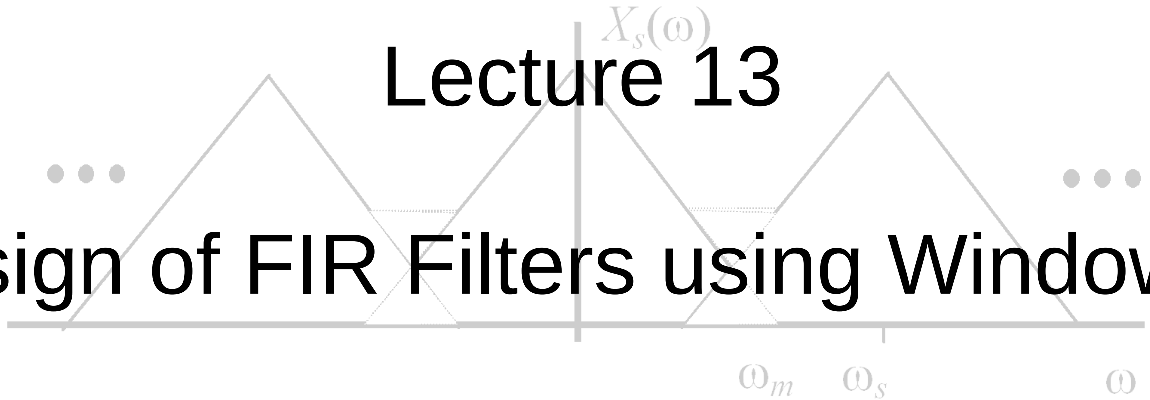
Lecture 13

Design of FIR Filters using Windowing



Design through truncation of the ideal impulse response

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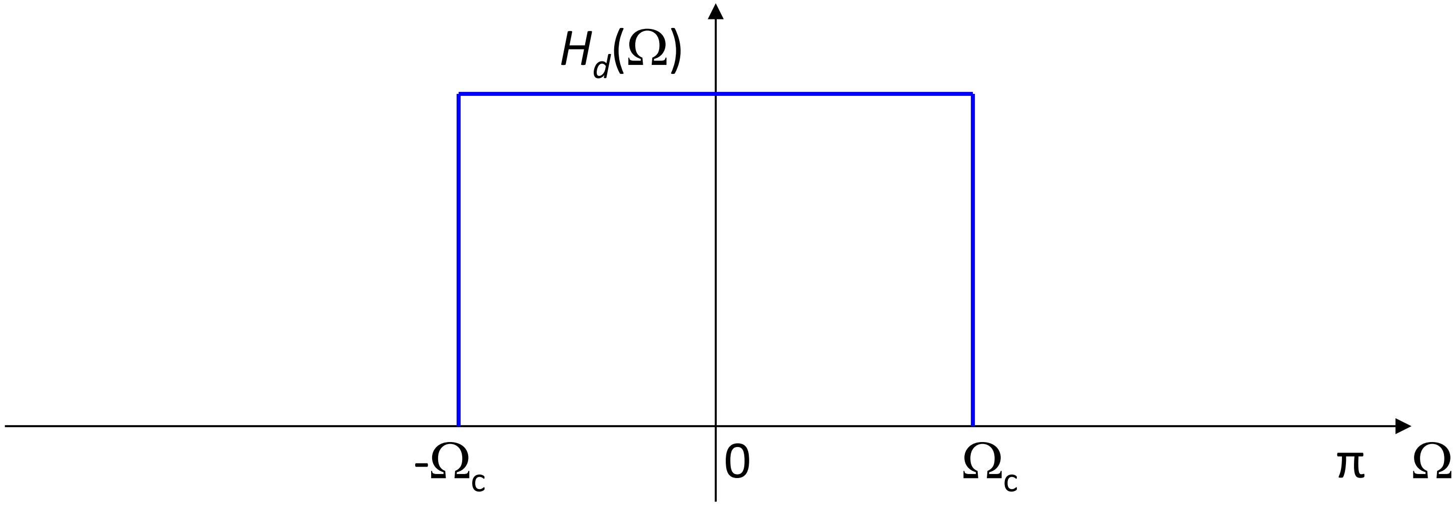
Design of FIR Filters using Windowing

Ideal Response

The ideal frequency response

$$H_d(\Omega) = \begin{cases} 1 & |\Omega| \le \Omega_c \\ 0 & \Omega_c < |\Omega| \le \pi \end{cases}$$



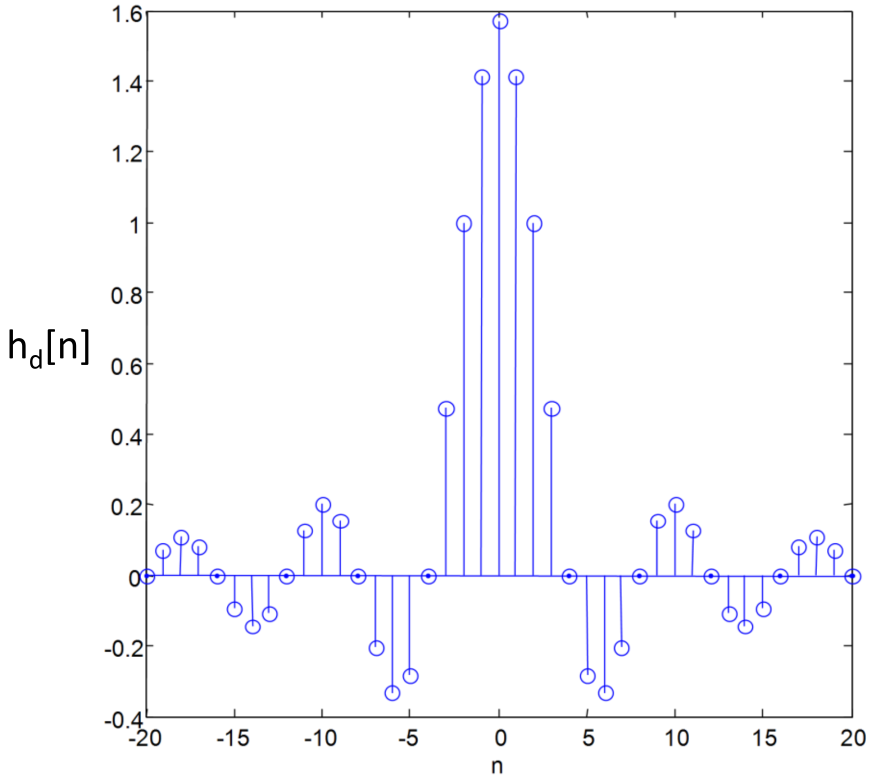
Ideal Low Pass Filter

Design of FIR Filters using Windowing

Ideal Response

The ideal impulse response

$$H_d(\Omega) = \begin{cases} 1 & |\Omega| \le \Omega_c \\ 0 & \Omega_c < |\Omega| \le \pi \end{cases} \xrightarrow{IDTFT} h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\Omega) e^{j\Omega n} d\Omega \quad = \boxed{\frac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)}$$

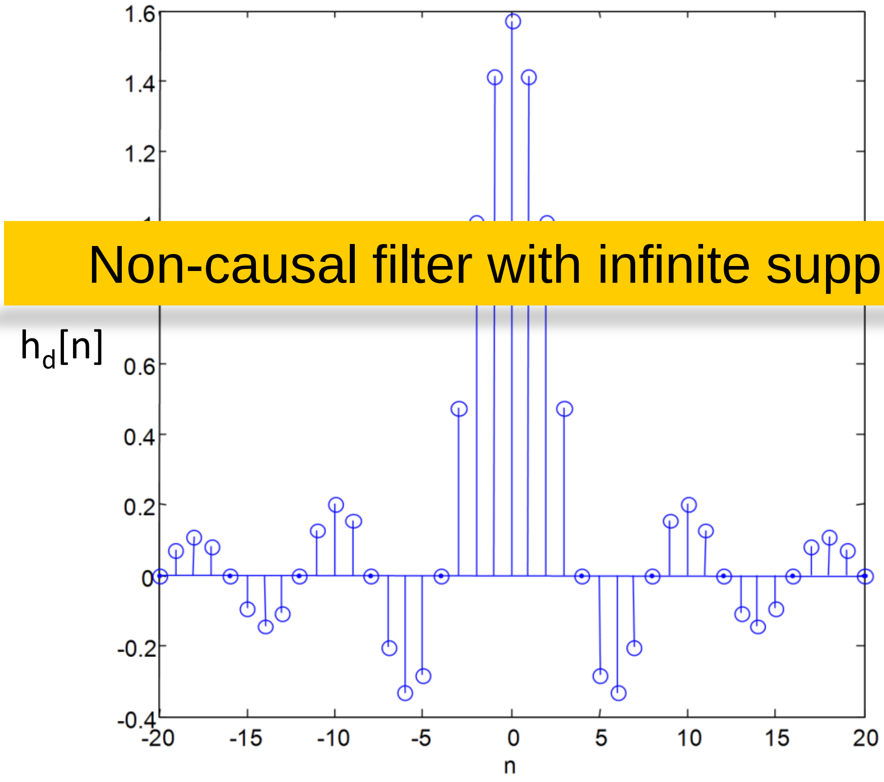


Design of FIR Filters using Windowing

Ideal Response

The ideal impulse response - problem

$$H_d(\Omega) = \begin{cases} 1 & |\Omega| \le \Omega_c \\ 0 & \Omega_c < |\Omega| \le \pi \end{cases} \xrightarrow{IDTFT} h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\Omega) e^{j\Omega n} d\Omega = \boxed{\frac{\Omega_c}{\pi}\mathrm{sinc}(\Omega_c n)}$$



Non-causal filter with infinite support

Design of FIR Filters using Windowing

Practical Impulse Response

Truncate the ideal impulse response to make it finite – Add delay for causality

Finite impulse response

- Multiply with finite symmetric window of length L (L=M+1, where M is the order)

Causality

- Add delay (L-1)/2 to make filter causal or …
- modify ideal response to include linear phase factor $e^{-j\Omega(L-1)/2}$ prior to windowing

Impulse Response

$$h_d[n] = \frac{1}{n\pi}\sin(\Omega_c n) = \frac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)$$

Rectangular window

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

Multiply

Truncated impulse response (FIR) $h_w[n]$

Frequency Response

$$H_d(\Omega)$$

Spectrum of rectangular window

$$W(\Omega) = e^{-jM/2}\frac{\sin(\Omega(M+1)/2)}{\sin(\Omega/2)}$$

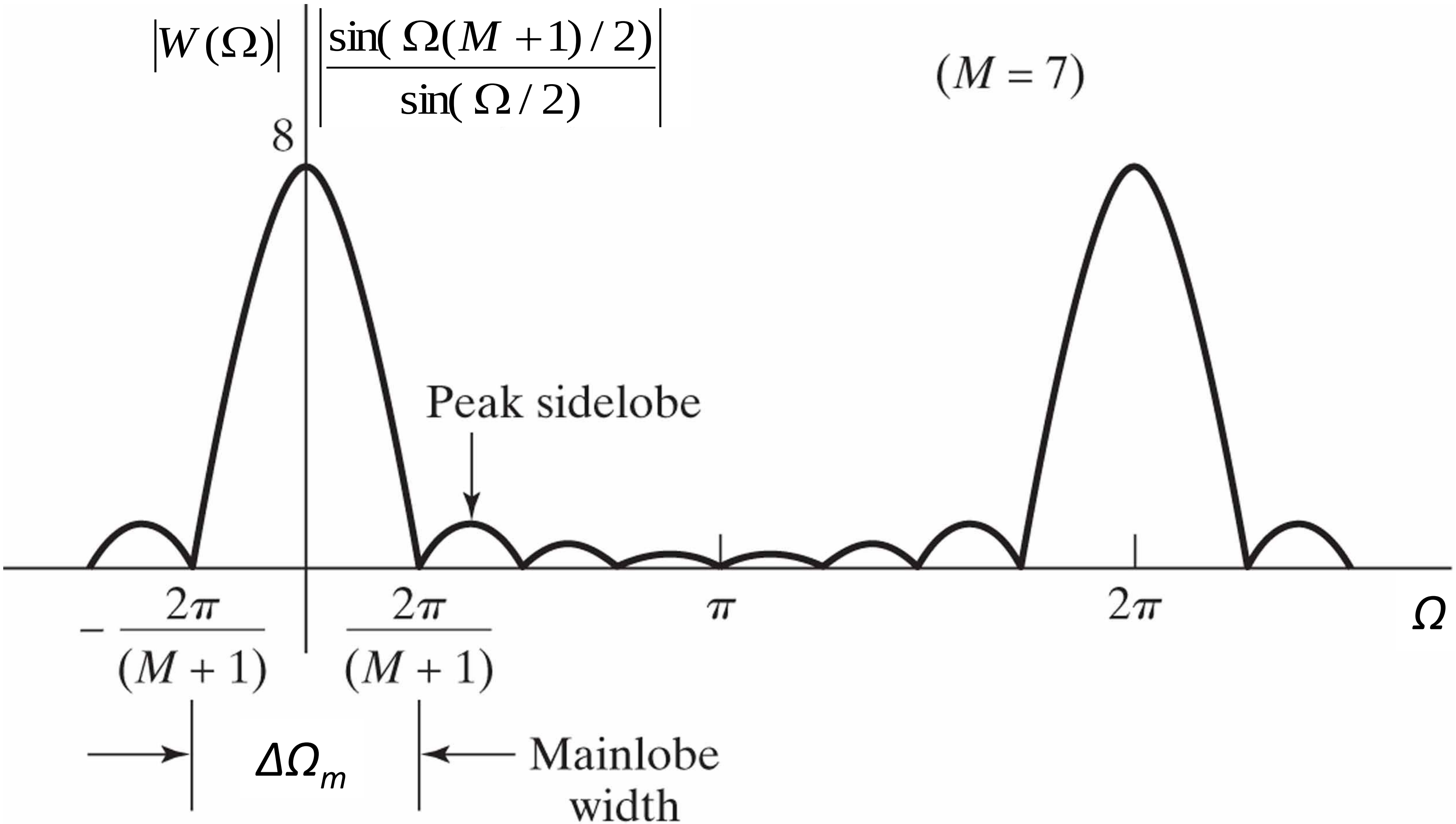
Convolve

$$H(\Omega) = H_d(\Omega) \otimes W(\Omega)$$

Design of FIR Filters using Windowing

Windowing in the Frequency Domain

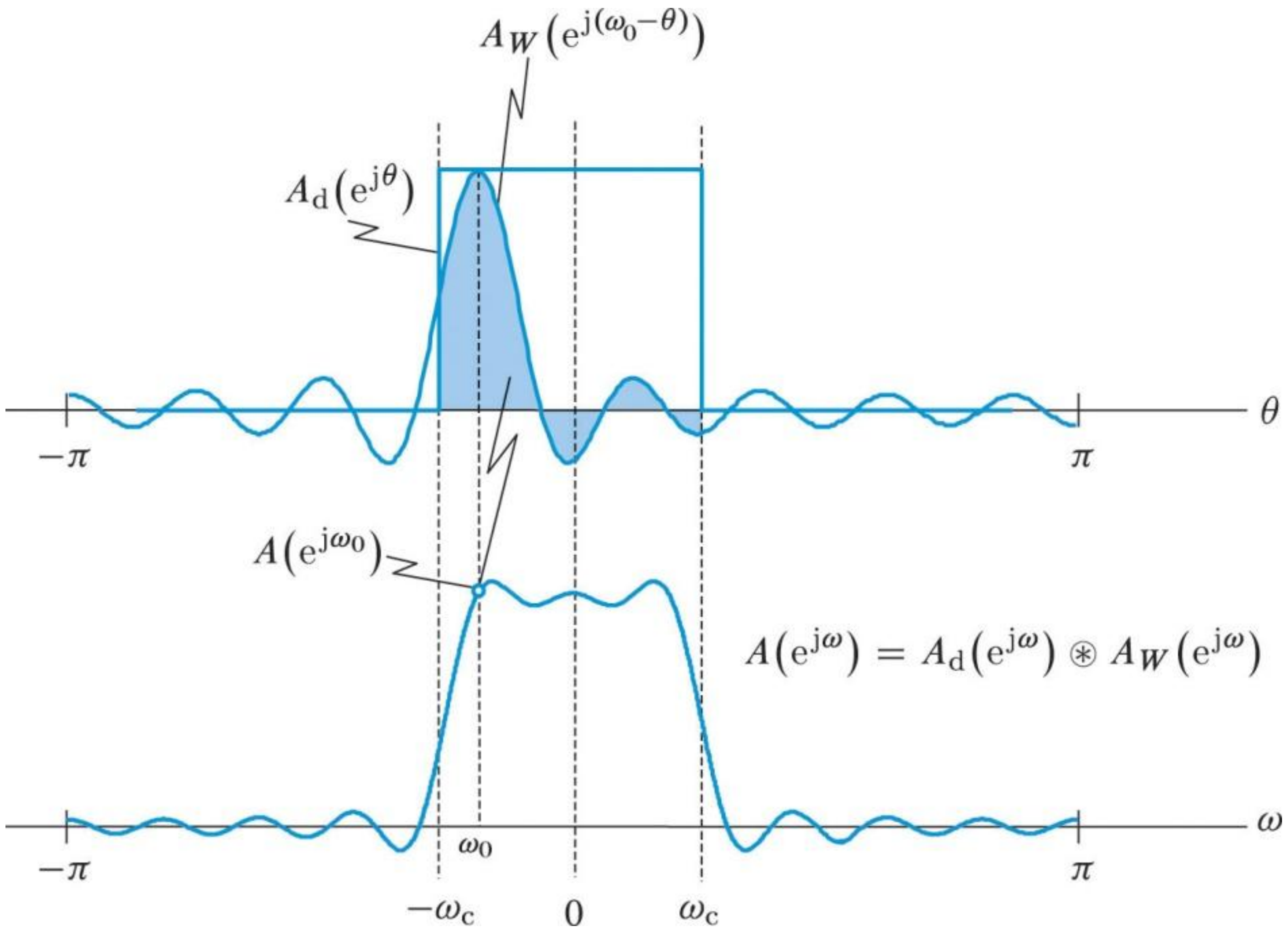
DTFT of Rectangular Window



Design of FIR Filters using Windowing

Windowing in the Frequency Domain

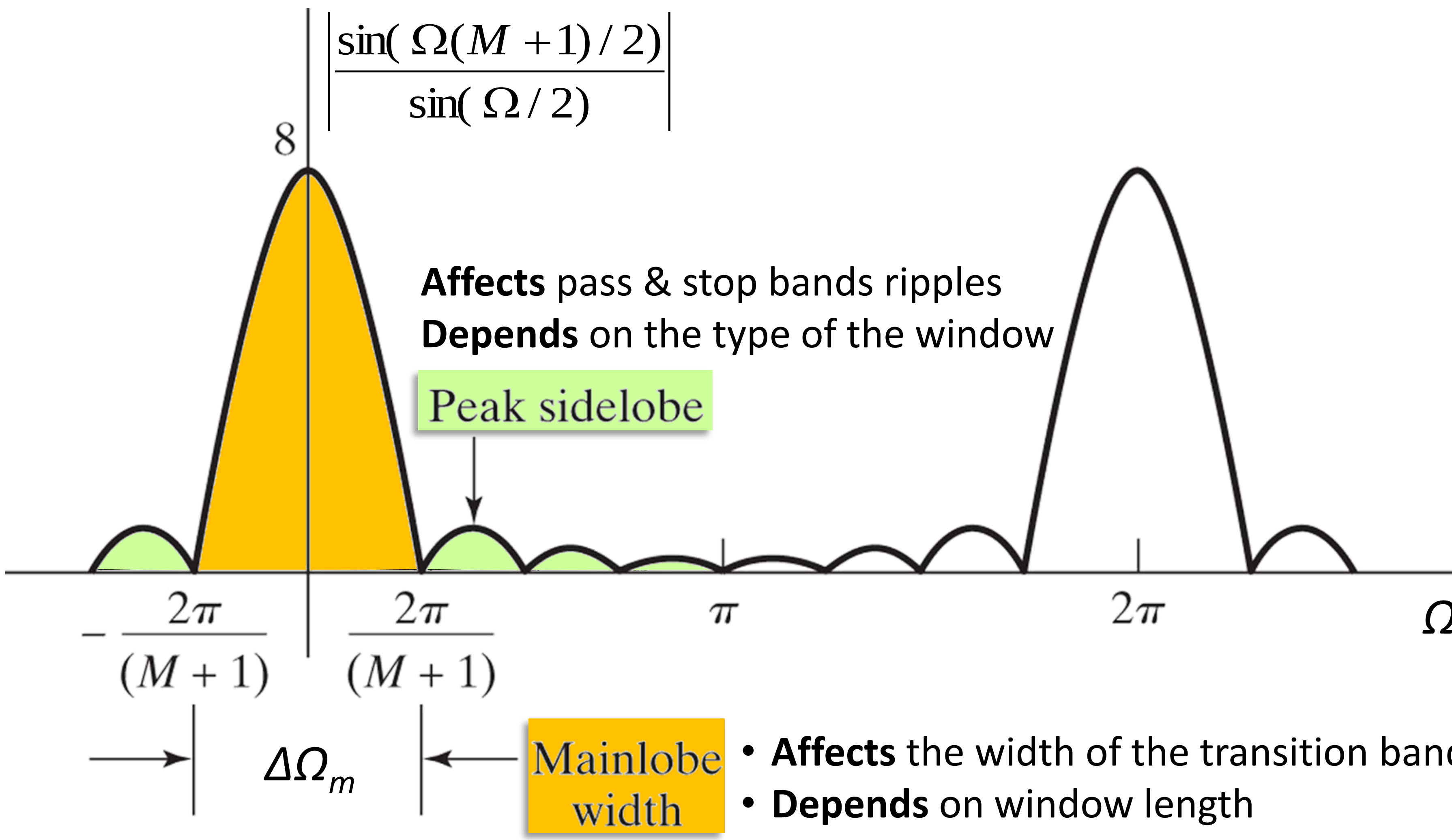
Convolution in frequency: ripples in passband & stopband, gradual transition



Design of FIR Filters using Windowing

Windowing in the Frequency Domain

Factors affecting the frequency response of the filter



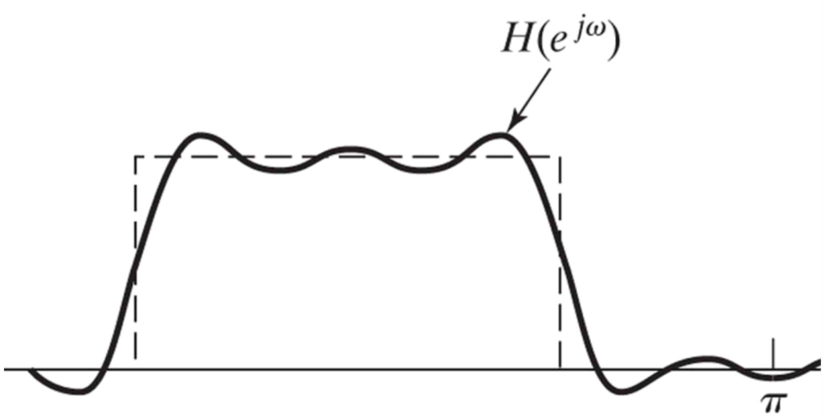
- **Affects** the width of the transition band
- **Depends** on window length

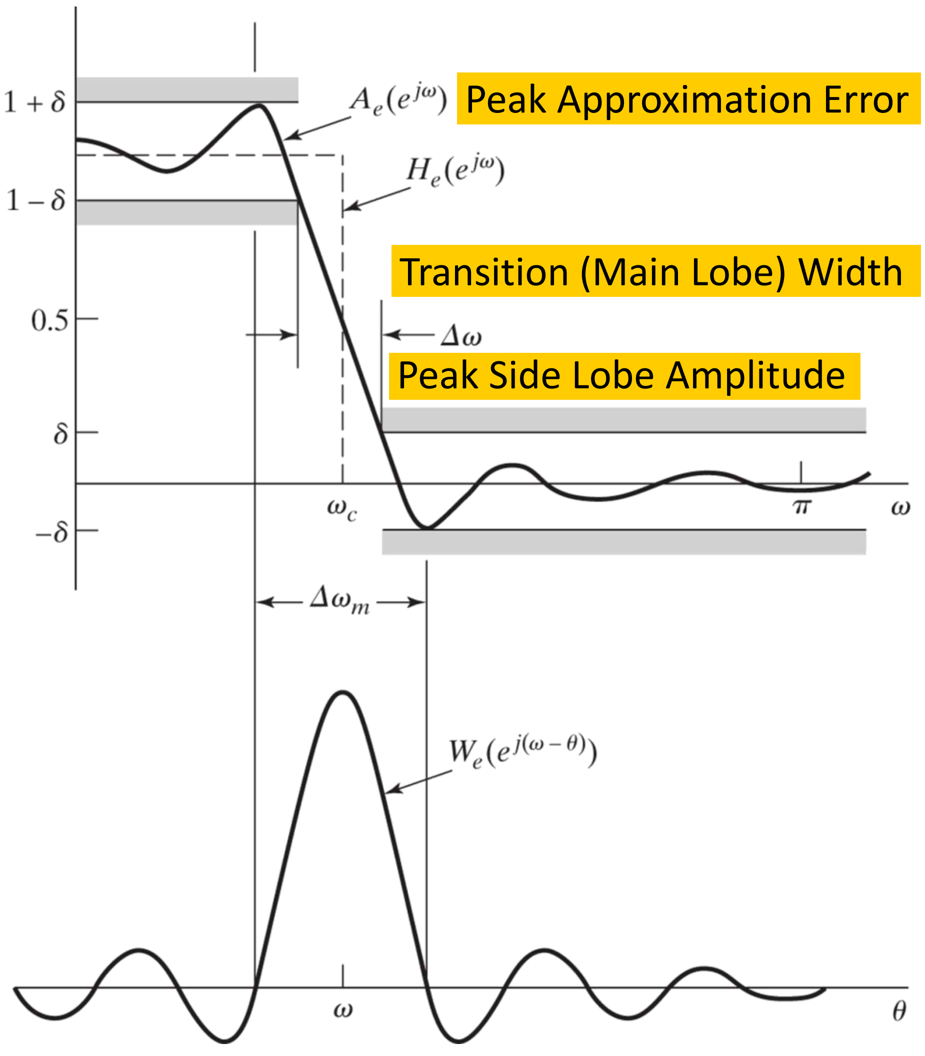
Windowing in the Frequency Domain

Effects of the convolution process & filter specification

1. Oscillations (high close to discontinuity) due to windowing (Gibbs Phenomenon)
2. Transition region due to main lobe width
3. Non zero side lobe amplitude

Filter Performance / Design Specifications



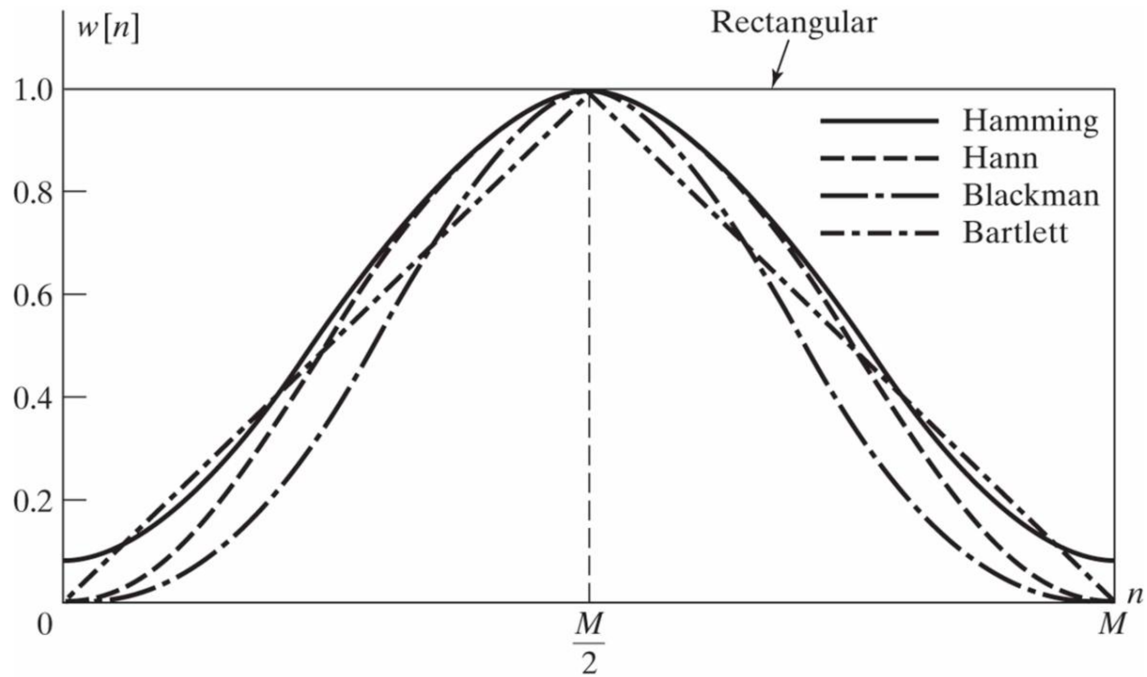


Design of FIR Filters using Windowing

Comparison of commonly used windows

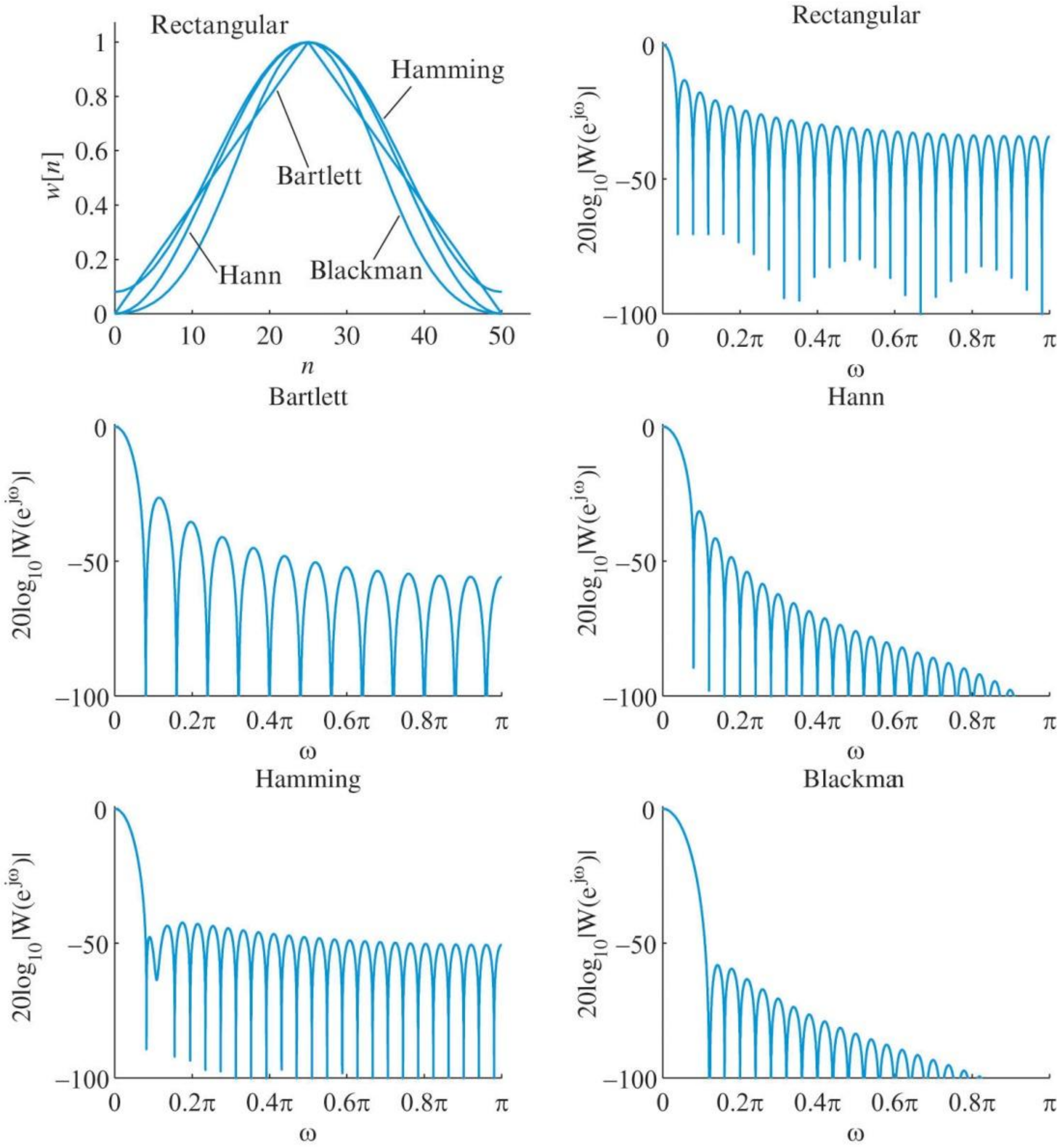
- Tapering the window smoothly to zero reduces the side-lobe amplitude and the peak approximation error
- Increasing the order M of the filter reduces the width of the main lobe
- Choosing a "smoother" window can result in larger main lobe width
- All windows are symmetric leading to linear phase filters

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)
Rectangular	−13	$4\pi/(M+1)$	−21
Bartlett	−25	$8\pi/M$	−25
Hann	−31	$8\pi/M$	−44
Hamming	−41	$8\pi/M$	−53
Blackman	−57	$12\pi/M$	−74



Design of FIR Filters using Windowing

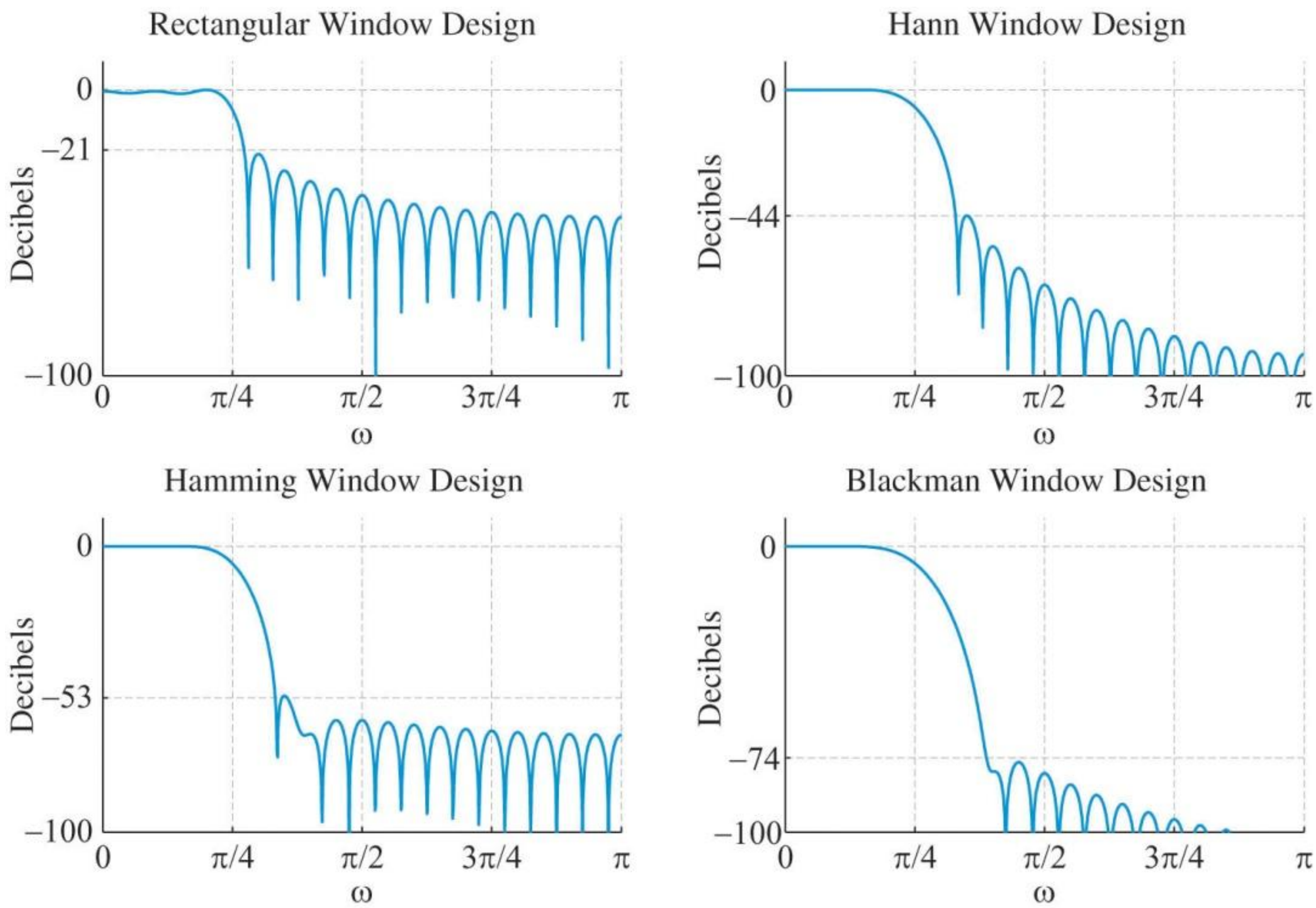
Comparison of commonly used windows



Design of FIR Filters using Windowing

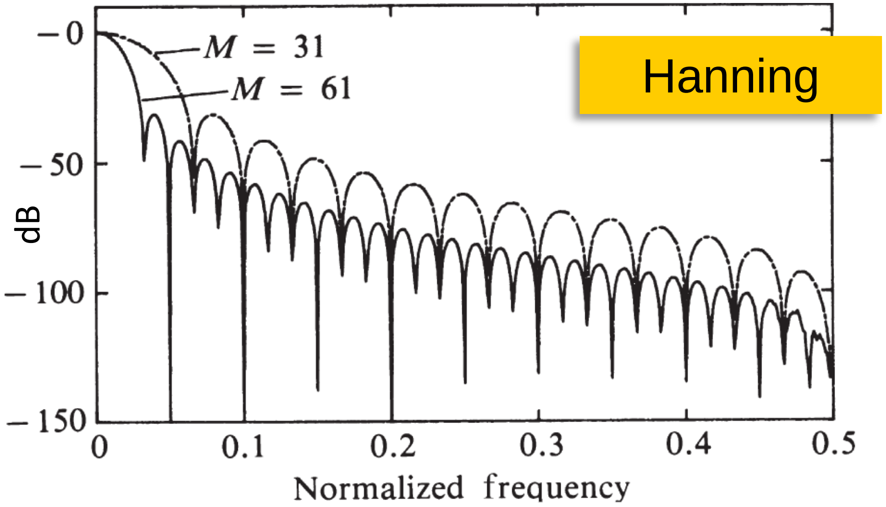
Comparison of commonly used windows

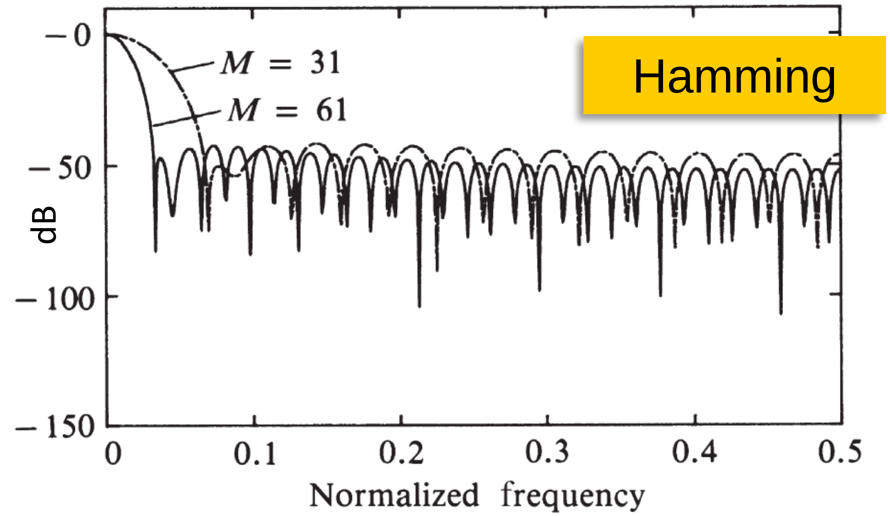
Magnitude response: 40th order FIR lowpass filter with cut-off frequency π/4

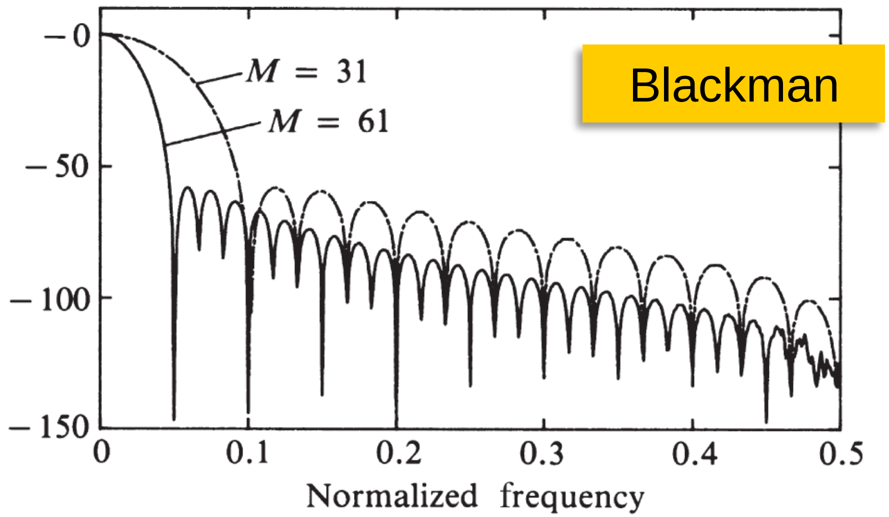


Comparison of commonly used windows

Effect of window length







Design of FIR Filters using Windowing

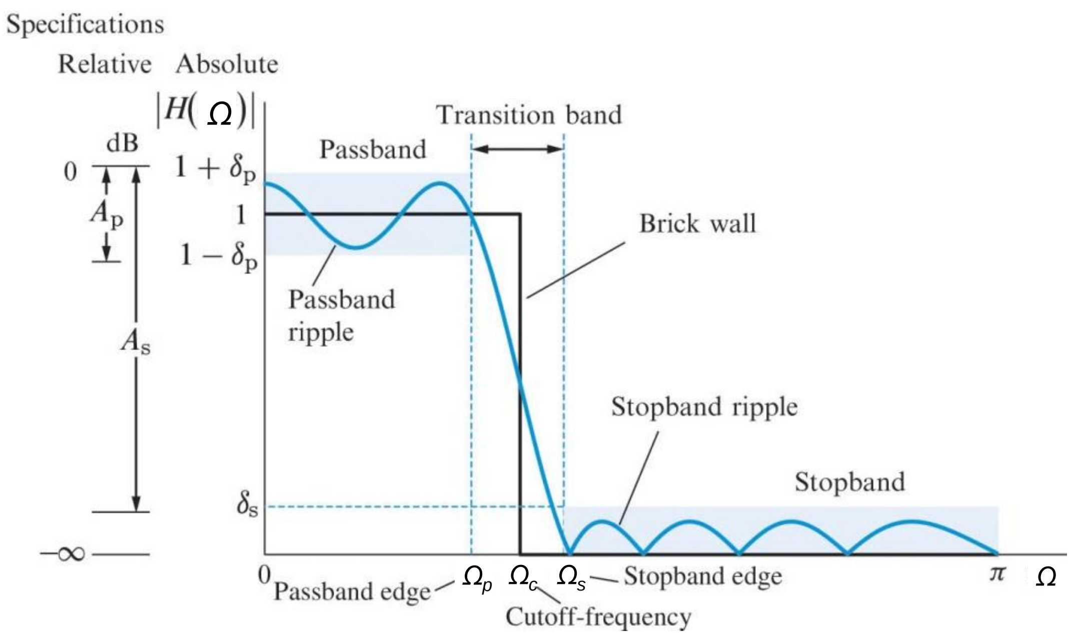
Filter Design Using Fixed Windows

Rectangular, Bartlett, Hann, Hamming, Blackman

Fixed Windows (stopband attenuation is independent of window length - fixed)

- Passband and stopband ripple are equal and independent of window length (only depend on shape of window)
- Width of transition band depends on length and shape of the window

Complexity Trade-offs



Design of FIR Filters using Windowing

Filter Design Using Fixed Windows

Rectangular, Bartlett, Hann, Hamming, Blackman

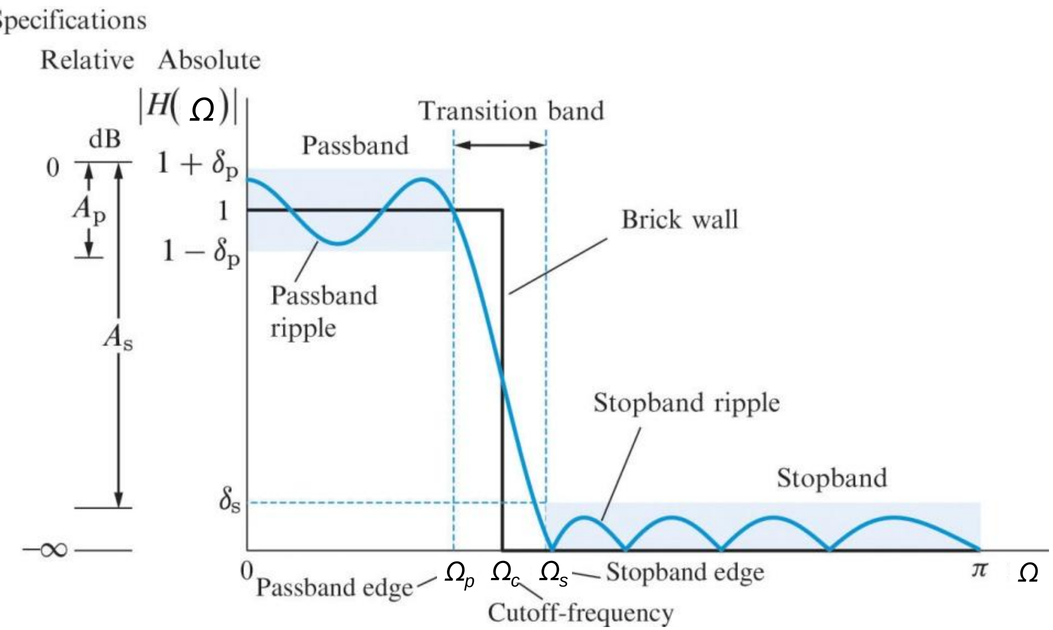
Window	Side lobe level (dB)	Approx. $\Delta\Omega$	A_p (dB)	A_s (dB)
Rectangular	-13	$4\pi/L$	0.75	21
Bartlett	-25	$8\pi/L$	0.45	26
Hann	-31	$8\pi/L$	0.055	44
Hamming	-41	$8\pi/L$	0.019	53
Blackman	-57	$12\pi/L$	0.002	74

Rectangular:
- Narrow transition band
- Big passband ripple
- Weak stopband attenuation

Bartlett, Hann, Hamming:
- Trade-off btwn width of transition band passband ripple & stopband attenuation

Blackman:
- Small passband ripple
- Strong stopband attenuation
- Wide transition band

Complexity



Design of FIR Filters using Windowing

Filter Design Using Fixed Windows

Rectangular, Bartlett, Hann, Hamming, Blackman

Window	Side lobe level (dB)	Approx. $\Delta\Omega$	A_p (dB)	A_s (dB)
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Narrow transition band
Big passband ripple
Weak stopband attenuation

Trade-off btwn width of transition band passband ripple & stopband attenuation

Small passband ripple
Strong stopband attenuation
Wide transition band

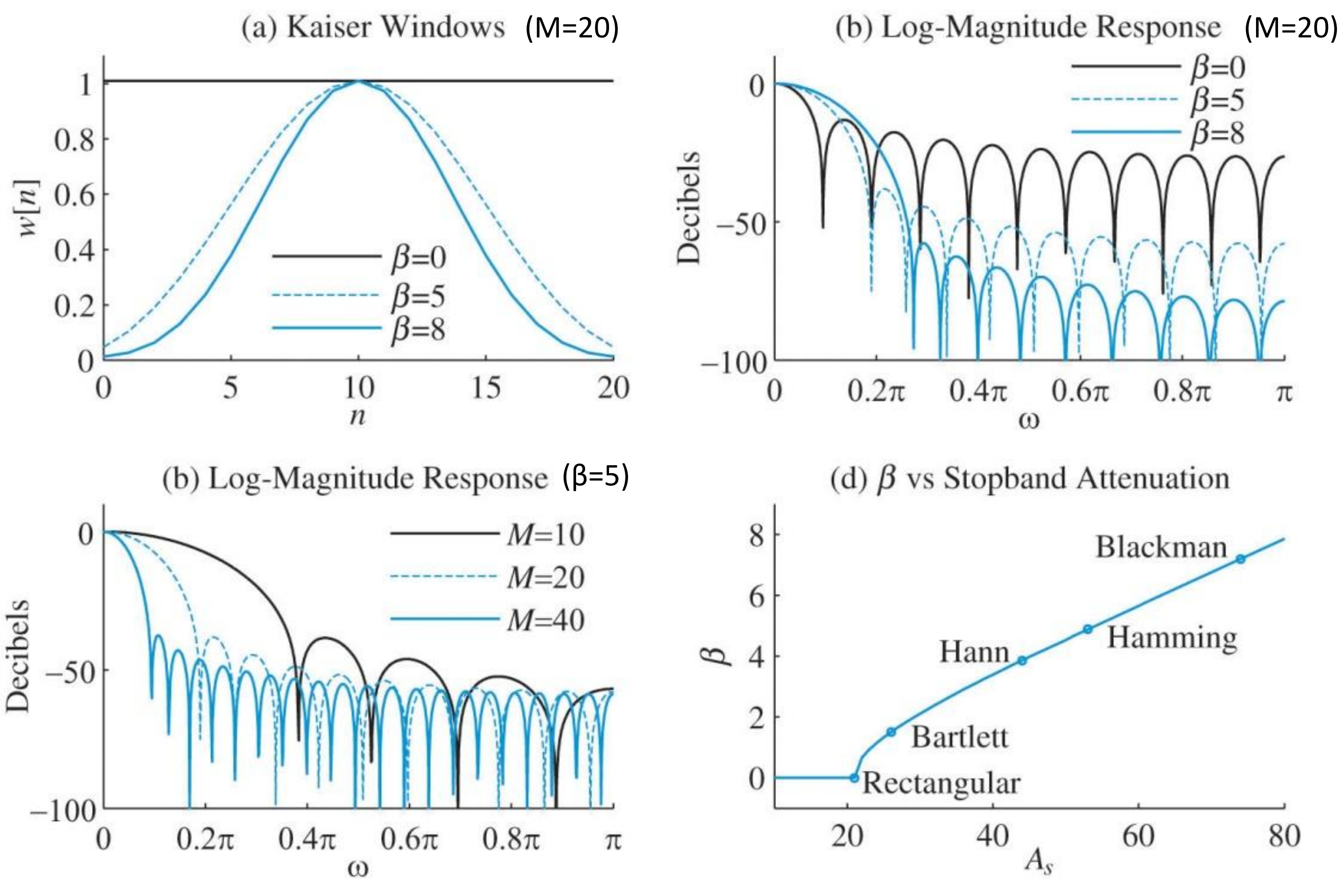
Complexity

1. Check the design specifications (Ω_p, Ω_s, A_p, A_s)
2. Determine the cut-off frequency of the ideal low pass prototype $\Omega_c = (\Omega_p + \Omega_s)/2$
3. Using table above choose the window function that provides the smallest stopband attenuation greater than A_s
4. Determine the required filter order (M=L-1) for the selected window that will give the desired transition bandwidth
5. Determine the impulse response of the ideal low pass filter with cut-off Ω_c
6. Compute the impulse response $h[n] = h_d[n]\, w[n]$ using the chosen window
7. Check if filter satisfies design specifications and if not increase the order and go to step 5

Design of FIR Filters using Windowing

Flexible Window

Keiser window



$$w[n]=\begin{cases}\dfrac{I_0\left[\beta(1-[(n-a)/a]^2)^{1/2}\right]}{I_0(\beta)}, & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}$$

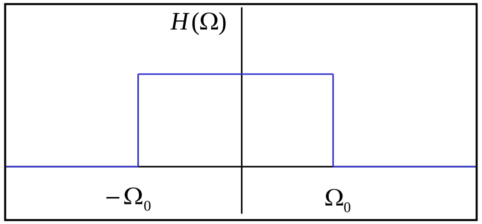
Trade-off between main lobe width (α=M/2) and side-lobe relative amplitude (β) possible

Design of FIR Filters using Windowing

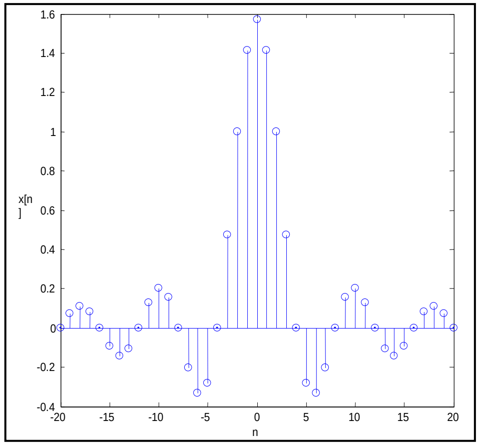
Windowing

Design Steps

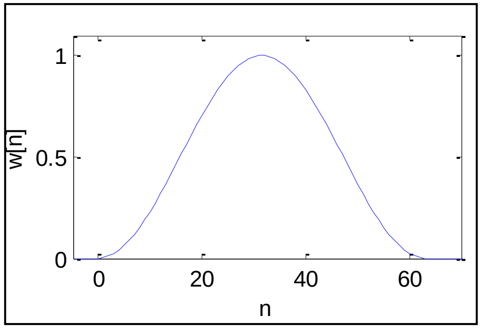
1. Specify desired Frequency response



2. Use inverse DTFT to obtain ideal impulse response



3. Multiply by window function to obtain a finite FIR approximation



For odd length (even order) filters it is often easier to design a non-causal zero phase filter and then add a M/2 sample delay term to obtain a causal filter (cannot do this for odd order)

M is the filter order

Design of FIR Filters using Windowing

Example 1: Odd Length – Rectangular Window

Design an 8th order (9 tap) linear phase low pass filter with a cut off of π/4 using a rectangular window.

Desired filter has odd length => design using delayed zero-phase filter

1. Desired (zero-phase) frequency response :

$$H(\Omega)=\begin{cases}1 & |\Omega|\le \frac{\pi}{4}\\ 0 & \frac{\pi}{4}<|\Omega|\le \pi\end{cases}$$

2. Use inverse DTFT to find the (infinite) impulse response:

$$h[n]=\frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\Omega n}.d\Omega=\left[\frac{e^{j\Omega n}}{2\pi jn}\right]_{-\pi/4}^{\pi/4}=\frac{1}{4}\operatorname{sinc}\left(\frac{n\pi}{4}\right)$$

3. Define a 9-tap rectangular window (<u>symmetric about n=0</u>)

$$w[n]=\begin{cases}1 & |n|\le 4\\ 0 & \text{elsewhere}\end{cases}$$

4. Multiply h[n] with w[n] to obtain the finite zero-phase filter

$$h_w[n]=h[n]w[n]=\begin{cases}\frac{1}{4}\operatorname{sinc}\left(\frac{n\pi}{4}\right) & |n|\le 4\\ 0 & \text{elsewhere}\end{cases}$$

Element-wise Multiplication

Design of FIR Filters using Windowing

Example 1: Odd Length – Rectangular Window

Design an 8th order (9 tap) linear phase low pass filter with a cut off of π/4 using a rectangular window.

$$h_w[n] = h[n]\,w[n] = \begin{cases} \frac{1}{4}\text{sinc}\left(\frac{n\pi}{4}\right) & |n| \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

5. Add a time delay of 4 samples to achieve a causal filter (h[n]=0 for n<0)

$$h_{wl}[n] = h_w[n-4] = \begin{cases} \frac{1}{4}\text{sinc}\left(\frac{(n-4)\pi}{4}\right) & 0 \le n \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

6. Evaluate for n=0 to 8 to find filter taps

$$h_{wl}[n] = \left\{0, \frac{\sqrt{2}}{6\pi}, \frac{1}{2\pi}, \frac{\sqrt{2}}{2\pi}, \frac{1}{4}, \frac{\sqrt{2}}{2\pi}, \frac{1}{2\pi}, \frac{\sqrt{2}}{6\pi}, 0\right\}$$

In this case the 1st and last taps are 0 =>this filter reduces to 6th order (7 taps)

Design of FIR Filters using Windowing

Example 1: Odd Length – Rectangular Window – Take 2

Design an 8th order (9 tap) linear phase low pass filter with a cut off of π/4 using a rectangular window.

Design causal filter directly noting that point of symmetry for a 9 tap filter is about n=4 (L-1/2)

1. Desired frequency response :

$$H(\Omega) = \begin{cases} e^{-j4\Omega} & |\Omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\Omega| \le \pi \end{cases}$$

Desired frequency response includes term $e^{-j4\Omega}$

2. Use inverse DTFT to find the (infinite) impulse response:

$$h[n] = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\Omega(n-4)}\, d\Omega = \left[\frac{e^{j\Omega(n-4)}}{2\pi j(n-4)} \right]_{-\pi/4}^{\pi/4} = \frac{1}{4}\operatorname{sinc}\left(\frac{(n-4)\pi}{4} \right)$$

IDTFT
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n}\, d\Omega$$

3. Define a 9-tap rectangular window (symmetric about n=4)

$$w[n] = \begin{cases} 1 & 0 \le n \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

4. Multiply h[n] with w[n] to obtain the finite linear phase filter

$$h_{wl}[n] = h[n]\, w[n] = \begin{cases} \frac{1}{4}\operatorname{sinc}\left(\frac{(n-4)\pi}{4} \right) & 0 \le n \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

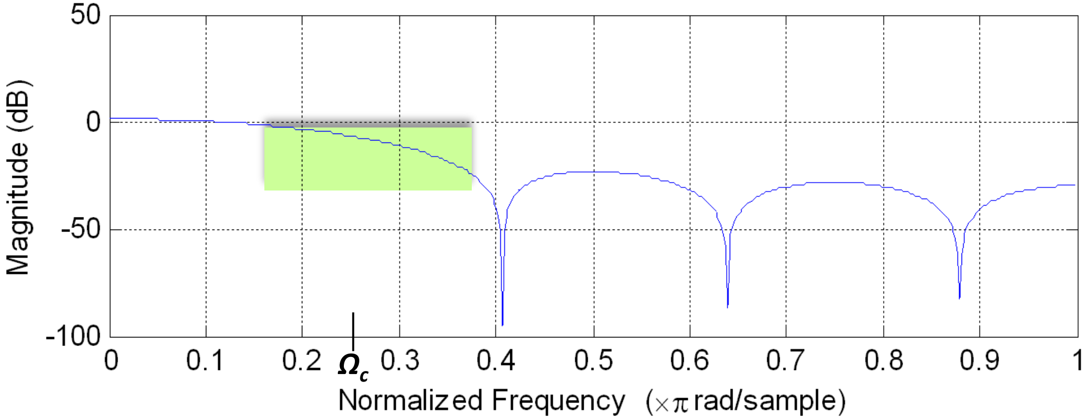
Same result as in previous slide

Design of FIR Filters using Windowing

Example 1: Odd Length – Rectangular Window

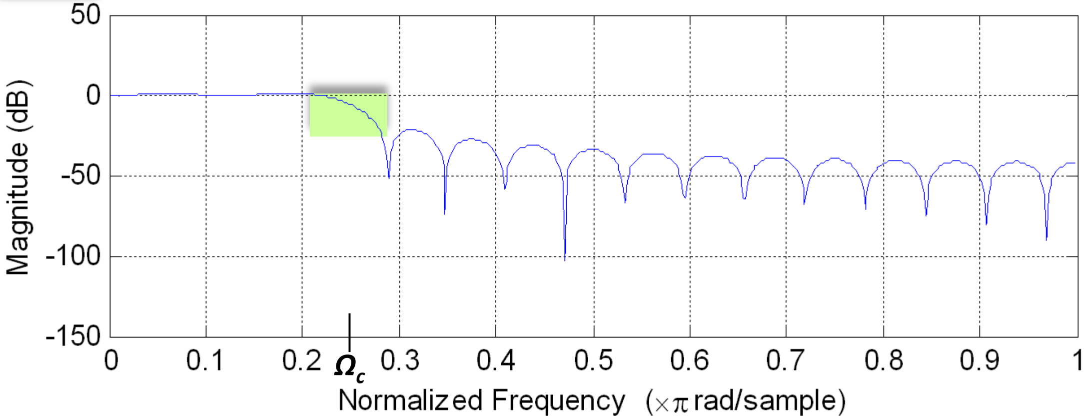
Effect of window length

8th order filter using rectangular window



$$h_{wl}[n]=\left\{0,\frac{\sqrt{2}}{6\pi},\frac{1}{2\pi},\frac{\sqrt{2}}{2\pi},\frac{1}{4},\frac{\sqrt{2}}{2\pi},\frac{1}{2\pi},\frac{\sqrt{2}}{6\pi},0\right\}$$

32nd order filter using rectangular window



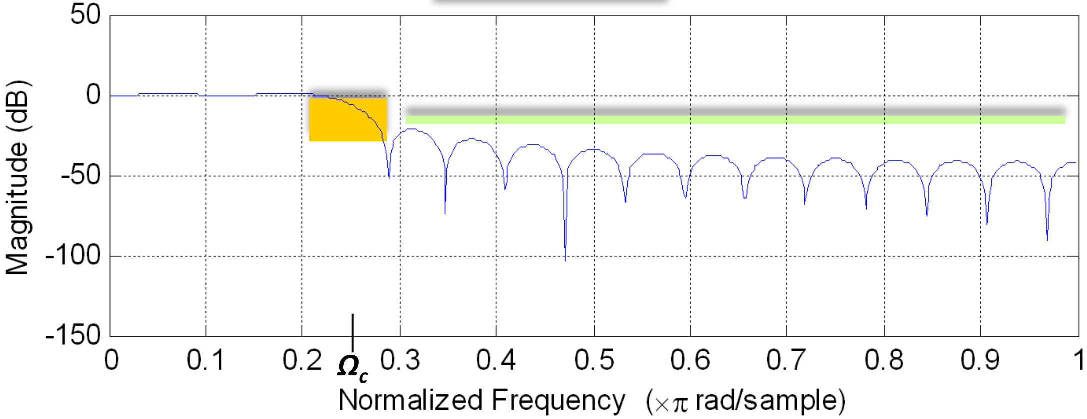
Width of transition band reduced due to increased window length

Complexity increased

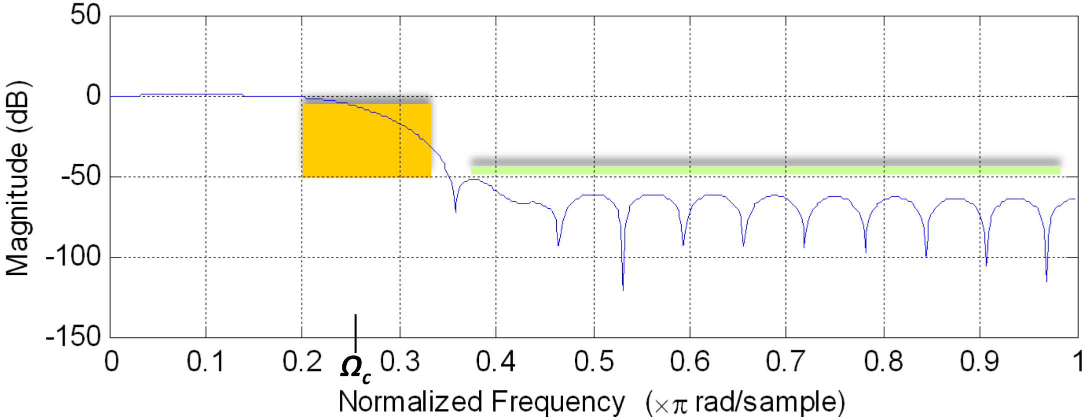
Example 1: Odd Length – Rectangular Window

Effect of window type

32nd order filter using **rectangular** window



32nd order filter using **Hamming** window



Stronger stopband attenuation for same complexity

Wider transition band

Design of FIR Filters using Windowing

Example 2: Even Length – Rectangular Window

Design an 7th order (8 tap) linear phase low pass filter with a cut off of π/4 using a rectangular window.

Even length filter => cannot represent as delayed version of zero-phase filter

Point of symmetry M/2 = 3½

1. Desired frequency response :

$$H(\Omega) = \begin{cases} e^{-j7\Omega/2} & |\Omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\Omega| \le \pi \end{cases}$$

2. Use inverse DTFT to find the (infinite) impulse response:

$$h[n] = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\Omega(n-\frac{1}{2})}\, d\Omega = \left[\frac{e^{j\Omega(n-\frac{7}{2})}}{2\pi j n} \right]_{-\pi/4}^{\pi/4} = \frac{1}{4}\,\text{sinc}\left(\frac{(n-\frac{7}{2})\pi}{4} \right)$$

3. Define a 8-tap rectangular window (<u>symmetric about n=7/2</u>)

$$w[n] = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{elsewhere} \end{cases}$$

4. Multiply h[n] with w[n] to obtain the finite linear phase filter

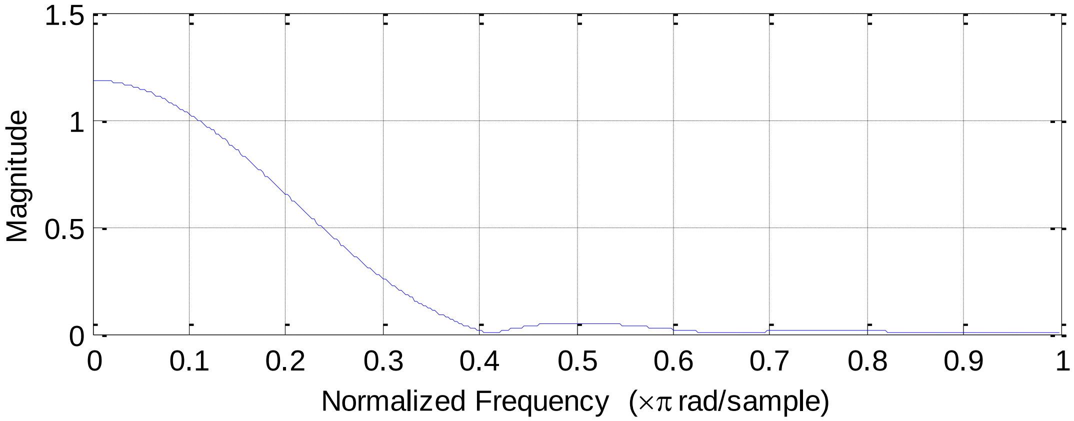
$$h_{wl}[n] = h[n]\,w[n] = \begin{cases} \frac{1}{4}\,\text{sinc}\left(\frac{(n-\frac{7}{2})\pi}{4} \right) & 0 \le n \le 7 \\ 0 & \text{elsewhere} \end{cases}$$

$$h_{wl}[n] = \{0.0348, \quad 0.1176, \quad 0.1961, \quad 0.2436, \quad 0.2436, \quad 0.1961, \quad 0.1176, \quad 0.0348\}$$

Example 2: Even Length – Rectangular Window

Design an 7^{th} order (8 tap) linear phase low pass filter with a cut off of π/4 using a rectangular window.

7^{th} order filter using rectangular window



$$h_{wl}[n] = \{0.0348, \quad 0.1176, \quad 0.1961, \quad 0.2436, \quad 0.2436, \quad 0.1961, \quad 0.1176, \quad 0.0348\}$$

Design of FIR Filters using Windowing

Example 3: Odd Length – Hamming Window

Use a Hamming window to design a 6th order (7 tap) low pass FIR filter with a cut-off of 100Hz and sampling frequency 1200Hz.

1. Desired frequency response :

$$|H(\Omega)| = \begin{cases} 1 & |\Omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < |\Omega| \le \pi \end{cases} \qquad H(\Omega) = \begin{cases} e^{-j3\Omega} & |\Omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < |\Omega| \le \pi \end{cases}$$

1.Find normalised cut-off freq: $\Omega_c = 2\pi f / f_s = 2\pi 100/1200 = \pi/6$

2.Find point of symmetry (group delay) : M/2 =3

2. Use inverse DTFT to find the (infinite) impulse response:

$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{-j3\Omega}e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{j(n-3)\Omega}d\Omega = \frac{1}{2\pi}\left[\frac{e^{j\Omega(n-3)}}{j(n-3)}\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{e^{j(n-3)\pi/6} - e^{-j(n-3)\pi/6}}{2\pi j(n-3)} = \frac{\sin\left((n-3)\pi/6\right)}{\pi(n-3)} = \frac{1}{6}\operatorname{sinc}\left((n-3)\pi/6\right)$$

3. Define a Hamming window with 7 points

$$w[n] = \alpha - (1-\alpha)\cos\left(\frac{2\pi n}{L-1}\right), \quad 0 \le n \le L-1$$

$a = 0.54$

Example 3: Odd Length – Hamming Window

Use a Hamming window to design a 6th order (7 tap) low pass FIR filter with a cut-off of 100Hz and sampling frequency 1200Hz.

4. Multiply $h[n]$ with $w[n]$ to obtain filter

$$h_{wl}[n] = h[n] \times w[n] \quad , \quad h[n] = \frac{1}{6}\,\mathrm{sinc}\big((n-3)\pi/6\big) \quad , \quad w[n] = 0.54 - (1-0.54)\cos\left(\frac{2\pi n}{6}\right)$$

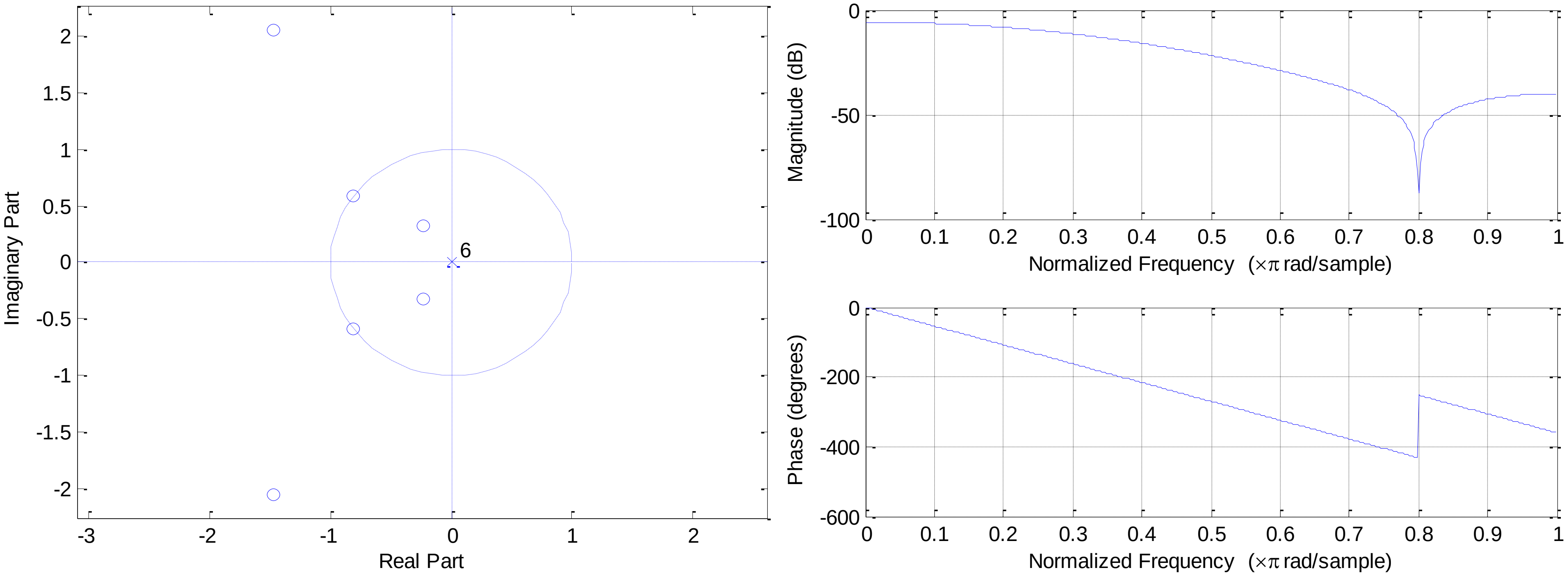
n	…	0	1	2	3	4	5	6	…
$h[n]$	…	0.1061	0.1378	0.1592	0.1667	0.1592	0.1378	0.1061	…
$w[n]$	0	0.08	0.31	0.77	1	0.77	0.31	0.08	0
$h_{wl}[n]$	0	0.0085	0.0427	0.1225	0.1667	0.1225	0.0427	0.0085	0

$$H_{wl}(z) = 0.0085\,(z^{0} + z^{-6}) + 0.0427\,(z^{-1} + z^{-5}) + 0.1225\,(z^{-2} + z^{-4}) + 0.1667\,z^{-3}$$

Design of FIR Filters using Windowing

Example 3: Odd Length – Hamming Window

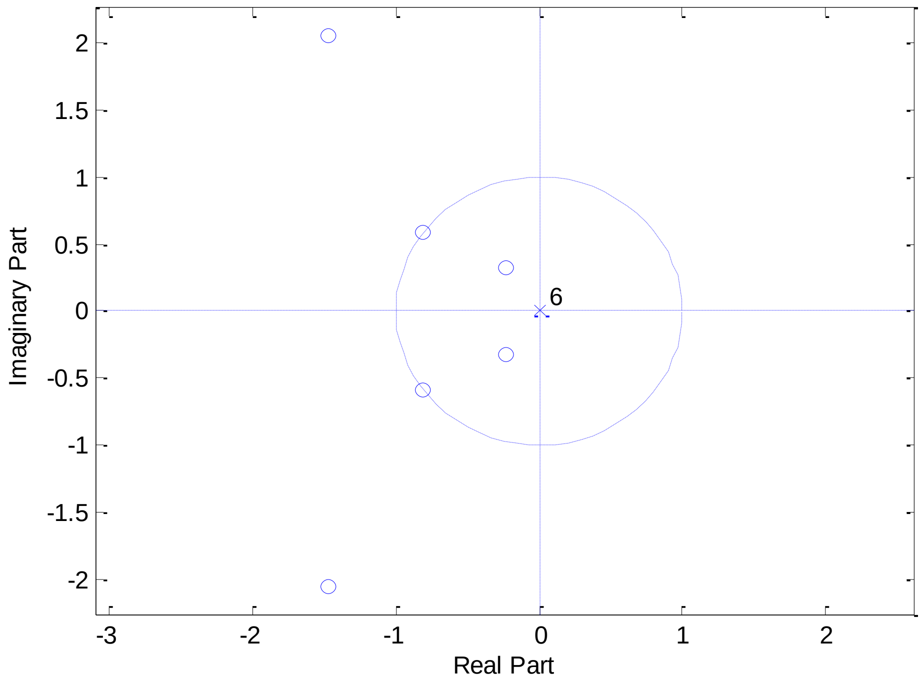
Use a Hamming window to design a 6th order (7 tap) low pass FIR filter with a cut-off of 100Hz and sampling frequency 1200Hz.

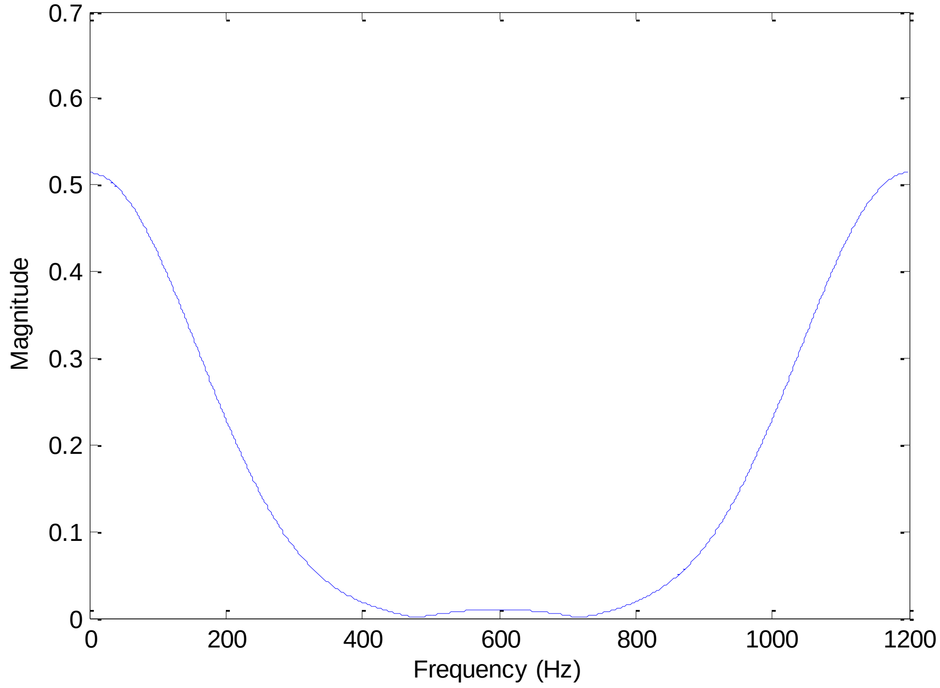


$$H_{wl}(z) = 0.0085\,(z^{0} + z^{-6}) + 0.0427\,(z^{-1} + z^{-5}) + 0.1225\,(z^{-2} + z^{-4}) + 0.1667\,z^{-3}$$

Example 3: Odd Length – Hamming Window

Use a Hamming window to design a 6th order (7 tap) low pass FIR filter with a cut-off of 100Hz and sampling frequency 1200Hz.





$$H_{wl}(z) = 0.0085\,(z^{0} + z^{-6}) + 0.0427\,(z^{-1} + z^{-5}) + 0.1225\,(z^{-2} + z^{-4}) + 0.1667\,z^{-3}$$

Design of FIR Filters using Windowing

Example 4: Band Pass Filter – Rectangular Window

Use a Rectangular window to design a 7th order (8 tap) band pass FIR filter with a pass-band between 100Hz-300Hz and sampling frequency 1200Hz.

1. Find normalised cut-off freq

$$\Omega_1 = 2\pi f / f_s = 2\pi 100 \ / 1200 \ = \pi / 6$$

$$\Omega_2 = 2\pi f / f_s = 2\pi 300 \ / 1200 \ = \pi / 2$$

2. Find point of symmetry (group delay) : M/2 = 3.5

1. Desired frequency response:

$$H(\Omega) = \begin{cases} e^{-j3.5\Omega} & \frac{\pi}{6} \le |\Omega| \le \frac{\pi}{2} \\ 0 & elsewhere \end{cases}$$

2. Construct band pass filter from 2 low pass filters

$$H(\Omega) = H_{LP(\Omega_2)}(\Omega) - H_{LP(\Omega_1)}(\Omega), \quad \Omega_2 = \pi / 2, \Omega_1 = \pi / 6$$

3. Inverse DTFT to find impulse response of generic low pass

$$H_{LP}(\Omega) = \begin{cases} e^{-j\tau_{gd}\Omega} & |\Omega| \le \Omega_c \\ 0 & \Omega_c < |\Omega| \le \pi \end{cases} \xrightarrow{IDTFT} h_{LP}[n] = \frac{1}{2\pi} \int_{2\pi} H_{LP}(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j(n-\tau_{gd})\Omega} d\Omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\Omega(n-\tau_{gd})}}{j(n-\tau_{gd})} \right]_{-\Omega_c}^{\Omega_c} = \frac{e^{j(n-\tau_{gd})\Omega_c} - e^{-j(n-\tau_{gd})\Omega_c}}{2\pi j(n-\tau_{gd})} = \frac{\sin\left((n-\tau_{gd})\Omega_c\right)}{\pi(n-\tau_{gd})} = \frac{\Omega_c}{\pi} \mathrm{sinc}\left((n-\tau_{gd})\Omega_c\right)$$

Example 4: Band Pass Filter – Rectangular Window

Use a Rectangular window to design a 7th order (8 tap) band pass FIR filter with a pass-band between 100Hz-300Hz and sampling frequency 1200Hz.

4. Construct band pass impulse response from low pass filters' impulse responses

$$h[n] = \frac{\Omega_2}{\pi}\,\mathrm{sinc}\big((n-3.5)\Omega_2\big) - \frac{\Omega_1}{\pi}\,\mathrm{sinc}\big((n-3.5)\Omega_1\big)$$

5. Multiply with rectangular window $h_w[n] = h[n]\,w[n]$ $\quad w[n]=1 \quad 0 \le n \le L-1$

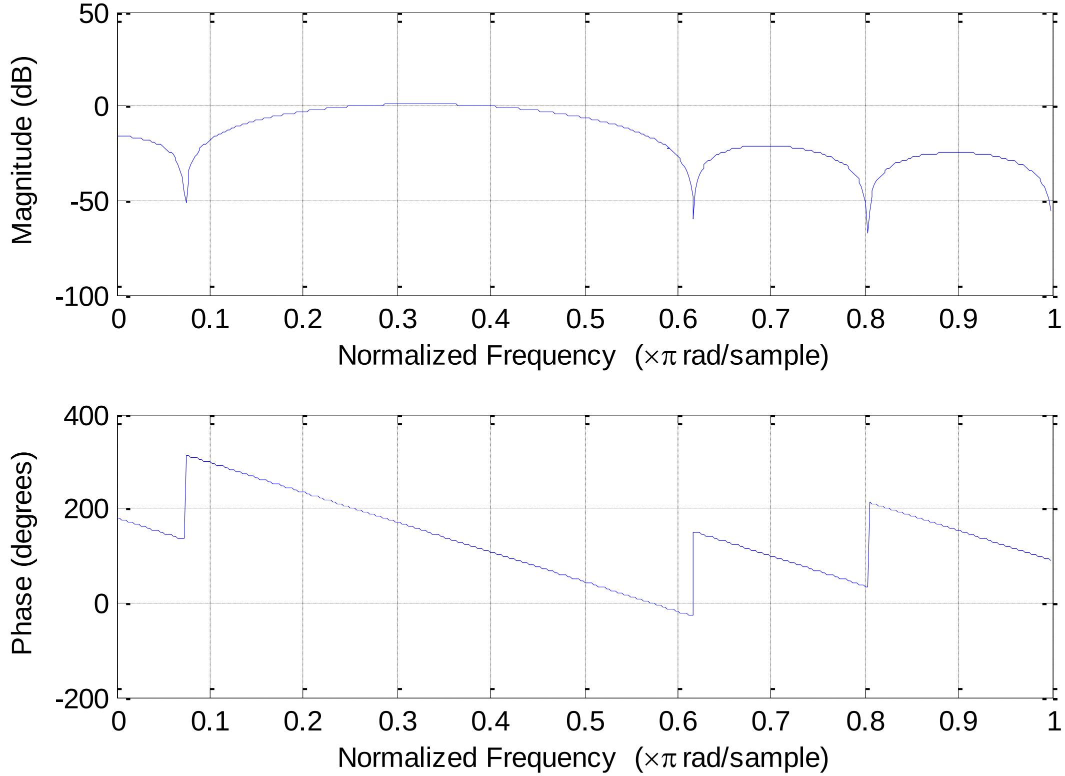
n	…	0	1	2	3	4	5	6	7	…
h[n]	…	-0.1522	-0.2130	0	0.2854	0.2854	0	-0.2130	-0.1522	…
w[n]	0	1	1	1	1	1	1	1	1	0
$H_w[n]$	0	-0.1522	-0.2130	0	0.2854	0.2854	0	-0.2130	-0.1522	0

$$H_w(z) = -0.1522(1+z^{-7}) - 0.2130(z^{-1}+z^{-6}) + 0.2854(z^{-3}+z^{-4})$$

Design of FIR Filters using Windowing

Example 4: Band Pass Filter – Rectangular Window

Use a Rectangular window to design a 7th order (8 tap) band pass FIR filter with a pass-band between 100Hz-300Hz and sampling frequency 1200Hz.



$$H_w(z) = -0.1522(1+z^{-7}) - 0.2130(z^{-1}+z^{-6}) + 0.2854(z^{-3}+z^{-4})$$

Filter Design in Matlab

Filter design functions

- `W = rectwin(N),hamming(N),bartlett(N), blackman(N)`

  Returns the N-point window

- `B = fir1(N,Wn,WIN)`

  Designs an N-th order FIR filter with cut-off frequency `Wn`, using the N+1 length vector `WIN` to window the impulse response. If empty or omitted, fir1 uses a Hamming window of length N+1. For a complete list of available windows, see the help for the `WINDOW` function. A `KAISER` window can be specified with an optional trailing argument. For example, `B = fir1(N,Wn,kaiser(N+1,4))` uses a Kaiser window with beta=4.

Design of FIR Filters using Windowing

Filter Design in Matlab

Filter design and analysis tool <fdatool>