Chapter 2

List of exercises

2.1 Sampling random points within D- dimensional domains by hit and miss

Rectangle Generate random points uniformly distributed within a rectangle $[a, b] \times [c, d]$ and compare the analytic value of the area $A = L_{ab} L_{cd}$ with the Monte Carlo estimate based on the hit-miss method as a function of the number of “throws”.

Disk Do the same for a unit radius disk.

2.2 Sampling random numbers from a given distribution

2.2.1 Inversion method

Exercise Use the inversion method to design an algorithm that samples random numbers according to the power law probability distribution

$$\rho(x) = c\, x^n, \quad \text{with } x \in [0, 1] \tag{2.1}$$

for some constant c that normalise $\rho(x)$. Simulate the cases $n = 3, 4$ and compare the histograms with the analytical expressions.

Exercise Use the inversion method to sample random numbers according to the probability distribution $\rho(x) = cx^2$ with $x \in [0, 2]$

Additional exercises Use the inversion method to generate random numbers with the following PDF

1. $\rho(x) = \mu\, e^{-\mu\, x}$, for $x \geq 0$;
2. $\rho(x) = 2x\, e^{-x^2}$, for $x \geq 0$.
3. $\rho(x) = \frac{1}{(a+bx)^n}$ for $x \geq 0$ and $n > 1$

> **Note.** *For all the exercises proposed above first compute the F, F^{-1} and the map $x_i = f(\xi_i)$ and then implement and run the corresponding algorithm. Compute the histogram of the sampled points and compare it with the expected PDF.*

2.3 Sampling via transformation of coordinates

Exercise Sampling uniformly points within a unit radius disk The obvious approach to sample points within the unit disk corresponds to considering $r = \xi_1$ and $\theta = 2\pi\xi_2$ with ξ_1, ξ_2 uniformly distributed in $[0, 1]$.

- Show by simulation that this algorithm does not sample points uniformly within the disk. Explain which is the conceptual mistake of this algorithm.
- Design an algorithm that does it correctly. (**Hint:** One way is to first perform the transformation into polar coordinates and then use the marginal $p(r)$ and the conditional $p(\theta|r)$ PDFs to do the sampling by applying in turn the 1D inversion method.)

Exercise A way to generate numbers from a 2D (normalised) 2D Gaussian PDF, $\mathcal{N}(0, 1)$ is the so-called Box-Muller transformation. This is based on the idea presented during the lecture in which one first makes a coordinate transformation to factorize the 2-point PDF

$$\rho(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \tag{2.2}$$

into a product of two one-point PDFs and then performs two separate samplings, one for each PDF.

- Write an algorithm that does this sampling by first performing the analytical calculations necessary to find the correct transformation;
- How one can extend the algorithm to sample from $\mathcal{N}(\mu, \sigma^2)$?

2.3.1 Rejection method

Exercise Use the rejection method to generate random numbers that are distributed according to the pdf

$$f(x) = \sqrt{2/\pi}\, e^{-x^2}. \tag{2.3}$$

Hint: One may use a function $g(x) = A$ for $0 \leq x \leq p$ and $g(x) = (A/p)\, x \, \exp\left(p^2 - x^2\right)$ for $x > p$. See how good the performance is for a few values of p (use a reasonable value N of "darts".

2.4 Importance sampling

Exercise Let us consider the following function $f(x) = e^{-x^2} g(x)$ in $[0, \infty]$ where $g(x)$ is a slowly varying function. Compute the integral both with the crude method and by using the importance sampling technique. Hint: For the importance sampling method use a Gaussian random number generator with density $W(x) = \sqrt{2/\pi} e^{-x^2}$ With this choice one has

$$I = \int_0^\infty f(x)\, dx \sim \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{W(x_i)} = \frac{1}{N} \sqrt{\pi/2} \sum_{i=1}^{N} g(x_i). \tag{2.4}$$

Exercise Estimate the integral

$$\int_0^{\pi/2} \cos\, x\, dx \tag{2.5}$$

using the importance sampling technique with $g(x)$ proportional to $a + bx^2$. Determine the optimal values of the parameters a and b to generate samples according to $g(x)$ and establish the number of iterations needed to get an accuracy of $\%1$.