}{8}, & \frac{1}{2} \le t < 1 \\ \frac{1}{2}t - \frac{1}{8}, & 0 \le t < \frac{1}{2} \end{cases}$$

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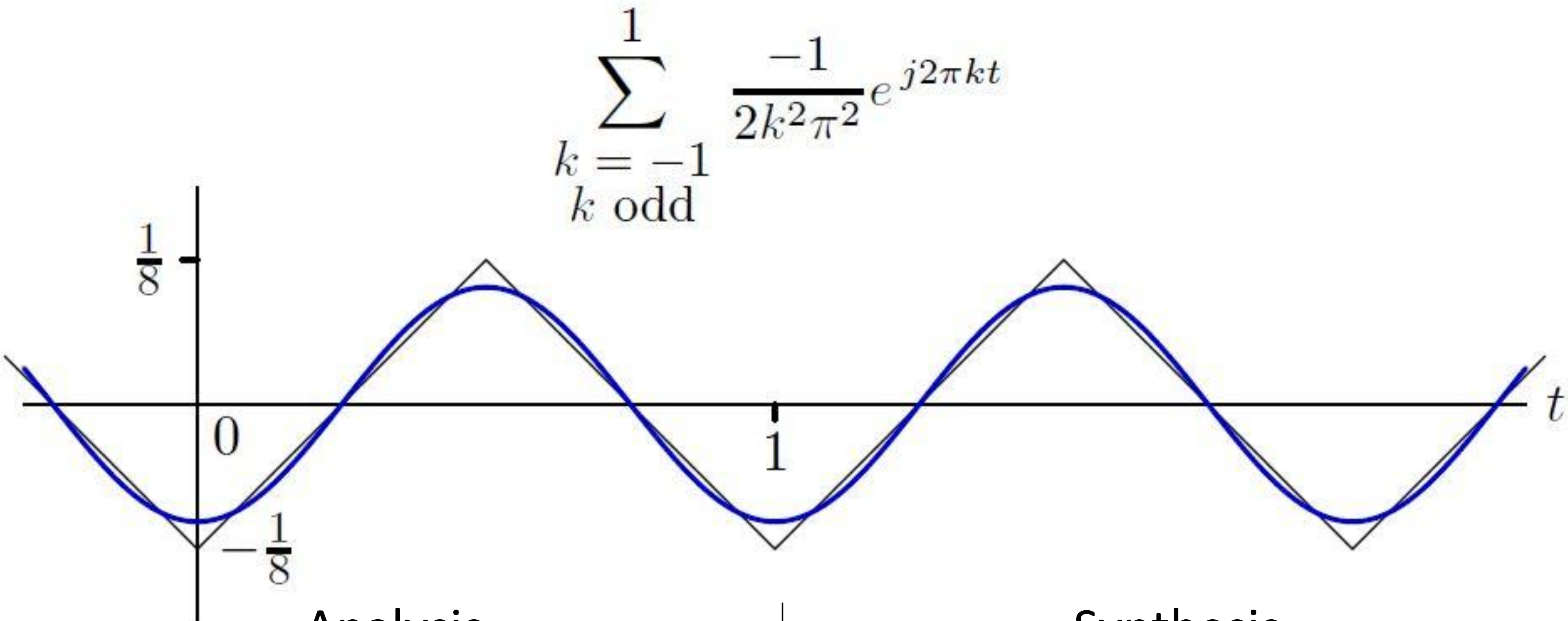
$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

2. Synthesise triangular wave by adding together harmonics weighted by $c_k$

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence



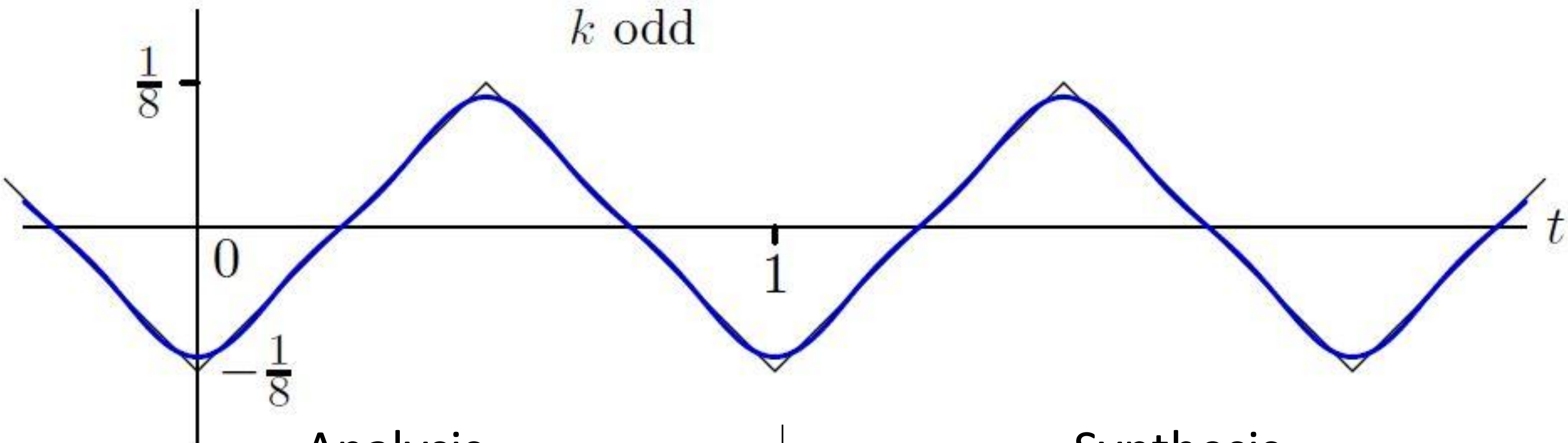

| Analysis | Synthesis |
| --- | --- |
| $c_k = \dfrac{-1}{2k^2\pi^2}, k\ odd$ | $x(t) = \sum_{k=-\infty, k\ odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence

$$\sum_{\substack{k=-3 \\ k \text{ odd}}}^{3} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



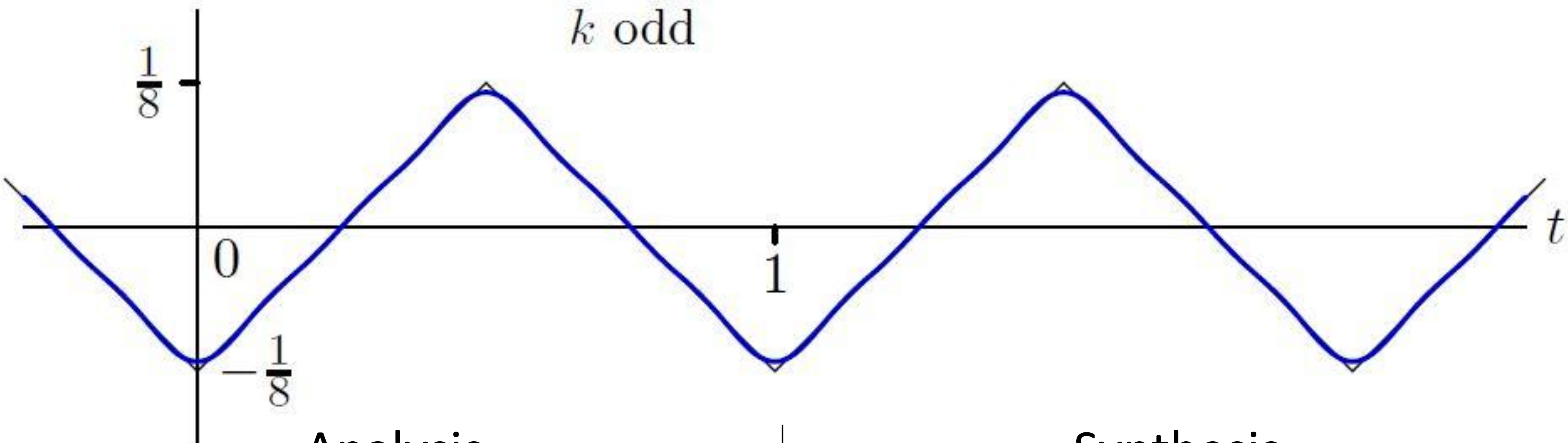

| Analysis | Synthesis |
|---|---|
| $c_k = \dfrac{-1}{2k^2\pi^2}, k\ odd$ | $x(t) = \displaystyle\sum_{k=-\infty, k\,odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



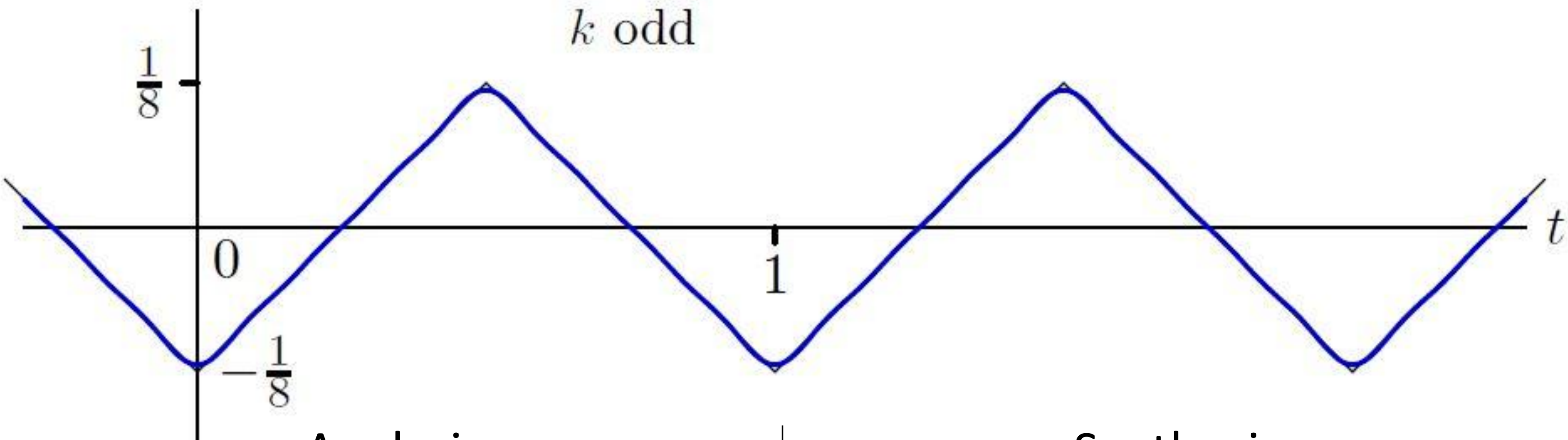

| Analysis | Synthesis |
| --- | --- |
| $c_k = \dfrac{-1}{2k^2\pi^2}, k\ odd$ | $x(t) = \sum_{k=-\infty, k\ odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series 

## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

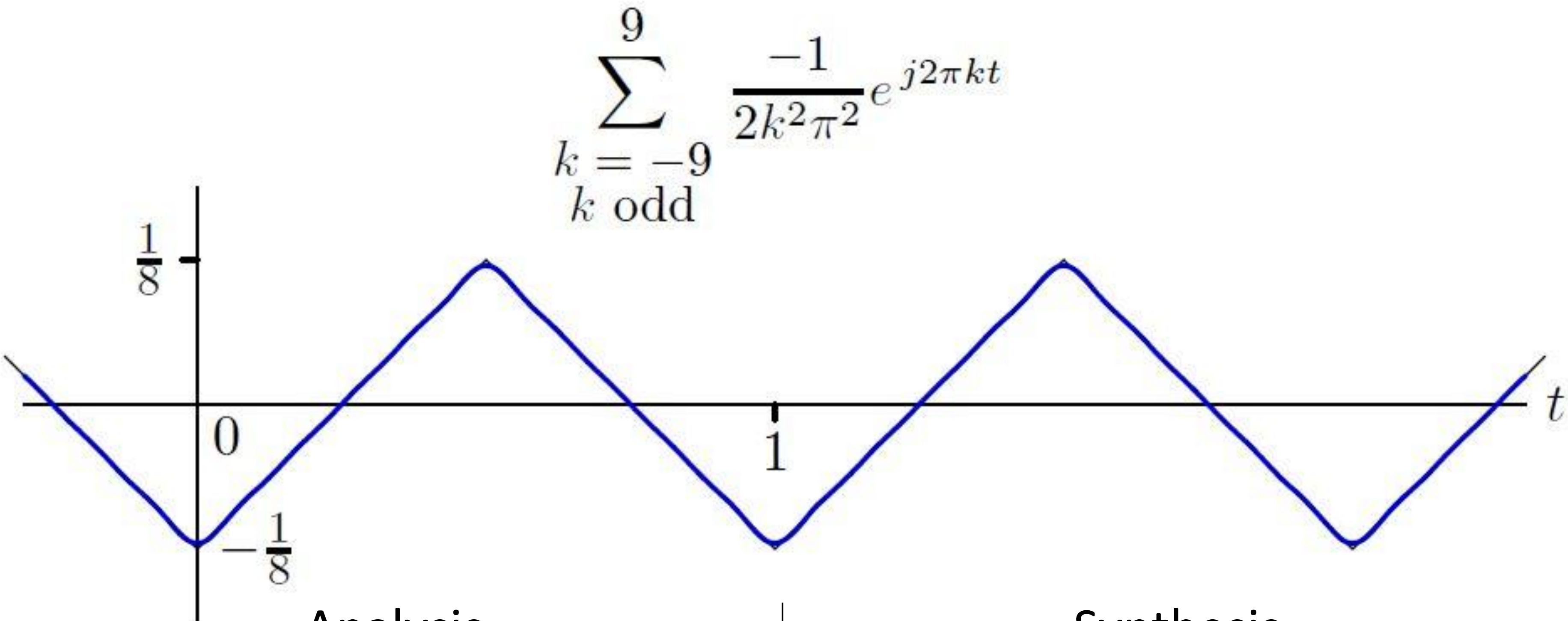



| Analysis | Synthesis |
| --- | --- |
| $c_k = \dfrac{-1}{2k^2\pi^2}, k\ odd$ | $x(t) = \displaystyle\sum_{k=-\infty, k\,odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence

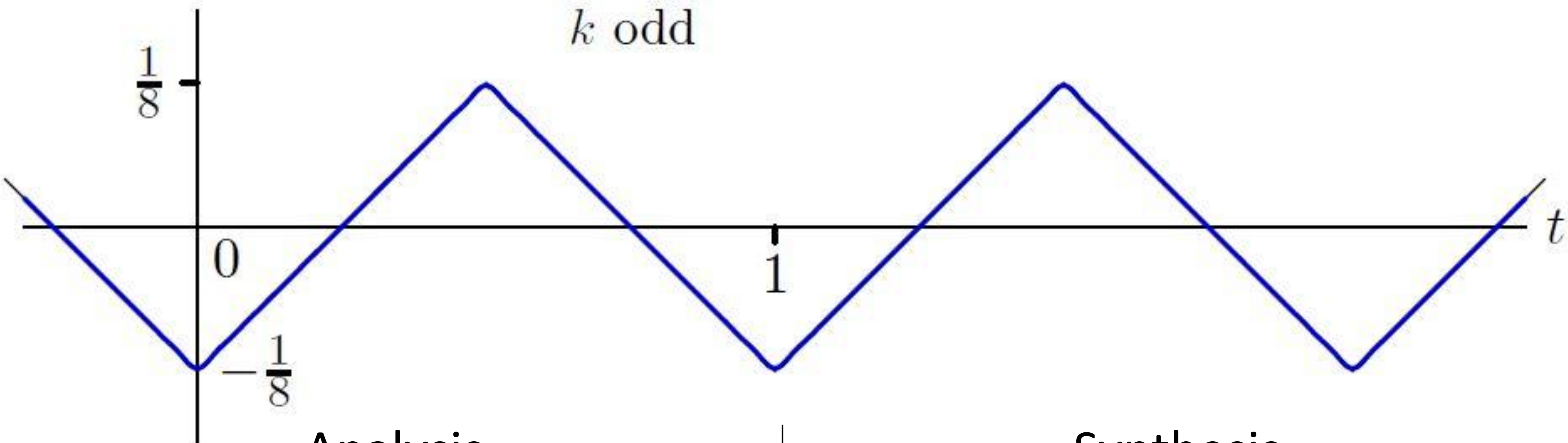



| Analysis | Synthesis |
| --- | --- |
| $c_k = \dfrac{-1}{2k^2\pi^2}, k\ odd$ | $x(t) = \displaystyle\sum_{k=-\infty, k\ odd}^{\infty} c_k e^{j2\pi kt}$ |

## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence

$$\sum_{\substack{k=-19 \\ k\ \text{odd}}}^{19} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



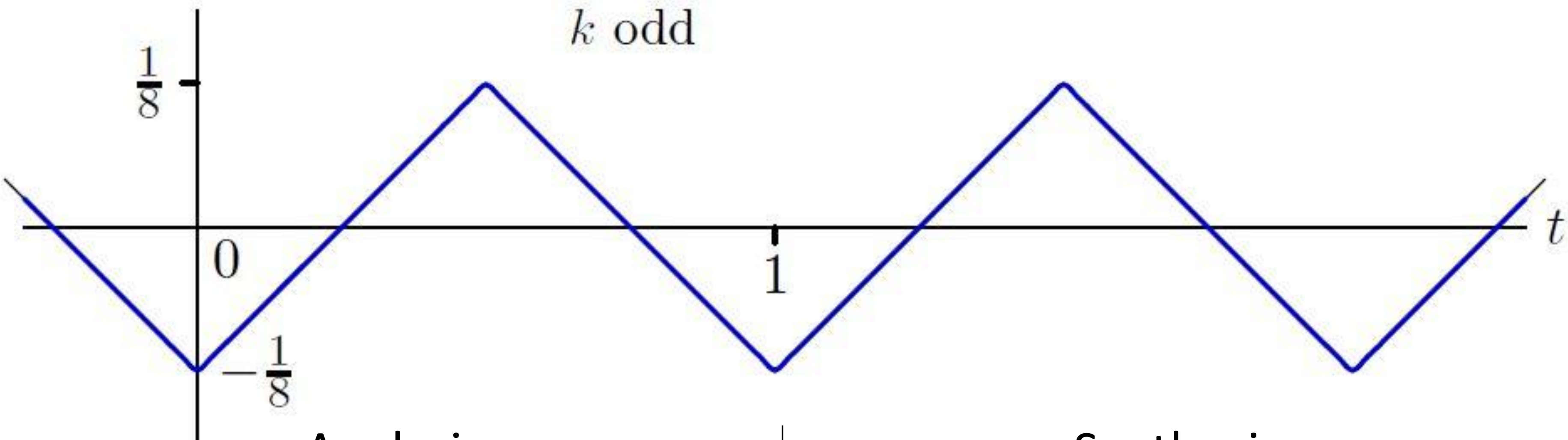

| Analysis | Synthesis |
|---|---|
| $c_k = \dfrac{-1}{2k^2\pi^2}, k\ odd$ | $x(t) = \sum_{k=-\infty, k\ odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence

$$\sum_{\substack{k=-29 \\ k \text{ odd}}}^{29} \frac{-1}{2k^2\pi^2} e^{j2\pi k t}$$




| Analysis | Synthesis |
|---|---|
| $c_k = \dfrac{-1}{2k^2\pi^2}, k\ odd$ | $x(t) = \displaystyle\sum_{k=-\infty, k\, odd}^{\infty} c_k e^{j2\pi k t}$ |

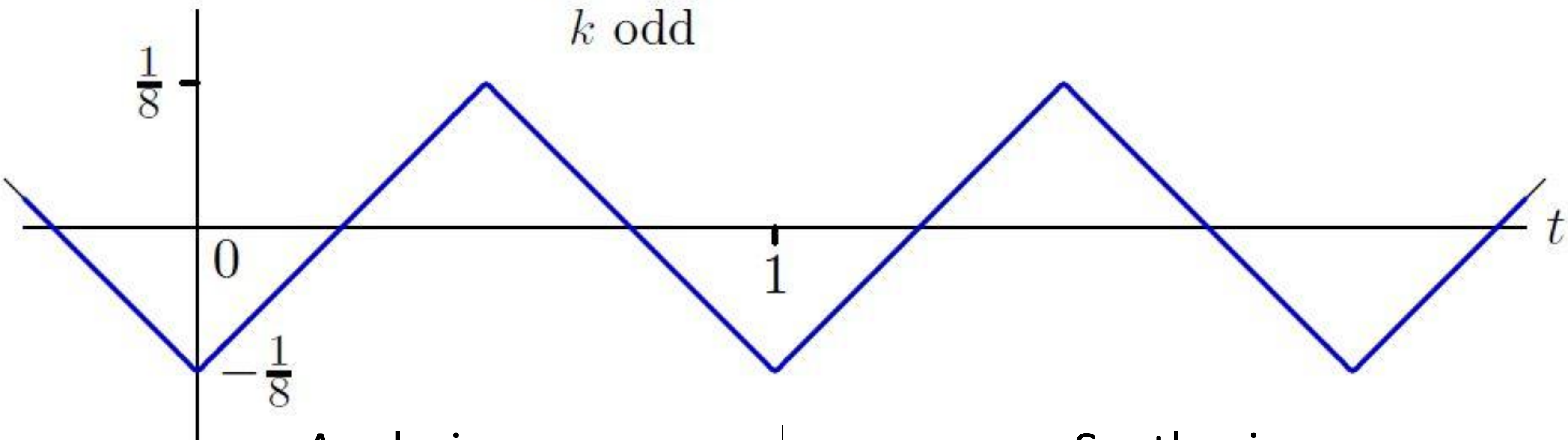
# Fourier Series



## Can we represent all periodic signals with harmonics?

### Triangle Wave – Fourier Series Convergence

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



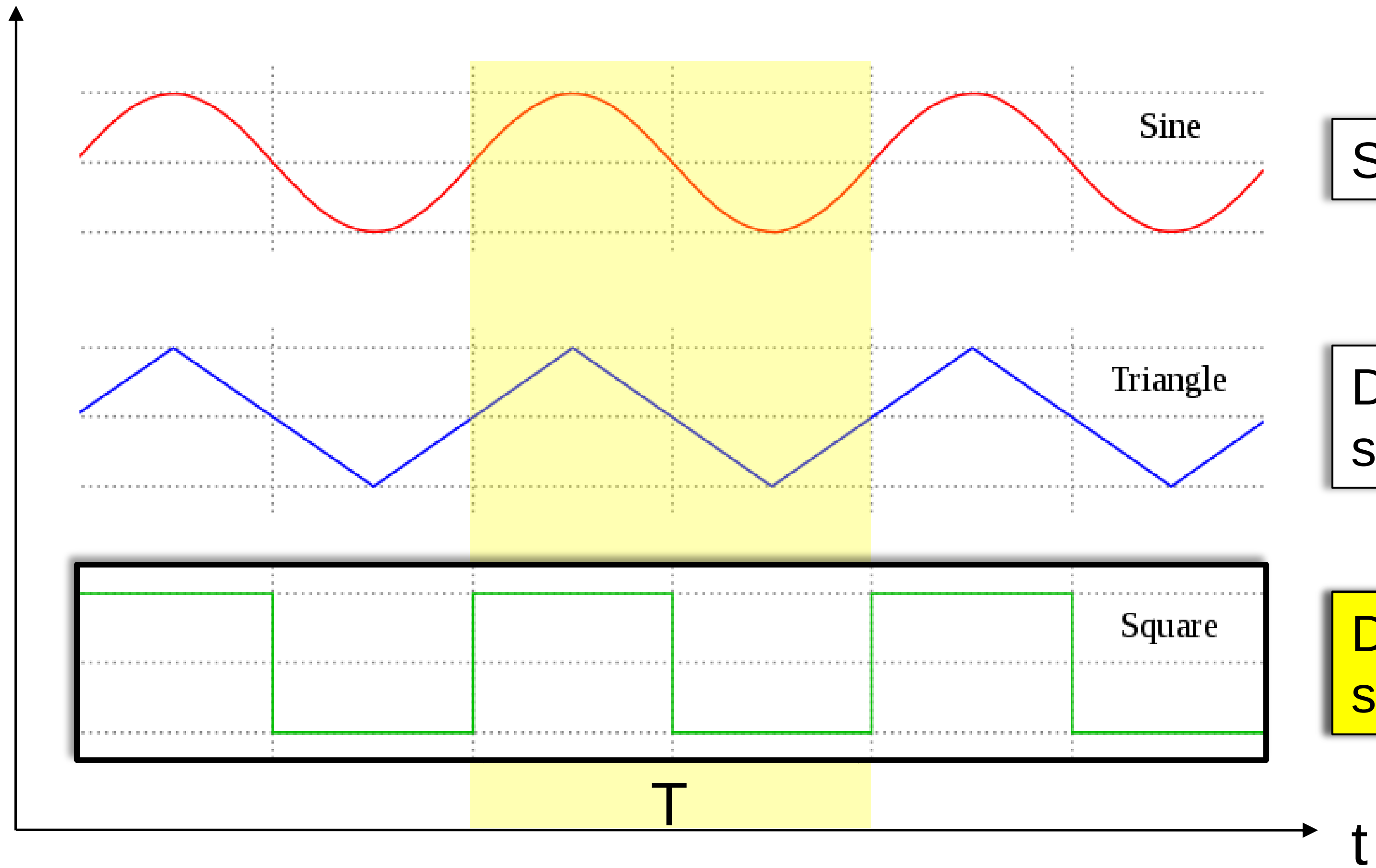

| Analysis | Synthesis |
|---|---|
| $c_k = \dfrac{-1}{2k^2\pi^2}, k \; odd$ | $x(t) = \displaystyle\sum_{k=-\infty, k\,odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series



## Can we represent all periodic signals with harmonics?

### Square Wave



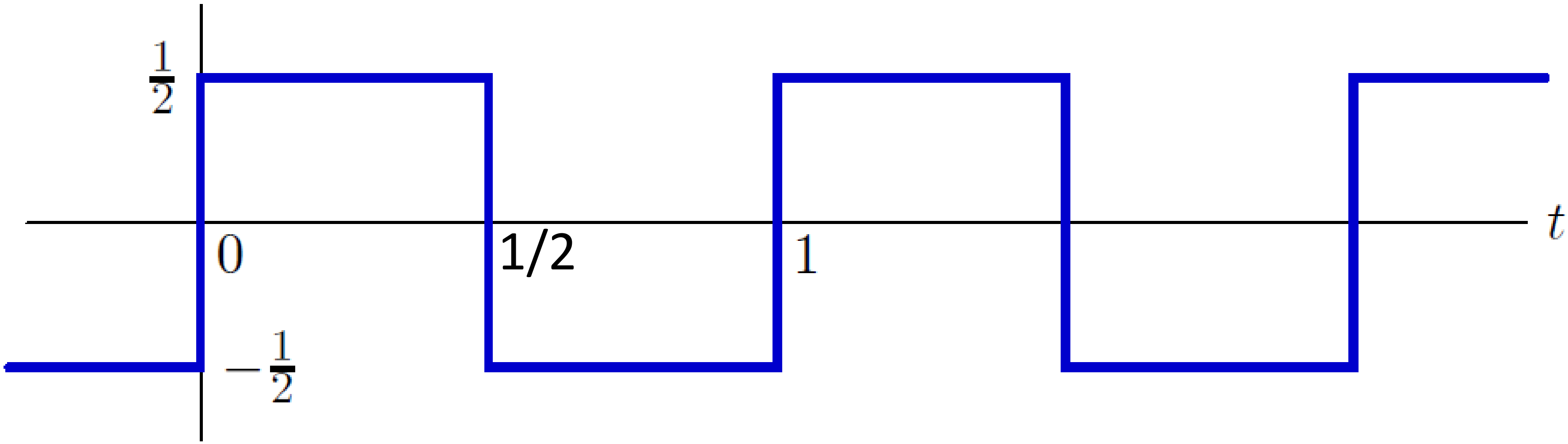

Smooth signal

Discontinuous slopes

Discontinuous signal

## Can we represent all periodic signals with harmonics?

### Square Wave




$$x(t) = \begin{cases} \frac{1}{2}, & 0 \le t < \frac{1}{2} \\ -\frac{1}{2}, & -\frac{1}{2} \le t < 0 \end{cases}$$

1. Apply the forward transform

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

2. Synthesise square wave by adding together harmonics weighted by $c_k$

# Fourier Series 

## Can we represent all periodic signals with harmonics?

### Square Wave

$$c_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt = -\frac{1}{2}\int_{-\frac{1}{2}}^{0} e^{-j2\pi kt}dt + \frac{1}{2}\int_0^{\frac{1}{2}} e^{-j2\pi kt}dt$$

$$= \frac{1}{j4\pi k}\left(2 - e^{j\pi k} - e^{-j\pi k}\right) = \begin{cases} \frac{1}{j\pi k}\ ; & \text{if } k \text{ is odd} \\ 0\ ; & \text{otherwise} \end{cases}$$

$$x(t) = \left\{ \begin{array}{ll} \frac{1}{2}, & 0 \le t < \frac{1}{2} \\ -\frac{1}{2}, & -\frac{1}{2} \le t < 0 \end{array} \right\}$$

1. Apply the forward transform

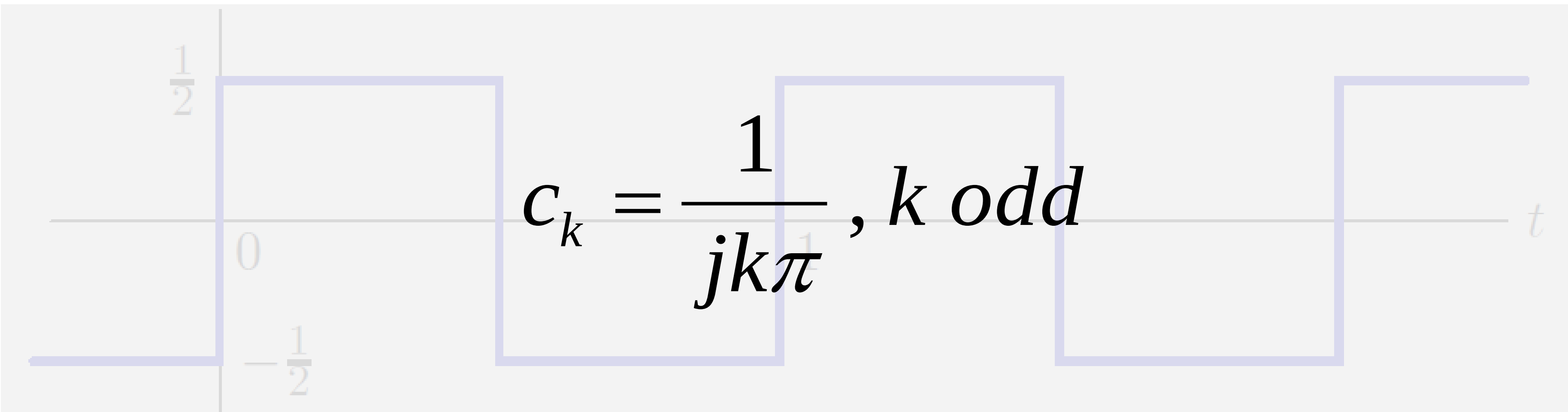
$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

2. Synthesise square wave by adding together harmonics weighted by $c_k$

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Square Wave




$$c_k = \frac{1}{jk\pi}, \; k \; odd$$

$$x(t) = \begin{cases} \frac{1}{2}, & 0 \le t < \frac{1}{2} \\ -\frac{1}{2}, & -\frac{1}{2} \le t < 0 \end{cases}$$

1. Apply the forward transform

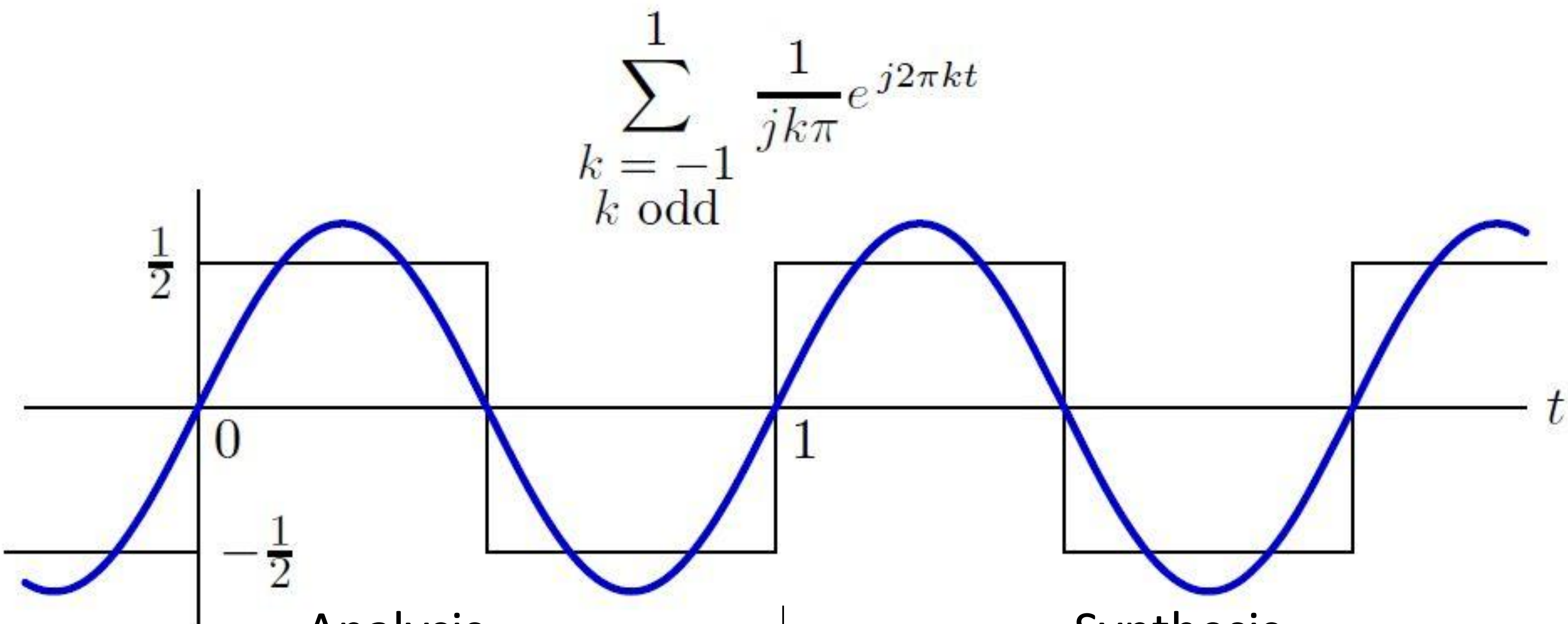
$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

2. Synthesise square wave by adding together harmonics weighted by $c_k$

# Fourier Series

## Can we represent all periodic signals with harmonics?

### Square Wave – Fourier Series Convergence



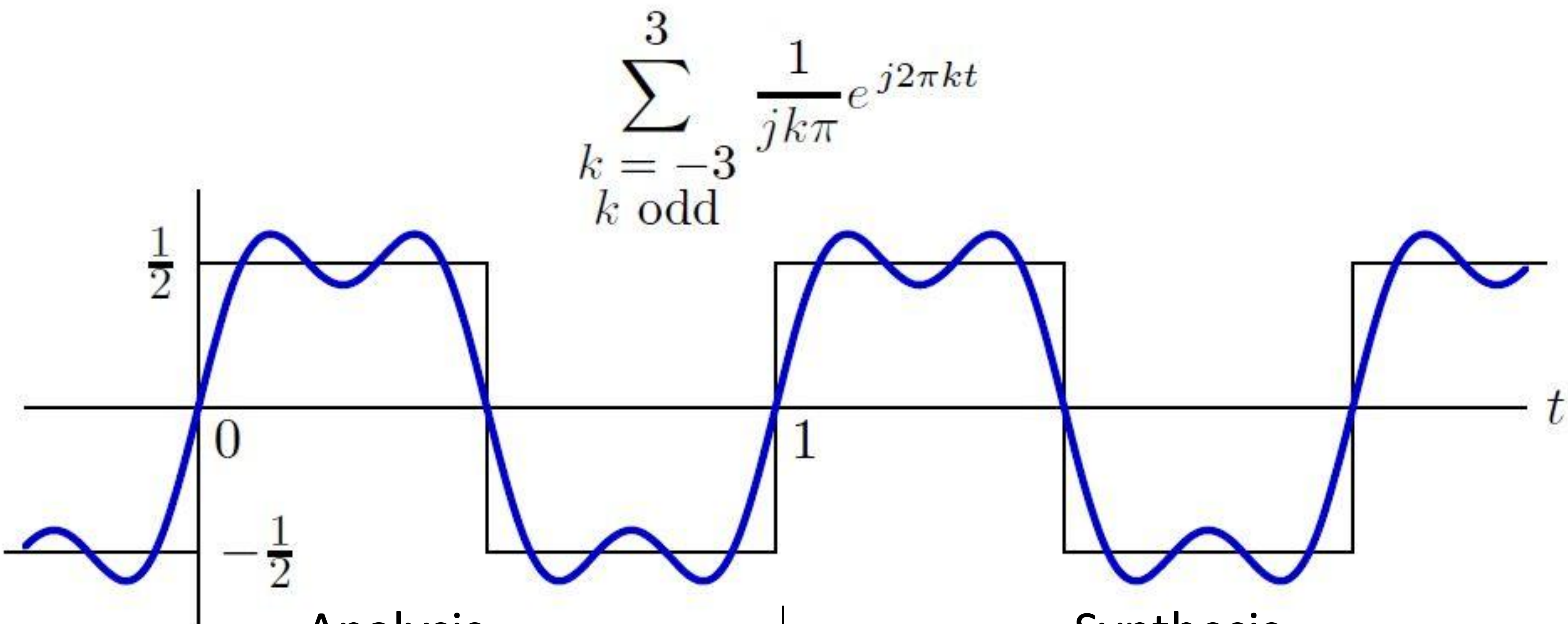

| Analysis | Synthesis |
| --- | --- |
| $c_k = \frac{1}{jk\pi}, k\ odd$ | $x(t) = \sum_{k=-\infty, k\ odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series



## Can we represent all periodic signals with harmonics?

### Square Wave – Fourier Series Convergence




| Analysis | Synthesis |
|---|---|
| $c_k = \frac{1}{jk\pi}, k \; odd$ | $x(t) = \sum_{k=-\infty, k \, odd}^{\infty} c_k e^{j2\pi kt}$ |

# Fourier Series

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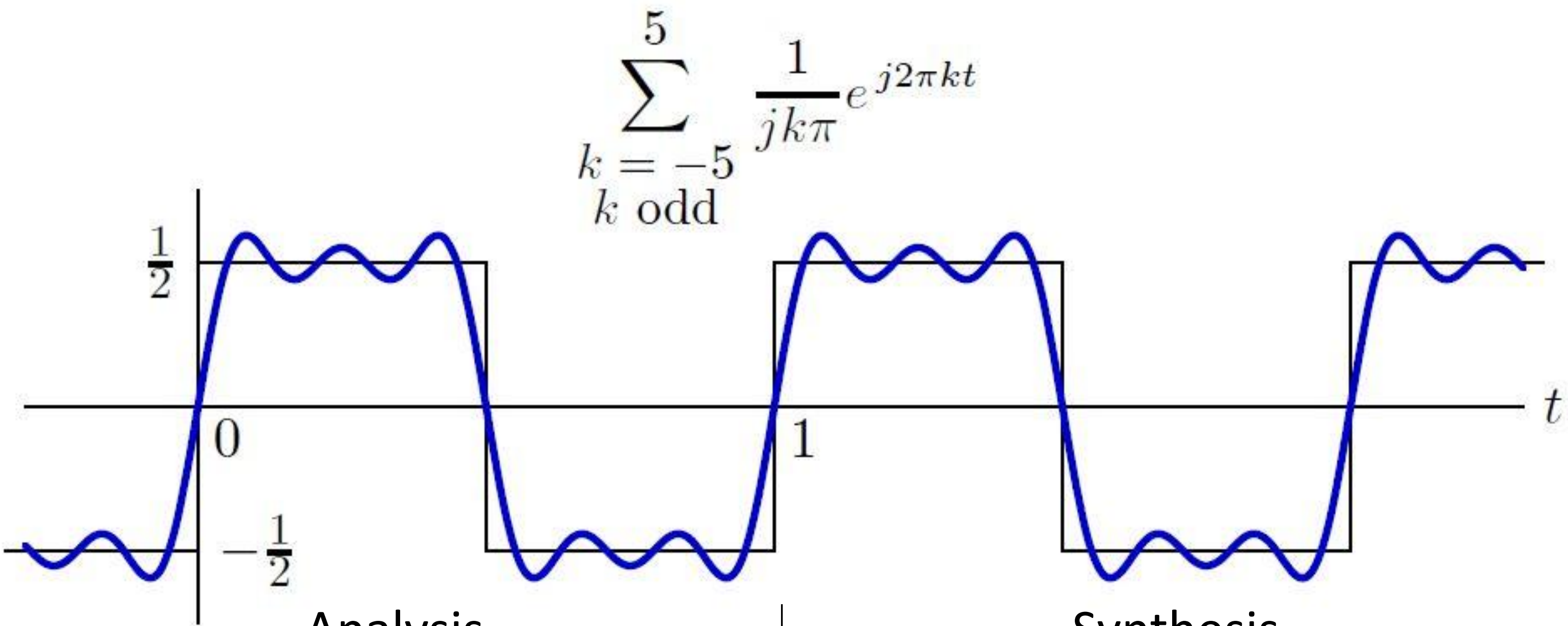


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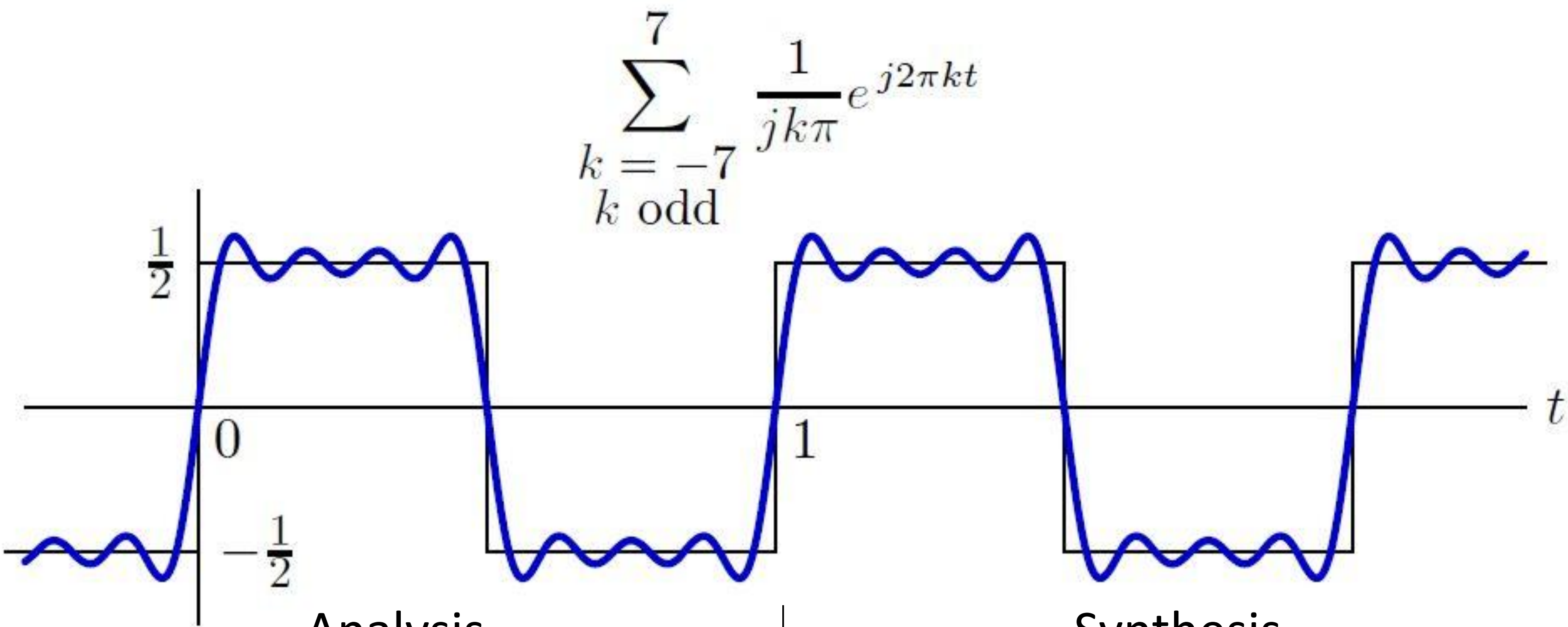


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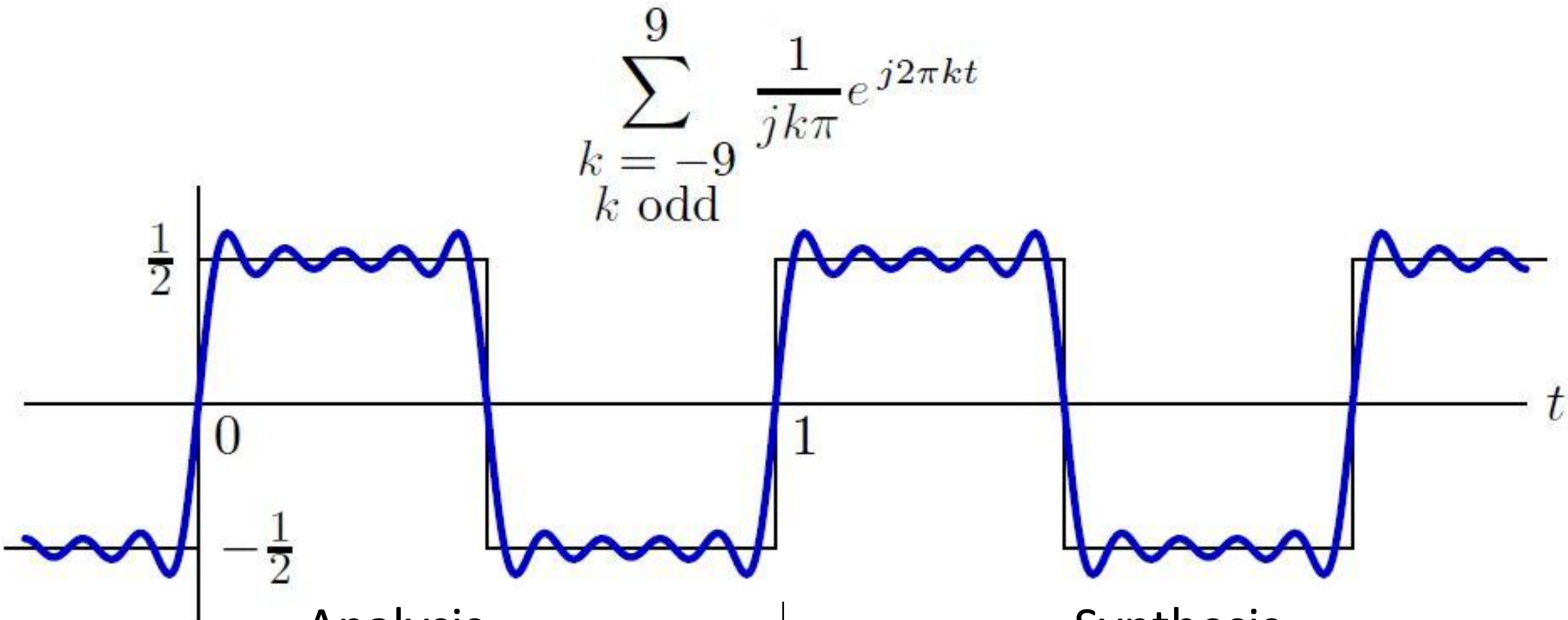


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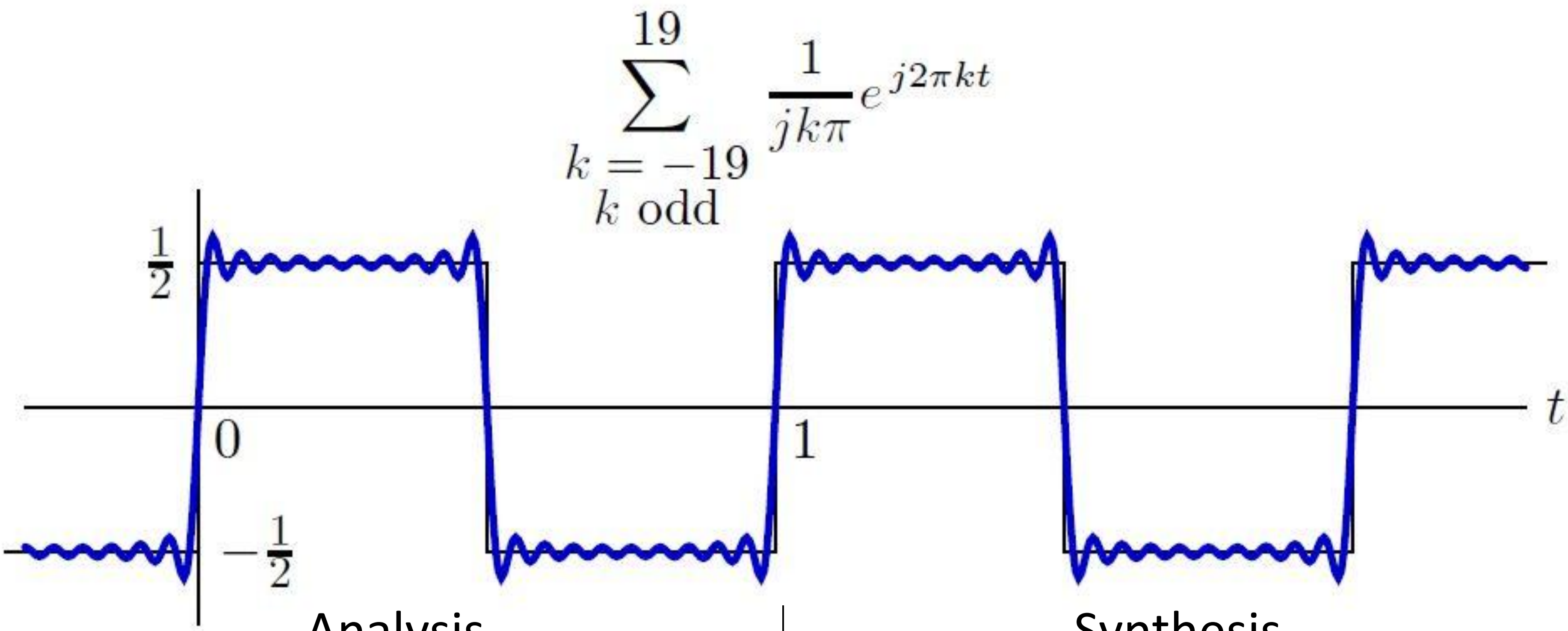


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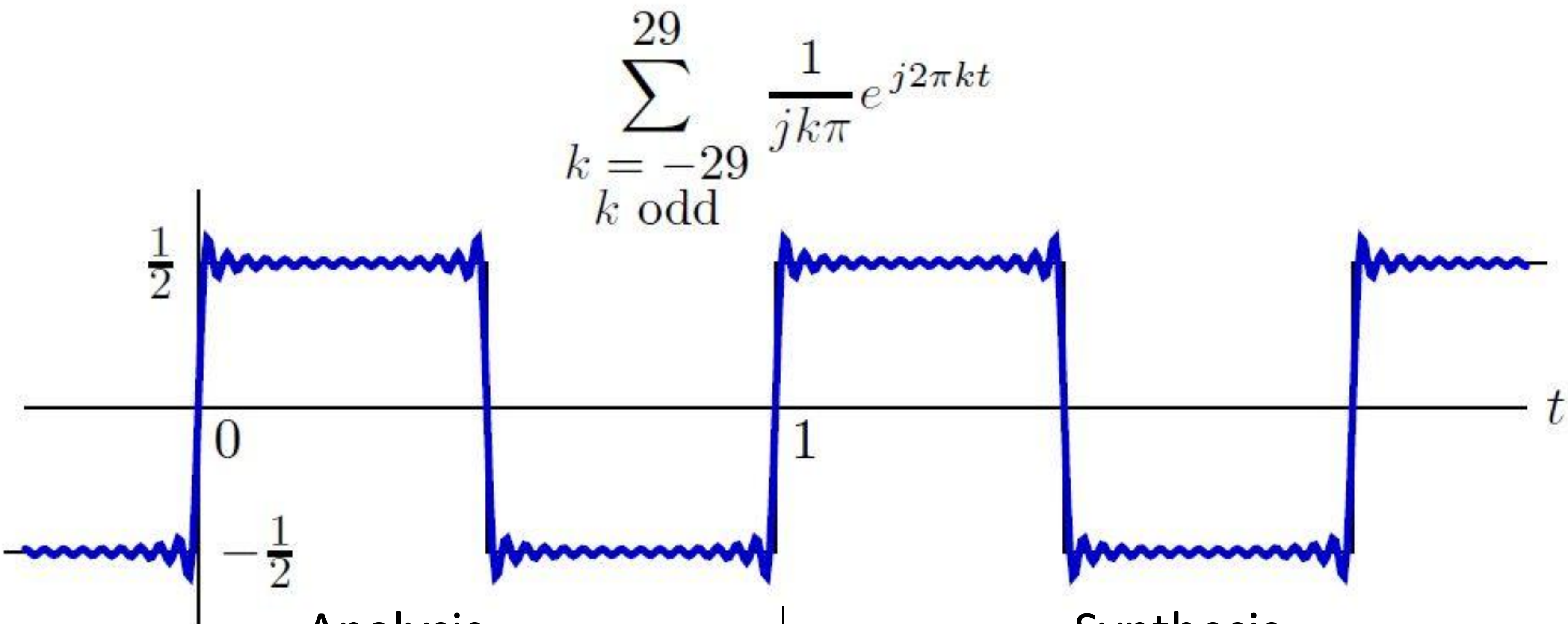


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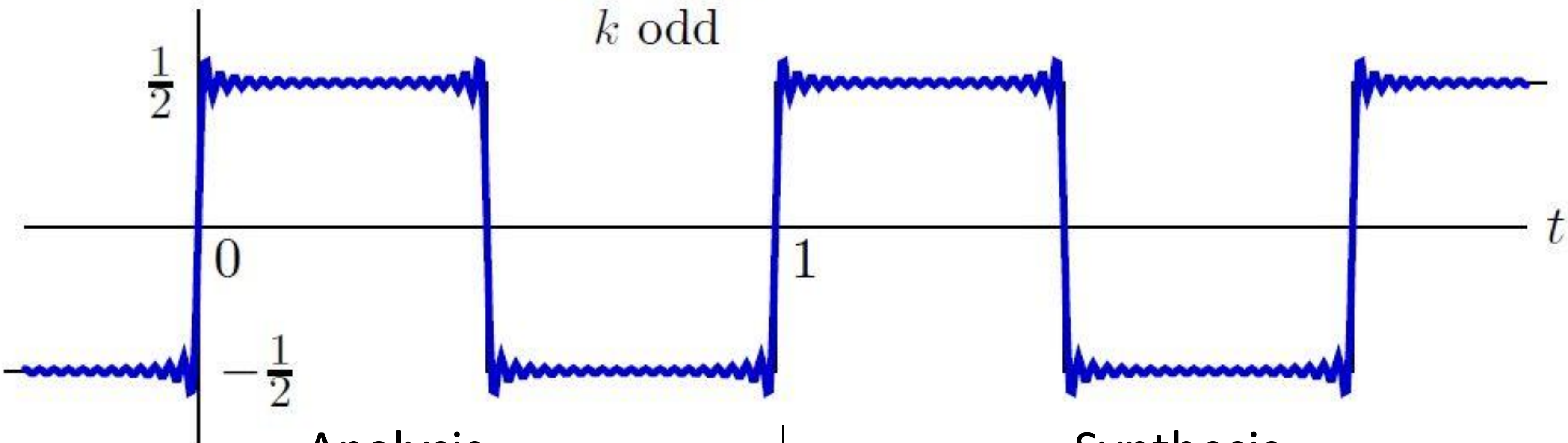
# Fourier Series

## Can we represent all periodic signals with harmonics?

### Square Wave – Fourier Series Convergence

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{1}{jk\pi} e^{j2\pi kt}$$

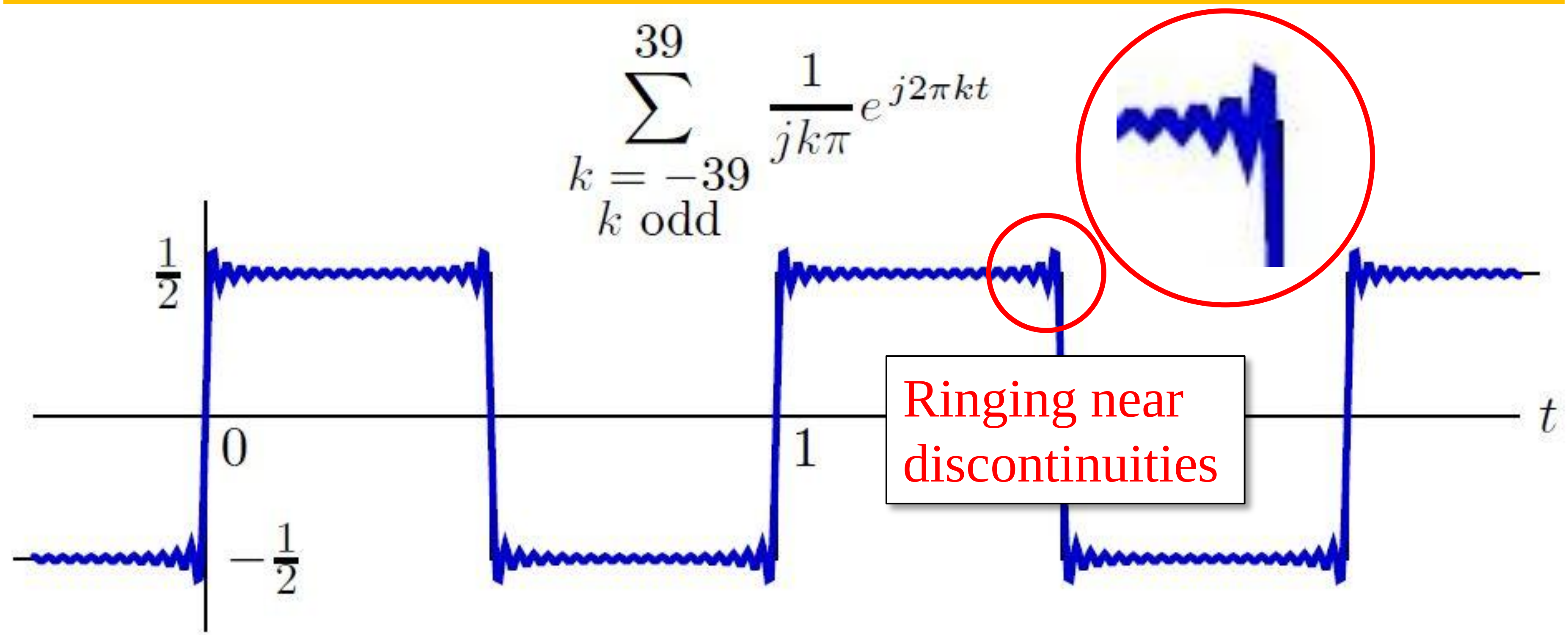



| Analysis | Synthesis |
|---|---|
| $c_k = \dfrac{1}{jk\pi}, k\ odd$ | $x(t) = \displaystyle\sum_{k=-\infty, k\ odd}^{\infty} c_k e^{j2\pi kt}$ |

## Ringing

### Discontinuous signals

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{1}{jk\pi} e^{j2\pi kt}$$



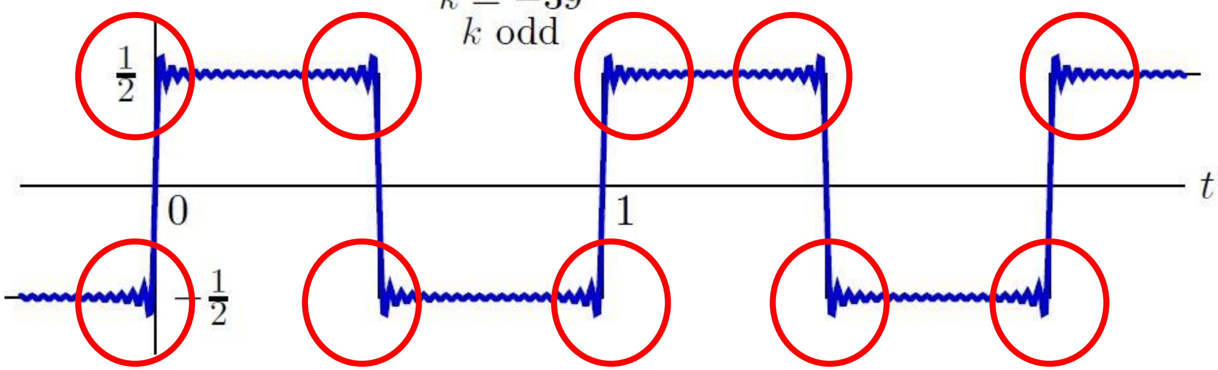

**Gibb's phenomenon** : Partial sums of Fourier series of discontinuous functions ring near discontinuities

# Fourier Series



## Ringing

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{1}{jk\pi} e^{j2\pi kt}$$

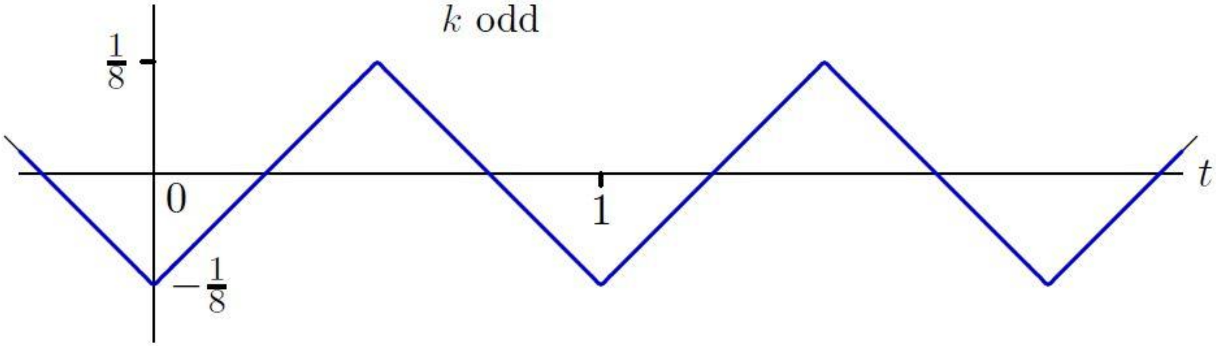



$$c_k = \frac{1}{jk\pi}, k \; odd$$

**Square Wave**: Magnitude of FS coefficients decreases as

$$1/k$$

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

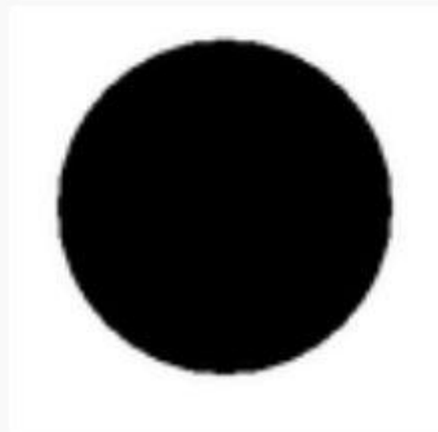
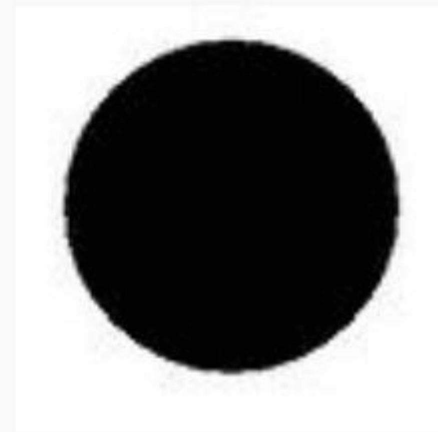
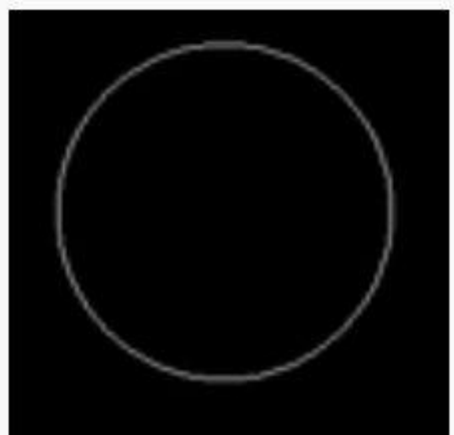
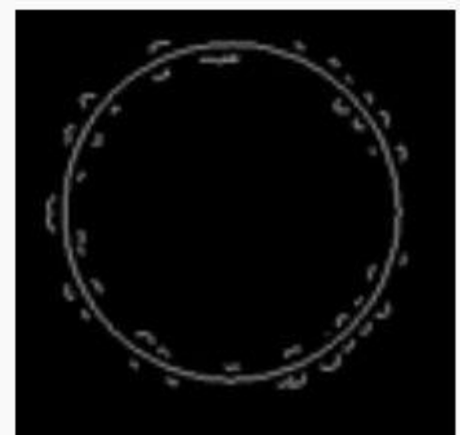



$$c_k = \frac{-1}{2k^2\pi^2}, k \; odd$$

**Triangle Wave**: Magnitude of FS coefficients decreases as

$$1/k^2$$

## Ringing – Image Compression

| Image | Lossless Compression | Lossy Compression |
|---|---|---|
| **Original** |  |  |
| **Processed by Canny edge detector, highlighting artifacts.** |  |  |

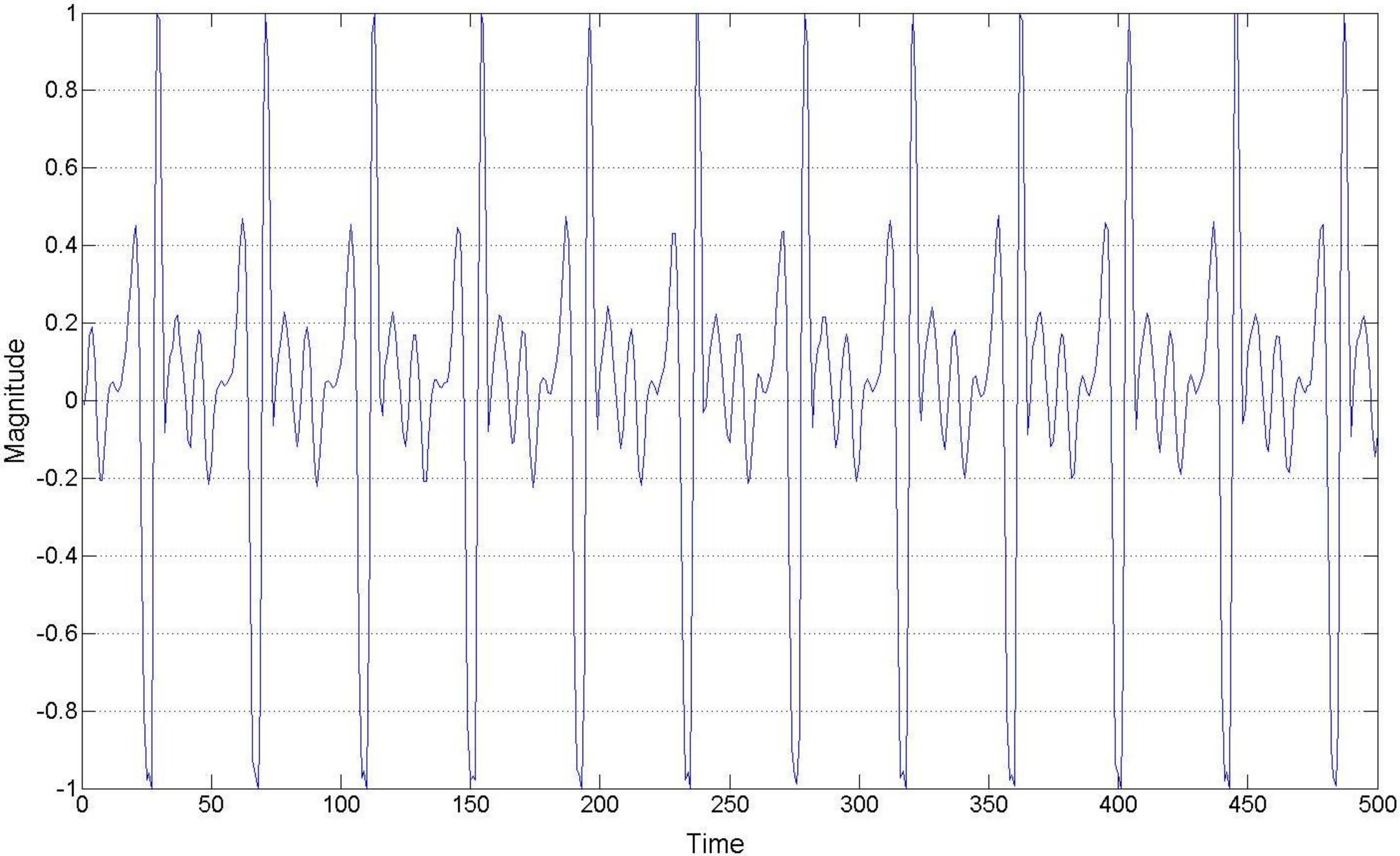
## Ringing – Image Compression

Compressed

Original

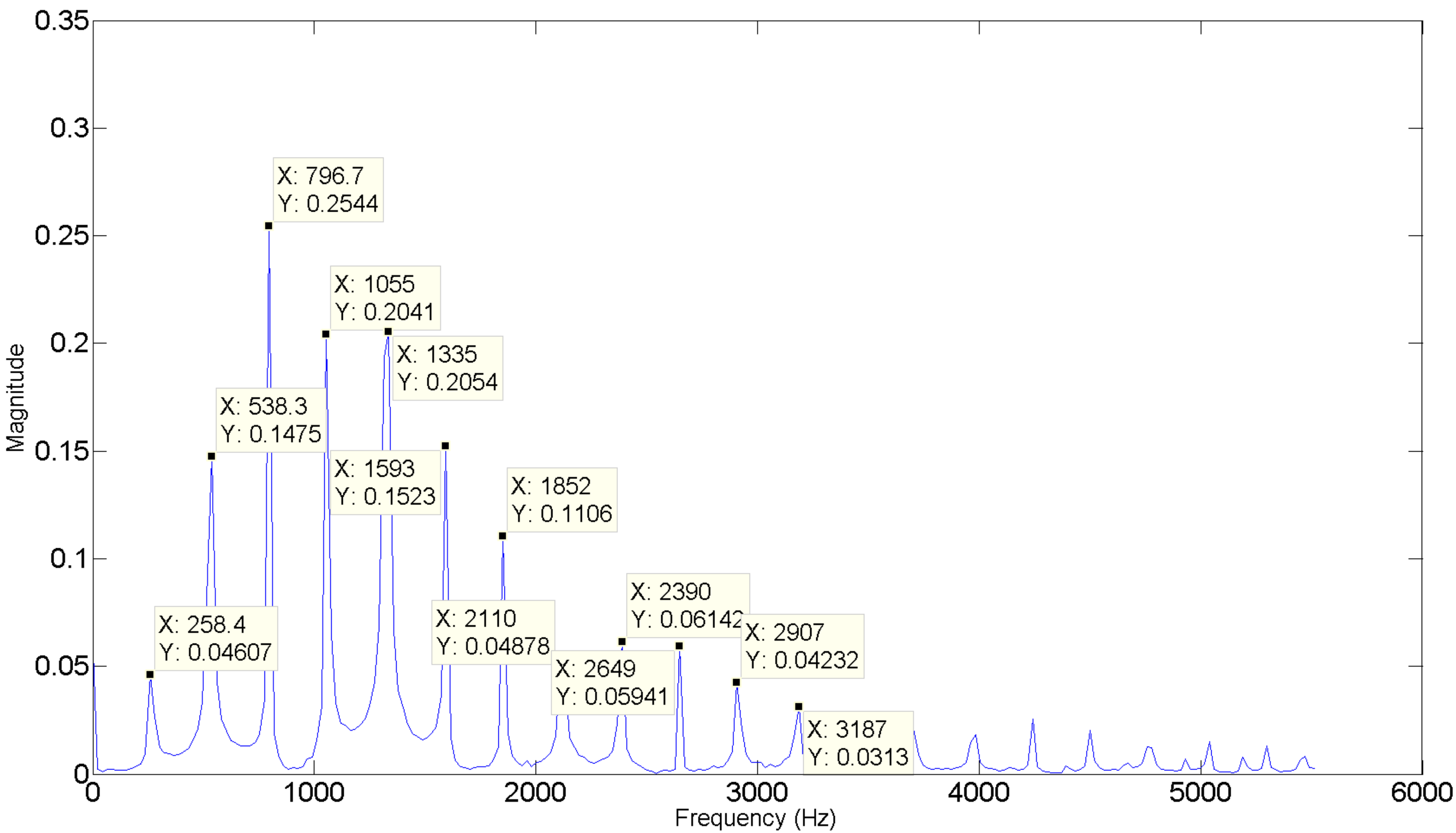
Compressed

## Synthesise Sound



# Fourier Series

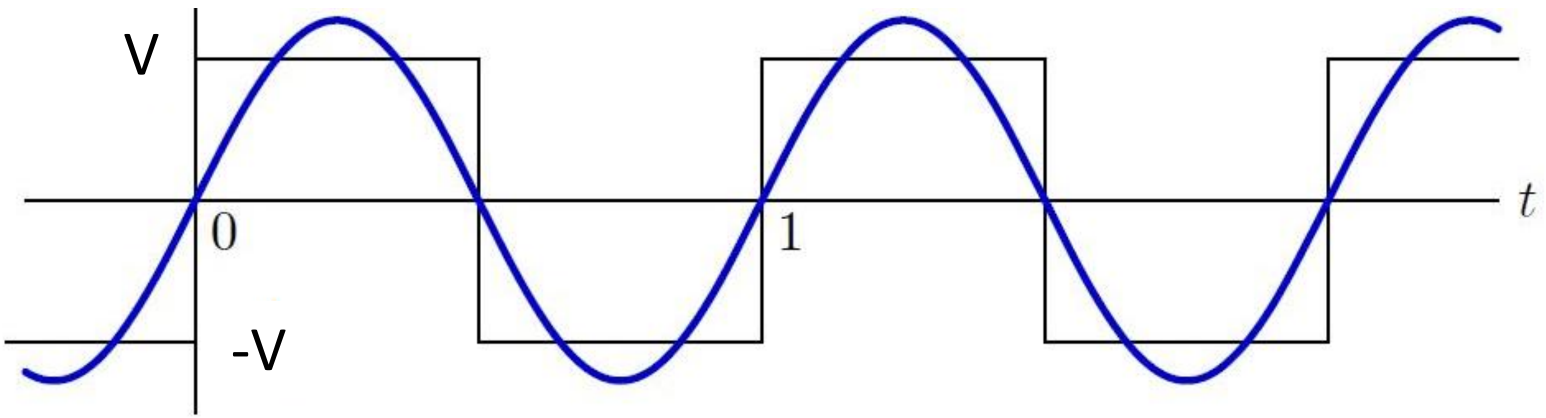
## Synthesise Sound



# Fourier Series



## Application : Generating sine waves from square waves




$$x(t) = \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$c_k = \frac{2V}{jk\pi}, k\ odd \quad \Rightarrow c_k = -\frac{j2V}{k\pi}, k\ odd$$

$$c_k = \begin{cases} a_0/2 & k = 0 \\ (a_k - jb_k)/2 & k > 0 \\ (a_k + jb_k)/2 & k < 0 \end{cases}$$

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t), b_k = \frac{4V}{k\pi}, k\ odd$$

**Band pass filter** with resonant frequency of $\omega_0$ (fundamental frequency)

# Fourier Series

## Application : Generating sine waves from square waves

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### SIMPLE FILTER TURNS SQUARE WAVES INTO SINE WAVES

*by R. Mark Stitt (602) 746-7445*

Many signals are digitally generated or transmitted as square waves. It is often desirable to convert these signals into sine waves. For example, the 350Hz, 440Hz, 480Hz, and 620Hz telephone supervisory tones transmitted over fiber-optics may appear at curb-side as square waves. To be used in telephone equipment it is desirable to convert the square waves into low-distortion sine waves. This can be done with a simple filter.

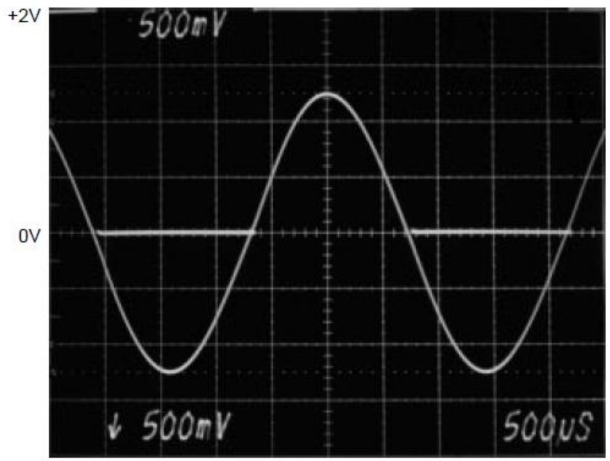
According to its Fourier series, a 50% duty-cycle square wave consists of odd order harmonic sine waves with the fundamental at the same frequency as the square wave.

Fourier Series for a Square Wave

$$\frac{4k}{\pi}\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots\right)$$

where k = peak amplitude of the square wave

A sine wave with the same frequency as the square wave can be gleaned by filtering out the harmonics above the fundamental. A "tuned-circuit" bandpass filter with a Q of 10 attenuates signals at three times the bandpass frequency by 28.4dB. Since the amplitude of the third harmonic is 1/3 that of the fundamental, the total attenuation of the third harmonic compared to the fundamental is nearly 40dB. The result is a low distortion sine wave as shown in Figure 1A. Notice that although the filter has unity gain, the amplitude of the sine wave output signal is greater than that of the square wave. This is because the fundamental has an amplitude of $4/\pi$ times that of the square wave as shown by the Fourier series. The bandpass filter will also filter out any DC




1a. A square wave passed through a simple "tuned-circuit" bandpass filter produces a low distortion sine wave.