Lecture 7

(Part 1)

The Discrete Fourier Transform

Digital spectral analysis of discrete-time signals

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Discrete Fourier Transform

Performing spectral analysis on a digital signal processor

Why do we need the DFT

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\Omega n} \qquad \Bigg| \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Time Domain
Infinite summation

Frequency Domain
Continuous representation

Computationally infeasible

Infinite memory required

Cannot be realised numerically

Discrete Fourier Transform

Performing spectral analysis on a digital signal processor

Why do we need the DFT

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Use a finite portion of the signal with N samples

(Windowing)

Sample the Frequency Domain taking N samples

(Discrete Frequency)

DFT

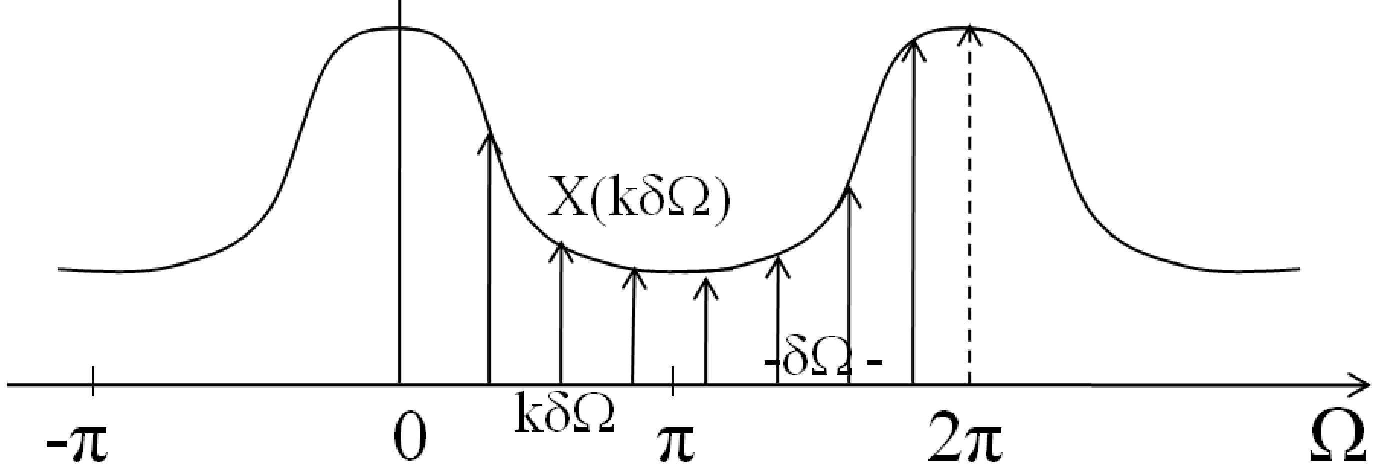
Discrete Fourier Transform

From the DTFT to the DFT

Frequency domain sampling

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$



N equidistant samples

$$\delta\Omega = \frac{2\pi}{N} \quad , \; 0 \le \Omega < 2\pi$$

Discrete Fourier Transform

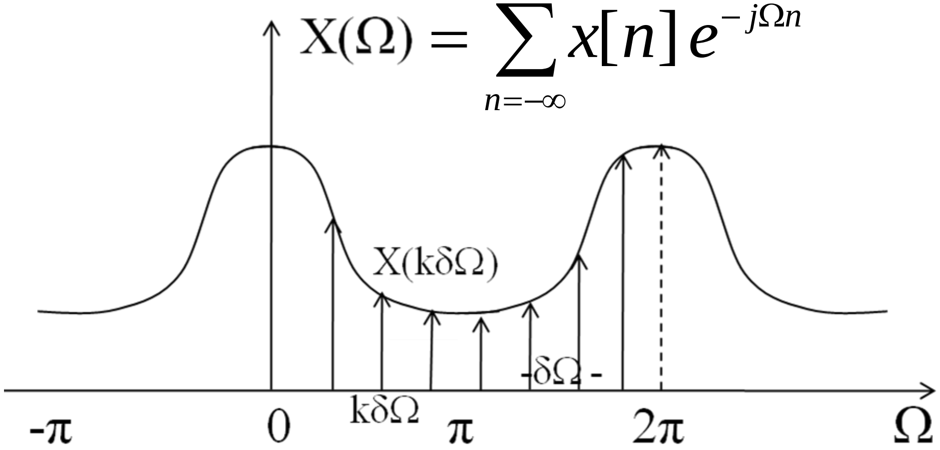
From the DTFT to the DFT

Frequency domain sampling – Can we recover the original DT signal?

$$\Omega_k = \frac{2\pi}{N} k, \; k = 0,1,\ldots, N-1$$

$$X(\Omega_k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega_k n}$$

$$X(\frac{2\pi}{N} k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi}{N}kn}$$



$$= \ldots + \sum_{n=-N}^{-1} x[n].e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x[n].e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x[n].e^{-j2\pi kn/N} + \ldots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n] \; e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN] \, e^{-j2\pi kn/N}$$

1. $n \rightarrow n - lN$
2. Change order of summation

$$e^{-j2\pi k(n-lN)/N} = e^{-j2\pi kn/N} e^{j2\pi klN/N} = e^{-j2\pi kn/N} e^{j2\pi kl}$$

Discrete Fourier Transform

From the DTFT to the DFT

Frequency domain sampling – Can we recover the original DT signal?

$$\Omega_k = \frac{2\pi}{N}k, \; k = 0,1,\ldots,N-1$$

$$X(\Omega_k) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega_k n}$$

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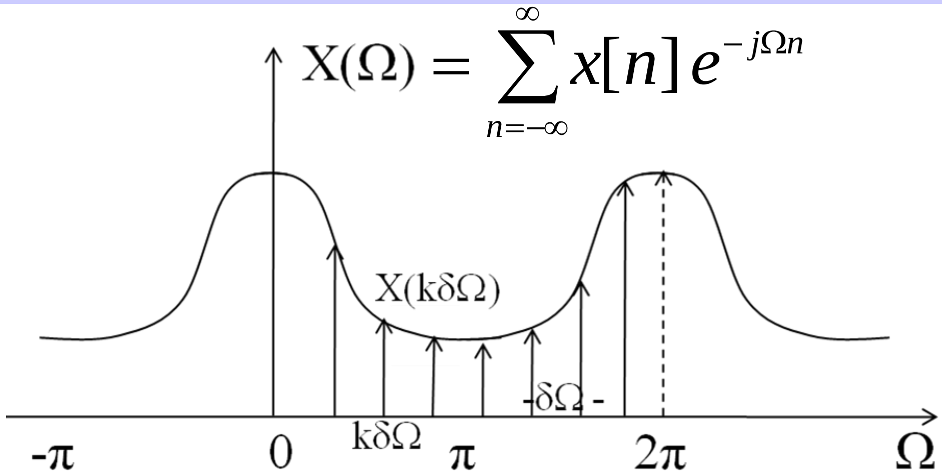


$$X(\frac{2\pi}{N}k) = \sum_{n=0}^{N-1} \boxed{\sum_{l=-\infty}^{\infty} x[n-lN]} e^{-j2\pi kn/N}, \; k = 0,1,\ldots,N-1$$

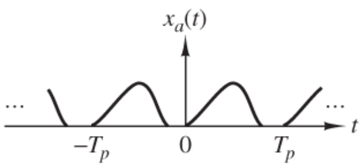
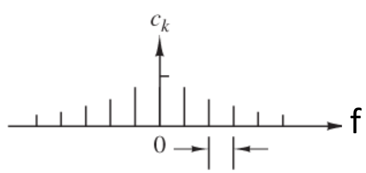
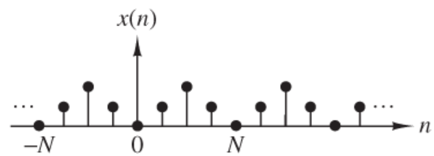
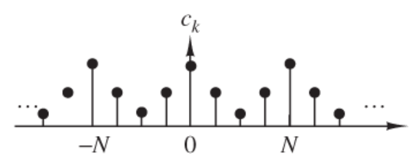
$$x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN], n = 0,1,..,N-1$$ Periodic version of x[n] with period N

$$X(\frac{2\pi}{N}k) = \sum_{n=0}^{N-1} x_p(n)e^{-j2\pi kn/N}, \; k = 0,1,\ldots,N-1 \qquad (1)$$

Discrete Fourier Transform

Discrete Fourier Series

Fourier series for discrete-time periodic signals

	Continuous-time signals		Discrete-time signals	
	Time-domain	Frequency-domain	Time-domain	Frequency-domain
Periodic signals / Fourier series				
	$c_k = \frac{1}{T}\int_T x_a(t) e^{-jk\omega_0 t} dt \Rightarrow$	$\Leftarrow$ $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \Rightarrow$	$\Leftarrow$ $x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$
	Continuous and periodic	Discrete and aperiodic	Discrete and periodic	Discrete and periodic

$$T_p \rightarrow N \ , \quad \omega_0 = 2\pi / T_p \rightarrow 2\pi / N \ , \quad t \rightarrow n \ , 0 \le k \le N-1$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x[n - lN] \, , n = 0,1,.., N-1$$

Periodic version of x[n] with period N

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \, , \ n = 0,1,..., N-1$$

Fourier Series expansion of $x_p[n]$

Discrete Fourier Transform

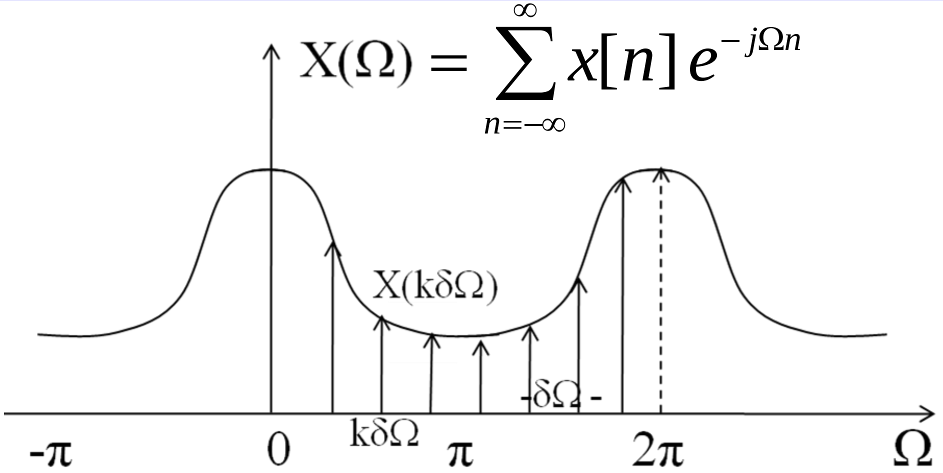
From the DTFT to the DFT

Frequency domain sampling – Can we recover the original DT signal?

$$\Omega_k = \frac{2\pi}{N}k, \quad k = 0,1,\ldots, N-1$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x[n-lN], \quad n = 0,1,\ldots, N-1$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N} \qquad (1)$$



$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, \quad n = 0,1,\ldots, N-1$$

Fourier Series expansion of $x_p[n]$ (2)

$$c_k = \frac{1}{N}\boxed{\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}} \quad \overset{(1)}{\Rightarrow} c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right), \quad k = 0,1,\ldots, N-1 \qquad (3)$$

$$\overset{(2,3)}{\Rightarrow} x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}, \quad n = 0,1,\ldots, N-1 \qquad (4)$$

From the DTFT to the DFT

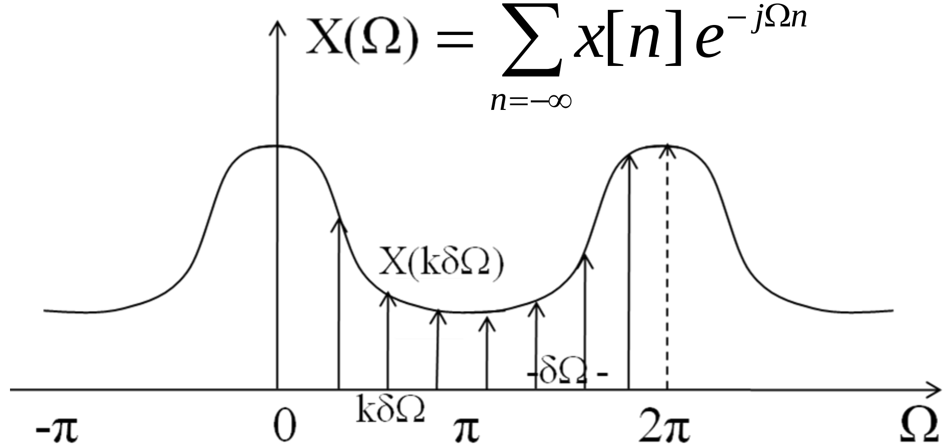
Can we recover the aperiodic DT signal x[n] from the samples of X(Ω)?

From the sampled spectrum we can recover a periodic extension of *x[n]* , $x_p[n]$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x[n-lN],\; n = 0,1,\ldots,N-1$$

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N} \qquad (4)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$



1. Sampling in the frequency domain leads to a periodic representation in the time domain
2. Sampling in the frequency domain can lead to overlapping copies in the time domain

Time Domain Aliasing

1. If aliasing is not present then we can recover x[n] from the periodic signal $x_p[n]$:

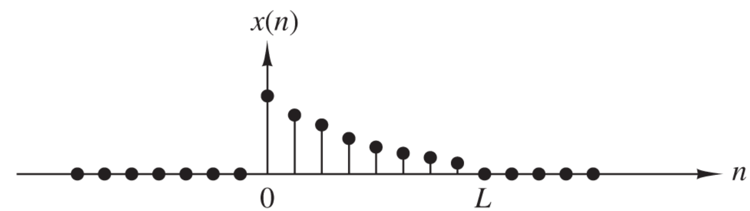
$$x[n] = \begin{cases} x_p[n], & 0 \le n \le N-1 \\ 0\;, & elsewhere \end{cases}$$

Discrete Fourier Transform

Time Domain Aliasing

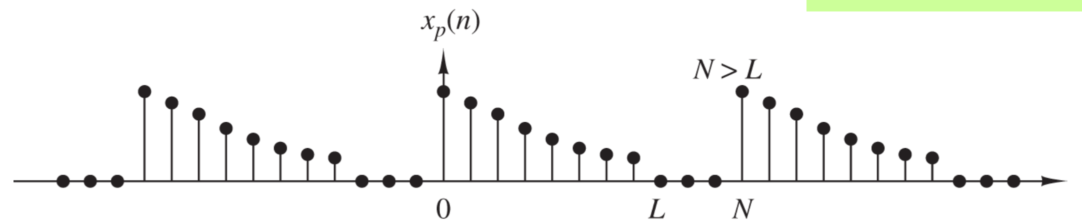
Can we recover the aperiodic signal x[n] from the samples of X(Ω)?

a) x[n] Original discrete time signal of length L

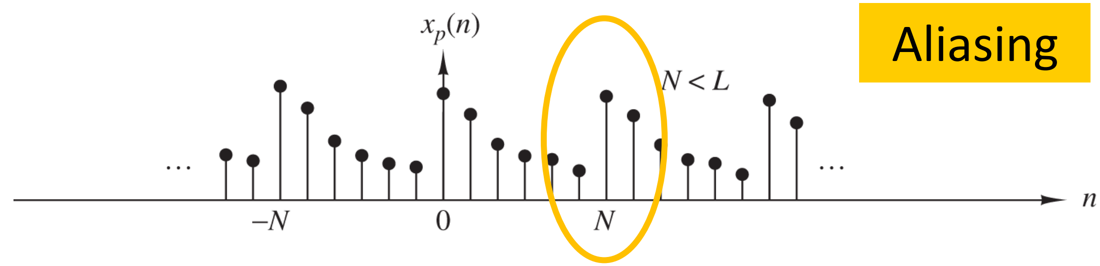


No Aliasing

b) x_p[n] Periodic discrete time signal recovered from the sampled spectrum when **N > L**



c) x_p[n] Aliased periodic discrete time signal recovered from the sampled spectrum when **N < L**



Aliasing

To avoid aliasing the number of samples N in the frequency domain should be greater than the number of samples L in the time domain (length of DFT > length of signal)

Discrete Fourier Transform

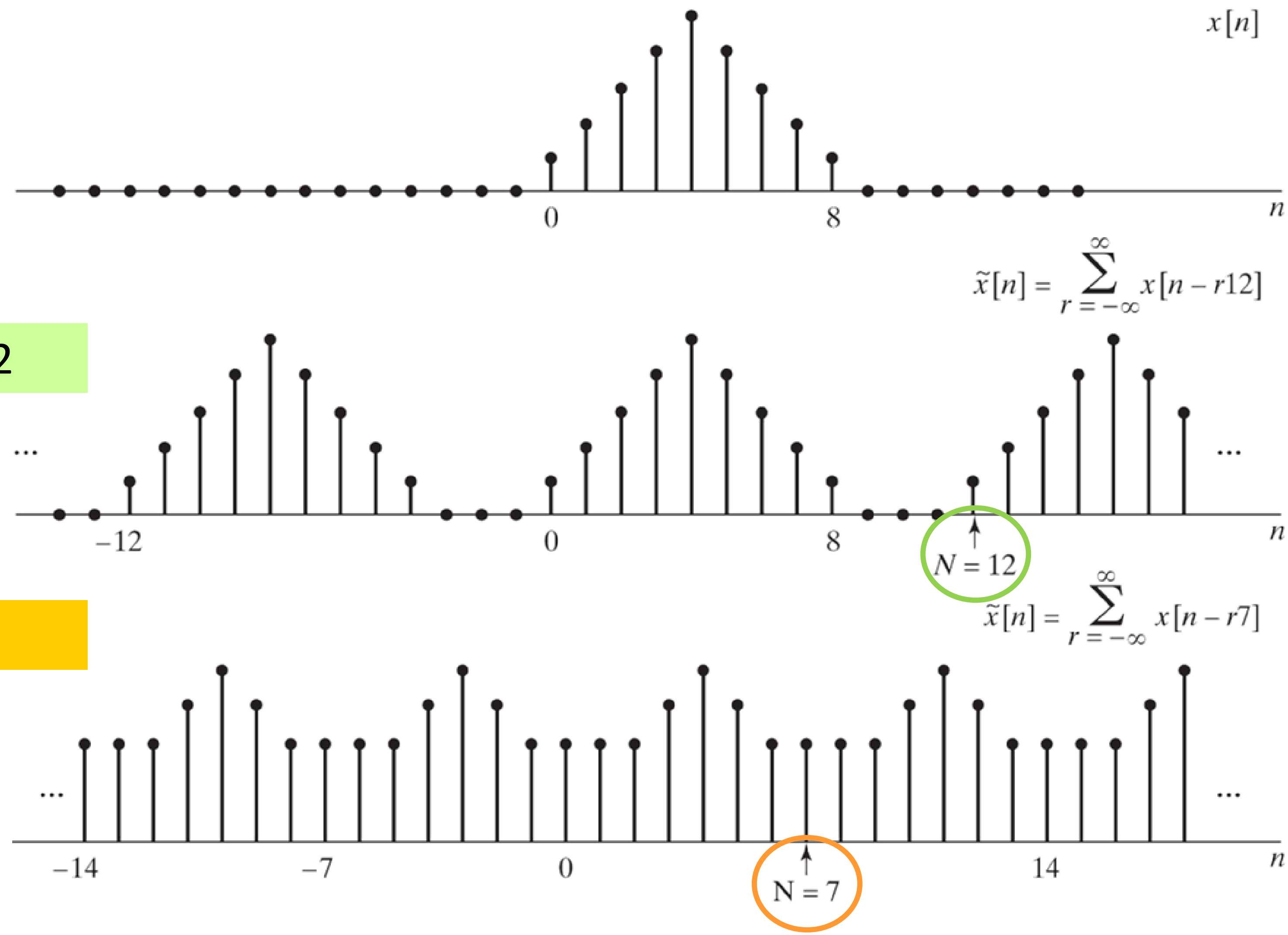
Time Domain Aliasing

Avoiding overlapping copies in the time domain due to spectrum sampling

L=8

N=12

N=7



Discrete Fourier Transform

From the DTFT to the DFT

From infinite-extent periodic signals to finite extent aperiodic ones

$$x_p(n) = \sum_{l=-\infty}^{\infty} x[n - lN]\,,\ n = 0,1,\ldots,\ N-1 \quad = \quad x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi kn/N}$$

Infinite extent periodic signal

N>L => No Aliasing

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi kn/N}\,,\ 0 \le n \le N-1$$

Inverse Discrete Fourier Transform (IDFT)

Discrete Fourier Transform

From the DTFT to the DFT

From infinite-extent periodic signals to finite extent aperiodic ones

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \; k = 0,1,2,\ldots,N-1$$

Discrete Fourier Transform (DFT)

N>L => No Aliasing

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(\frac{2\pi}{N} k) e^{j2\pi kn/N}, \; 0 \le n \le N-1$$

Inverse Discrete Fourier Transform (IDFT)

Discrete Fourier Transform

Improving the visualisation of the DFT

DTFT of Rectangular Pulse

Signal Length L =5

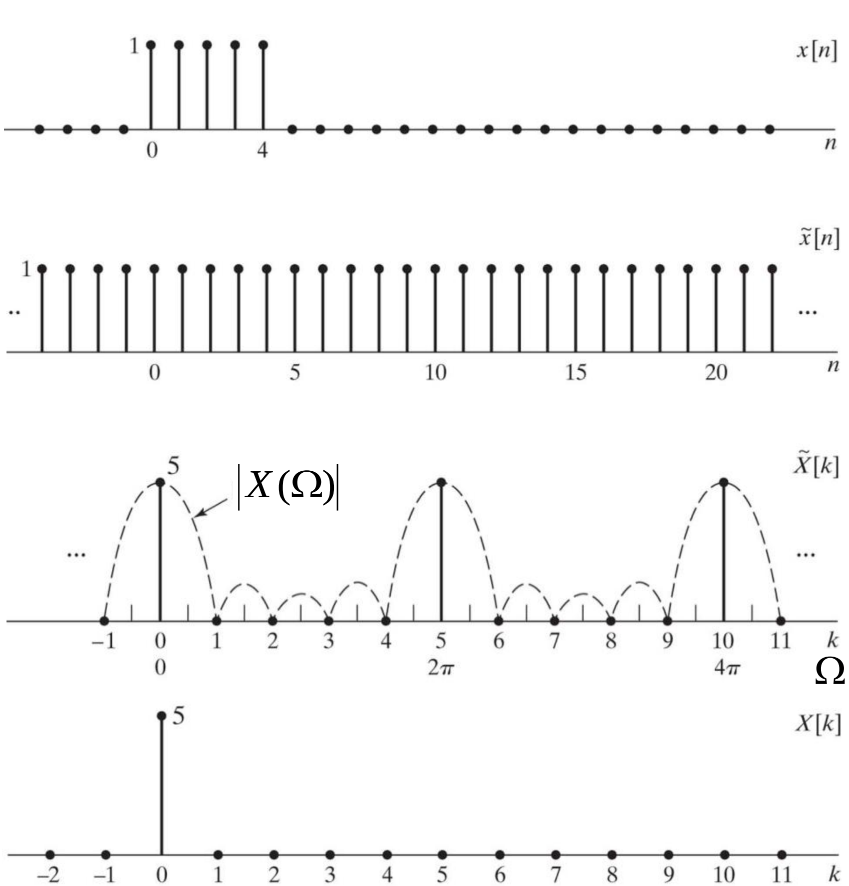
$$x[n]=\begin{cases}1 & 0\le n\le L-1\\ 0 & \text{elsewhere}\end{cases}\quad\overset{DTFT}{\leftrightarrow}\quad X(\Omega)=\frac{\sin(\Omega(L/2))}{\sin(\Omega/2)}e^{-j(\Omega/2)(L-1)}$$

Discrete Fourier Transform

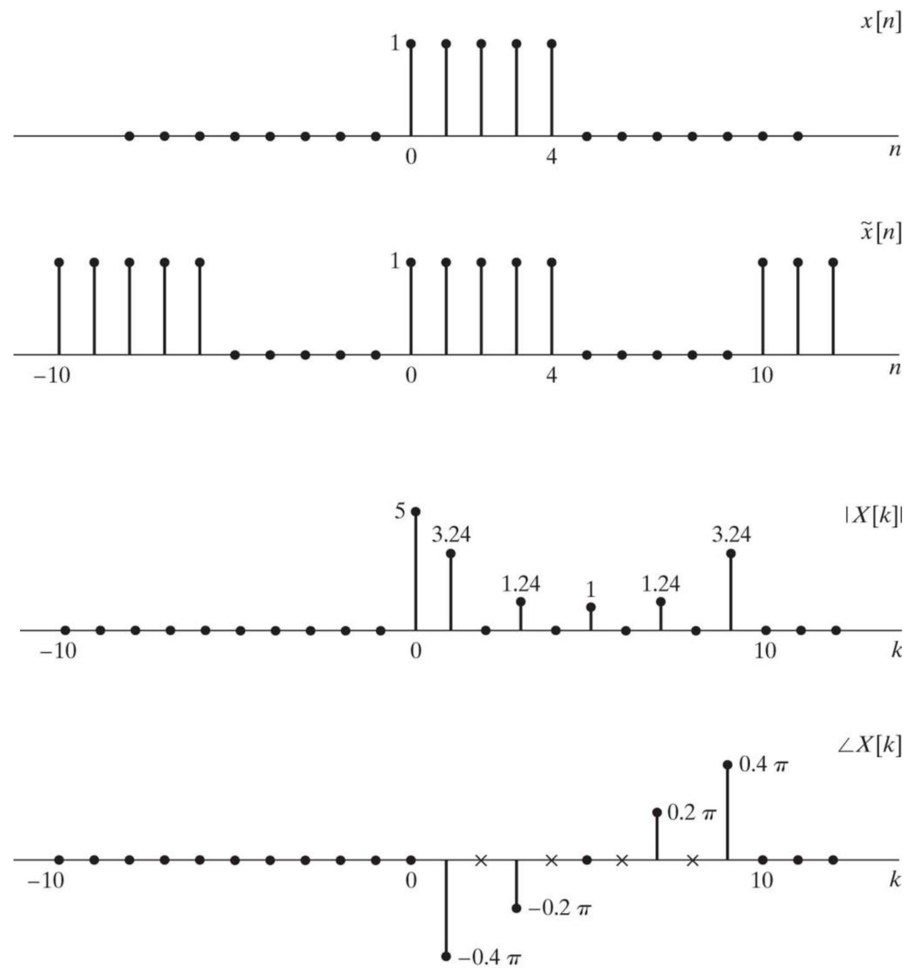
Improving the visualisation of the DFT

Zero Padding

Signal Length L=5 – DFT Length N=5



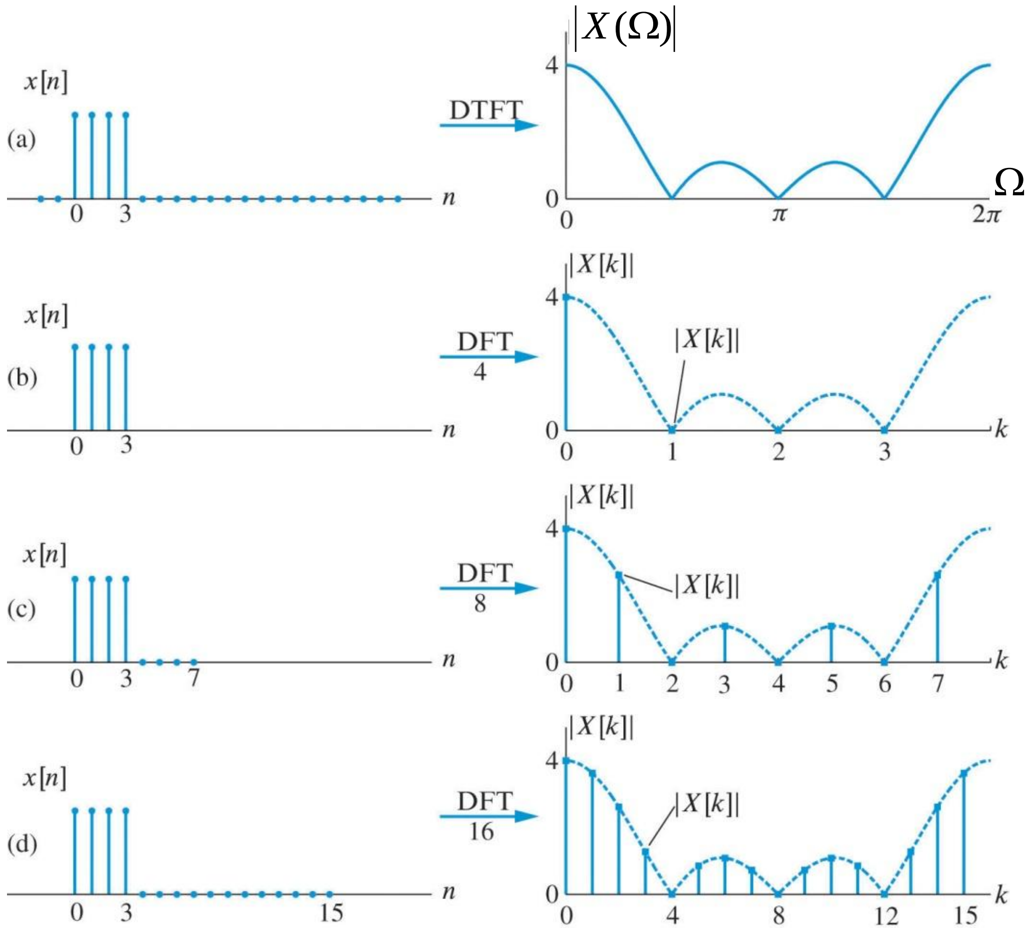
Signal Length L=5 – DFT Length N=10



Improving the visualisation of the DFT

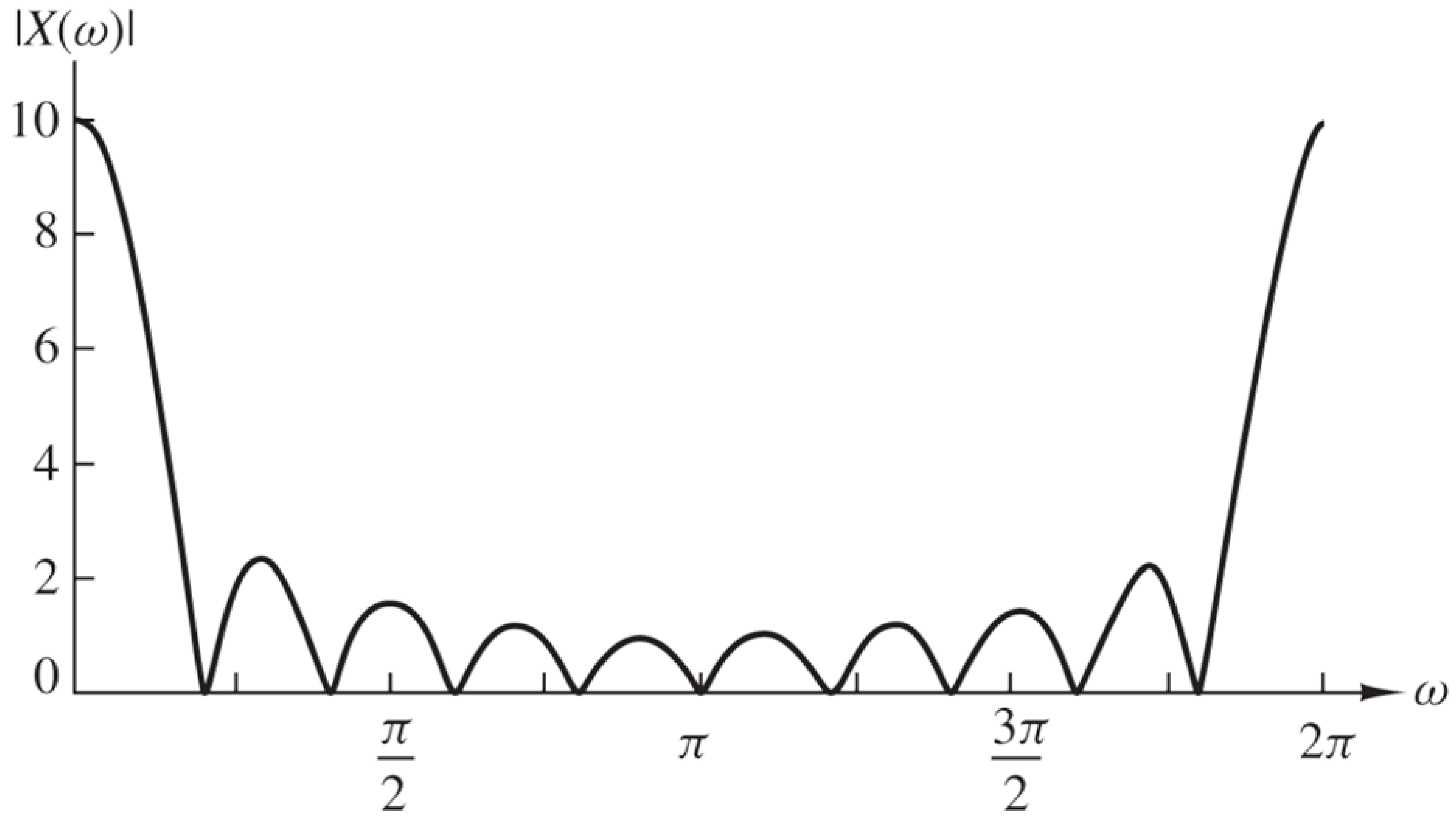
Zero Padding

Zero Padding helps make the shape of the spectrum more evident (better visualisation)



Improving the visualisation of the DFT

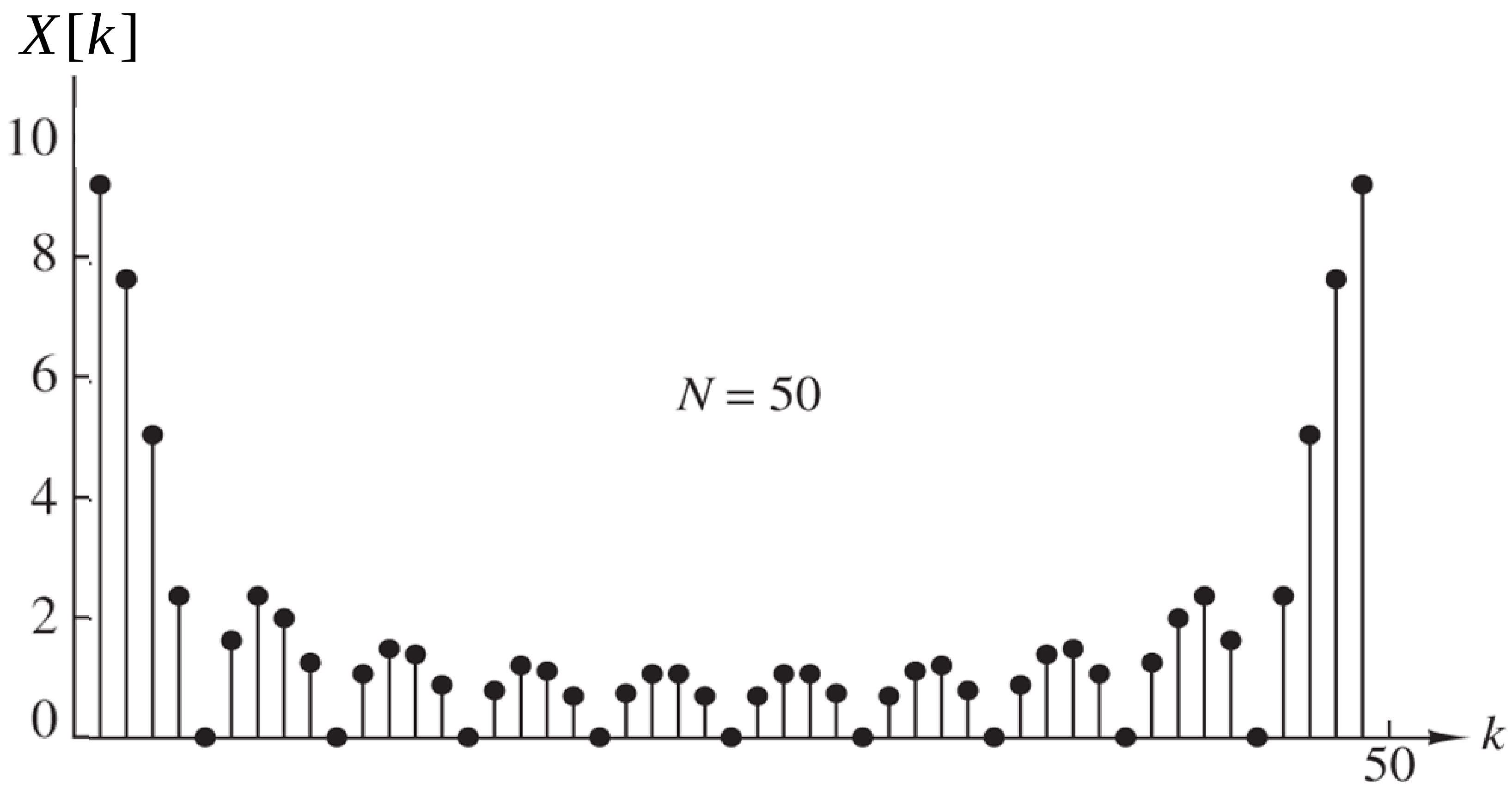
Zero Padding



Discrete Fourier Transform

Improving the visualisation of the DFT

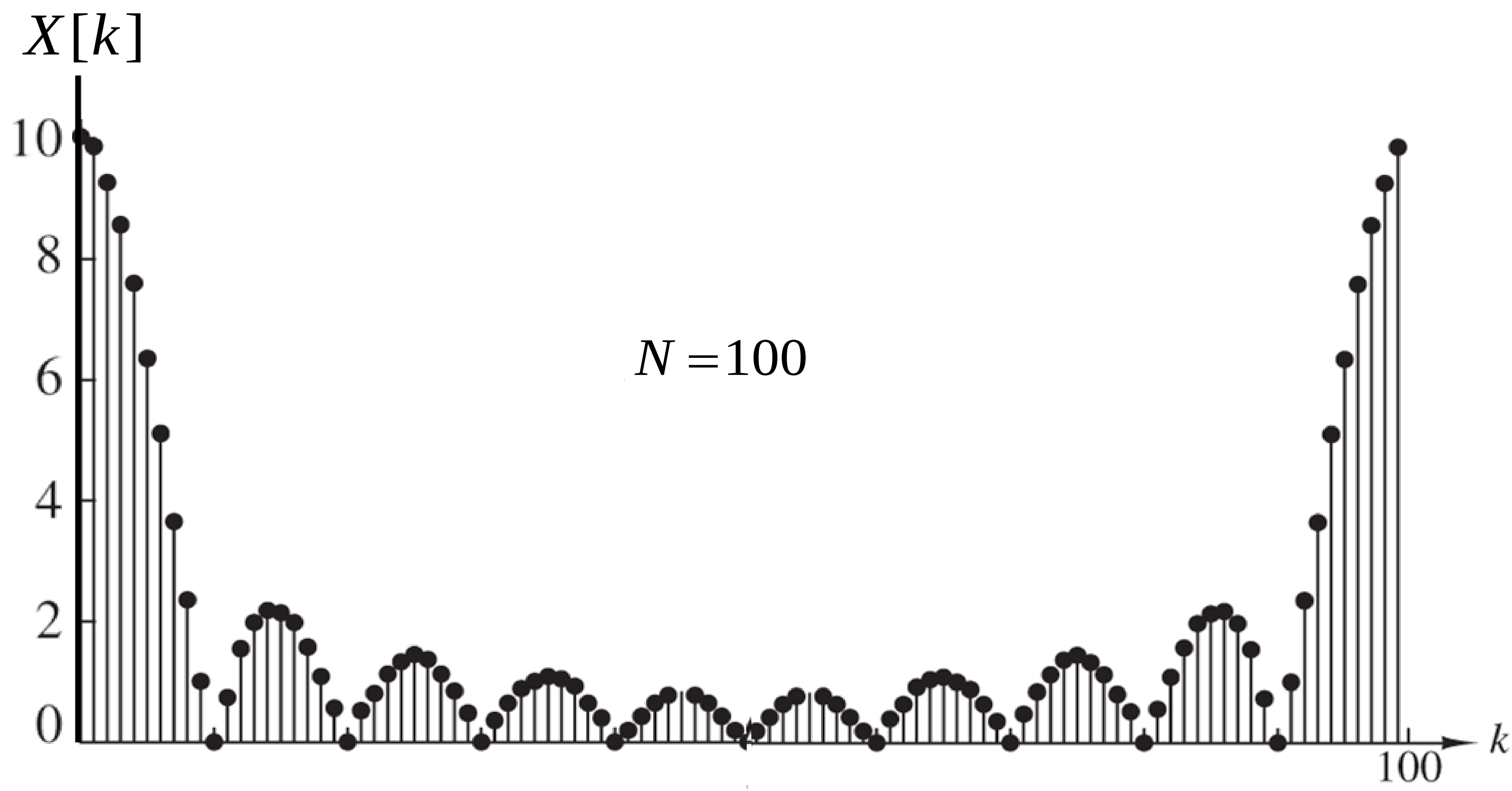
Zero Padding



Discrete Fourier Transform

Improving the visualisation of the DFT

Zero Padding

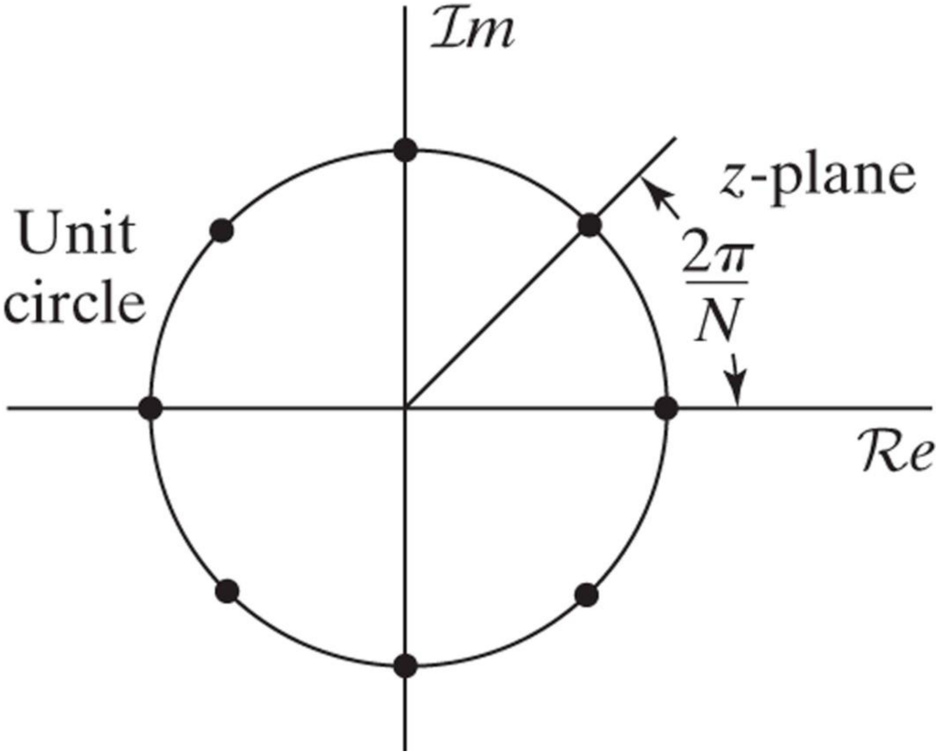


Discrete Fourier Transform

Relationship to the Z transform

$$z = re^{j\Omega}$$

DTFT equal to Z transform evaluated on the unit circle



X(z) is sampled to obtain the periodic sequence *X*[*k*] *(N* = 8*)*.

DFT can be obtained by sampling X(z) at N equally spaced samples on the unit circle