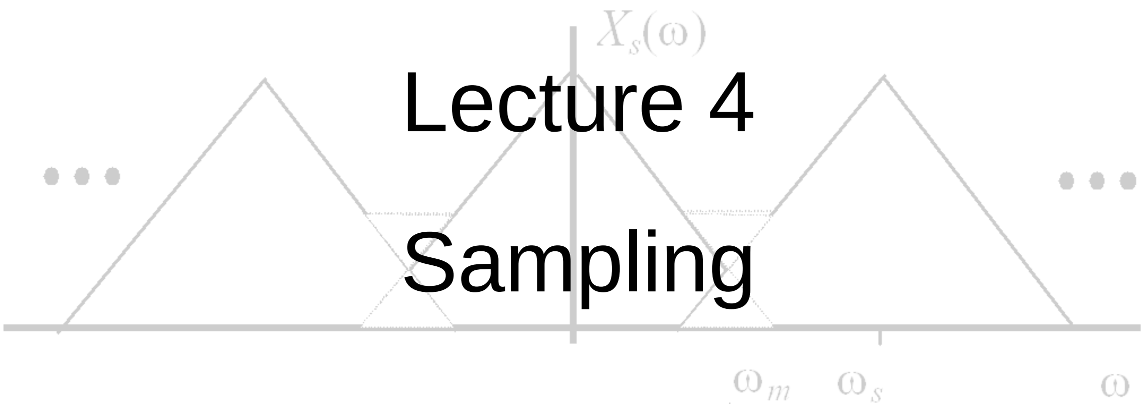


Lecture 4

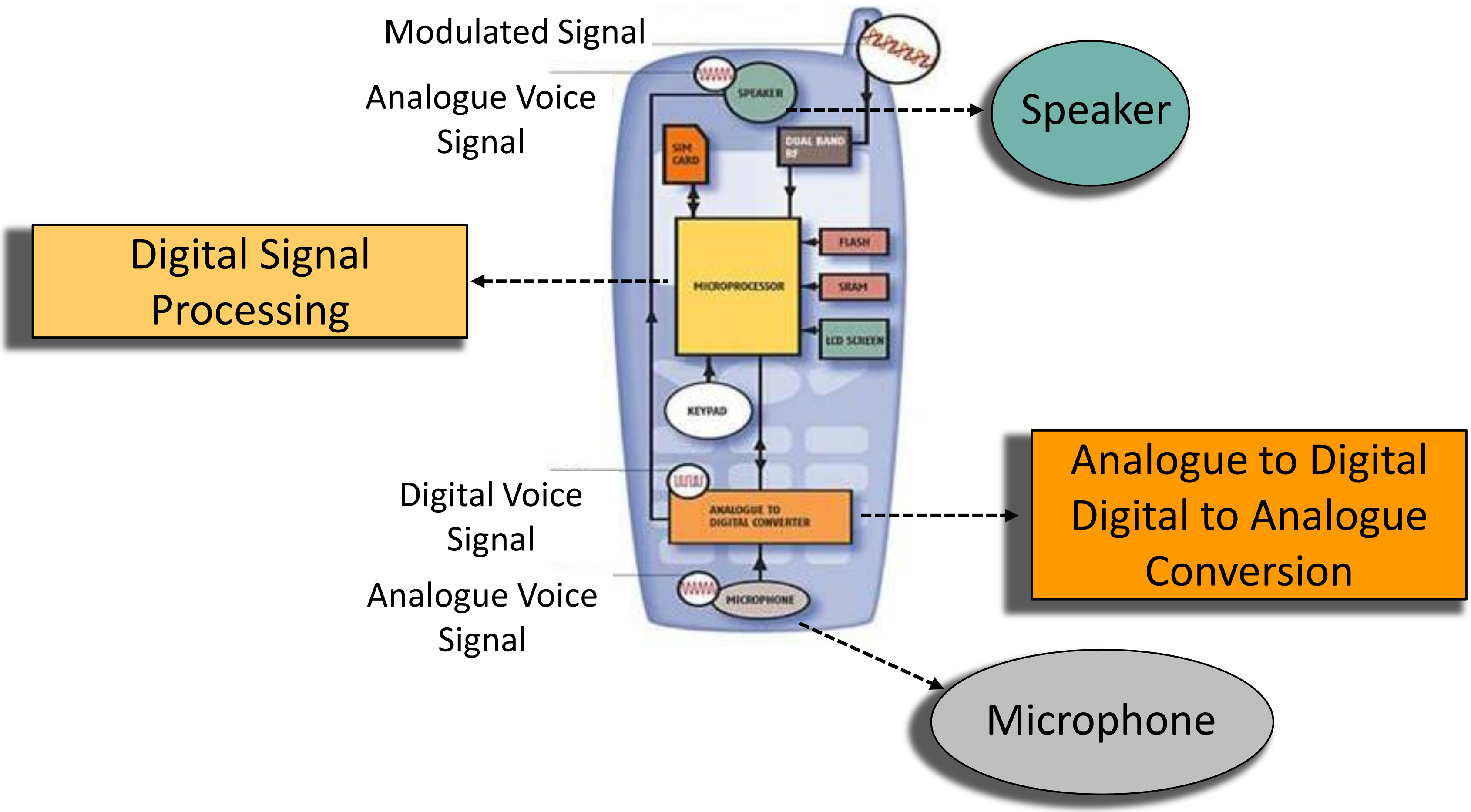
Sampling

Converting continuous-time signals to discrete-time signals

Dr. Dimitris Agrafiotis, Dept. Of Electrical & Electronic Engineering

Digital Signal Processing

1D Sampling Example (Audio)

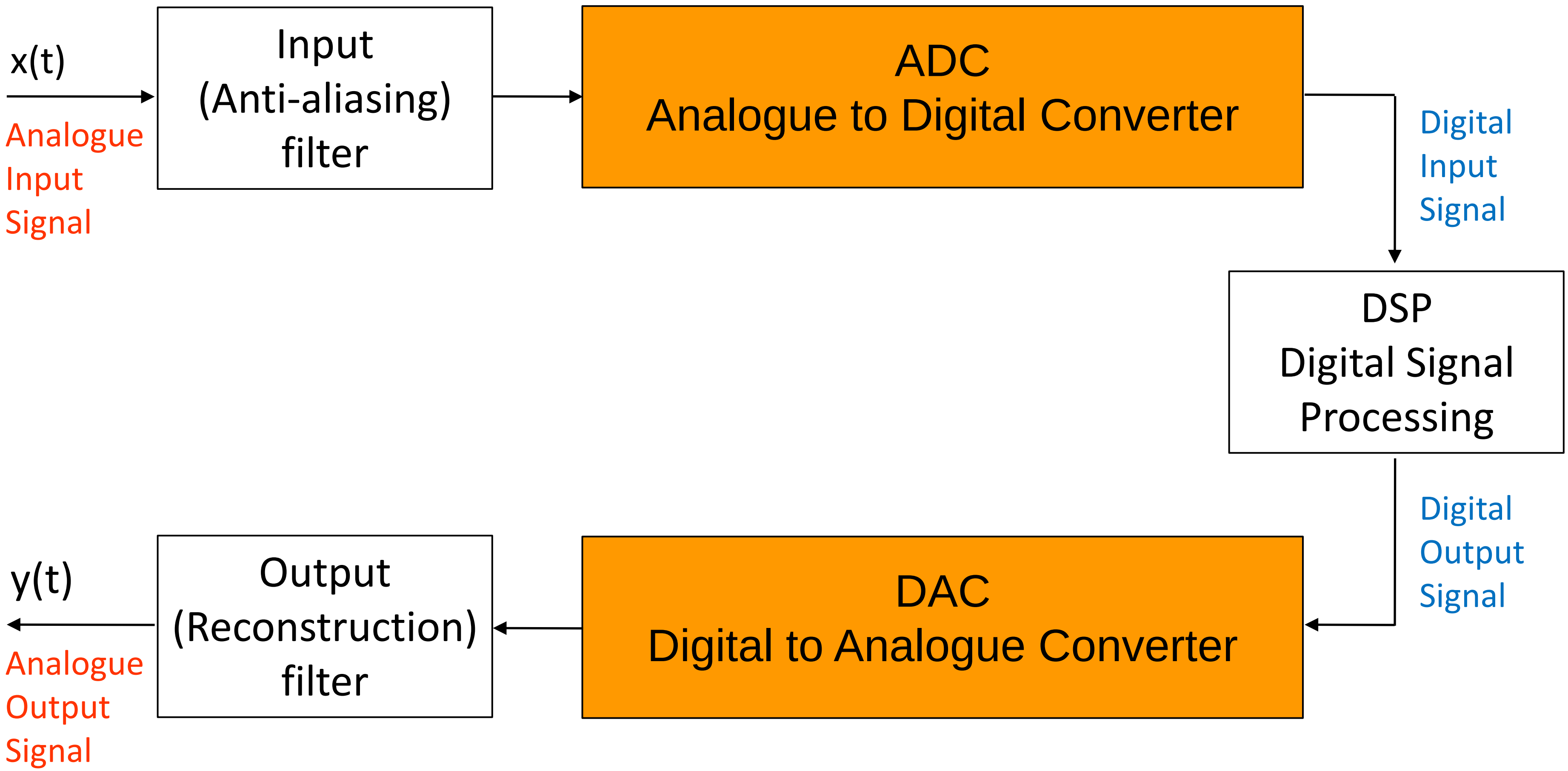


Digital Signal Processing

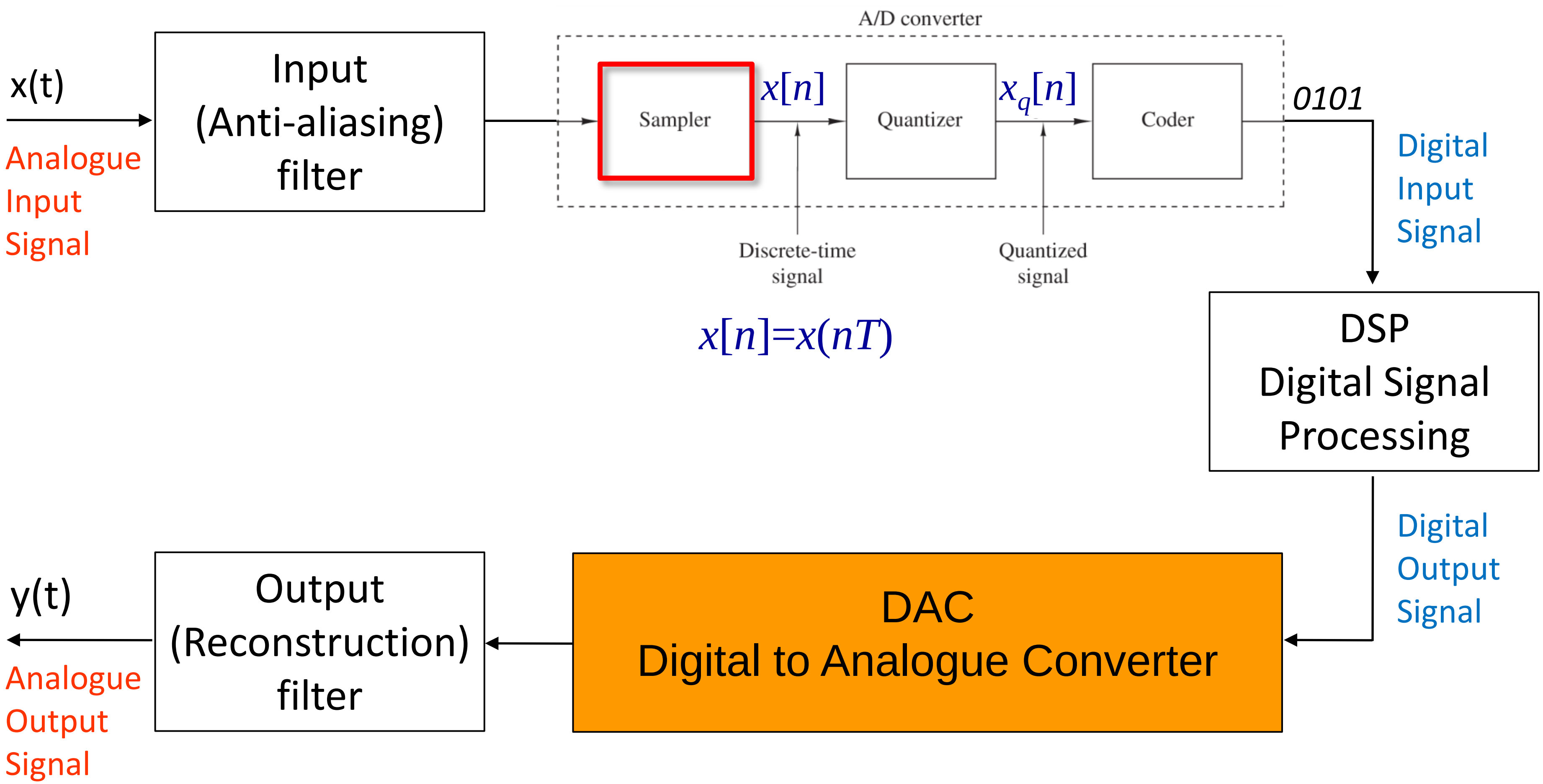
2D Sampling Example (Image)

Sampling

Digital Signal Processing

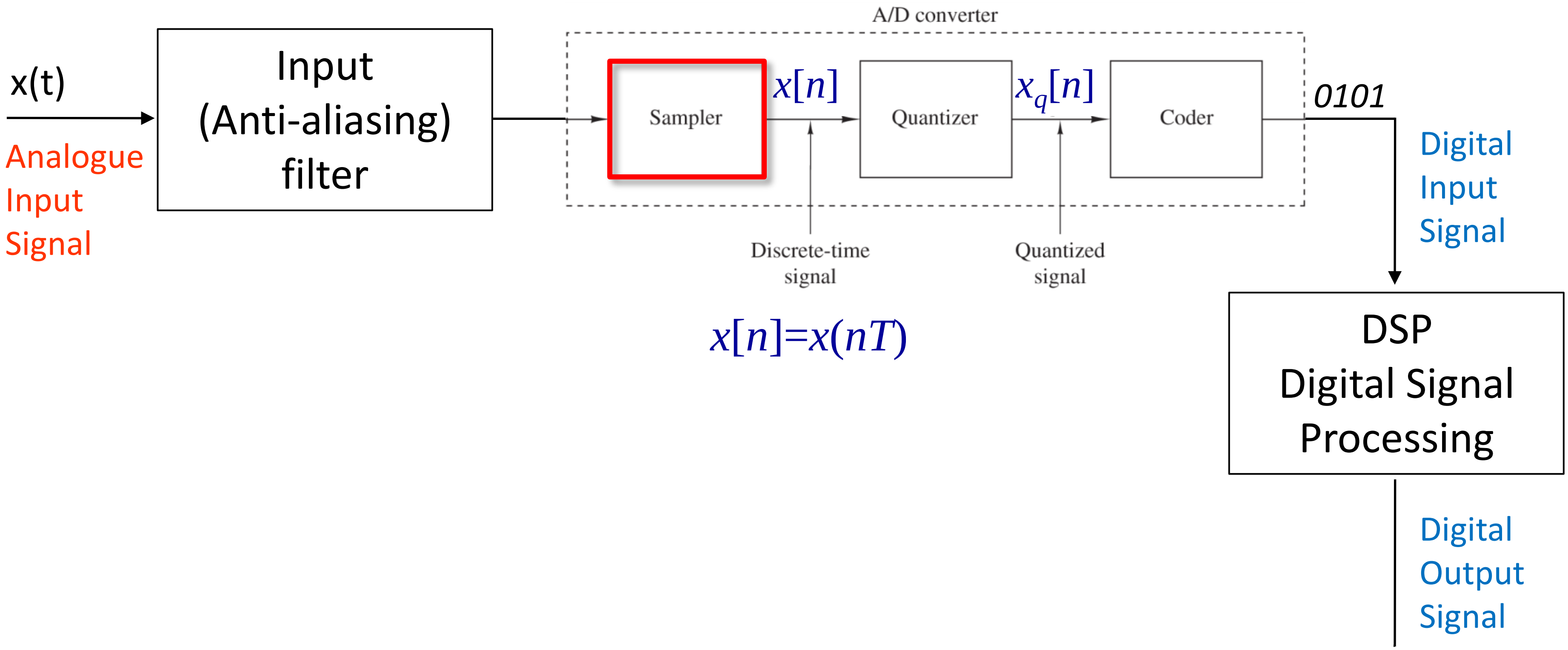


Digital Signal Processing



$$x[n]=x(nT)$$

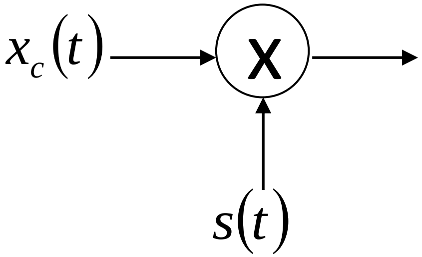
Digital Signal Processing



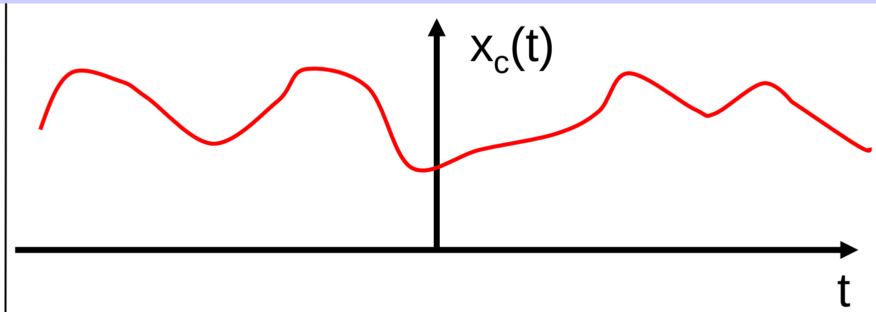
Sampling

Periodic Sampling

Mathematical Representation of Sampling

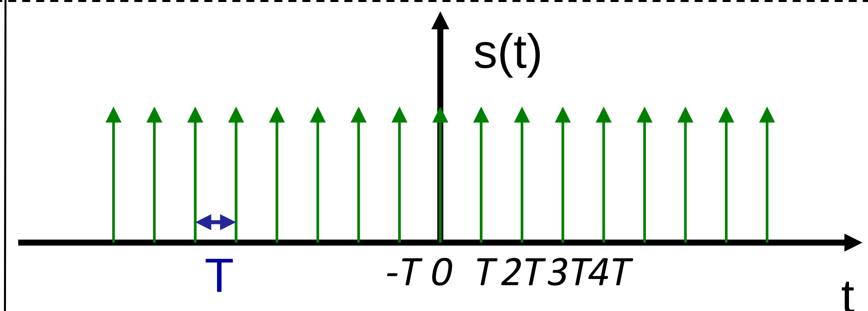


$$x_s(t) = x_c(t) \times s(t)$$



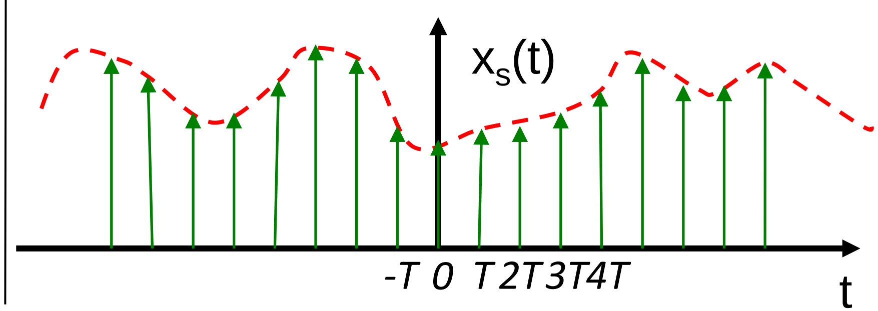
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)\ ,\ f_s = \frac{1}{T}$$

Sampling Frequency



$$x_s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(t) \delta(t - nT)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT)$$



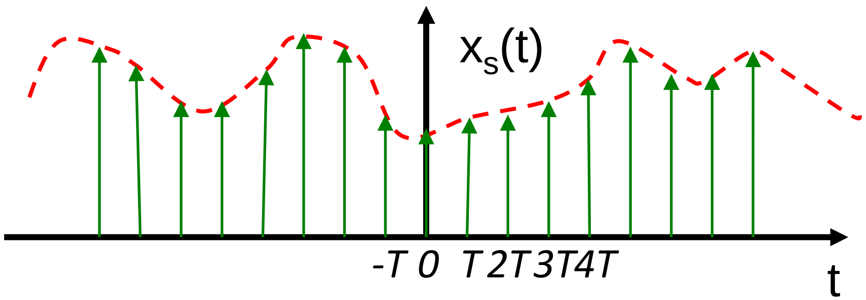
Sampling

Periodic Sampling

Mathematical Representation of Sampling

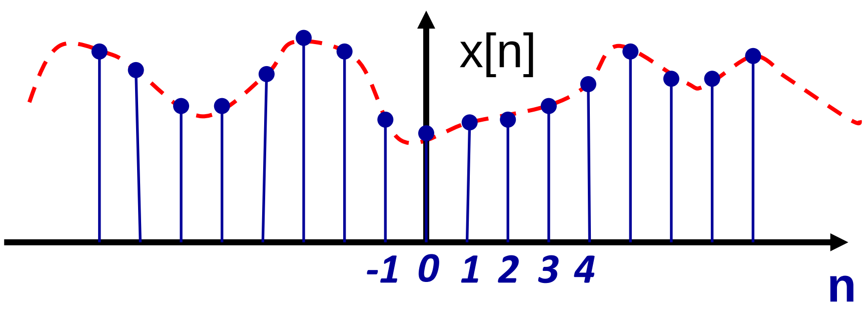
$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

Continuous time signal



$$x[n] = x_c(nT)$$

Discrete time signal



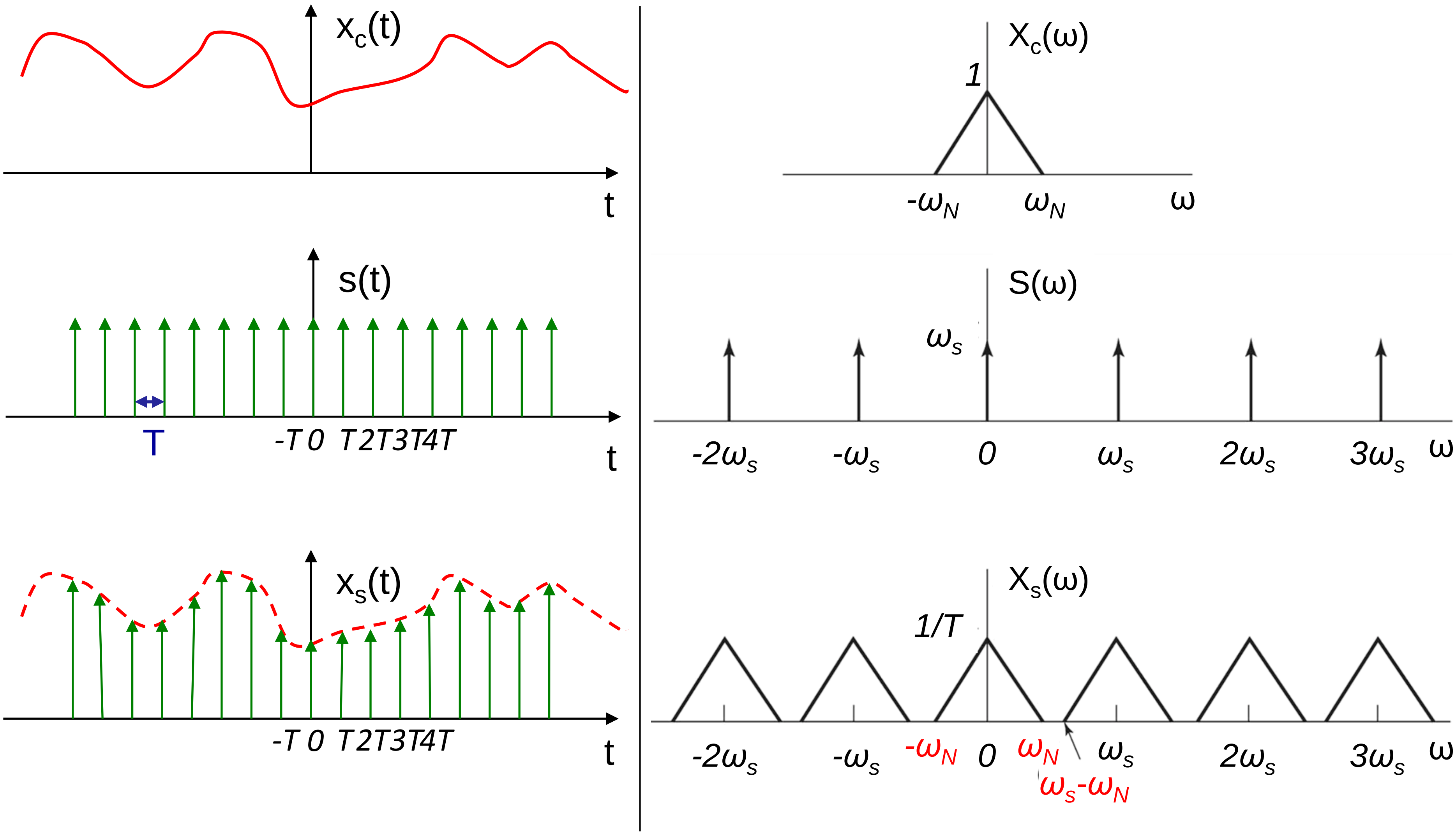
- Time is normalised (*n* contains no explicit information about the sampling period *T*)
- Normalisation by a factor of T ([n] = (nT))

What happens to the spectrum of the sampled signal?

$x_s(t) = x_c(t) \times s(t) \quad \Rightarrow X_s(\omega) = \frac{1}{2\pi} X_c(\omega) * S(\omega)$	FT Frequency Convolution: Multiplication in time ⇔ convolution in frequency
$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \overset{F}{\leftrightarrow} S(\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$	FT of an Impulse train
$X_s(\omega) = \frac{1}{2\pi} X_c(\omega) * S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X_c(\omega - k\omega_s)$	Convolution of a signal with a sifted impulse => shifting of the signal at the location of the impulse
$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(\omega - k\omega_s)$	$\omega_s = \frac{2\pi}{T_s}$
$X_s(\omega) = \frac{1}{T_s} \left\{ \begin{array}{l} \ldots + X(\omega + 2\omega_s) + X(\omega + \omega_s) + X(\omega) \ldots \\ + X(\omega - \omega_s) + X(\omega - 2\omega_s) + \ldots \end{array} \right\}$	Copies of original spectrum at integer multiples of the sampling frequency
$X_s(\omega + m\omega_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega + m\omega_s - k\omega_s)$ $= \frac{1}{T_s} \sum_{k-m=-\infty}^{\infty} X(\omega - (k-m)\omega_s) = X_s(\omega)$	Spectrum of sampled signal is periodic

Frequency Domain Representation of Sampling

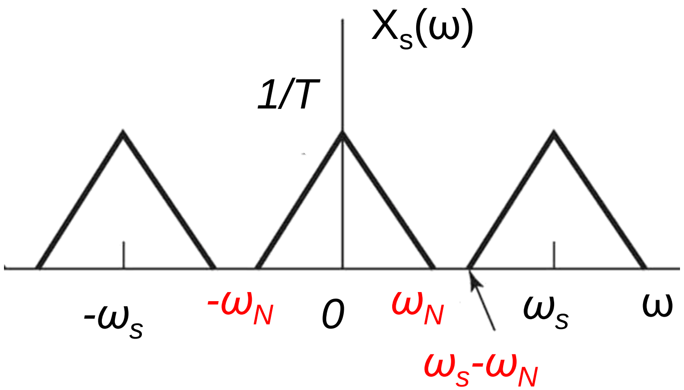
What happens to the spectrum of the sampled signal ?

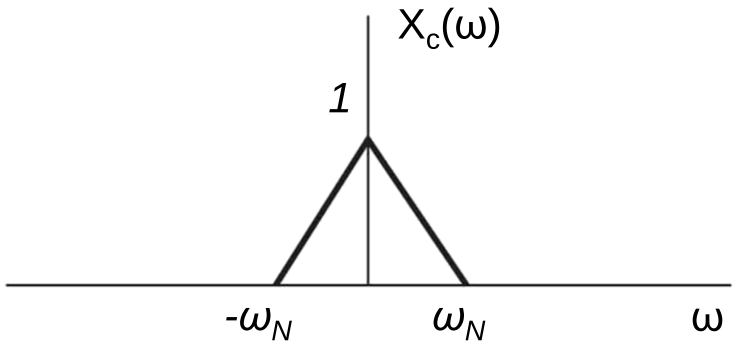


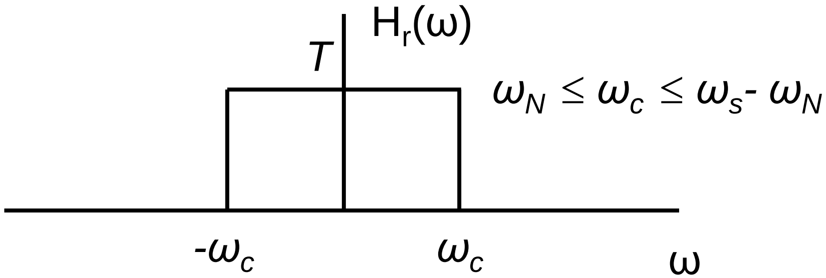
Sampling

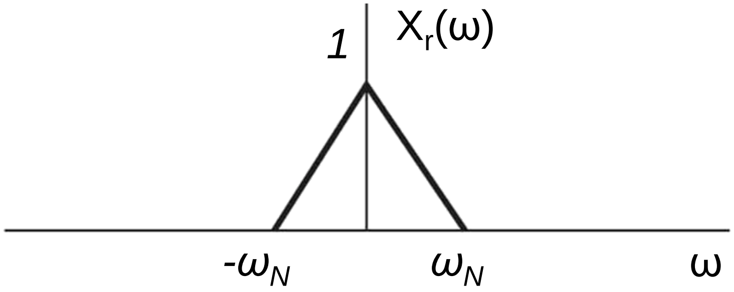
Frequency Domain Representation of Sampling

Can we recover the original signal (spectrum) ?

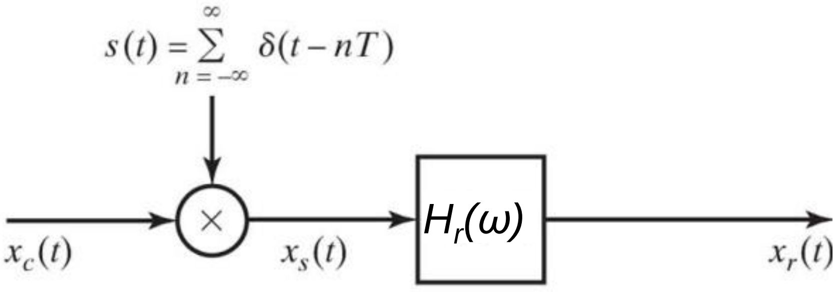








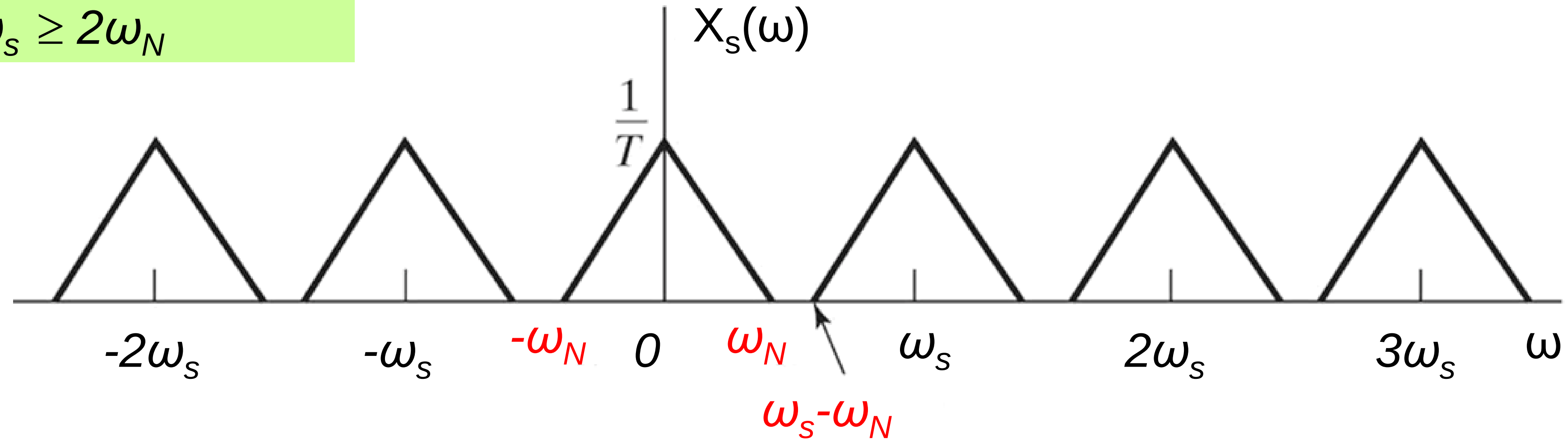
Filter the sampled signal with an ideal low pass filter to recover the original



Frequency Domain Representation of Sampling

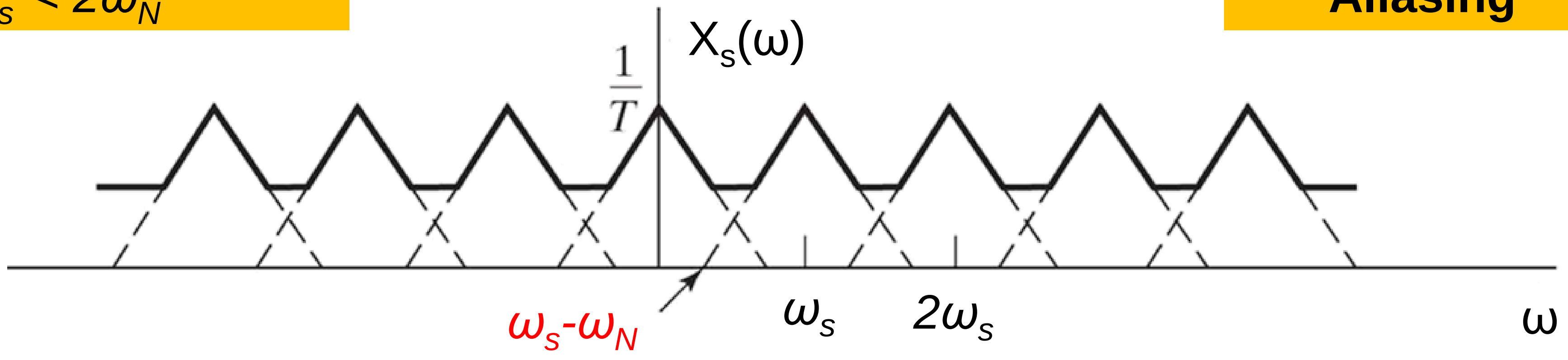
Is the choice of sampling frequency important ?

$\omega_s \geq 2\omega_N$



$\omega_s < 2\omega_N$

Aliasing



Aliasing

a) Spectrum of band-limited signal $x_c(t)$

$X_c(f) = 0, \ |X(f)| \geq B$

b) Sampling with

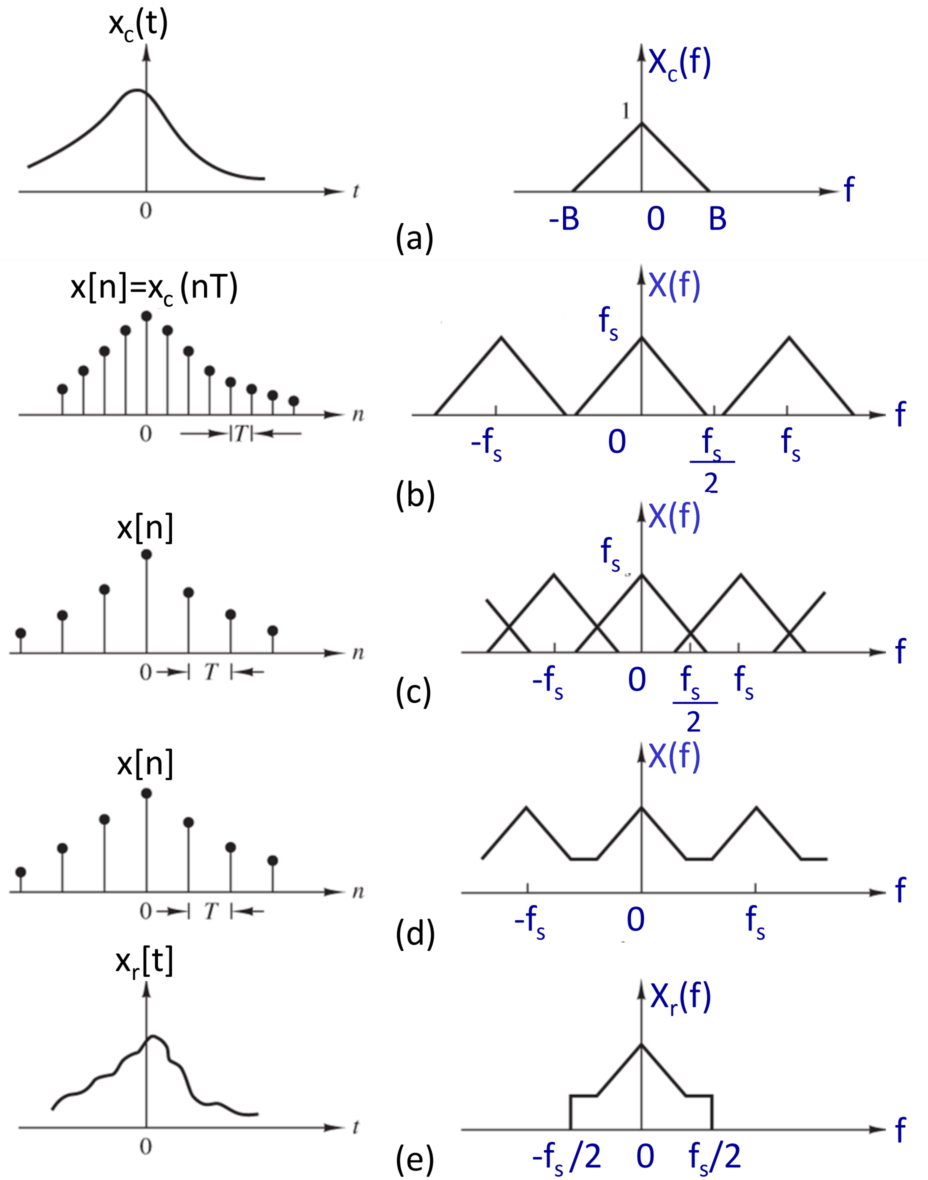
$f_s \geq 2B$

c) Sampling with

$f_s < 2B$

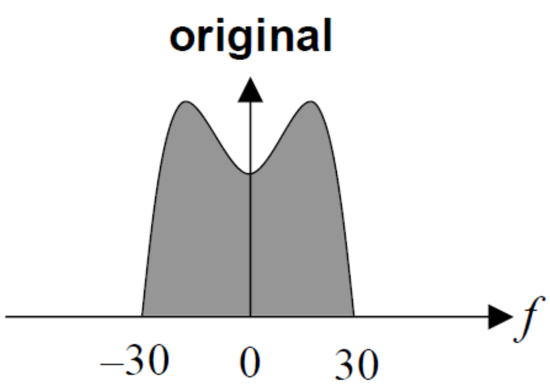
d) Aliasing - overlap of spectral images

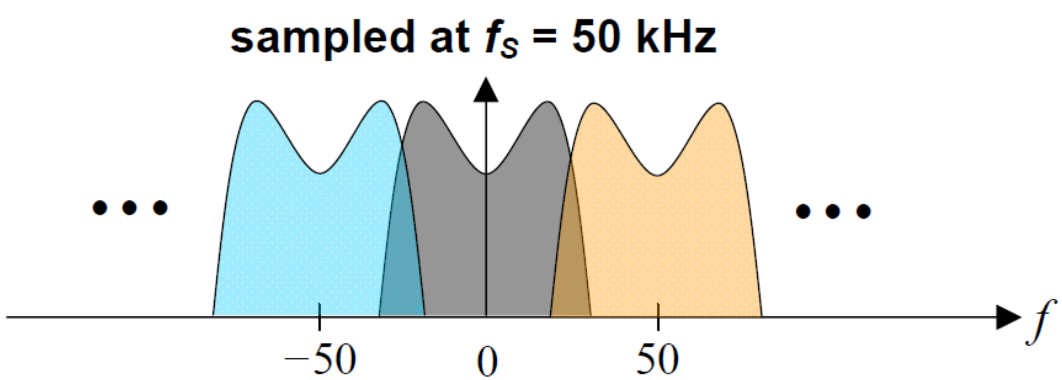
e) Impossible to recover original signal after low pass (reconstruction) filtering (aliasing is irreversible)



Aliasing

Just to make clear that the spectrum does not always look like a triangle…..



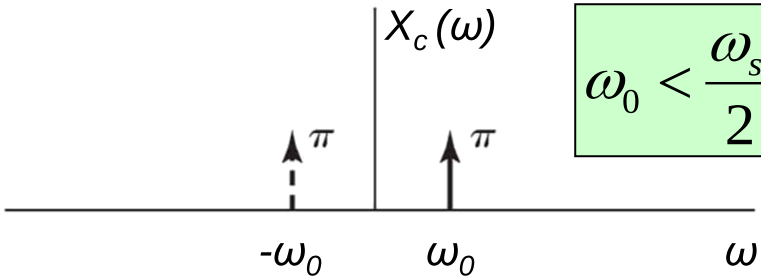


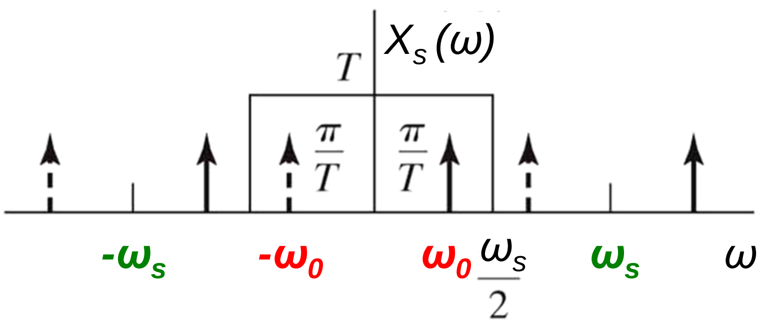
Sampling

Aliasing

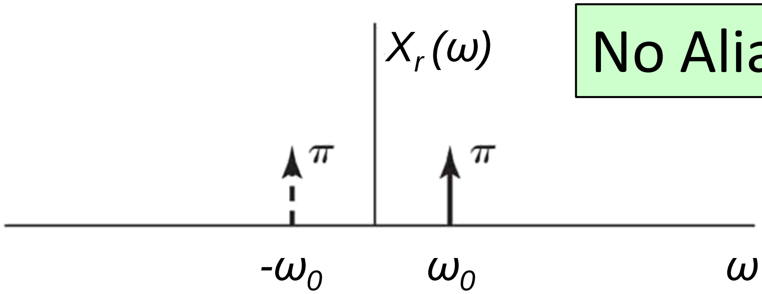
Sampling a sinusoidal signal (frequency domain representation)

$$\omega_0 < \frac{\omega_s}{2} = \frac{\pi}{T}$$

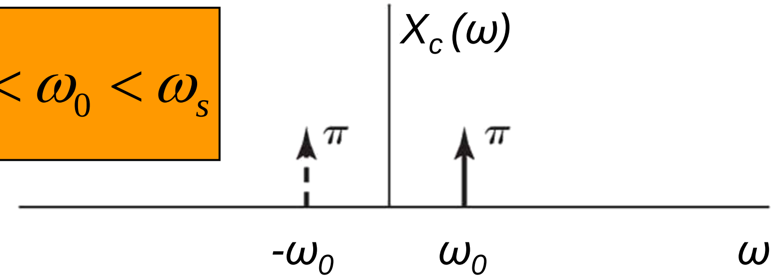


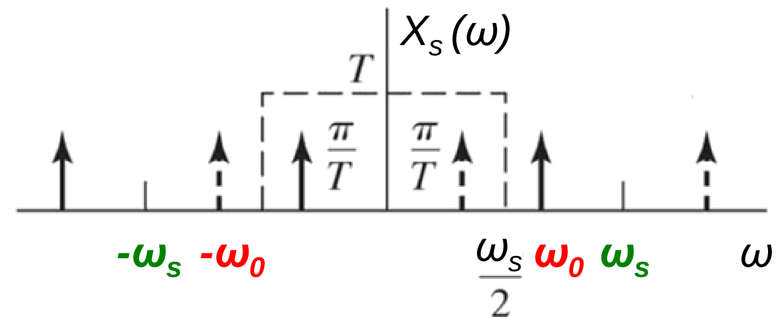


No Aliasing

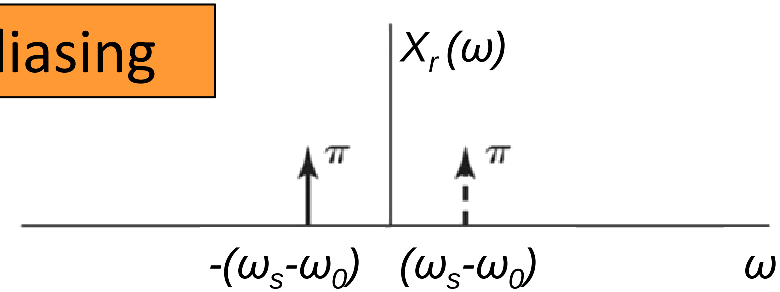


$$\frac{\omega_s}{2} < \omega_0 < \omega_s$$





Aliasing



Aliasing

Sampling a sinusoidal signal (time & frequency domain representation)

Aliasing in the time domain

$$x_r(t) = \cos(2\pi f t)$$

$$T = 3s$$

$$f = 1/T = 0.333Hz$$

Aliasing in the frequency domain

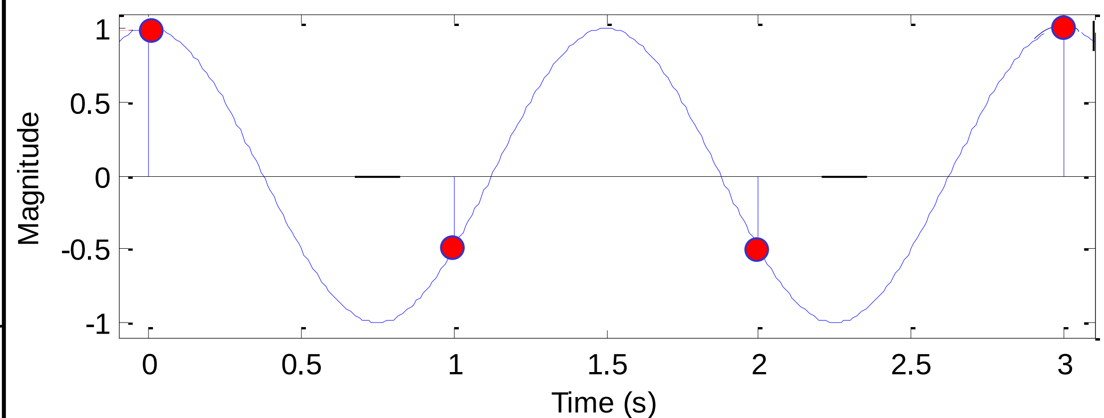
$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

$$k = 0 \Rightarrow \begin{Bmatrix} \delta(f) = \delta(0.666Hz) \\ \delta(-f) = \delta(-0.666Hz) \end{Bmatrix}$$

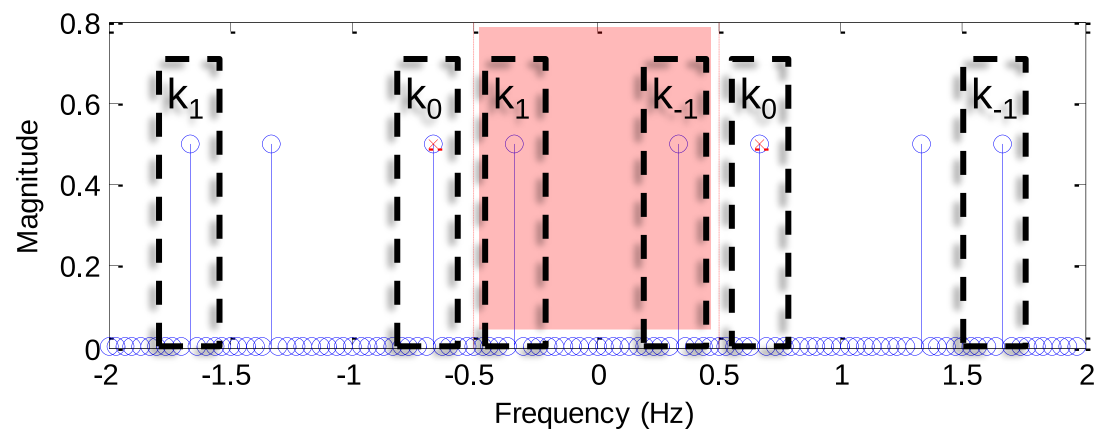
$$k = 1 \Rightarrow \begin{Bmatrix} \delta(f-1) = \delta(-0.333Hz) \\ \delta(-f-1) = \delta(-1.666Hz) \end{Bmatrix}$$

$$k = -1 \Rightarrow \begin{Bmatrix} \delta(f+1) = \delta(1.666Hz) \\ \delta(-f+1) = \delta(0.333Hz) \end{Bmatrix}$$

$$x(t) = \cos(2\pi f t),\ f = 0.666Hz\ (T = 3/2s)$$



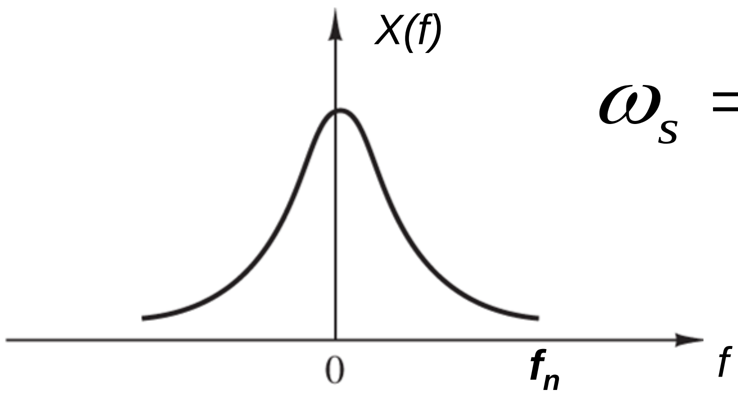
$f_s = 1Hz$ (1 sample per second) , $f_s < 2f$



Avoiding Aliasing

Nyquist – Shannon Sampling Theorem

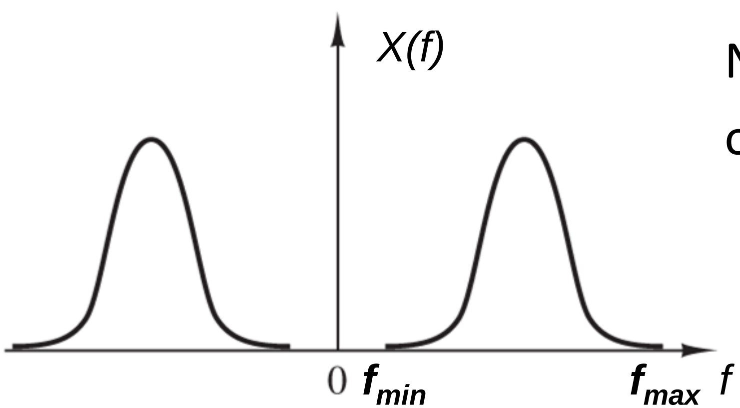
If a signal, *x*(*t*), is band-limited with $X(\omega) = 0,\ \forall |\omega| > \omega_n$ then *x*(*t*) is uniquely determined by its samples, *x[n]* = *x*(*nT*), provided that:



$$\omega_s = \frac{2\pi}{T} \geq \boxed{2\omega_n} \Leftrightarrow f_s \geq 2\boxed{f_n}$$

Nyquist rate

Nyquist frequency



Not all signals are low pass signals. For other types of signals we talk about bandwidth

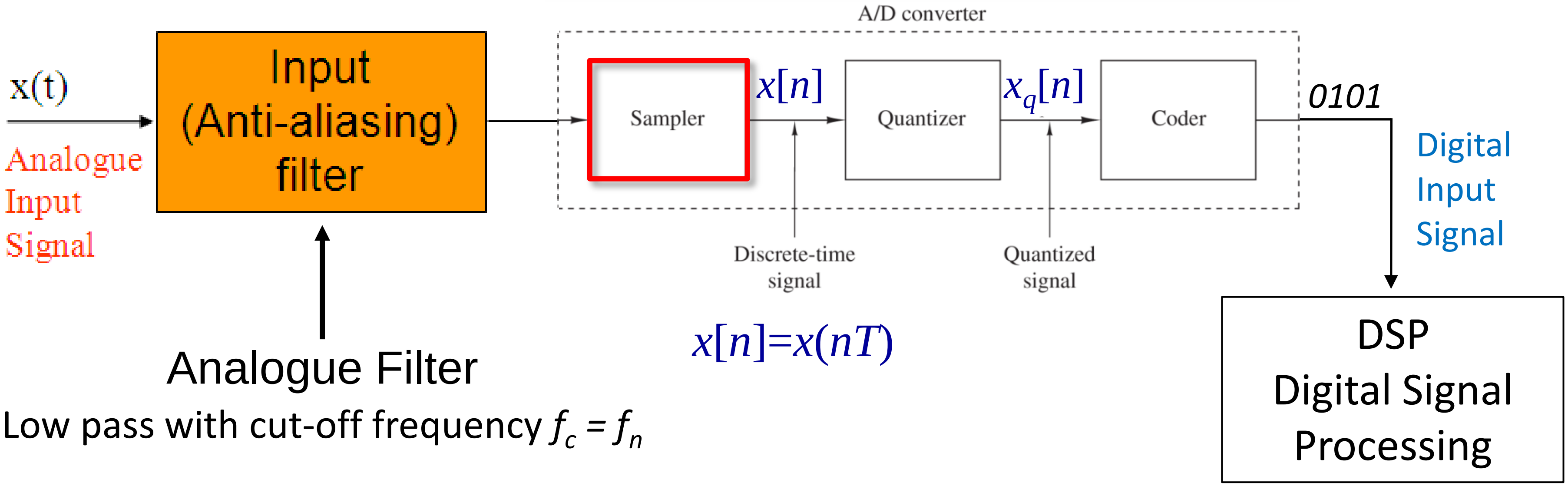
$$f_s \geq \boxed{2B}\ ,\ B = f_{\max} - f_{\min}$$

Sampling

Avoiding Aliasing

Anti-aliasing Filter

If the sampling rate is fixed then the input continuous time signal $x(t)$ will have to be band-limited if we want to avoid aliasing

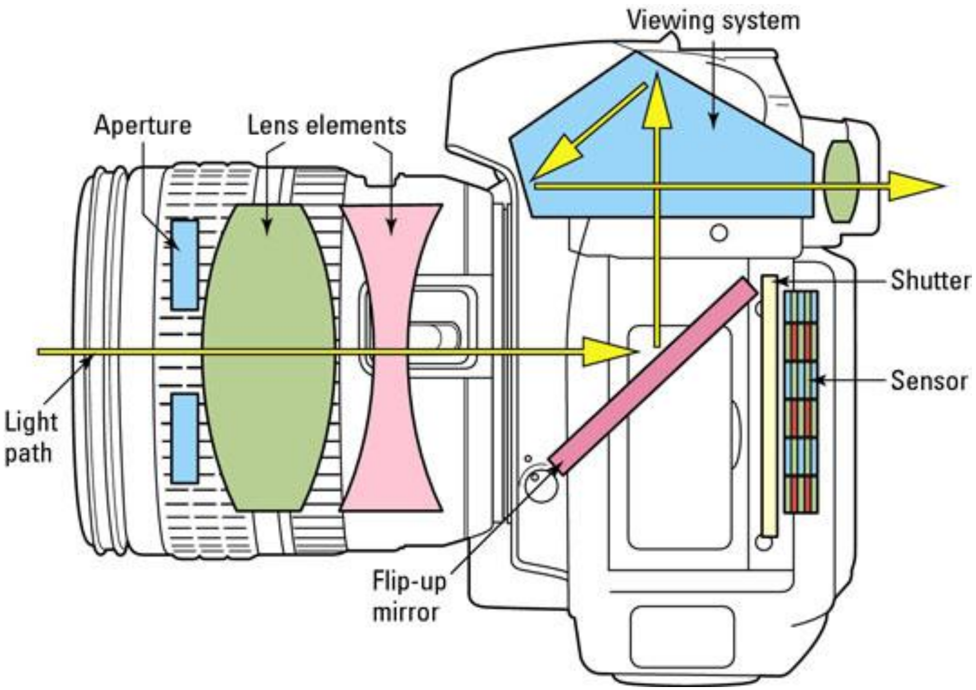


Apply low-pass filtering prior to sampling in order to ensure that the input CT signal is band-limited below the Nyquist frequency i.e. $X(f)=0,\ \forall|f|> f_n,\quad f_n = f_s/2$

Avoiding Aliasing

Anti-aliasing Filter Example – Digital camera

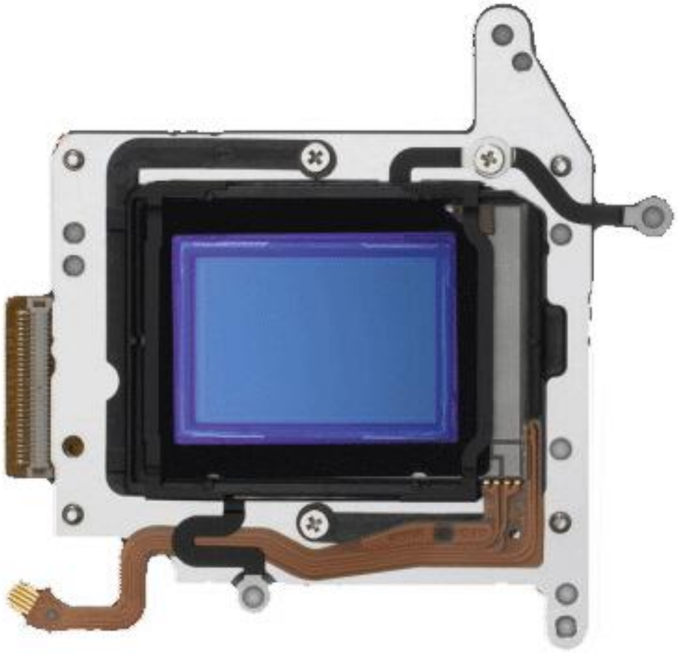
The resolution and hence the sampling frequency of the imaging sensor is fixed. The bandwidth of the input analogue signal (the real world scene seen through the lens) can vary.

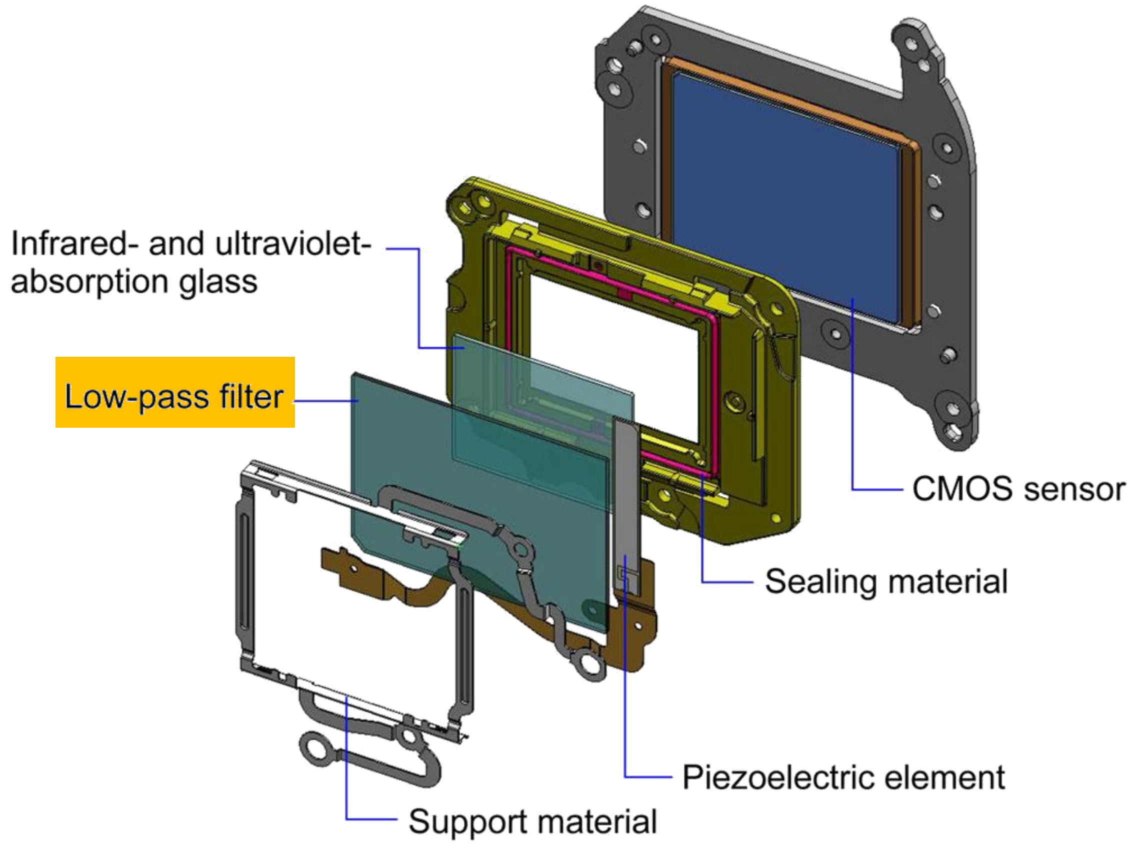


Avoiding Aliasing

Anti-aliasing Filter Example – Digital camera

In most digital cameras an optical low pass filter is used for band-limiting the input analogue signal (the imaged scene) below the Nyquist frequency (half the sensor's sampling frequency)

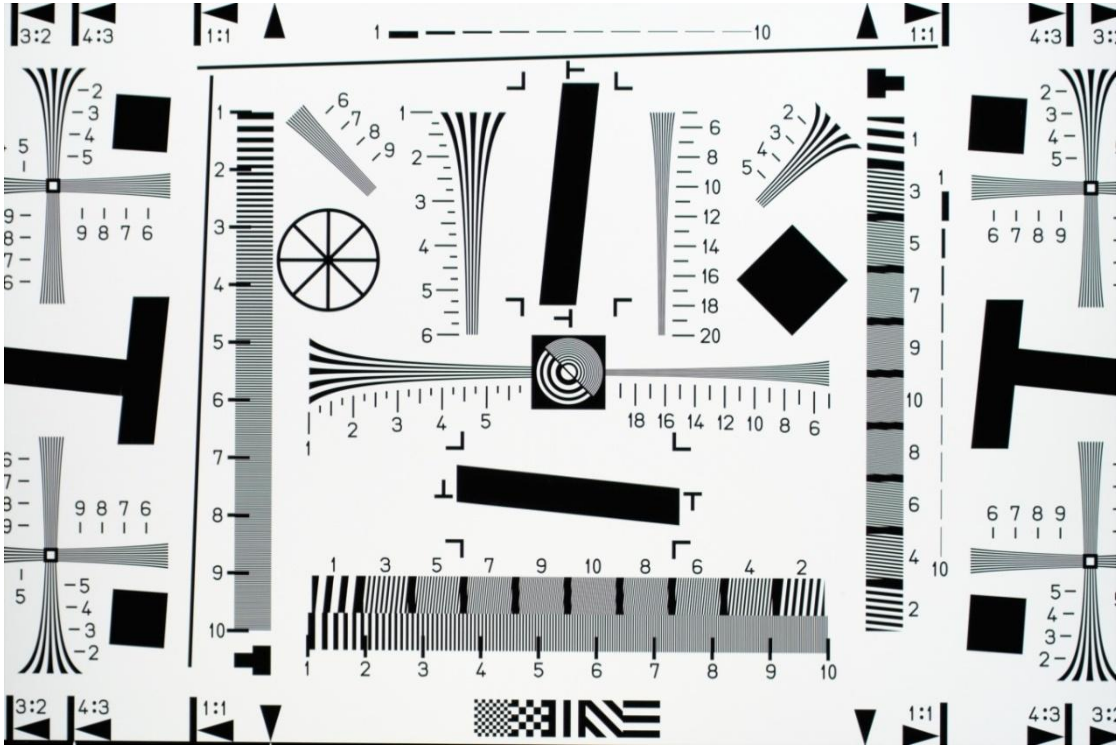




Avoiding Aliasing

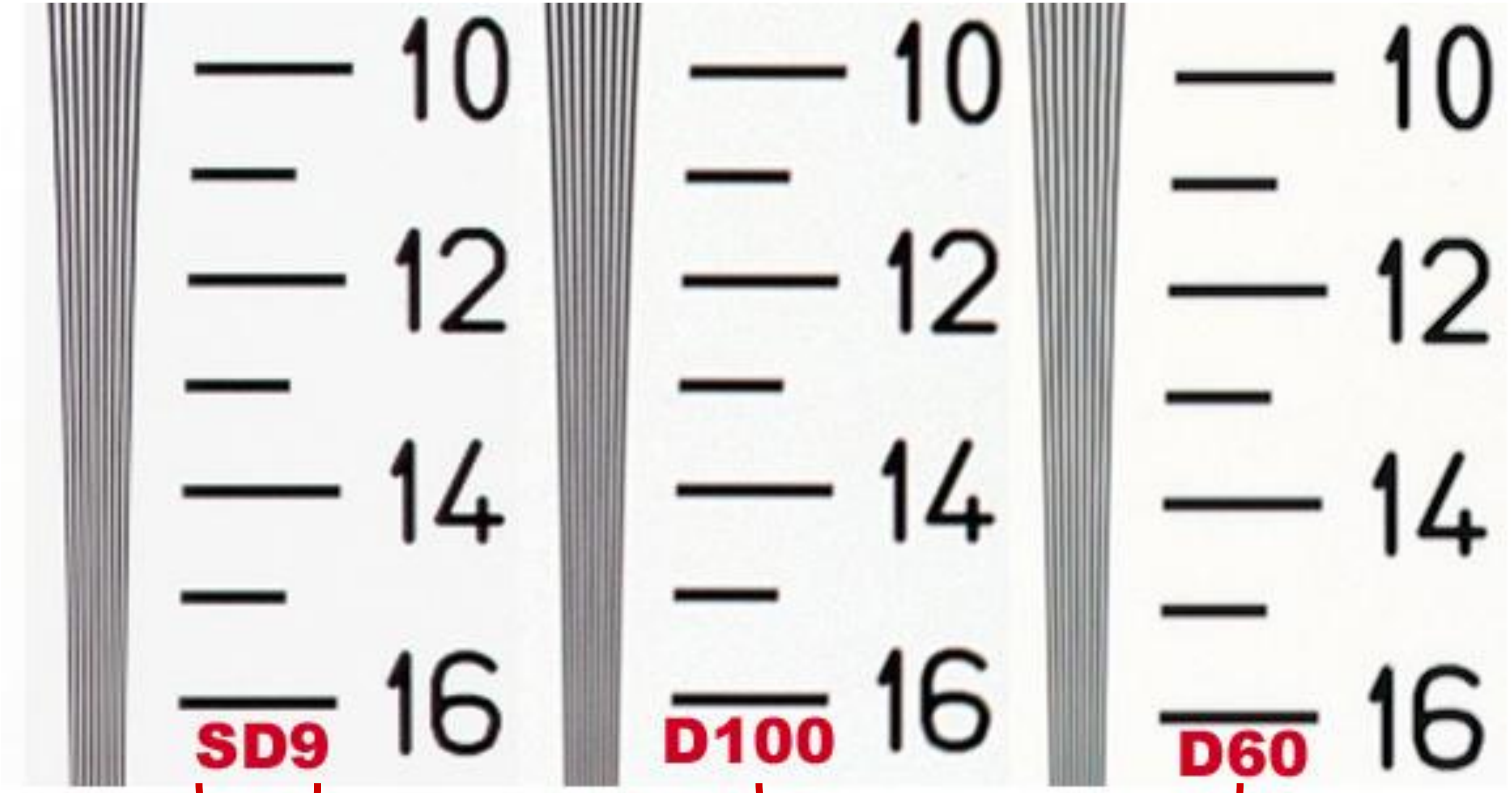
Anti-aliasing Filter Example – Digital camera

The “sharpness” of the camera will depend largely on the lens (which acts as a filter) and the sampling process performed at the sensor (which depends on the resolution and the anti-aliasing filter used). To test the sharpness the ISO 12233 Test Chart shown below is often used



Avoiding Aliasing

Anti-aliasing filter – Image example

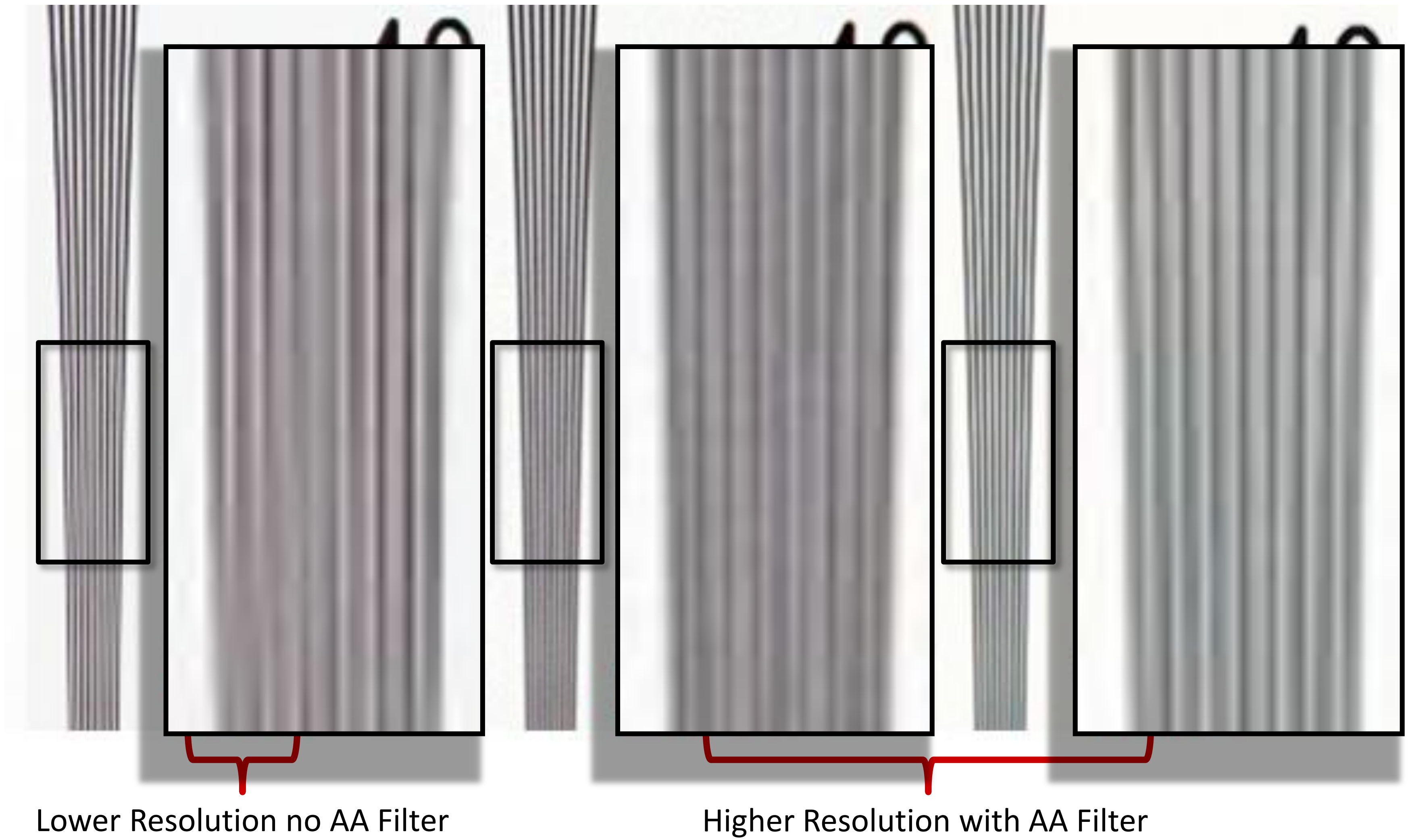


Lower Resolution no AA Filter

Higher Resolution with AA Filter

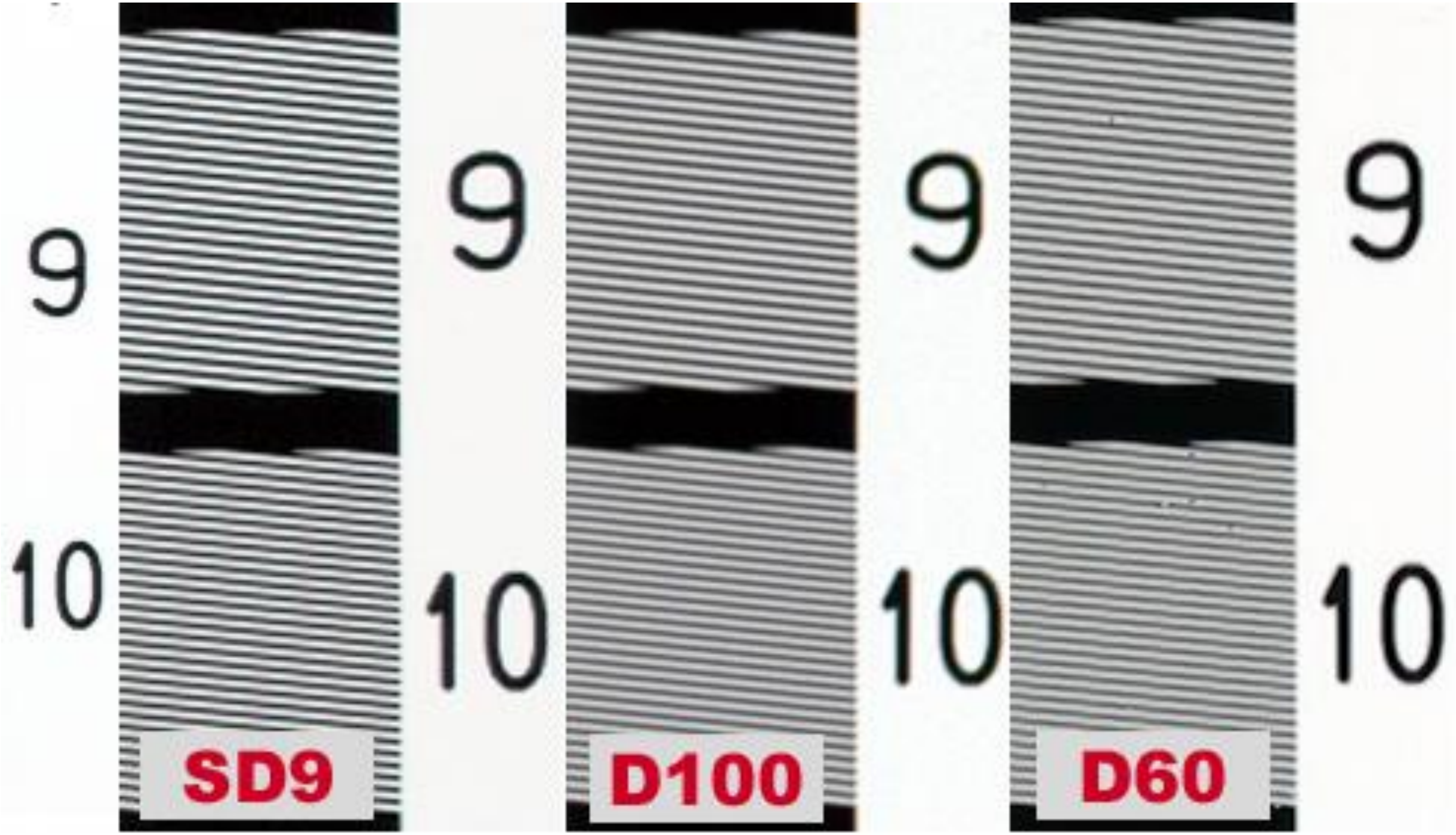
Avoiding Aliasing

Anti-aliasing filter – Image example



Avoiding Aliasing

Anti-aliasing filter – Image example



Avoiding Aliasing

Anti-aliasing Filter Example – Digital camera with (left) & without filter (right)

Avoiding Aliasing

Anti-aliasing filter – Image example

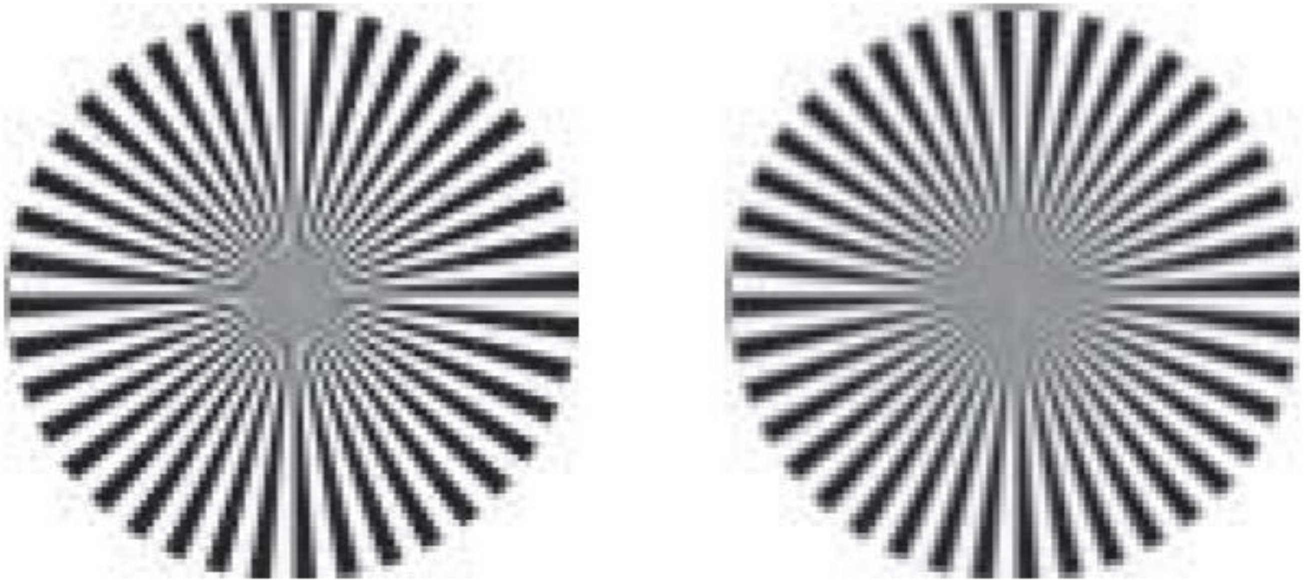


Image acquisition without anti alias filter (left) and with filter (right)

Sampled Signal Frequency

Relationship between continuous time and discrete time signal frequency

$$\left.\begin{aligned} x[n] &= \cos(2\pi f n T_s) \\ t = nT_s \ , \ T_s &= \frac{1}{f_s} \end{aligned}\right\} \quad \left.\begin{aligned} x[n] &= \cos\left(2\pi n \frac{f}{f_s}\right) \\ x(t) &= \cos(2\pi f t) \end{aligned}\right\} \quad \boxed{F = \frac{f}{f_s} \Leftrightarrow \Omega = \omega T_s}$$

The normalised (or digital) frequency ***F*** is related to the analogue frequency ***f*** through the sampling frequency $\boldsymbol{f_s}$:

$$F = \frac{f}{f_s} \Leftrightarrow \Omega = \omega T_s \Leftrightarrow f = F \times f_s$$

Sampling

Sampled Signal Frequency

Relationship between continuous time and discrete time signal frequency

The frequency $\boldsymbol{f}$ of a continuous time sinusoid when sampled at a rate $\boldsymbol{f_s}$ falls within the range $-\frac{f_s}{2} < f < \frac{f_s}{2}$ (**Fundamental Frequency Range**). Anything outside this range is an alias.

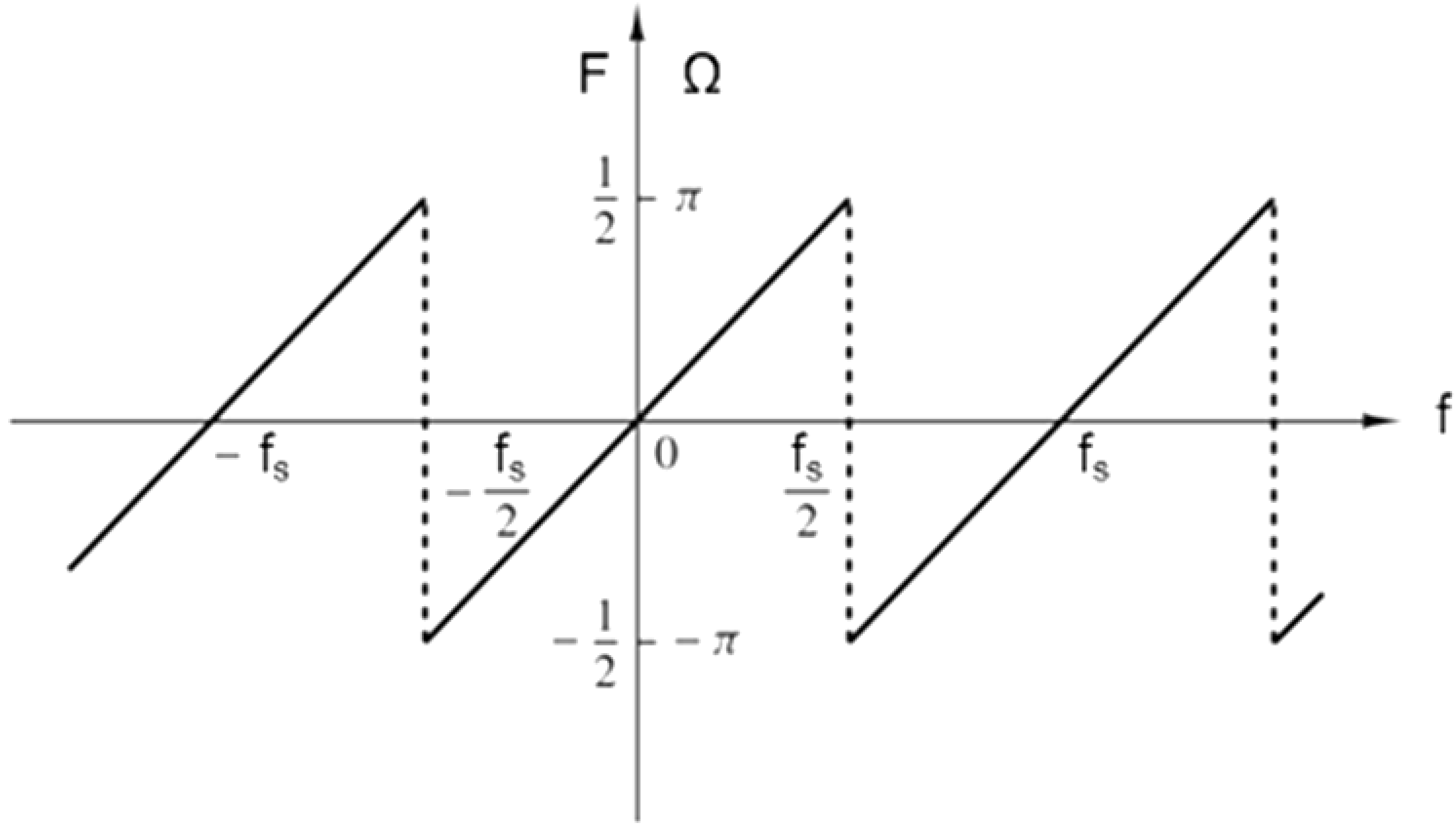
$$\left.\begin{array}{l} -\frac{f_s}{2} < f < \frac{f_s}{2} \\ f = F \times f_s \end{array}\right\} \left.\begin{array}{c} -\frac{1}{2} < F < \frac{1}{2} \\ \Omega = 2\pi F \end{array}\right\} -\pi < \Omega < \pi$$

The **highest rate of oscillation** of a discrete time sinusoid is attained when **Ω=π** (or equivalently **F=1/2**)

Sampled Signal Frequency

Relationship between continuous time and discrete time signal frequency

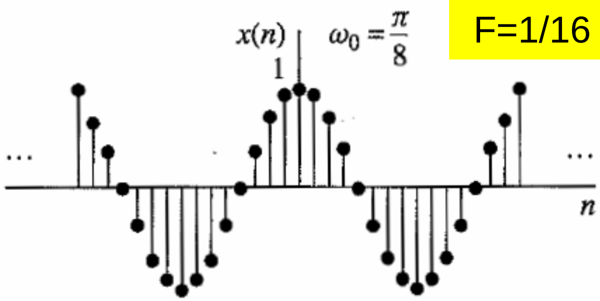
Digital Frequency (F) vs. Analogue Frequency (f)



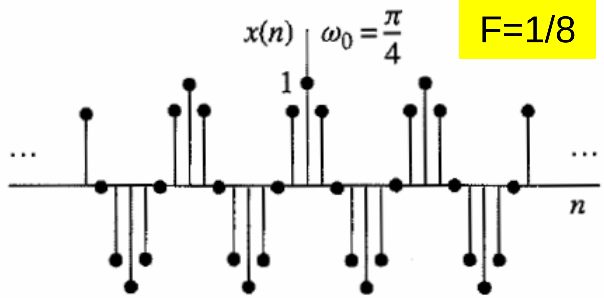
Sampled Signal Frequency

Relationship between continuous time and discrete time signal frequency

F=1/16

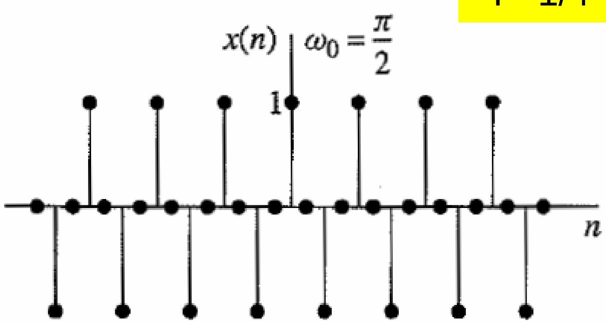


F=1/8

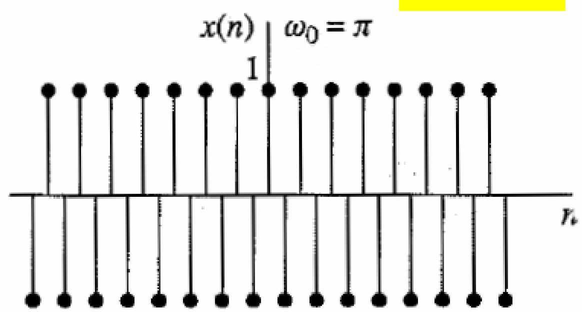


$x(n) = \cos(2\pi Fn)$

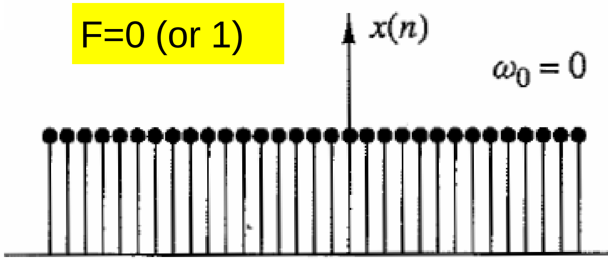
F=1/4



F=1/2

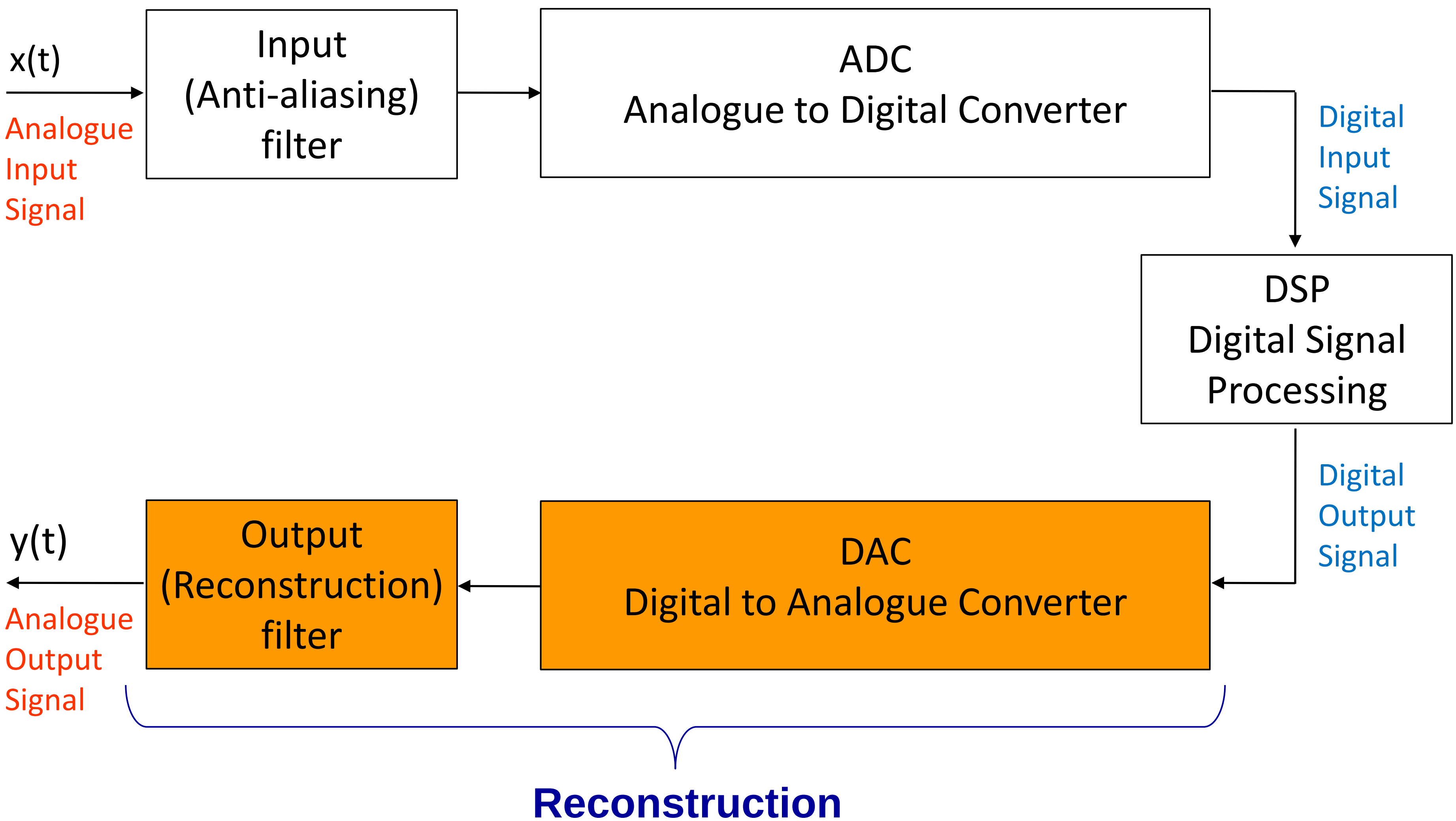


F=0 (or 1)



With a sampled signal we measure the **frequency in cycles per sample**. With a continuous time signal we measure it in cycles per second (Hz)

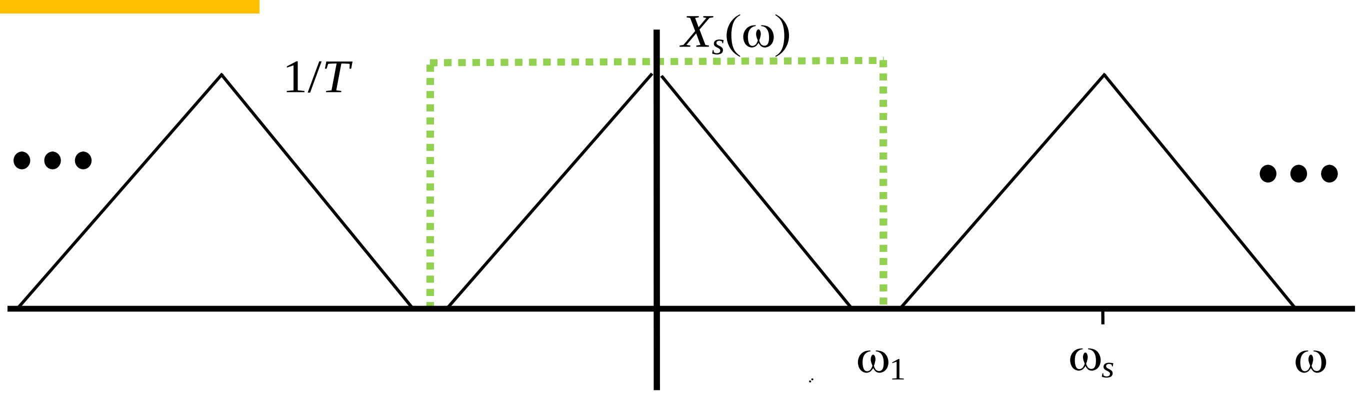
Digital Signal Processing



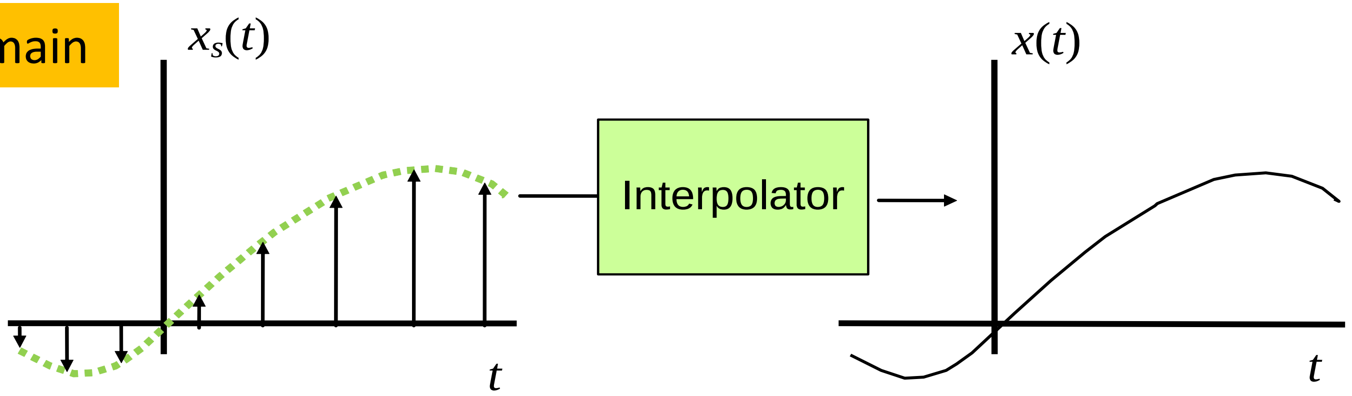
Reconstruction

Ideal reconstruction

Frequency Domain



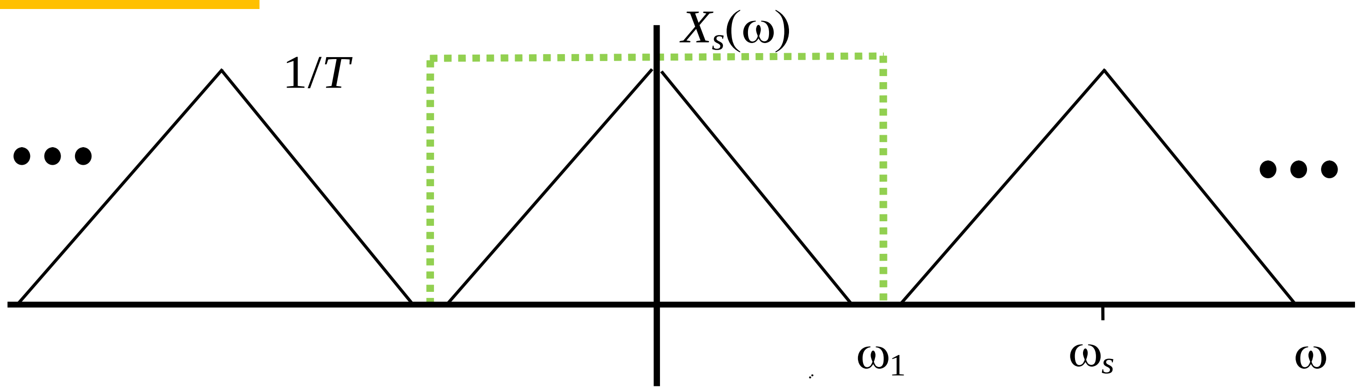
Time Domain



Reconstruction

Ideal reconstruction filter

Frequency Domain



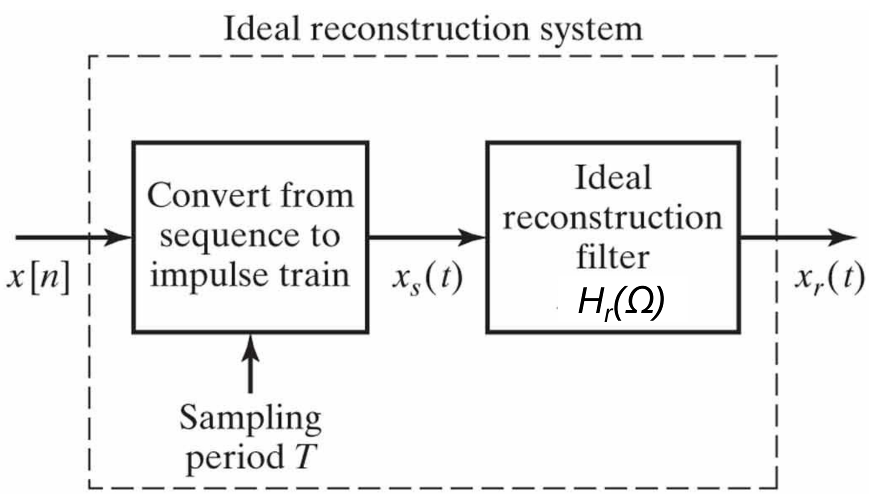
Time Domain

$$\left.\begin{array}{l} h(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega \\ H(\omega) = \begin{cases} 1 & \omega_1 \le \omega \le \omega_1 \\ 0 & \text{Elsewhere} \end{cases} \end{array}\right\} \Rightarrow h(t) = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} e^{j\omega t} d\omega = \frac{\omega_1}{\pi}\left(\frac{\sin(\omega_1 t)}{\omega_1 t}\right) = \frac{\omega_1}{\pi}\operatorname{sinc}(\omega_1 t)$$

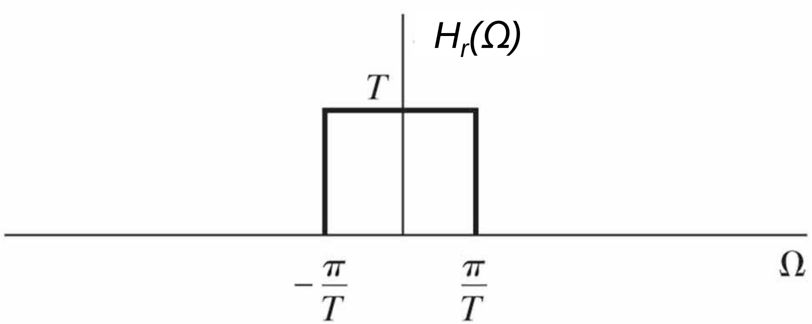
Reconstruction

Ideal reconstruction filter

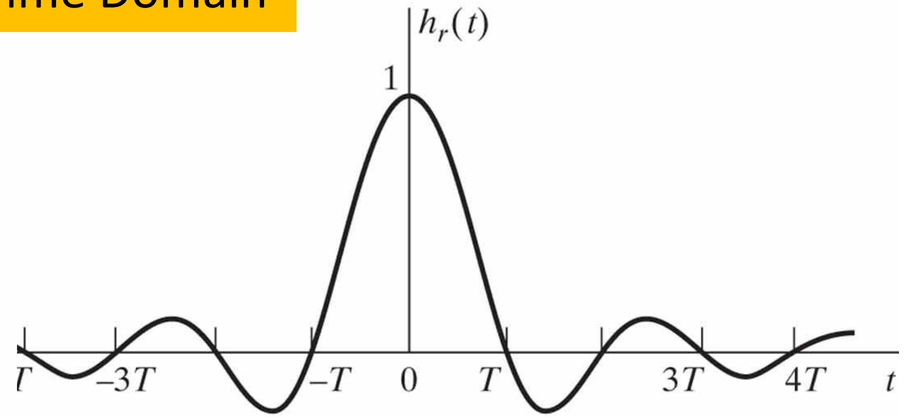
- In the frequency domain we multiply the spectrum of the ideal reconstruction filter with the spectrum of the sampled sign
- In the time domain this multiplication becomes a convolution between a sinc function and a train of impulses (the sampled signal)



Frequency Domain

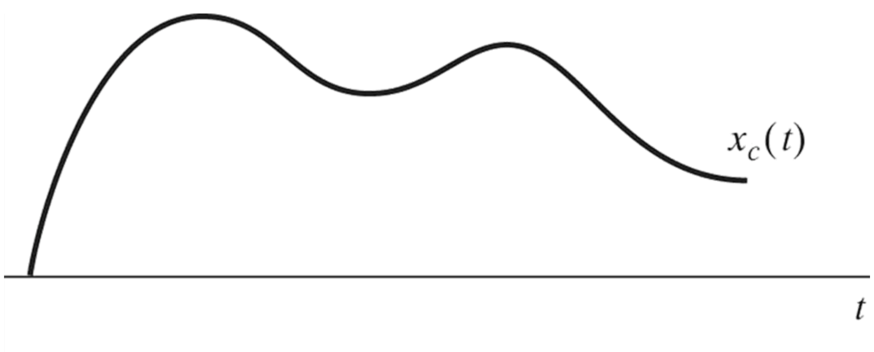


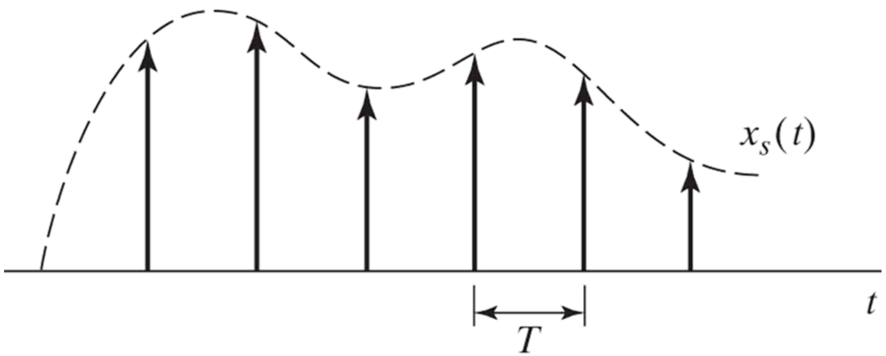
Time Domain



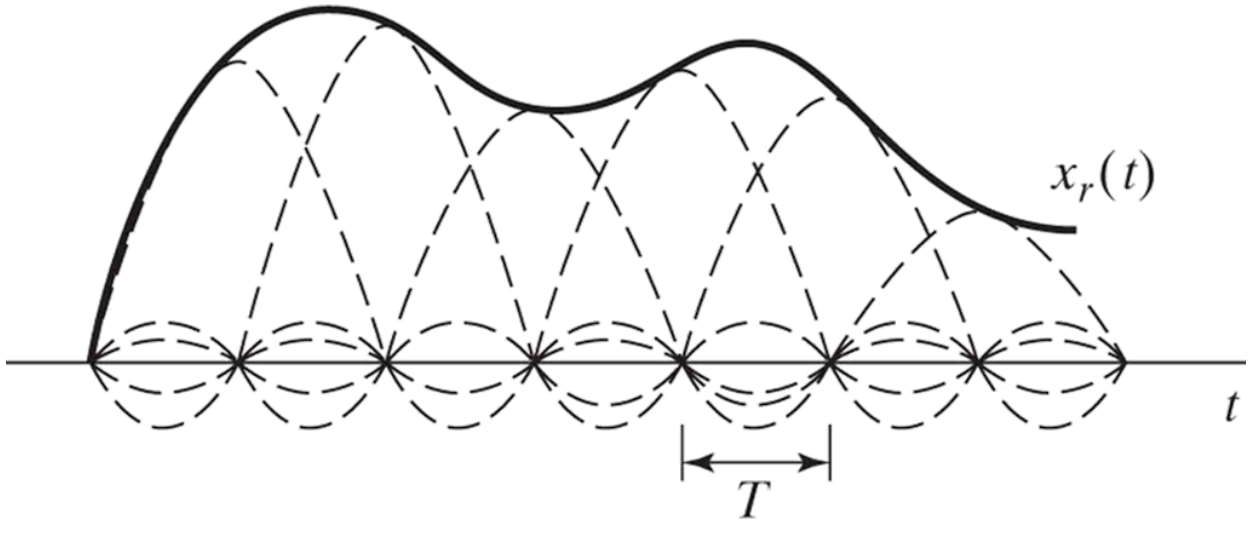
Reconstruction

Ideal reconstruction filter



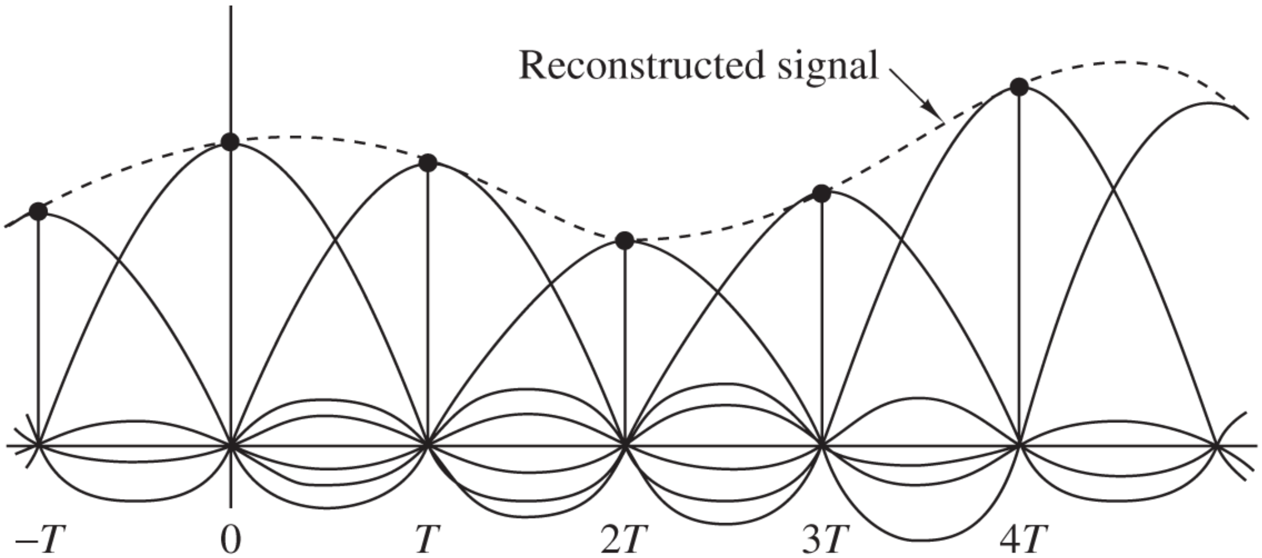


Result of convolution of ideal reconstruction filter (sinc) with sampled signal (train of impulses)



Reconstruction

Ideal reconstruction filter

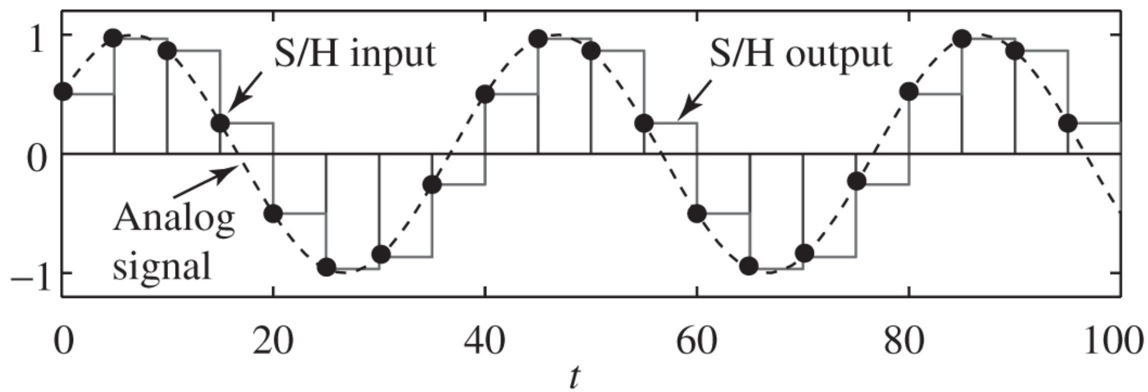


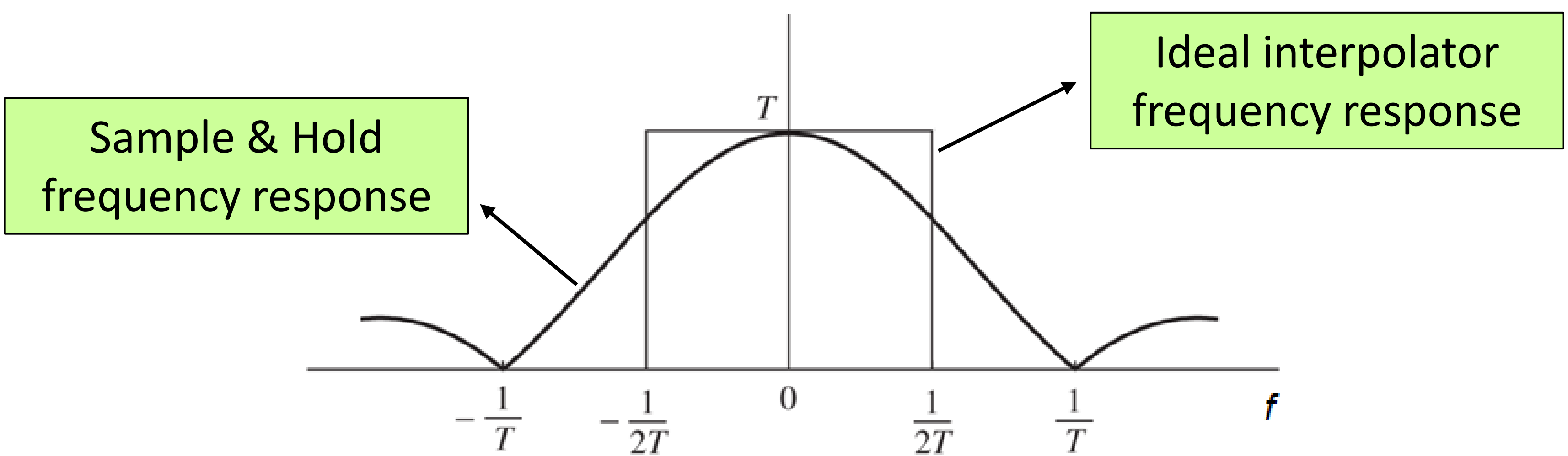
$$x_r(t) = \frac{\omega_1}{\pi} \sum_{n=-\infty}^{\infty} x_s(nT_s)\operatorname{sinc}\left(\omega_1(t - nT_s)\right)$$

Sampling

Reconstruction

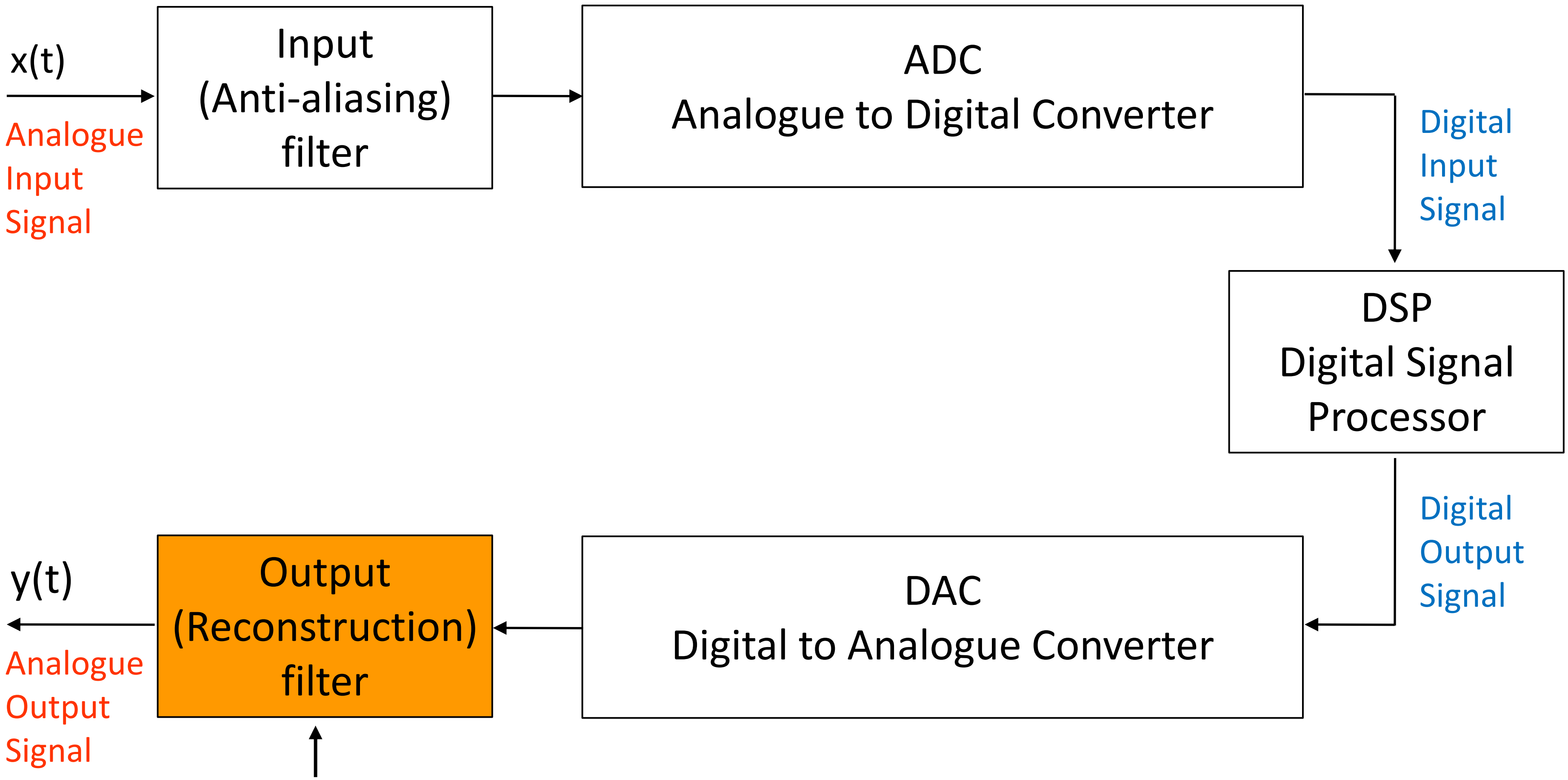
Realistic reconstruction





Reconstruction

Realistic reconstruction



Low pass filter: Smoothens edges of sample & hold operation in time domain ⇔ Attenuates high frequencies let through by sample & hold filter in frequency domain