Lecture 5

Discrete Time Fourier Transform

Spectral Analysis of Discrete Time Signals

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Discrete Time Fourier Transform

Effect of Sampling on Spectrum

Use of the continuous time FT to find the spectrum of discrete time signals

$$x_s(t) = x_c(t) \times s(t) \Rightarrow X_s(\omega) = \frac{1}{2\pi} X_c(\omega) * S(\omega) \Rightarrow X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(\omega - k\omega_s)$$

1. Sampled signal = Continuous time signal x Impulse train
2. Fourier Transform : Multiplication in time ⇔ Convolution in frequency
3. FT of Impulse train = train of impulses ω_s apart; Convolution with Impulse train=> Periodic spectrum with period ω_s

Can we calculate the spectrum of a Discrete Time signal *x[n]* directly ?

Discrete Time Fourier Transform

From the CTFT to the DTFT

Continuous Time Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Take the CT sampled signal

$$x_s(t) = \begin{cases} x[n] & t = nT_s \\ 0 & t \neq nT_s \end{cases}$$

Use Normalised Frequency

$$F = f / f_s \Leftrightarrow \Omega = \omega T_s$$

Apply the Fourier Transform

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j\omega\, t}\, dt$$

$$\sum_{n=-\infty}^{\infty} x[n]\;\; e^{-j\omega\, nT_s}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\;\; e^{-j\Omega\, n}$$

Periodic spectrum (period = ω_s)

$$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c(\omega - k\omega_s)$$

Periodic spectrum (period = 2π)

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k2\pi)$$

Discrete Time Fourier Transform

$$X(F) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi Fn}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$e^{j\Omega n}$ periodic with 2π

$$x[n] = \int_{1} X(F) e^{j2\pi Fn} dF$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Discrete Time Fourier Transform

Discrete-Time Signal Periodicity

Discrete time complex exponentials & sinusoids

$$x[n] = e^{j(\Omega_0 + 2\pi r)n} = e^{j\Omega_0 n} \underbrace{e^{j2\pi r n}}_{1} = e^{j\Omega_0 n}$$

$$x[n] = \cos[(\Omega_0 + 2\pi r)n + \phi] = \cos(\Omega_0 n + \varphi)$$

$$-\pi < \Omega < \pi \Leftrightarrow -1/2 < F < 1/2$$

$$0 < \Omega < 2\pi \Leftrightarrow 0 < F < 1$$

1. DT complex exponentials (or sinusoids) with frequencies *($\Omega_0 + 2\pi r$)* are indistinguishable

CT periodicity $x(t) = x(t+T)$	$\cos(\omega_0 (t+T)) = \cos(\omega_0 t + 2\pi) = \cos(\omega_0 t)$
DT periodicity $x[n] = x[n+N]$ N integer	$\cos(\Omega_0 n) = \cos[\Omega_0 (n+N)] = \cos(\Omega_0 n + \Omega_0 N)$ $\Omega_0 N = 2\pi k \Rightarrow \Omega_0 = 2\pi \frac{k}{N}$ same applies to $e^{j\Omega_0 n} = e^{j\Omega_0 (n+N)}$

2. Complex exponentials and sinusoidal sequences are not necessarily periodic in *n*
   This depends on the value of Ω_0 i.e. it depends on ω_0 and the sampling period T_s ($\Omega_0 = \omega_0 T_s$)
3. Complex exponentials and sinusoidal sequences that are periodic in *n* don't necessarily have the same period $T = 2\pi/\omega_0$ as their continuous time counterparts

Discrete-Time Signal Periodicity

Discrete time complex exponentials & sinusoids

Frequency Component Periodicity

$$e^{j(\Omega_0+2\pi r)n} = e^{j\Omega_0 n}$$

Frequencies repeat every *2π* ⇔ Frequencies are unique only in the range *–π* to *π*

Time Domain Signal Periodicity

$$e^{j\Omega_0 n} = e^{j\Omega_0 (n+N)} \quad \text{only if} \quad \Omega_0 = 2\pi \frac{k}{N}$$

Not all DT sinusoids are periodic in *n* (equivalent of time t)

If they are they don't necessarily have the same period with their CT counterpart

Discrete Time Fourier Transform

Discrete-Time Signal Periodicity: Examples

Frequency component periodicity

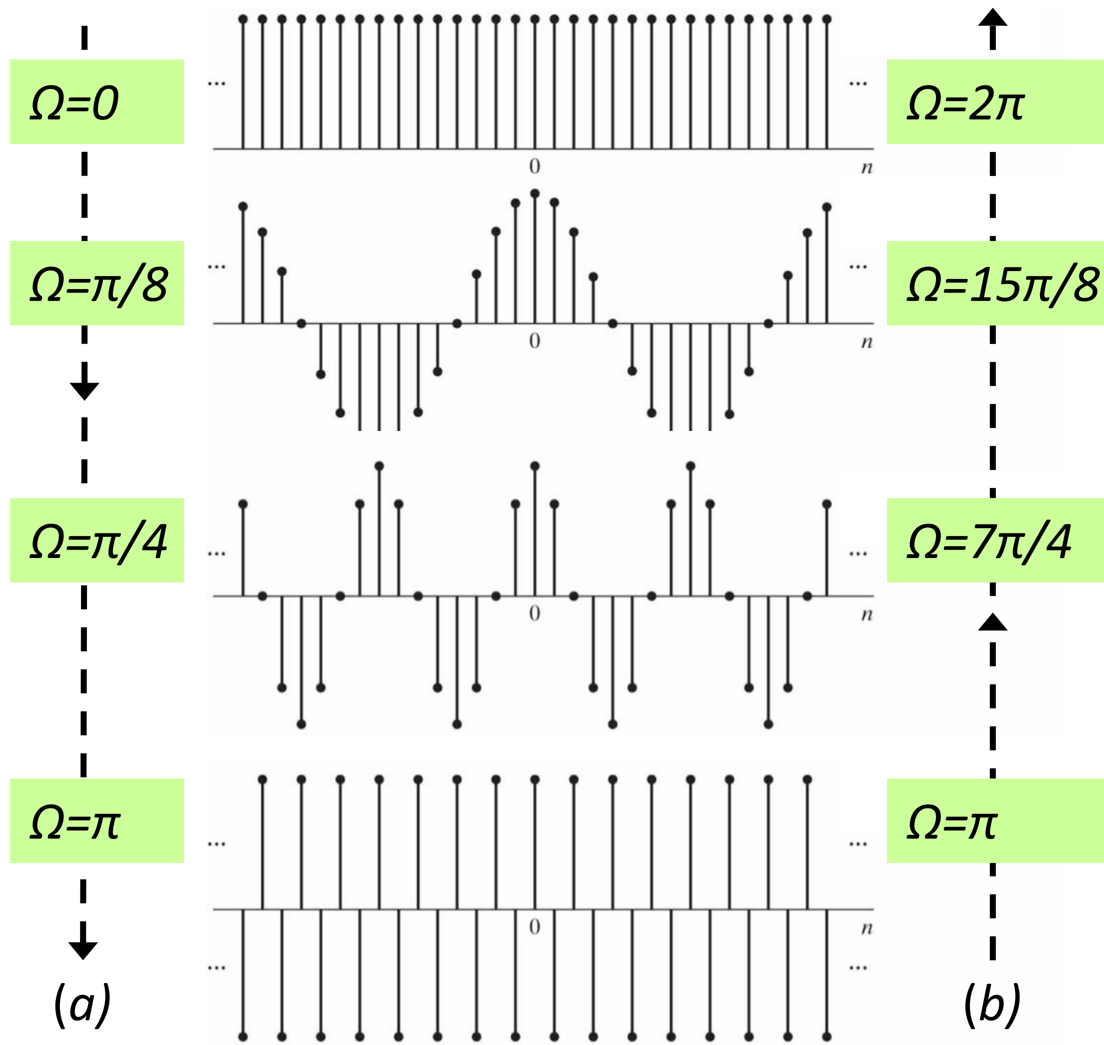
$$x[n] = \cos(\Omega n)$$

a) As the frequency increases from 0 to π, *x[n] oscillates* more rapidly

b) As the frequency increases from π to 2π, *x[n] oscillates* more slowly

- Values of Ω close to $\boldsymbol{2\pi k}$ are **low frequencies.**

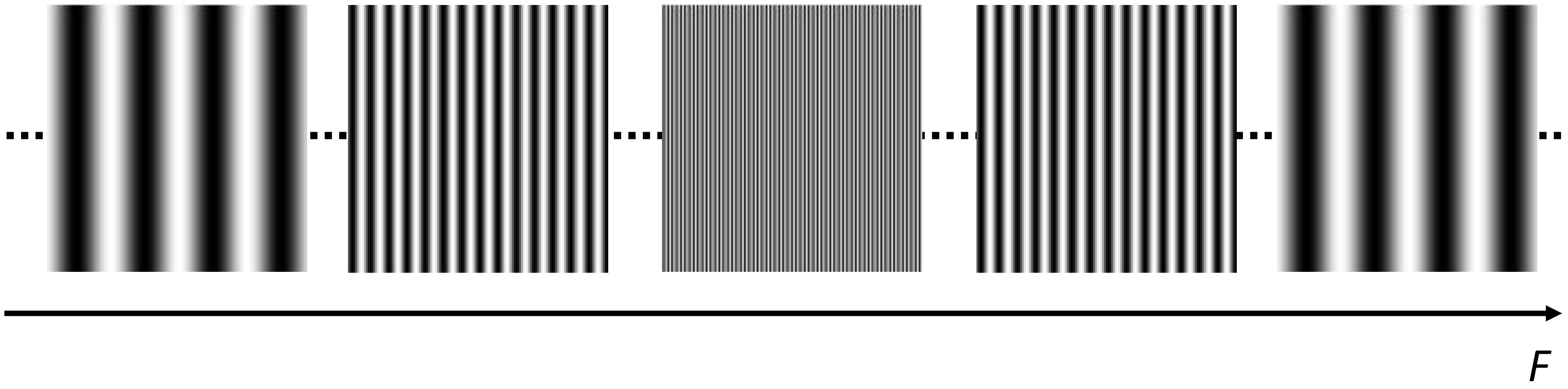
- Values of Ω close to $\boldsymbol{\pi k}$ are **high frequencies**



Discrete-Time Signal Periodicity: Examples

Frequency component periodicity

Sinusoidal image with the horizontal frequency increasing from 0 to just under 1 cycle/pixel



The perceived frequency begins to drop after the halfway point
F = 0.5 cycles/pixel ($\Omega=\pi$ radians/pixel)

Discrete-Time Signal Periodicity: Examples

Time domain periodicity

1. What is the period of the discrete-time cosine $x_1[n] = \cos(\pi n / 4)$

$$\cos(\Omega n) = \cos(\frac{\pi}{4} n) \Rightarrow \Omega = \frac{\pi}{4}$$

$$\text{periodicity} \Rightarrow \Omega N = 2\pi k, \ N \text{ integer} \Rightarrow N = 2\pi k / \Omega$$

$$N = 8$$

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2. What happens to the period *N* of the above cosine if we increase the frequency *Ω* from π/4 to 3π/8 , i.e. what is the period of $x_2[n] = \cos(3\pi n / 8)$

s

$$\Omega = \frac{3\pi}{8} > \frac{\pi}{4} = \frac{2\pi}{8},$$

$$N = \frac{2\pi}{\Omega} k$$

$$N = \frac{16}{3} k$$

$$N \text{ integer}$$

$$k = 3 \Rightarrow N = 16$$

Increasing the frequency *Ω* increased the period *N*

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3. Is this discrete-time cosine periodic: $x_1[n] = \cos(n)$

$$\cos(\Omega n) = \cos(n) \Rightarrow \Omega n = n \Rightarrow \Omega = 1$$

$$\cos(\Omega n) \text{ is periodic only if } \ \Omega = 2\pi \frac{k}{N}$$

$$\text{with } N \text{ integer}$$

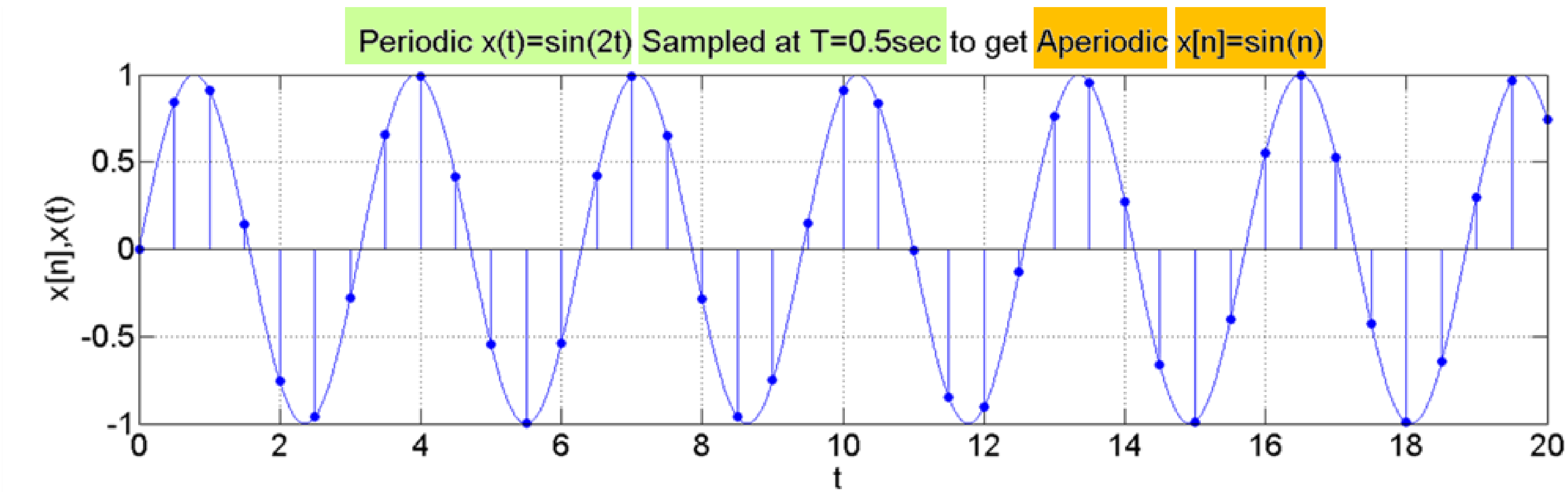
$$N = 2\pi k$$

$$\pi = 3.14$$

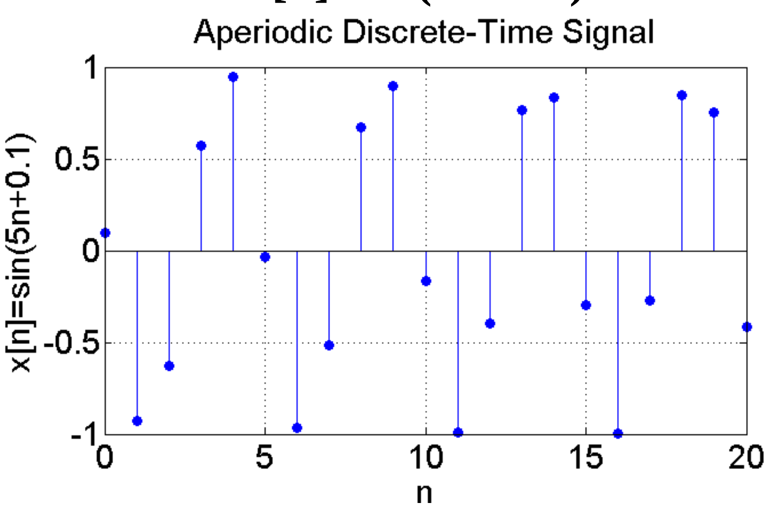
There is no value of *k* resulting in an integer value for *N* => *there* is no integer *N* for which : *cos(n) = cos(n+N)* $\forall$ *n*

Discrete-Time Signal Periodicity: Examples

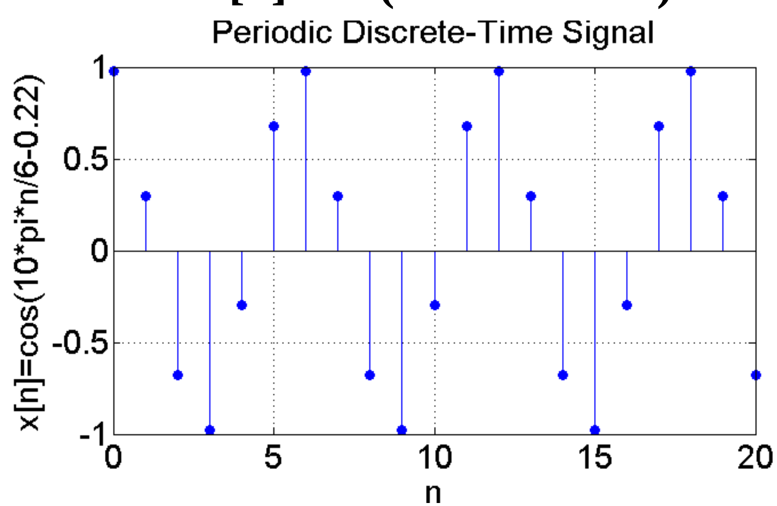
Time domain periodicity



$x[n]=\sin(5n+0.1)$



$x[n]=\cos(10\pi n/6 - 0.22)$



Discrete-Time Signal Periodicity: Examples

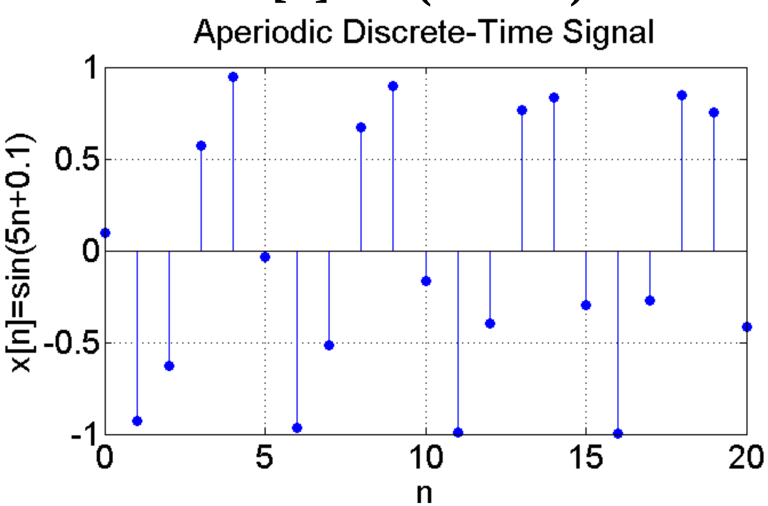
Time domain periodicity

$$\left.\begin{aligned}&\Omega N = 2\pi k,\ N \text{ integer} \Rightarrow N = 2\pi k/\Omega\\ &\sin(\Omega n+\phi)=\sin(5n+0.1)\Rightarrow \Omega = 5\end{aligned}\right\}$$

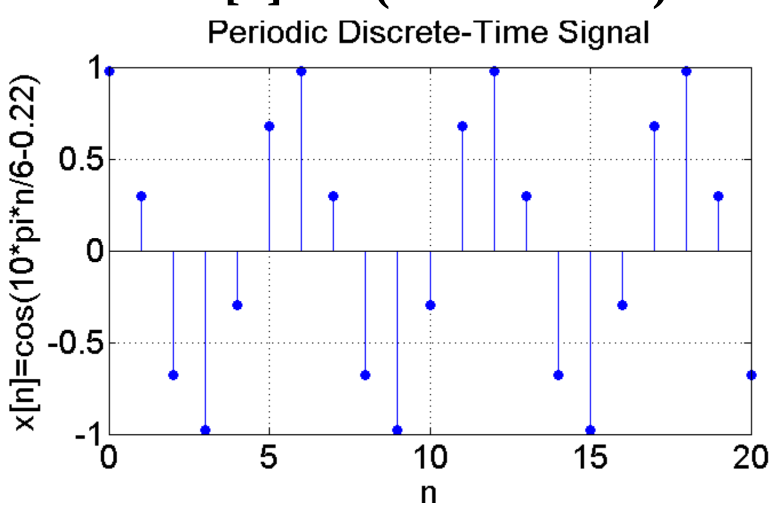
There is no integer *N* for which *sin(5n+0.1) = sin(5(n+N)+0.1*) for every *n*

$$\cos(\Omega n+\phi)=\cos(\frac{10\pi}{6}n-0.22)\Rightarrow \Omega=\frac{10\pi}{6},\ N=2\pi k/\Omega, N \,\text{integer} \Rightarrow N=6\ (\text{for}\, k=5)$$

x[n]=sin(5n+0.1)



x[n]=cos(10πn/6 - 0.22)



Discrete Time Fourier Transform

Properties & Theorems

	Time	Frequency
Complex exponential	1 $e^{j\Omega_0 n}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$ $2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - k2\pi)$ Note 1 impulse in range $0 \rightarrow 2\pi$
Cosine	$\cos(\Omega_0 n) = \dfrac{e^{j\Omega_0 n} + e^{-j\Omega_0 n}}{2}$	$\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - k2\pi)$ $+\pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - k2\pi)$
Sine	$\sin(\Omega_0 n) = \dfrac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j}$	$j\pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - k2\pi)$ $-j\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - k2\pi)$
Impulse train	$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$\dfrac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\frac{2\pi}{N})$

Discrete Time Fourier Transform

Relationship to the Z transform

$$z = re^{j\Omega}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\Omega n}$$

$$e^{j\Omega} \leftrightarrow z$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n].z^{-n}$$

Discrete Time Fourier Transform

Z transform evaluated on the unit circle

Bilateral Z Transform (non-causal system)

Discrete Time Fourier Transform

Periodicity and Discrete Nature

Discrete representation in time/frequency domain ⇔ Periodicity in frequency/time domain

	Analysis	Synthesis	Time	Frequency
FS	$c_k = \frac{1}{T}\int_T x(t).e^{-jkt\frac{2\pi}{T}}.dt$	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkt\frac{2\pi}{T}}$	Continuous Periodic	Non-periodic Discrete
FT	$X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}.dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	Continuous Non-periodic	Non-periodic Continuous
DTFT	$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\Omega n}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X[\Omega]e^{j\Omega n}d\Omega$	Discrete Non-periodic	Periodic Continuous

Discrete Time Fourier Transform

Periodicity and Discrete Nature

Discrete representation in time/frequency domain ⇔ Periodicity in frequency/time domain

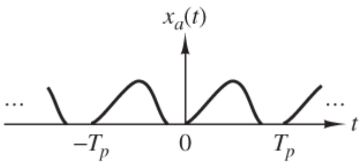
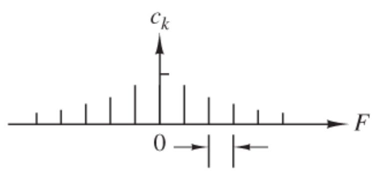
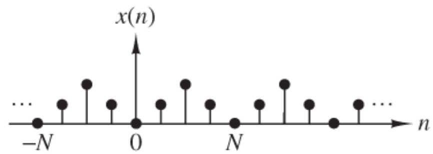
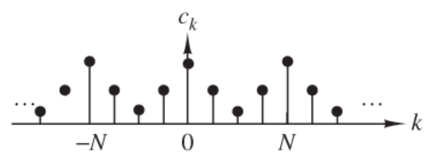
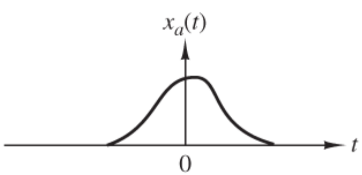
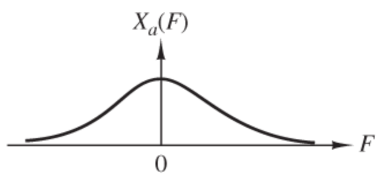
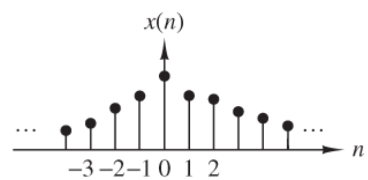
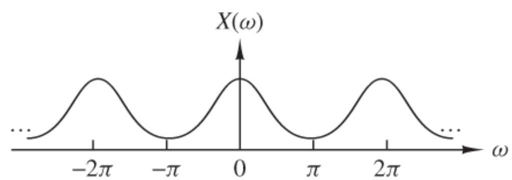
		Continuous-time signals		Discrete-time signals	
		Time-domain	Frequency-domain	Time-domain	Frequency-domain
Periodic signals	Fourier series	 $c_k = \frac{1}{T_p}\int_{T_p} x_a(t)e^{-j2\pi kF_0 t}\,dt \Longrightarrow$ $\Longleftarrow$	 $F_0 = \frac{1}{T_p}$ $x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$	 $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn} \Longrightarrow$ $\Longleftarrow$	 $x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$
		Continuous and periodic	Discrete and aperiodic	Discrete and periodic	Discrete and periodic
Aperiodic signals	Fourier transforms	 $X_a(F) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi Ft}\,dt \Longrightarrow$ $\Longleftarrow$	 $x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}\,dF$	 $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \Longrightarrow$ $\Longleftarrow$	 $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$
		Continuous and aperiodic	Continuous and aperiodic	Discrete and aperiodic	Continuous and periodic

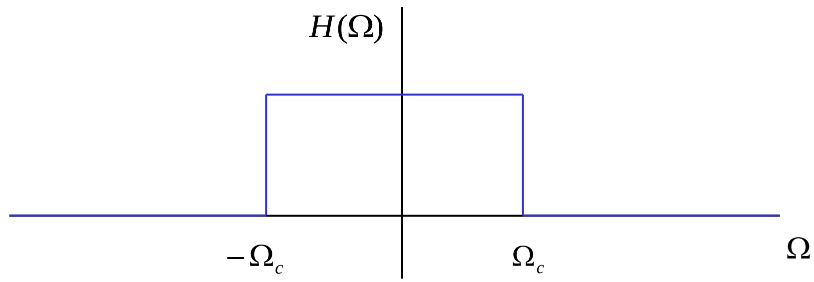
Table from : J. Proakis and D. Manolakis, 'Digital Signal Processing: Principles, Algorithms and Applications', Macmillan

NOTE : opposite notation to lectures: Capital letters used to denote analogue frequency and small letters for normalised (digital) frequency

Discrete Time Fourier Transform

Impulse response of an ideal low pass filter

$$H(\Omega)=\begin{cases}1 & |\Omega|\le\Omega_c\\ 0 & \Omega_c<|\Omega|\le\pi\end{cases}$$

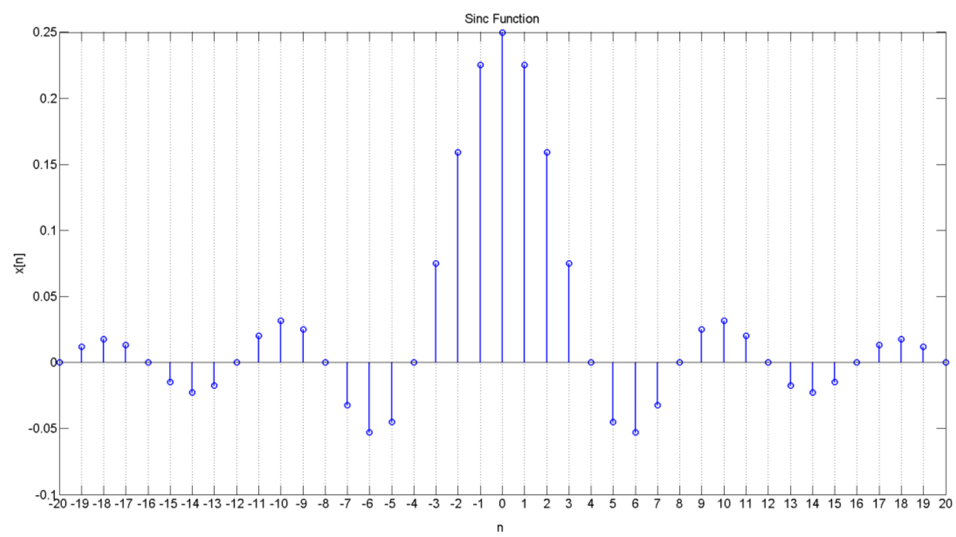


$$h[n]=\frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c}e^{j\Omega n}d\Omega \;=\frac{1}{2\pi}\left[\frac{e^{j\Omega n}}{jn}\right]_{-\Omega_c}^{\Omega_c}$$

$$=\frac{1}{2\pi}\frac{e^{j\Omega_c n}-e^{-j\Omega_c n}}{jn} \;=\frac{1}{n\pi}\frac{e^{j\Omega_c n}-e^{-j\Omega_c n}}{2j}$$

$$=\frac{1}{n\pi}\sin(\Omega_c n)\Rightarrow \boxed{h[n]=\frac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)}$$

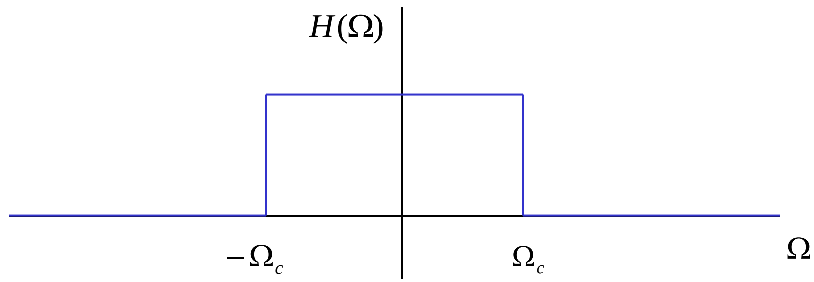
$$h[n]=\frac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n),\;\; \text{for } \Omega_c=\pi/4$$



Discrete Time Fourier Transform

Impulse response of an ideal low pass filter

$$H(\Omega)=\begin{cases}1 & |\Omega|\le\Omega_c\\ 0 & \Omega_c<|\Omega|\le\pi\end{cases}$$



$$h[n]=\frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\,d\Omega \;=\frac{1}{2\pi}\left[\frac{e^{j\Omega n}}{jn}\right]_{-\Omega_c}^{\Omega_c}$$

For any Ω_c

$$=\frac{1}{2\pi}\,\frac{e^{j\Omega_c n}-e^{-j\Omega_c n}}{jn} \qquad =\frac{1}{n\pi}\,\frac{e^{j\Omega_c n}-e^{-j\Omega_c n}}{2j}$$

$$=\frac{1}{n\pi}\sin(\Omega_c n)\Rightarrow \boxed{h[n]=\frac{\Omega_c}{\pi}\,\mathrm{sinc}(\Omega_c n)}$$