Positive maps and extendibility hierarchies from copositive matrices

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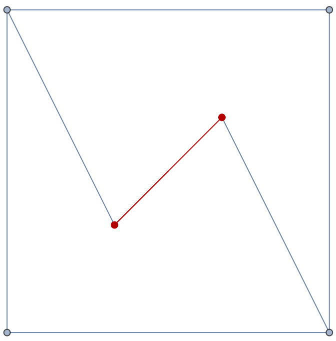
This notebook contains numerical routines for computing the quantity σ(G) defined in the Section 6: Maps associated to graphs of the paper. The function MaxSKn0 computes returns the quantity σ for a graph G given by its adjacency matrix A. It is used to find all the graphs G on n=6 and n=7 vertices with the property that σ(G) < 1 + 1/(ω(G)-1), where ω(G) is the clique number of G. Such graphs yield positive indecomposable maps for a non-trivial range of parameters using the construction in the paper.

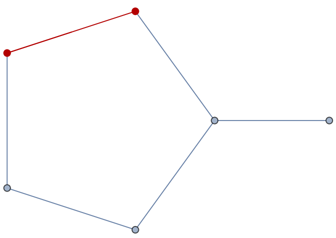
```
In[1]:= MaxSKn0[A_] := Module[
    {n, J, P, pvars, W, wvars, cons, obj, sdp, sopt, Popt, Wopt, i, j, vars, s},
    n = Dimensions[A][[1]];
    J = ConstantArray[1, {n, n}];
    P = Table[p[i, j], {i, 1, n}, {j, 1, n}];
    For[i = 1, i ≤ n, i++,
     For[j = i + 1, j ≤ n, j++,
      P[[j, i]] = P[[i, j]];
     ]];
    pvars = {};
    For[i = 1, i ≤ n, i++,
     For[j = i, j ≤ n, j++,
      AppendTo[pvars, P[[i, j]]];
     ]];
    W = Table[w[i, j], {i, 1, n}, {j, 1, n}];
    For[i = 1, i ≤ n, i++,
     For[j = i + 1, j ≤ n, j++,
      W[[j, i]] = W[[i, j]];
     ]];
    wvars = {};
    For[i = 1, i ≤ n, i++,
     For[j = i, j ≤ n, j++,
      AppendTo[wvars, W[[i, j]]];
     ]];
    (*W//MatrixForm
       P//MatrixForm*)
    vars = Join[{s}, wvars, pvars];
    (*Print[vars];*)
    cons = {
      VectorGreaterEqual[{P, 0}, {"SemidefiniteCone", n}],
      VectorGreaterEqual[{W, 0}],
      J - s * A == P + W
     };
    obj = -s;
    sdp = SemidefiniteOptimization[obj, cons, vars];
    sopt = s /. sdp;
    Popt = P /. sdp;
    Wopt = W /. sdp;
    {sopt, Popt, Wopt}
   ]
```

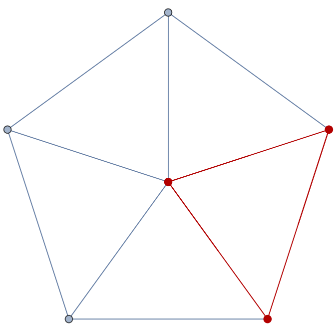
Computation of σ(G) for graphs of order 6

In[47]:=
```
n = 6;
gs = GraphData[n];
Print["Number of graphs on ", n, " vertices: ", Length[gs]]
nG = 0;
For[i = 1, i ≤ Length[gs], i++,
  g = GraphData[gs⟦i⟧, "Graph"];
  If[ConnectedGraphQ[g],
   A = AdjacencyMatrix[g];
   {sopt, Popt, Wopt} = MaxSKn0[A];
   mc = FindClique[g];
   ω = mc⟦1⟧ // Length;
   If[sopt < 1 + 1 / (ω - 1) - 10 ^ (-6),
    Print[{HighlightGraph[g, Subgraph[g, mc]], sopt, 1 + 1 / (ω - 1)}];
    nG++;
   ];
  ]];
Print["Number of graphs having gap: ", nG]
```

```
Number of graphs on 6 vertices: 156
```

{  , 1.80902, 2}

{  , 1.80902, 2}

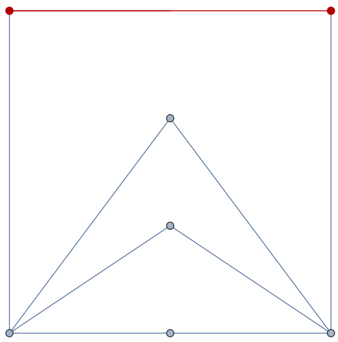
{  , 1.44721, $\frac{3}{2}$}

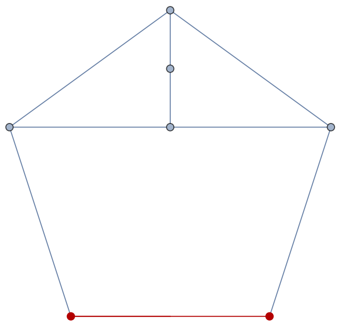
```
Number of graphs having gap: 3
```

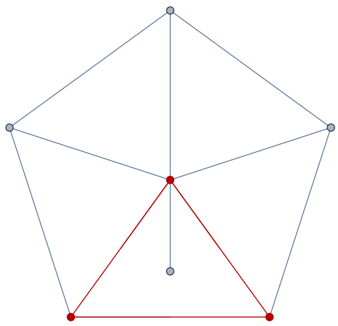
Computation of σ (G) for graphs of order 7

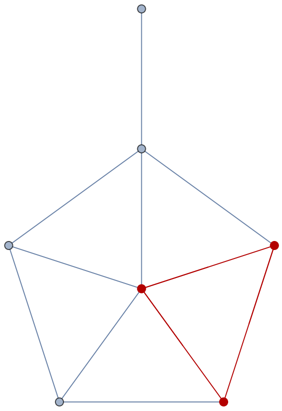
In[41]:=
```
n = 7;
gs = GraphData[n];
Print["Number of graphs on ", n, " vertices: ", Length[gs]]
nG = 0;
For[i = 1, i ≤ Length[gs], i++,
  g = GraphData[gs⟦i⟧, "Graph"];
  If[ConnectedGraphQ[g],
   A = AdjacencyMatrix[g];
   {sopt, Popt, Wopt} = MaxSKn0[A];
   mc = FindClique[g];
   ω = mc⟦1⟧ // Length;
   If[sopt < 1 + 1 / (ω - 1) - 10 ^ (-6),
    Print[{HighlightGraph[g, Subgraph[g, mc]], sopt, 1 + 1 / (ω - 1)}];
    nG++;
   ];
  ]];
Print["Number of graphs having gap: ", nG]
```

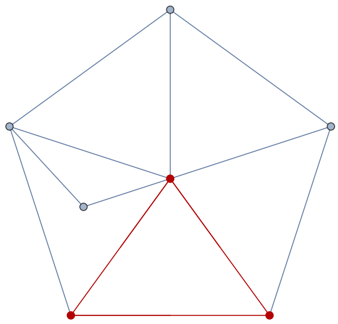
```
Number of graphs on 7 vertices: 1044
```

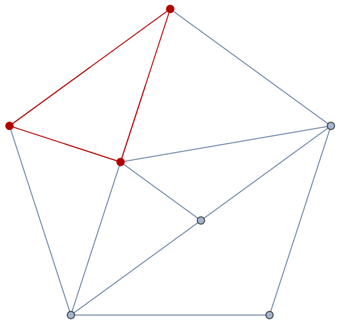
{, 1.80902, 2}

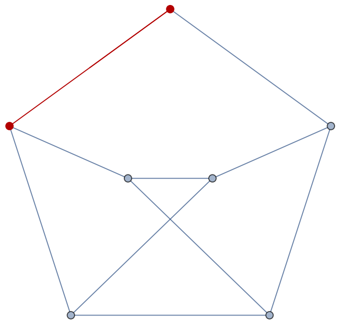
{, 1.80902, 2}

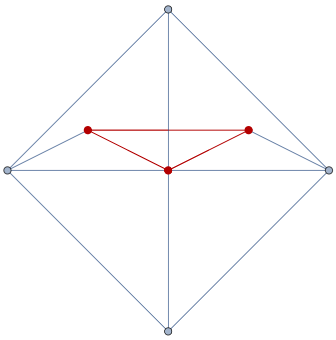
{  , 1.44721, $\frac{3}{2}$ }

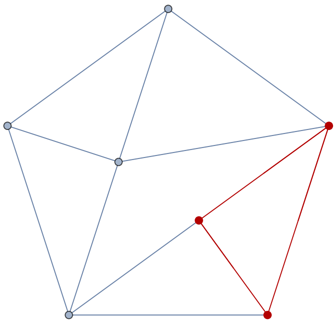
{  , 1.44721, $\frac{3}{2}$ }

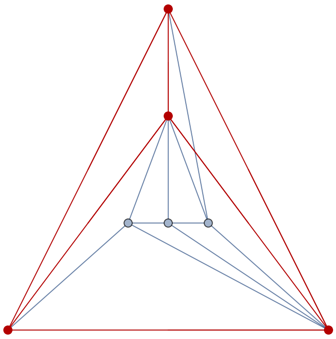
{  , 1.44721, $\frac{3}{2}$ }

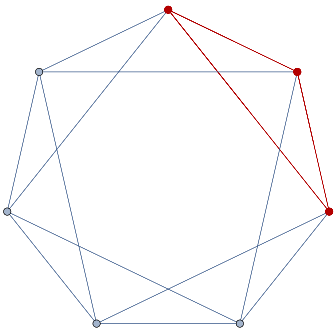
{  , 1.44721, $\frac{3}{2}$ }

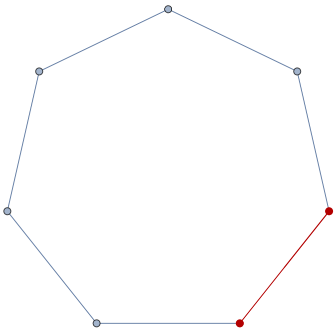
{  , 1.80902, 2 }

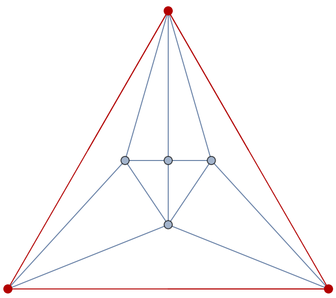
{  , 1.44721, $\frac{3}{2}$ }

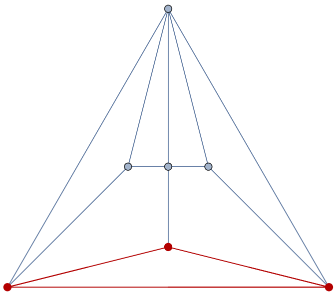
{  , 1.44721, $\frac{3}{2}$ }

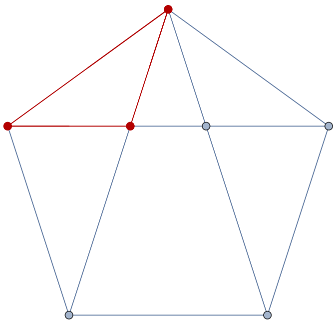
{  , 1.30902, $\frac{4}{3}$ }

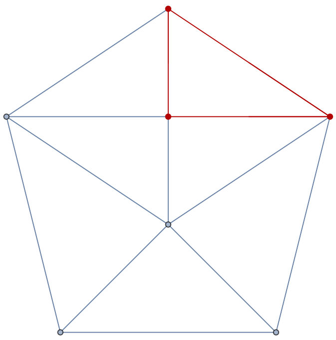
{  , 1.43147, $\frac{3}{2}$ }

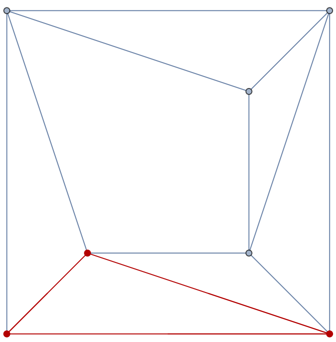
{  , 1.90097, 2}

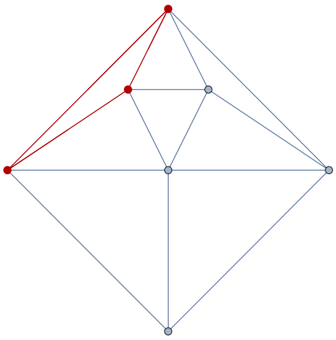
{, 1.44721, $\frac{3}{2}$}

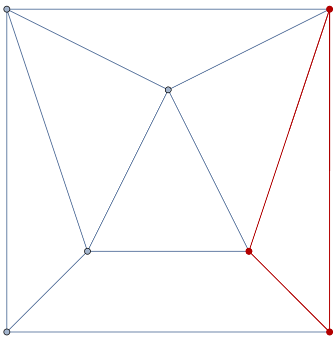
{, 1.44721, $\frac{3}{2}$}

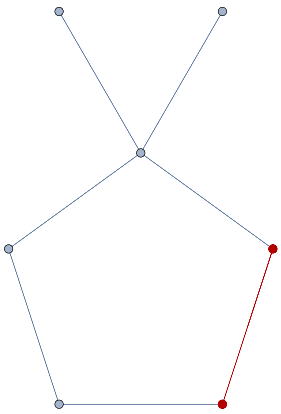
{, 1.45526, $\frac{3}{2}$}

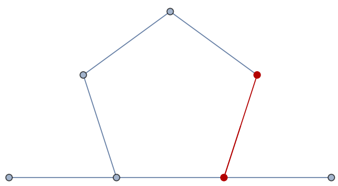
{, 1.44721, $\frac{3}{2}$}

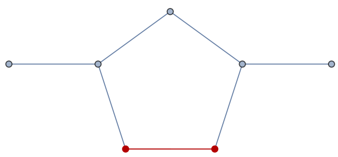
{, 1.44721, $\frac{3}{2}$}

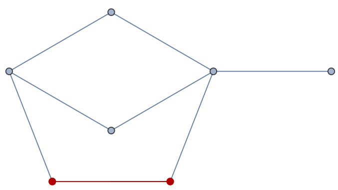
{  , 1.44721, $\frac{3}{2}$ }

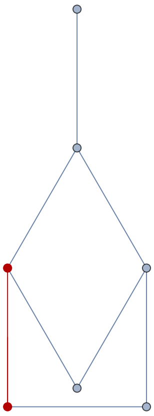
{  , 1.45526, $\frac{3}{2}$ }

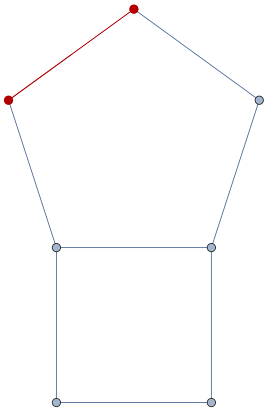
{  , 1.80902, 2 }

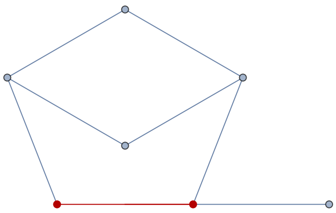
{  , 1.80902, 2 }

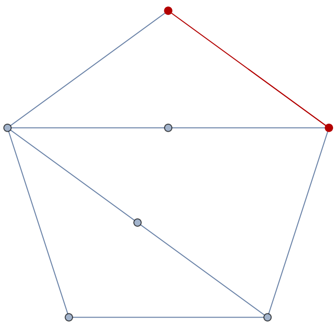
{  , 1.80902, 2 }

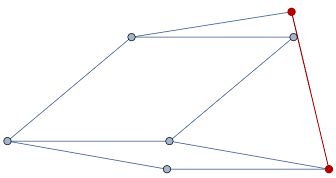
{  , 1.80902, 2 }

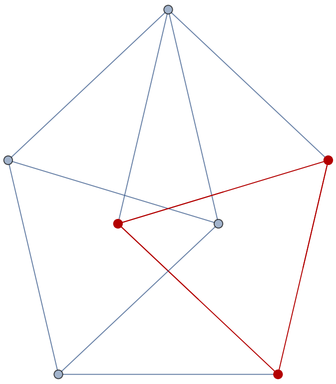
{  , 1.80902, 2}

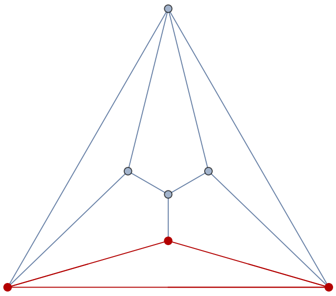
{  , 1.80902, 2}

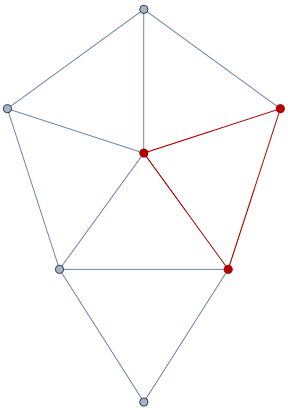
{  , 1.80902, 2}

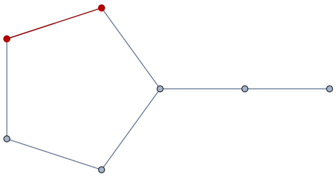
{  , 1.80902, 2}

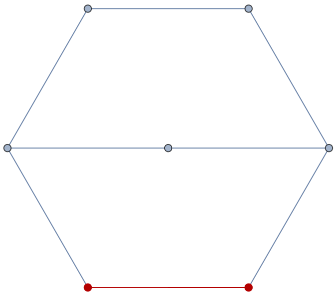
{  , 1.80902, 2}

{, 1.45526, $\frac{3}{2}$}

{, 1.48445, $\frac{3}{2}$}

{, 1.44721, $\frac{3}{2}$}

{, 1.80902, 2}

{, 1.80902, 2}

Number of graphs having gap: 33