Lecture 14

Design of FIR Filters using Frequency Sampling

Design through sampling of the desired frequency response

Dr. Dimitris Agrafiotis, Dept. Of Electrical & Electronic Engineering

FIR Filter Design using Windowing

What's good and what's not so good

Like

+ Simple to understand
+ Easy to use
+ Provides practically useful filters

Dislike

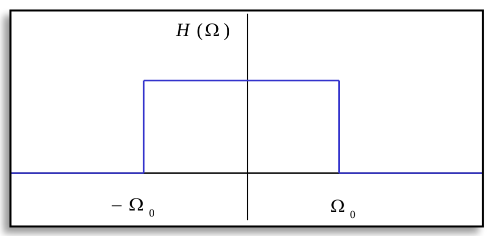
- Filters have approximately equal size ripples in each band i.e. passband and stopband)
- Band edges and maximum ripple size cannot be specified precisely (due to frequency convolution)
- Analytical expression required for the desired frequency response
- Analytical expression for the ideal impulse response corresponding to an arbitrary desired frequency response is not always easy to get
- Filters are suboptimal as no optimality criterion is satisfied

(apart from rectangular window the use of which minimises the mean squared error between desired and actual frequency response)

FIR Filter Design using Windowing

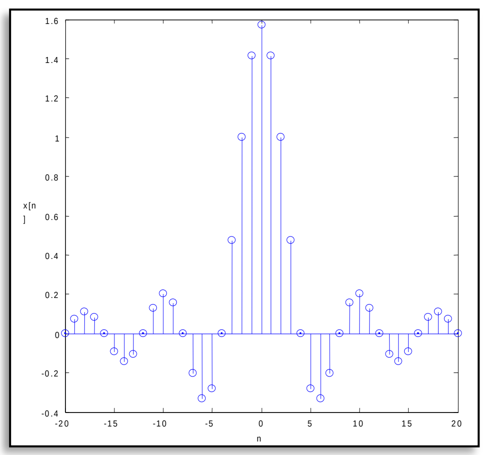
Design Steps

1. Specify the desired frequency response



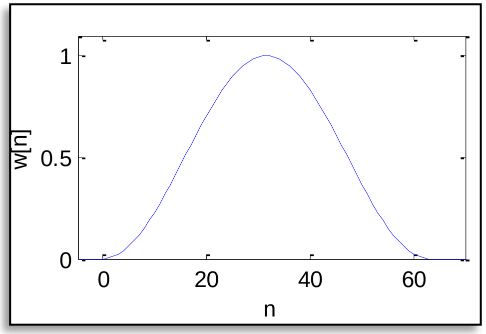
Analytical expression required for desired frequency response

2. Inverse DTFT to obtain ideal impulse response



The analytical expression for the ideal impulse response (IDTFT) of the desired frequency response is not always easy to get

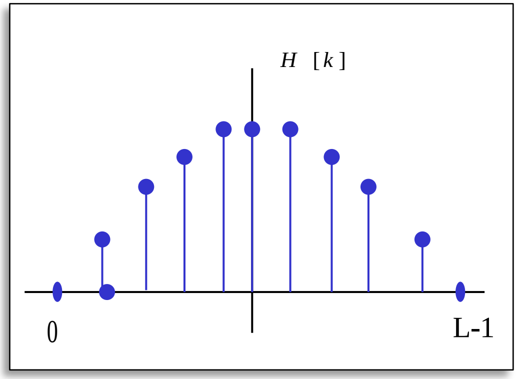
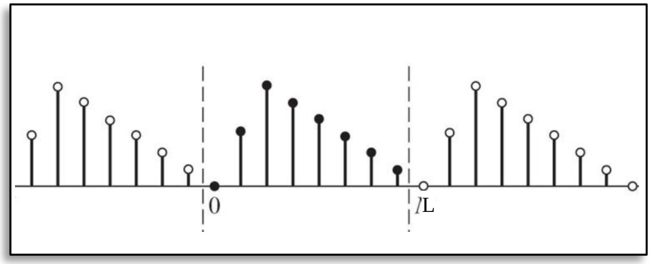
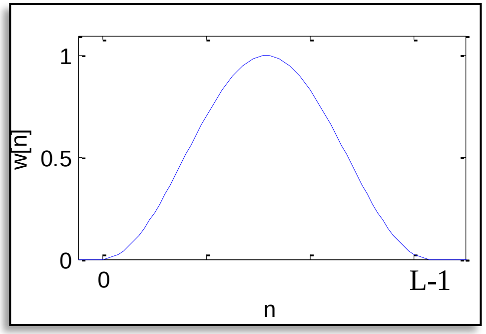
3. Multiply by window function to obtain FIR approximation



Band edges and maximum ripple size cannot be specified precisely due to convolution with the window spectrum in the frequency domain)

FIR Filter Design using **Frequency Sampling**

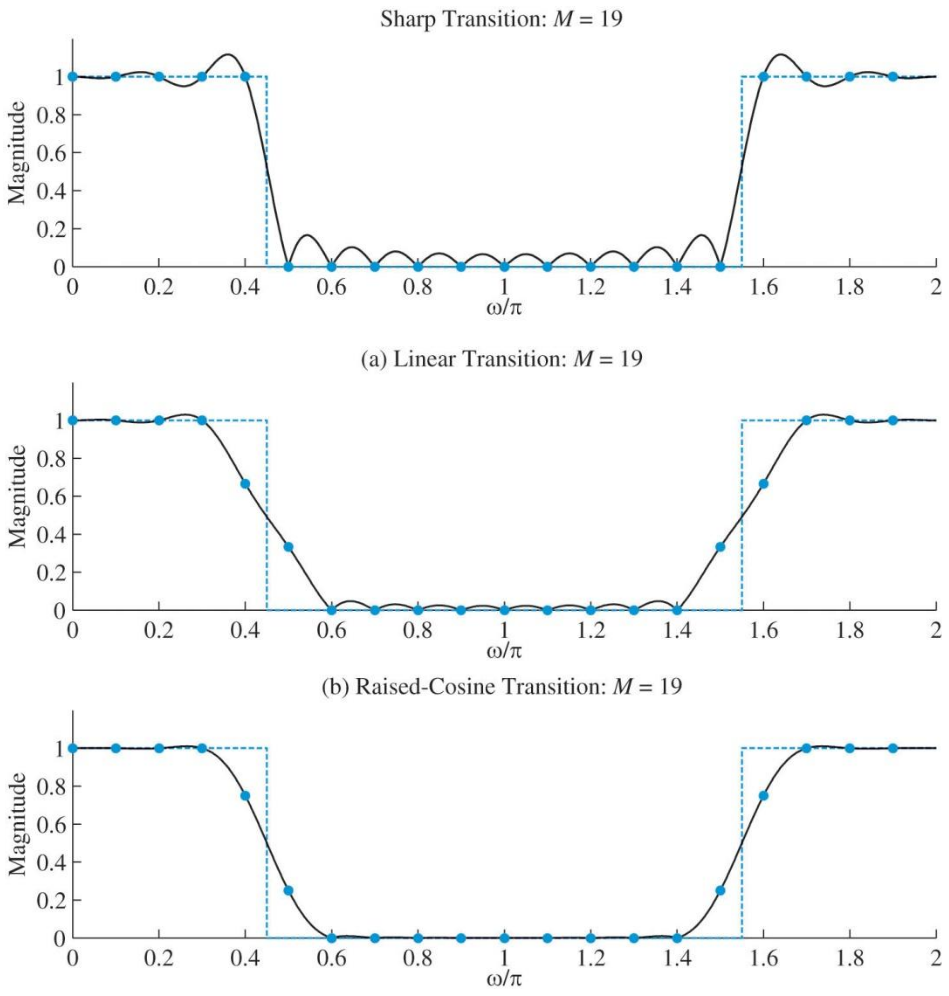
Design Steps

1. Sample the desired frequency response at L points (k2π/L) or use L given freq.samples	2. Inverse DFT of length L to obtain impulse response	3. Multiply with window function to keep only one period (FIR filter)
		
Unless care is taken how samples are specified, phase response will be arbitrary	Impulse response will be periodic and may suffer from time domain aliasing depending on length L	The DFT of the desired filter will be convolved with spectrum of window - approximation error at freq. samples will be zero or close to zero

FIR Filter Design using Frequency Sampling

Effect of transition band sharpness

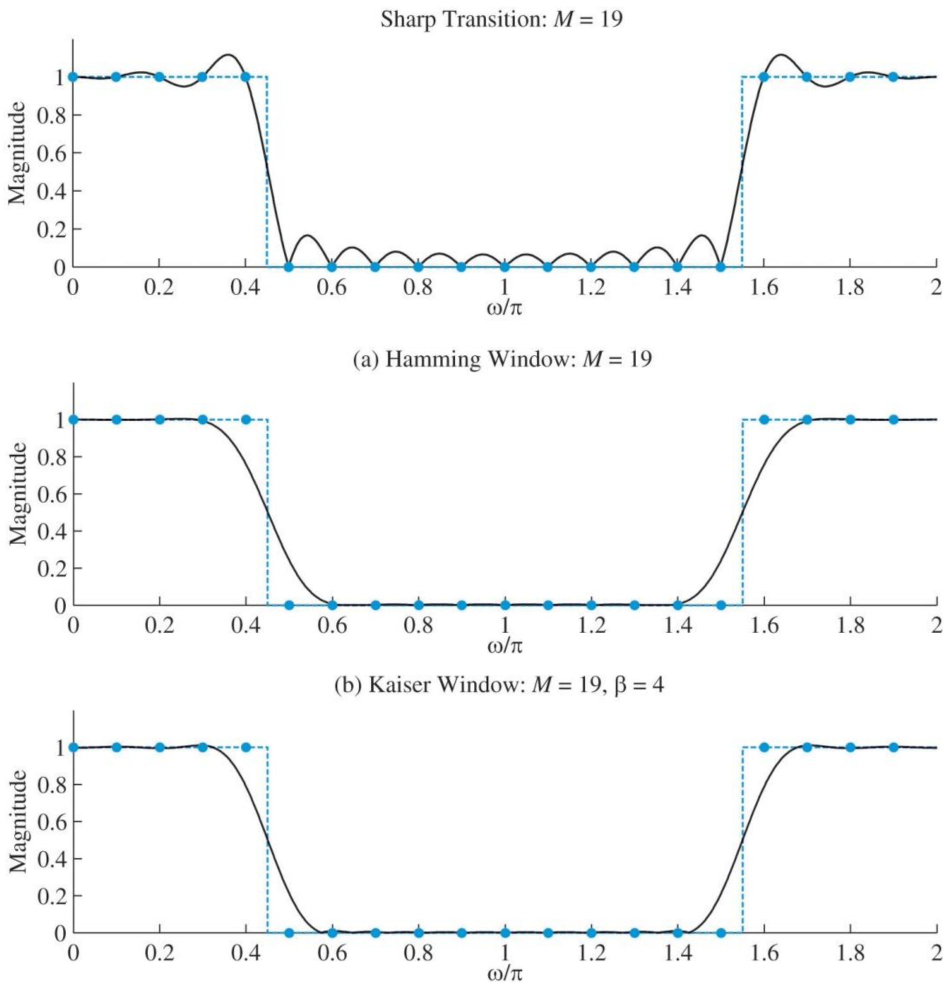
- Transition band sharpness affects passband and stopband ripple
- A sharp transition creates a discontinuity and leads to Gibbs phenomenon (ripples)
- A smooth transition band (linear or raised cosine) eliminates the Gibbs phenomenon (almost no ripple at passband and stoband)
- Penalty paid for ripple elimination is a wider transition band



FIR Filter Design using Frequency Sampling

Effect of window used

- Rectangular window leads to substantial ripple
- A smoother window (Hamming, Keiser) results in less ripple but wider transition band
- Note that the frequency response of the resulting filter doesn’t include the samples of the desired frequency response near the transition
- Matlab uses a Hamming window by default for frequency sampling



Example 1 – Odd Length FIR Filter

Use frequency sampling to design a 8th order (9 tap) low pass filter with cut-off of π/4

1. Desired frequency response:

$$H(\Omega) = \begin{cases} e^{-j4\Omega} & |\Omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\Omega| \le \pi \end{cases}$$

Point of symmetry (group delay) $\Big\}\; M/2 = 4$

2. Sample the desired frequency response at : $\Omega = k2\pi/L \qquad 0 \le k < L \qquad L = 9$

$$H(\Omega) \ne 0 \text{ for } |\Omega| \le \frac{\pi}{4} \Rightarrow H(\Omega) \ne 0 \text{ for } \Omega \ge 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} \text{ as well}$$

Samples at 2π/9

$$\left.\begin{array}{l} H(\Omega) \ne 0 \;\; for \;\; |\Omega| < \frac{9\pi}{36} \;\; and \;\; |\Omega| > \frac{63\pi}{36} \\ \Omega = k\frac{2\pi}{9} = k\frac{8\pi}{36},\; k = 0,1,2,3,\ldots, 8 \end{array}\right\} H(\Omega) \ne 0 \;\; for \;\; k = 0,1,8$$

$$H[k] = \{1.0000, \quad -0.9397 - 0.3420j, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad -0.9397 + 0.3420j\}$$

3. Inverse 9 point DFT to get impulse response

$$h[n] = \{-0.0977, \quad 0.0000, \quad 0.1497, \quad 0.2813, \quad 0.3333, \quad 0.2813, \quad 0.1497, \quad 0.0000, \quad -0.0977\}$$

Design of FIR Filters - Frequency Sampling

Example 1 – Odd Length FIR Filter – Take 2

Use frequency sampling to design a 8th order (9 tap) low pass filter with cut-off of π/4

Point of symmetry (group delay) $M/2 = 4$

1. Desired frequency response:

$$\left|H(\Omega)\right| = \begin{cases} 1 & \left|\Omega\right| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < \left|\Omega\right| \le \pi \end{cases}$$

2. Sample the desired frequency response at : $\Omega = k2\pi / L \qquad 0 \le k < L \qquad L = 9$

$$H[k] = \{1.0000, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 1\}$$

3. Inverse 9 point DFT to get impulse response

$$h[n] = \{0.3333, \quad 0.2813, \quad 0.1497, \quad 0, \quad -0.0977, \quad -0.0977, \quad 0, \quad 0.1497, \quad 0.2813\}$$

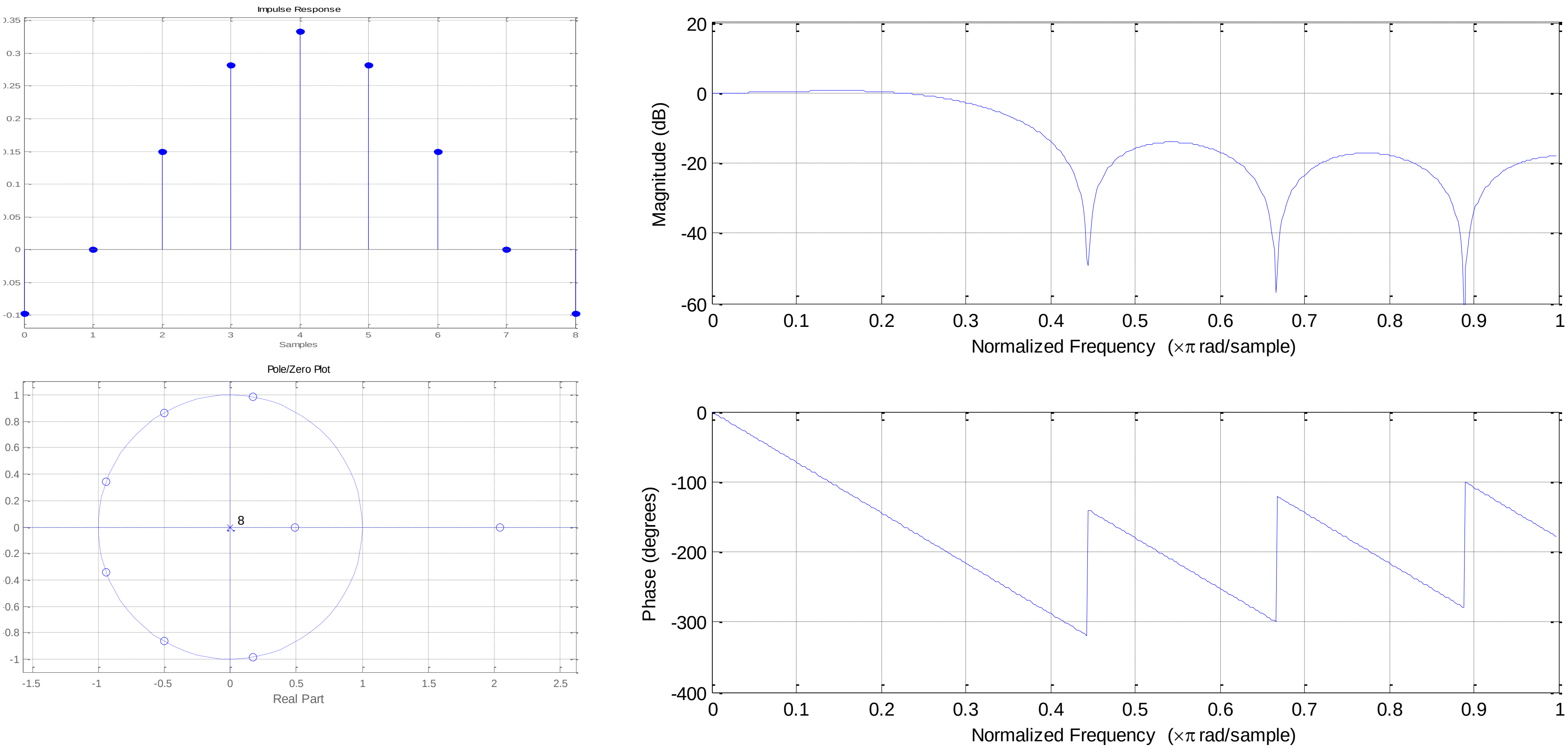
4. Cycle shift by 4 samples

$$h[n] = \{-0.0977, \quad 0.0000, \quad 0.1497, \quad 0.2813, \quad 0.3333, \quad 0.2813, \quad 0.1497, \quad 0.0000, \quad -0.0977\}$$

Example 1 – Odd Length FIR Filter

Use frequency sampling to design a 8th order (9 tap) low pass filter with cut-off of π/4

$$h[n] = \{-0.0977, \quad 0.0000, \quad 0.1497, \quad 0.2813, \quad 0.3333, \quad 0.2813, \quad 0.1497, \quad 0.0000, \quad -0.0977\}$$

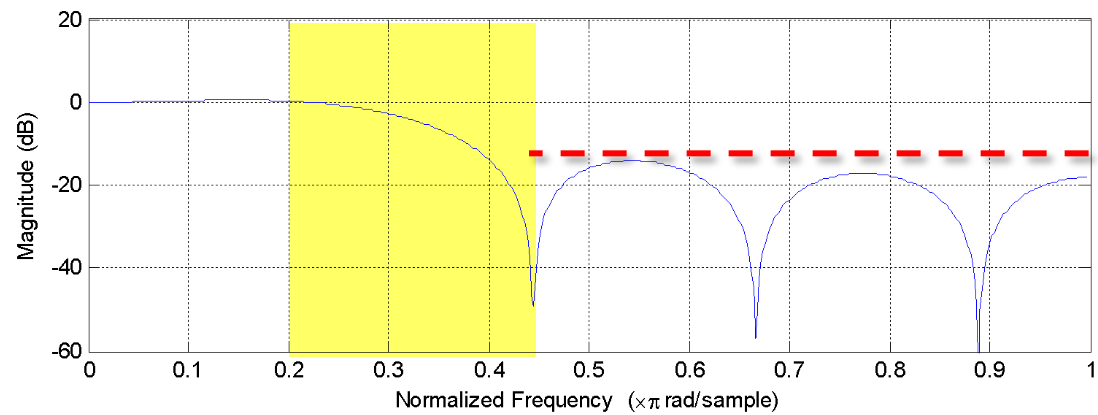


Example 1 – Odd Length FIR Filter

Use frequency sampling to design a 8th order (9 tap) low pass filter with cut-off of π/4

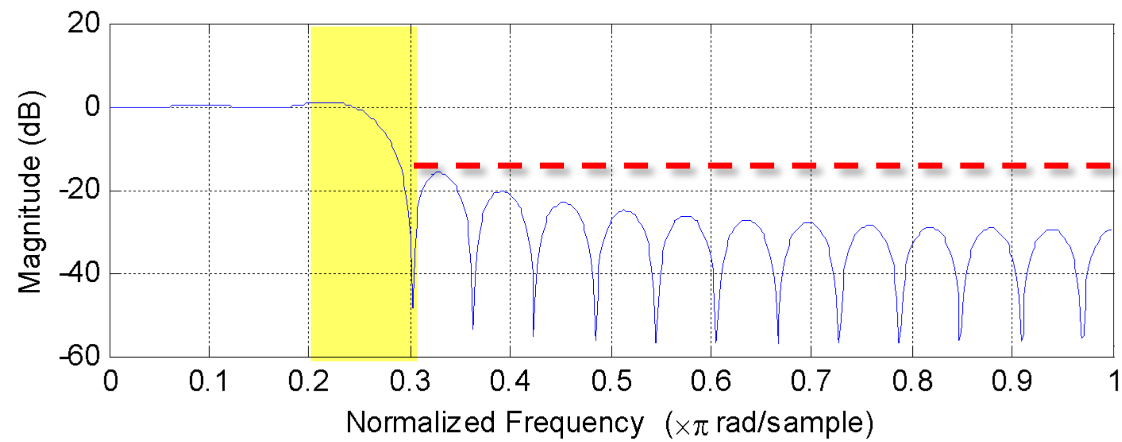
$$h[n] = \{-0.0977, \quad 0.0000, \quad 0.1497, \quad 0.2813, \quad 0.3333, \quad 0.2813, \quad 0.1497, \quad 0.0000, \quad -0.0977\}$$

8th order filter



32nd order filter

- Reduced Transition Band Width
- No change in peak side lobe amplitude



Design of FIR Filters - Frequency Sampling

Example 2 – Odd Length FIR Filter

Use frequency sampling to design a 6th order (7 tap) low pass filter with a cut-off of 100Hz and sampling frequency 1200Hz

1. Desired frequency response:

$$\left|H(\Omega)\right| = \begin{cases} 1 & \left|\Omega\right| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < \left|\Omega\right| \le \pi \end{cases} \qquad H(\Omega) = \begin{cases} e^{-j3\Omega} & \left|\Omega\right| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < \left|\Omega\right| \le \pi \end{cases}$$

Find normalised frequency

$$\Omega = 2\pi f / f_s = 2\pi 100 / 1200 = \pi / 6$$

Point of symmetry (group delay) $M/2 = 3$

2. Sample the desired frequency response at : $\Omega = k 2\pi / L \qquad 0 \le k < L \qquad L = 7$

$$H(\Omega) \ne 0 \text{ for } \left|\Omega\right| \le \frac{\pi}{6} \Rightarrow H(\Omega) \ne 0 \text{ for } \Omega \ge 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \text{ as well}$$

Samples at 2π/7

$$H(\Omega) \ne 0 \; for \; \left|\Omega\right| < \tfrac{7\pi}{42} \; and \; \left|\Omega\right| > \tfrac{77\pi}{42}$$

$$\Omega = k\frac{2\pi}{7} = k\frac{12\pi}{42}, \; k = 0,1,2,3,...,6$$

$$H(\Omega) \ne 0 \; for \; k = 0 \; => \; H[k] = \{1\;0\;0\;0\;0\;0\;0\}$$

3. Inverse 7 point DFT to get impulse response

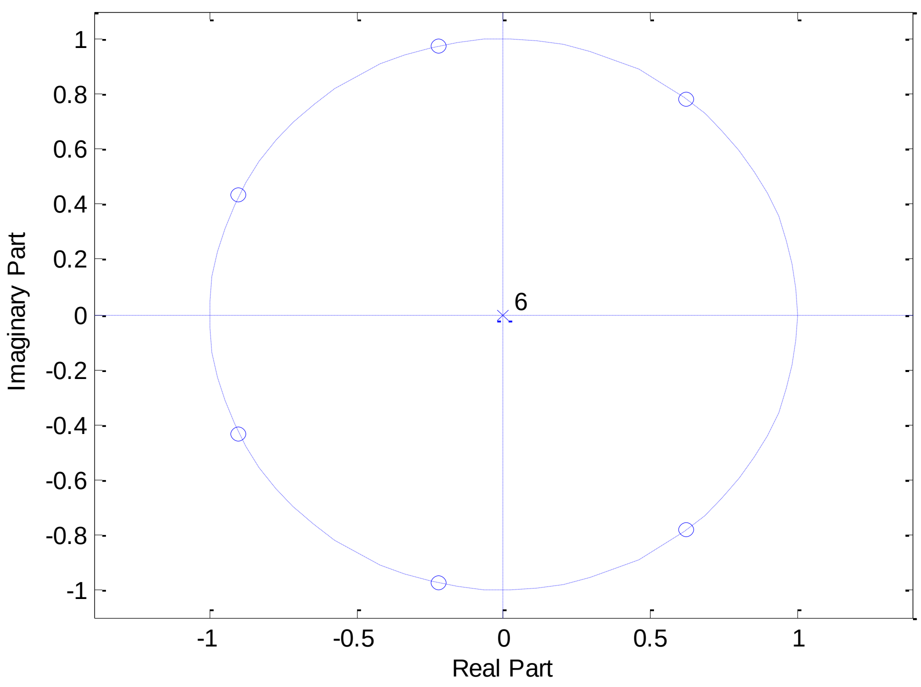
$$h[n] = \frac{1}{L}\sum_{k=0}^{L-1} H[k] e^{jkn\frac{2\pi}{L}} = \frac{1}{L} \forall \; n = 0,1,...6 \Rightarrow h[n] = 0.1429 \; \{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\}$$

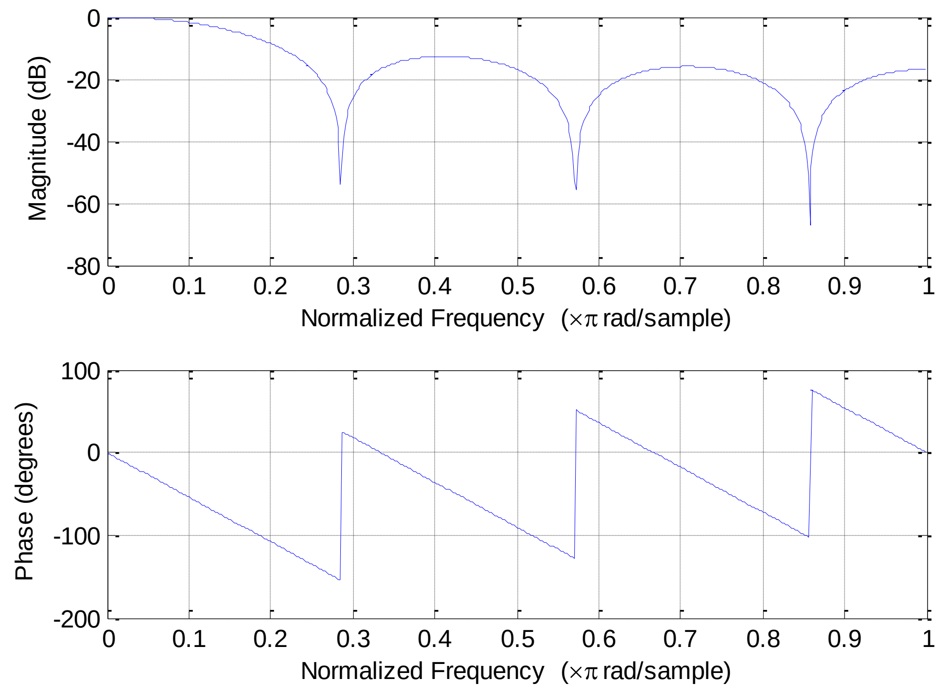
$$H(z) = 0.1429 \left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}\right)$$

Example 2 – Odd Length FIR Filter

Use frequency sampling to design a 6th order (7 tap) low pass filter with a cut-off of 100Hz and sampling frequency 1200Hz

$$H(z) = 0.1429\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}\right)$$





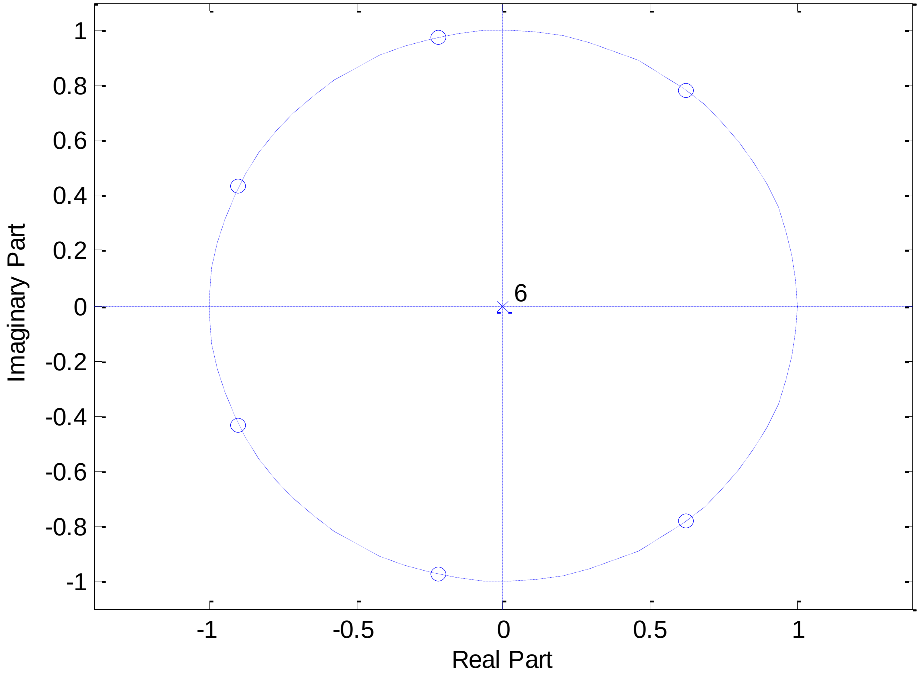
Design of FIR Filters - Frequency Sampling

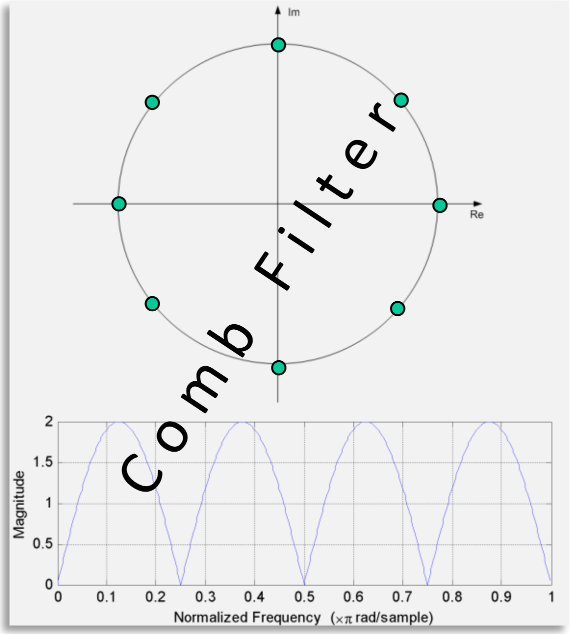
Example 2 – Odd Length FIR Filter

Use frequency sampling to design a 6th order (7 tap) low pass filter with a cut-off of 100Hz and sampling frequency 1200Hz

$$H(z) = 0.1429\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}\right)$$

The presence of multiple zeros on the unit circle means that the filter can be efficiently implemented as a cascade of an FIR comb filter and an IIR resonator





Frequency Sampling & Filter Implementation

Use frequency sampling to design a 6th order (7 tap) low pass filter with a cut-off of 100Hz and sampling frequency 1200Hz

$$H(z) = 0.1429\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}\right)$$

The presence of multiple zeros on the unit circle means that the filter can be efficiently implemented as a cascade of an FIR comb filter and an IIR resonator

$$H(z) = 0.1429\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}\right) = 0.1429 \times \boxed{\left(1 - z^{-7}\right)} \times \boxed{\frac{1}{(1 - z^{-1})}}$$

Comb filter Resonator

The zero at 1 will cancel the pole at 1 => The **filter is still FIR**

- Implementation can have stability problems why?
- This can be solved by moving both poles and zeros just inside the unit circle

$$H(z) \approx 0.1429\left(1 - r^{7} z^{-7}\right) \cdot \frac{1}{(1 - rz^{-1})} \qquad |r| < 1$$

Frequency Sampling & Filter Implementation

Efficient Implementation – General case

The presence of multiple zeros on the unit circle means that the filter can be efficiently implemented as a cascade of a FIR comb filter and a bank of IIR resonators

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{jkn\frac{2\pi}{N}} \right) z^{-n} \Leftrightarrow H(z) = \frac{1}{N} \sum_{k=0}^{N-1} H[k] \boxed{\left(\sum_{n=0}^{N-1} e^{jkn\frac{2\pi}{N}} z^{-n} \right)}$$

- The inner bracket is a geometric series and thus can be expressed as:

$$\sum_{n=0}^{N-1} \left(e^{jk\frac{2\pi}{N}} z^{-1} \right)^n = \frac{1 - \left(e^{jk\frac{2\pi}{N}} z^{-1} \right)^N}{1 - e^{jk\frac{2\pi}{N}} z^{-1}} = \frac{1 - z^{-N}}{1 - e^{jk\frac{2\pi}{N}} z^{-1}}$$

$$\boxed{H(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{jk\frac{2\pi}{N}} z^{-1}}}$$

Example 3 – Even Length FIR Filter

Use frequency sampling to design a 7th order (8 tap) band pass filter with a pass band between 100Hz-300Hz and sampling frequency 1200Hz.

1. Desired frequency response:

$$H(\Omega) = \begin{cases} e^{-j3.5\Omega} & \frac{\pi}{6} \le |\Omega| < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Find normalised frequencies

$$\Omega = 2\pi f / f_s = 2\pi 100 / 1200 = \pi / 6$$

$$\Omega = 2\pi f / f_s = 2\pi 300 / 1200 = \pi / 2$$

Point of symmetry (group delay) $N/2 = 3.5$

2. Sample the desired frequency response at : $\Omega = k2\pi / L \qquad 0 \le k < L \qquad L = 8$

$$H(\Omega) \ne 0 \text{ for } \frac{\pi}{6} \le |\Omega| < \frac{\pi}{2} \Rightarrow H(\Omega) \ne 0 \text{ for } 2\pi - \frac{\pi}{2} < |\Omega| \le 2\pi - \frac{\pi}{6} \Leftrightarrow \frac{3\pi}{2} < |\Omega| \le \frac{11\pi}{6} \text{ as well}$$

$$\left.\begin{array}{l} H(\Omega) \ne 0 \; for \; \frac{4\pi}{24} \le |\Omega| < \frac{12\pi}{24} \; and \; \frac{36\pi}{24} < |\Omega| \le \frac{44\pi}{24} \\ \Omega = k\frac{2\pi}{8} = k\frac{6\pi}{24}, \; k = 0,1,2,3,\ldots,7 \end{array}\right\} H(\Omega) \ne 0 \; for \;\; k = 1,7$$

$$H[k] = \left\{ 0 \quad e^{-j\frac{7}{2}\frac{\pi}{4}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad e^{-j\frac{7}{2}\frac{7\pi}{4}} \right\} = \left\{ 0 \quad e^{-j\frac{7\pi}{8}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad e^{-j\frac{49\pi}{8}} \right\} = \left\{ 0 \quad e^{-j\frac{7\pi}{8}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad e^{-j\frac{\pi}{8}} \right\}$$

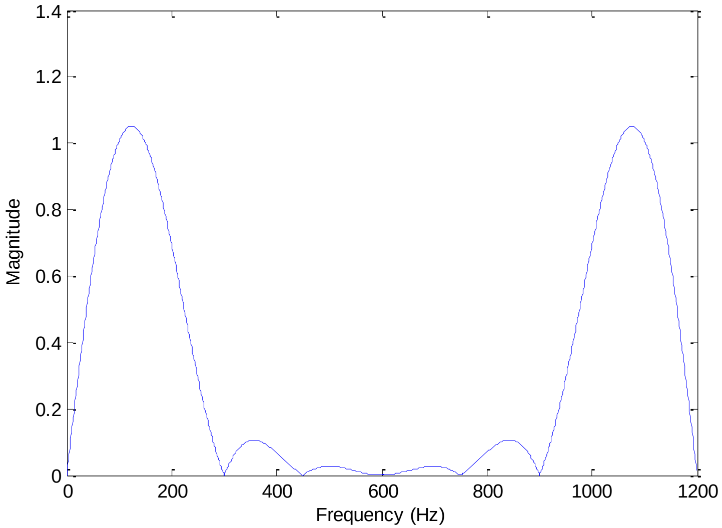
Example 3 – Even Length FIR Filter

Use frequency sampling to design a 7th order (8 tap) band pass filter with a pass band between 100Hz-300Hz and sampling frequency 1200Hz.

3. Inverse 8 point DFT to get impulse response $H(\Omega) \neq 0 \; for \;\; k = 1,7$

$$h[n] = \frac{1}{L}\sum_{k=0}^{L-1} H[k] e^{jkn\frac{2\pi}{L}} \qquad = \frac{1}{8}\left\{ e^{j(n\frac{\pi}{4} - 7\frac{\pi}{8})} + e^{j(7n\frac{\pi}{4} - \frac{\pi}{8})} \right\} = \frac{1}{8}\left\{ e^{j(2n-7)\frac{\pi}{8}} + e^{j(14n-1)\frac{\pi}{8}} \right\}$$

$\Rightarrow \; h[n] = j[$- 0.0957, - 0.2310, - 0.2310, - 0.0957, 0.0957, 0.2310, 0.2310, 0.0957 $]$



$$H[k] = \left\{ 0 \quad e^{-j\frac{7}{2}\frac{7\pi}{4}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad e^{-j\frac{7}{2}\frac{7\pi}{4}} \right\} = \left\{ 0 \quad e^{-j\frac{7\pi}{8}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad e^{-j\frac{49\pi}{8}} \right\} = \left\{ 0 \quad e^{-j\frac{7\pi}{8}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad e^{-j\frac{\pi}{8}} \right\}$$

Filter Design in Matlab

Frequency sampling filter design functions

- `B = fir2(N,F,A)`

  Designs an `Nth` order linear phase FIR digital filter with the frequency response specified by vectors `F` and `A` and returns the filter coefficients in length N+1 vector `B`. The vectors `F` and `A` specify the frequency and magnitude breakpoints for the desired frequency response. The frequencies in `F` must be given in increasing order with 0.0 < `F` < 1.0 and 1.0 corresponding to half the sample rate. The first and last elements of `F` must equal 0 and 1 respectively.