Markov Chains

(JRA)

A special type of difference equation arises in the study of Markov chains or Markov processes. We cannot go into the interesting theory of Markov chains, but we will give an example that illustrates some of the ideas.

EXAMPLE 5 An automobile rental company has three locations, which we designate as P, Q, and R. When an automobile is rented at one of the locations, it may be returned to any of the three locations.

Suppose, at some specific time, that there are p cars at location P, q cars at Q, and r cars at R. Experience has shown, in any given week, that the p cars at location P are distributed as follows: 10% are rented and returned to Q, 30% are rented and returned to R, and 60% remain at P (these either are not rented or are rented and returned to P). Similar rental histories are known for locations Q and R, as summarized below.

Weekly Distribution History

Location P: 60% stay at P, 10% go to Q, 30% go to R.

Location Q: 10% go to P, 80% stay at Q, 10% go to R.

Location R: 10% go to P, 20% go to Q, 70% stay at R.

Solution Let $\mathbf{x}_k$ represent the state of the rental fleet at the beginning of week k:

$$\mathbf{x}_k = \begin{bmatrix} p(k) \\ q(k) \\ r(k) \end{bmatrix}.$$

For the state vector $\mathbf{x}_k$, $p(k)$ denotes the number of cars at location P, $q(k)$ the number at Q, and $r(k)$ the number at R.

From the weekly distribution history, we see that

$$\begin{aligned} p(k+1) &= .6p(k) + .1q(k) + .1r(k) \\ q(k+1) &= .1p(k) + .8q(k) + .2r(k) \\ r(k+1) &= .3p(k) + .1q(k) + .7r(k). \end{aligned}$$

(For instance, the number of cars at P when week $k+1$ begins is determined by the 60% that remain at P, the 10% that arrive from Q, and the 10% that arrive from R.)

To the extent that the weekly distribution percentages do not change, the rental fleet is rearranged among locations P, Q, and R according to the rule $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $k = 0, 1, \ldots,$ where A is the (3×3) matrix

$$A = \begin{bmatrix} .6 & .1 & .1 \\ .1 & .8 & .2 \\ .3 & .1 & .7 \end{bmatrix}.$$ ■

Example 5 represents a situation in which a fixed population (the rental fleet) is rearranged in stages (week by week) among a fixed number of categories (the locations P, Q, and R). Moreover, in Example 5 the rules governing the rearrangement remain fixed from stage to stage (the weekly distribution percentages stay constant). In general, such problems can be modeled by a difference equation of the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \quad k = 0, 1, \ldots.$$

For such problems the matrix A is often called a ***transition*** matrix. Such a matrix has two special properties:

The entries of A are all nonnegative. (8a)

In each column of A, the sum of the entries has the value 1. (8b)

It turns out that a matrix having properties (8a) and (8b) always has an eigenvalue of $\lambda = 1$. This fact is established in Exercise 26 and illustrated in the next example.

EXAMPLE 6 Suppose the automobile rental company described in Example 5 has a fleet of 600 cars. Initially an equal number of cars is based at each location, so that $p(0) = 200$, $q(0) = 200$, and $r(0) = 200$. As in Example 5, let the week-by-week distribution of cars be governed by $\mathbf{x}_{k+1} = A\mathbf{x}_k, k = 0, 1, \ldots$, where

$$\mathbf{x}_k = \begin{bmatrix} p(k) \\ q(k) \\ r(k) \end{bmatrix}, \quad A = \begin{bmatrix} .6 & .1 & .1 \\ .1 & .8 & .2 \\ .3 & .1 & .7 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 200 \\ 200 \\ 200 \end{bmatrix}.$$

Find $\lim_{k\to\infty} \mathbf{x}_k$. Determine the number of cars at each location in the kth week, for $k = 1, 5$, and 10.

Solution If A is not defective, we can use Eq. (6) to express $\mathbf{x}_k$ as

$$\mathbf{x}_k = a_1(\lambda_1)^k\mathbf{u}_1 + a_2(\lambda_2)^k\mathbf{u}_2 + a_3(\lambda_3)^k\mathbf{u}_3,$$

where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for R^3, consisting of eigenvectors of A.

It can be shown that A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = .6$, and $\lambda_3 = .5$. Thus A has three linearly independent eigenvectors:

$$\lambda_1 = 1, \quad \mathbf{u}_1 = \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix}; \quad \lambda_2 = .6 \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix};$$

$$\lambda_3 = .5, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

The initial vector, $\mathbf{x}_0 = [200, 200, 200]^T$, can be written as

$$\mathbf{x}_0 = 30\mathbf{u}_1 - 150\mathbf{u}_2 - 80\mathbf{u}_3.$$

Thus the vector $\mathbf{x}_k = [p(k), q(k), r(k)]^T$ is given by

$$\begin{aligned} \mathbf{x}_k &= A^k\mathbf{x}_0 \\ &= A^k(30\mathbf{u}_1 - 150\mathbf{u}_2 - 80\mathbf{u}_3) \\ &= 30(\lambda_1)^k\mathbf{u}_1 - 150(\lambda_2)^k\mathbf{u}_2 - 80(\lambda_3)^k\mathbf{u}_3 \\ &= 30\mathbf{u}_1 - 150(.6)^k\mathbf{u}_2 - 80(.5)^k\mathbf{u}_3. \end{aligned} \tag{9}$$

From the expression above, we see that

$$\lim_{k\to\infty} \mathbf{x}_k = 30\mathbf{u}_1 = \begin{bmatrix} 120 \\ 270 \\ 210 \end{bmatrix}.$$

Therefore, as the weeks proceed, the rental fleet will tend to an equilibrium state with 120 cars at P, 270 cars at Q, and 210 cars at R. To the extent that the model is valid, location Q will require the largest facility for maintenance, parking, and the like.

Finally, using Eq. (9), we can calculate the state of the fleet for the kth week:

$$\mathbf{x}_1 = \begin{bmatrix} 160 \\ 220 \\ 220 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 122.500 \\ 260.836 \\ 216.664 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_{10} = \begin{bmatrix} 120.078 \\ 269.171 \\ 210.751 \end{bmatrix}.$$

4.8 EXERCISES

In Exercises 1–6, consider the vector sequence $\{\mathbf{x}_k\}$, where $\mathbf{x}_k = A\mathbf{x}_{k-1}$, $k = 1, 2, \ldots$. For the given starting vector $\mathbf{x}_0$, calculate $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ by using direct multiplication, as in Example 1.

1. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

2. $A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$

3. $A = \begin{bmatrix} .5 & .25 \\ .5 & .75 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 128 \\ 64 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

In Exercises 7–14, let $\mathbf{x}_k = A\mathbf{x}_{k-1}$, $k = 1, 2, \ldots$, for the given A and $\mathbf{x}_0$. Find an expression for $\mathbf{x}_k$ by using Eq. (6), as in Example 3. With a calculator, compute $\mathbf{x}_4$ and $\mathbf{x}_{10}$ from the expression. Comment on $\lim_{k\to\infty} \mathbf{x}_k$.

7. A and $\mathbf{x}_0$ in Exercise 1

8. A and $\mathbf{x}_0$ in Exercise 2

9. A and $\mathbf{x}_0$ in Exercise 3

10. A and $\mathbf{x}_0$ in Exercise 4

11. A and $\mathbf{x}_0$ in Exercise 5

12. A and $\mathbf{x}_0$ in Exercise 6

13. $A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 3 \\ -14 \\ 8 \end{bmatrix}$

14. $A = \begin{bmatrix} -6 & 1 & 3 \\ -3 & 0 & 2 \\ -20 & 2 & 10 \end{bmatrix}$, $\quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

In Exercises 15–18, solve the initial-value problem.

15. $u'(t) = 5u(t) - 6v(t), \quad u(0) = 4$
$v'(t) = 3u(t) - 4v(t), \quad v(0) = 1$

16. $u'(t) = u(t) + 2v(t), \quad u(0) = 1$
$v'(t) = 2u(t) + v(t), \quad v(0) = 5$

17. $u'(t) = u(t) + v(t) + w(t), \quad u(0) = 3$
$v'(t) = 3v(t) + 3w(t), \quad v(0) = 3$
$w'(t) = -2u(t) + v(t) + w(t), \quad w(0) = 1$

18. $u'(t) = -2u(t) + 2v(t) - 3w(t), \quad u(0) = 3$
$v'(t) = 2u(t) + v(t) - 6w(t), \quad v(0) = -1$
$w'(t) = -u(t) - 2v(t), \quad w(0) = 3$

19. Consider the matrix A given by

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Note that $\lambda = 1$ is the only eigenvalue of A.

a) Verify that A is defective.

b) Consider the sequence $\{\mathbf{x}_k\}$ determined by $\mathbf{x}_k = A\mathbf{x}_{k-1}$, $k = 1, 2, \ldots$, where $\mathbf{x}_0 = [1, 1]^T$. Use induction to show that

$\mathbf{x}_k = [2k+1, 1]^T$. (This exercise gives an example of a sequence $\mathbf{x}_k = A\mathbf{x}_{k-1}$, where $\lim_{k\to\infty} \|\mathbf{x}_k\| = \infty$, even though A has no eigenvalue larger than 1 in magnitude.)

In Exercises 20 and 21, choose a value α so that the matrix A has an eignevalue of $\lambda = 1$. Then, for $\mathbf{x}_0 = [1, 1]^T$, calculate $\lim_{k\to\infty} \mathbf{x}_k$, where $\mathbf{x}_k = A\mathbf{x}_{k-1}$, $k = 1, 2, \ldots$.

20. $A = \begin{bmatrix} .5 & .5 \\ .5 & 1+\alpha \end{bmatrix}$

21. $A = \begin{bmatrix} 0 & .3 \\ .6 & 1+\alpha \end{bmatrix}$

22. Suppose that $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ are sequences satisfying $\mathbf{u}_k = A\mathbf{u}_{k-1}$, $k = 1, 2, \ldots$, and $\mathbf{v}_k = A\mathbf{v}_{k-1}$, $k = 1, 2, \ldots$. Show that if $\mathbf{u}_0 = \mathbf{v}_0$, then $\mathbf{u}_i = \mathbf{v}_i$ for all i.

23. Let $B = (b_{ij})$ be an $(n \times n)$ matrix. Matrix B is called a stochastic matrix if B contains only nonnegative entries and if $b_{i1} + b_{i2} + \cdots + b_{in} = 1$, $1 \le i \le n$. (That is, B is a stochastic matrix if B^T satisfies conditions 8a and 8b.) Show that $\lambda = 1$ is an eigenvalue of B. [*Hint:* Consider the vector $\mathbf{w} = [1, 1, \ldots, 1]^T$.]

24. Suppose that B is a stochastic matrix whose entries are all positive. By Exercise 23, $\lambda = 1$ is an eigenvalue of B. Show that if $B\mathbf{u} = \mathbf{u}$, $\mathbf{u} \neq \boldsymbol{\theta}$, then $\mathbf{u}$ is a multiple of the vector $\mathbf{w}$ defined in Exercise 23. [*Hint:* Define $\mathbf{v} = \alpha\mathbf{u}$ so that $v_i = 1$ and $|v_j| \le 1$, $1 \le j \le n$. Consider the ith equations in $B\mathbf{w} = \mathbf{w}$ and $B\mathbf{v} = \mathbf{v}$.]

25. Let B be a stochastic matrix, and let λ by any eigenvalue of B. Show that $|\lambda| \le 1$. For simplicity, assume that λ is real. [*Hint:* Suppose that $B\mathbf{u} = \lambda\mathbf{u}$, $\mathbf{u} \neq \boldsymbol{\theta}$. Define a vector $\mathbf{v}$ as in Exercise 24.]

26. Let A be an $(n \times n)$ matrix satisfying conditions (8a) and (8b). Show that $\lambda = 1$ is an eigenvalue of A and that if $A\mathbf{u} = \beta\mathbf{u}$, $\mathbf{u} \neq \boldsymbol{\theta}$, then $|\beta| \le 1$. [*Hint:* Matrix A^T is stochastic.]

27. Suppose that $(A - \lambda I)\mathbf{u} = \boldsymbol{\theta}$, $\mathbf{u} \neq \boldsymbol{\theta}$, and there is a vector $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{u}$. Then $\mathbf{v}$ is called a ***generalized eigenvector***. Show that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set. [*Hint:* Note that $A\mathbf{v} = \lambda\mathbf{v} + \mathbf{u}$. Suppose that $a\mathbf{u} + b\mathbf{v} = \boldsymbol{\theta}$, and multiply this equation by A.]

28. Let A, $\mathbf{u}$, and $\mathbf{v}$ be as in Exercise 27. Show that $A^k\mathbf{v} = \lambda^k\mathbf{v} + k\lambda^{k-1}\mathbf{u}$, $k = 1, 2, \ldots$.

29. Consider matrix A in Exercise 19.

a) Find an eigenvector $\mathbf{u}$ and a generalized eigenvector $\mathbf{v}$ for A.

b) Express $\mathbf{x}_0 = [1, 1]^T$ as $\mathbf{x}_0 = a\mathbf{u} + b\mathbf{v}$.

c) Using the result of Exercise 28, find an expression for $A^k\mathbf{x}_0 = A^k(a\mathbf{u} + b\mathbf{v})$.

d) Verify that $A^k\mathbf{x}_0 = [2k+1, 1]^T$ as was shown by other means in Exercise 19.

SUPPLEMENTARY EXERCISES

1. Find all values x such that A is singular, where

$$A = \begin{bmatrix} x & 1 & 2 \\ 3 & x & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

2. For what values x does A have only real eigenvalues, where

$$A = \begin{bmatrix} 2 & 1 \\ x & 3 \end{bmatrix}?$$

3. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a + b = 2$ and $c + d = 2$. Show that $\lambda = 2$ is an eigenvalue for A. [*Hint:* Guess an eigenvector.]

4. Let A and B be (3×3) matrices such that $\det(A) = 2$ and $\det(B) = 9$. Find the values of each of the following.

a) $\det(A^{-1}B^2)$

b) $\det(3A)$

c) $\det(AB^2A^{-1})$