2.3 Sampling via transformation of coordinates

Exercise Sampling uniformly points within a unit radius disk The obvious approach to sample points within the unit disk corresponds to considering $r = \xi_1$ and $\theta = 2\pi\xi_2$ with ξ_1, ξ_2 uniformly distributed in $[0, 1]$.

- Show by simulation that this algorithm does not sample points uniformly within the disk. Explain which is the conceptual mistake of this algorithm.
- Design an algorithm that does it correctly. (**Hint:** One way is to first perform the transformation into polar coordinates and then use the marginal $p(r)$ and the conditional $p(\theta|r)$ PDFs to do the sampling by applying in turn the 1D inversion method.)

Exercise A way to generate numbers from a 2D (normalised) 2D Gaussian PDF, $\mathcal{N}(0, 1)$ is the so-called Box-Muller transformation. This is based on the idea presented during the lecture in which one first makes a coordinate transformation to factorize the 2-point PDF

$$\rho(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \tag{2.2}$$

into a product of two one-point PDFs and then performs two separate samplings, one for each PDF.

- Write an algorithm that does this sampling by first performing the analytical calculations necessary to find the correct transformation;
- How one can extend the algorithm to sample from $\mathcal{N}(\mu, \sigma^2)$?

2.3.1 Rejection method

Exercise Use the rejection method to generate random numbers that are distributed according to the pdf

$$f(x) = \sqrt{2/\pi}\, e^{-x^2}. \tag{2.3}$$

Hint: One may use a function $g(x) = A$ for $0 \le x \le p$ and $g(x) = (A/p)\, x \exp\left(p^2 - x^2\right)$ for $x > p$. See how good the performance is for a few values of p (use a reasonable value N of "darts".

2.4 Importance sampling

Exercise Let us consider the following function $f(x) = e^{-x^2} g(x)$ in $[0, \infty]$ where $g(x)$ is a slowly varying function. Compute the integral both with the crude method and by using the importance sampling technique. Hint: For the importance sampling method use a Gaussian random number generator with density $W(x) = \sqrt{2/\pi} e^{-x^2}$ With this choice one has

$$I = \int_0^\infty f(x)\, dx \sim \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{W(x_i)} = \frac{1}{N} \sqrt{\pi/2} \sum_{i=1}^{N} g(x_i). \tag{2.4}$$

Exercise Estimate the integral

$$\int_0^{\pi/2} \cos\, x\, dx \tag{2.5}$$

using the importance sampling technique with $g(x)$ proportional to $a + bx^2$. Determine the optimal values of the parameters a and b to generate samples according to $g(x)$ and establish the number of iterations needed to get an accuracy of $\%1$.

Exercise Let us consider the function $f(x)$ defined as

$$f(x) = \begin{cases} 0 & \text{for} \quad x < T \\ 1 & \text{for} \quad x \geq T \end{cases} \tag{2.6}$$

and compute the average

$$\langle f \rangle_\rho = \int_{\mathbb{R}} f(x)\, \rho(x)\, dx \tag{2.7}$$

with respect to the PDF $\rho(x) = e^{-x}$ defined for $x \geq 0$. Notice that while the function $f(x)$ is zero for $x < T(=5)$ the PDF $\rho(x)$ is very high in that region. This means that most values x sampled according to $\rho(x)$ will be zero. On the other hand $f(x) = 1$ for $x > T(=5)$ where the $\rho(x)$ is very small. We can try to get an estimate of $\langle f \rangle_\rho$ via importance sampling. Consider the function $g(a, x) = a \exp(-ax)$ defined for $x \geq 0$ and for $0 < a \leq 1$ where a is the parameter to optimise according to a minimum variance principle.

- Show analytically that

$$\begin{aligned} \langle f \rangle &= e^{-T} \\ \sigma^2(f) &= \langle f \rangle (1 - \langle f \rangle) \\ \sigma^2\left(a, f(x)\rho(x)/g(a,x)\right) &= \frac{e^{-T(2-a)}}{a(2-a)} - e^{-2T} \end{aligned} \tag{2.8}$$

- Find the value of a, a^* such that the variance $\sigma^2\left(a, f(x)\rho(x)/g(a,x)\right)$ is minimum.
- Discuss the improvement one can get in the statistical errors in the cases $T = 3, 5, 10$ and 20. This can be done by comparing the values $\sigma(f)/\langle f \rangle$, $\sigma\left(f(x)\rho(x)/g(a^*,x)\right)$, and $\sigma(f)/\sigma\left(f(x)\rho(x)/g(a^*,x)\right)$ for $T = 3, 5, 10$ and 20.

2.5 Markov chains

Exercise We have seen that the state probability vector of a Markov chain μ_n satisfies the recurrence relation

$$\mu_n = \mu_{n-1} \boldsymbol{\mathcal{P}} \tag{2.9}$$

Show that this equation is equivalent to write

$$\mu_n(i) = \left(1 - \sum_{j \in S\, j \neq i} p_{j,i}\right) \mu_{n-1}(i) + \sum_{j \in S\, j \neq i} p_{i,j} \mu_{n-1}(j). \tag{2.10}$$

What does it say this equation?

Exercise Draw the digraphs and classify the states of the Markov chains defined by the following transition matrices:

(A)

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} \tag{2.11}$$

(B)

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.12}$$

(C)

$$\mathcal{P} = \begin{pmatrix} 0.3 & 0.4 & 0 & 0 & 0.3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.13}$$

Exercise Given the two Markov chains defined by the following transition matrices

$$\begin{aligned} \mathcal{P}_1 &= \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \\ \mathcal{P}_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1/4 & 0 & 3/4 \end{pmatrix} \end{aligned} \tag{2.14}$$

verify whether they are irreducible and in case compute their period. Moreover,for both of them compute $(\mathcal{P})^n$ and the limit $\lim_{n\to\infty}$ (or the limit of its average).

Exercise Given the Markov chain defined by

$$\mathcal{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \tag{2.15}$$

show that is regular.

Exercise Is the Markov chain defined by

$$\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \tag{2.16}$$

regular?

Exercise Let $0 < p < 1$ and let us consider a Markov chain defined on the finite space $S = \{1, 2, 3, 4\}$ by the transition matrix

$$\mathcal{P} = \begin{pmatrix} p & 1-p & 0 & 0 \\ 0 & 0 & p & 1-p \\ p & 1-p & 0 & 0 \\ 0 & 0 & p & 1-p \end{pmatrix} \tag{2.17}$$

- Show that the Markov chain is recurrent irreducible
- Show that the Markov chain is aperiodic (consider for instance the term p_{11}
- Compute the fixed point π that, given the first two points, is the invariant unique distribution of the Markov chain.

Exercise Find the stationary distribution π of the Markov chain with transition matrix

$$\mathcal{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix} \tag{2.18}$$