Widely colorable graphs
and their multichromatic numbers

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Abstract: Answering a question of Claude Tardif we show that if a graph admits a so-called s-wide coloring using t colors then its s-fold chromatic number is at most $t + 2(s - 1)$. The talk is based on the paper [2].

Keywords: homomorphism, Kneser graphs, multichromatic number, wide coloring

1 Introduction

For every pair of positive integers n, k satisfying $n \geq 2k$ the Kneser graph $\mathrm{KG}(n, k)$ is defined in the following way. Its vertices are the $\binom{n}{k}$ k-element subsets of $[n] = \{1, \dots n\}$ and its edges are formed by pairs of disjoint subsets. The study of multichromatic numbers goes back to Stahl [8] whose conjecture about the multichromatic numbers of Kneser graphs (that can also be expressed by the existence and non-existence of graph homomorphisms between different Kneser graphs) is still wide open, see the book [5] for further information. In short we can say that the k-fold chromatic number $\chi_k(G)$ of a graph G is the smallest n for which using n colors in total we can assign k distinct colors to each vertex of G in a way that no color appears on adjacent vertices. It is easy to see that this is equivalent to say that n is the smallest positive integer for which G admits a homomorphism to the Kneser graph $\mathrm{KG}(n, k)$.

Wide colorings provide another graph coloring concept that turned out to be relevant in several contexts. For every $s \geq 1$ an s-wide coloring of a graph G is a coloring of its vertices in such a way that no walk of length $2s - 1$ can start and end in the same color class. In particular, a 1-wide coloring is just a proper coloring, a 2-wide coloring is a proper coloring with the additional property that the (first) neighborhood of any color class is also an independent set. In general, an s-wide coloring is a proper coloring where the first, second, $\dots, (s-1)^{\text{th}}$ neighborhood of any color class is also an independent set. It is obvious that if a graph G admits an s-wide coloring then its odd girth (the length of its shortest odd cycle) $g_o(G)$ should be at least $2s + 1$. On the other hand, if $g_o(G) \geq 2s + 1$ then a coloring assigning a different color to every vertex is certainly s-wide. The concept becomes more interesting if we do not need to use more colors for an s-wide coloring than for any proper coloring. In fact, it was a question of Harvey and Murty whether there exist t-chromatic graphs that admit a t-coloring which is 2-wide. This was answered affirmatively by Gyárfás, Jensen and Stiebitz in [3]. 3-wide colorings turned out to be relevant

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in connection to investigations of the local chromatic number, see [7]. More recently, s-wide colorability was also used in the context of finding counterexamples to Hedetniemi's conjecture with small chromatic number, cf. [12, 9, 13, 10].

In the talk, which is based on the paper [2], a result about the multichromatic numbers of s-wide-colorable graphs will be presented that answers a question asked by Claude Tardif in [9].

2 The graphs $W(s,t)$ and their multichromatic numbers

It can be shown that a graph G admits an s-wide coloring with t colors if and only if there exists a homomorphism from G into a certain universal graph we denote by $W(s,t)$. These graphs came up in different forms in the papers [3, 1, 7, 4, 12]. One of their possible definitions is as follows.

Definition 1

$$V(W(s,t)) = \{(x_1 \dots x_t) : \ \forall i \ x_i \in \{0,1,\dots,s\}, \exists! i \ x_i = 0, \ \exists j \ x_j = 1\},$$

$$E(W(s,t)) = \{\{(x_1 \dots x_t),(y_1 \dots y_t)\} : \forall i \ |x_i - y_i| = 1 \text{ or } x_i = y_i = s\}.$$

Using the topological method introduced by Lovász in his celebrated work [6] on Kneser graphs it is shown in the above mentioned papers that $\chi_1(W(s,t)) = \chi(W(s,t)) = t$ for all meaningful values of the parameters s and t.

The motivation for our work came from Tardif [9] who observed that $\chi_2(W(s,t)) = t+2$ when $s = 2$ and in general

$$\chi_k(W(s,t)) \geq t + 2(k-1)$$

holds for every k. He asked whether we will have strict inequality for $k = s = 3$. Our main result answers this in the negative, in fact, we proved the following more general theorem.

Theorem 2 ([2]) *If $k \leq s$, then*

$$\chi_k(W(s,t)) = t + 2(k-1).$$

Nevertheless, asymptotically Tardif's guess was correct as one can also prove (as he also noted [11]) that the following holds.

Proposition 3 *For all pairs of positive integers $t \geq 3$ and $s \geq 1$ there exists some threshold $k_0 = k_0(s,t) > s$ for which*

$$\chi_k(W(s,t)) > t + 2(k-1)$$

whenever $k \geq k_0$.

It would be interesting to know whether the smallest possible k_0 in Proposition 3 is $s+1$, as our result may suggest, or larger.

Finally, we remark that since a graph G admits an s-wide coloring with t colors if and only if there exists a homomorphism from G to $W(s,t)$, and a graph F has $\chi_k(F) \leq n$ if and only if it admits a homomorphism to the Kneser graph $\mathrm{KG}(n,k)$, Theorem 2 implies that if G is a graph that admits an s-wide coloring with t colors, then

$$\chi_k(G) \leq t + 2(k-1)$$

whenever $k \leq s$.

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