Middle East Technical University
Department of Statistics
2019-20 Fall Semester
STAT 497 Applied Time Series Analysis

HOMEWORK 1

(Due: October 21, 2019, Monday – 16.00)

- *You should work on your own. Please feel free to get help from me, but not from anyone else. Let me know if my wording in the questions is not clear. I will post the answers to this homework on the same day. Therefore, absolutely, no late homework will be accepted.*

1. Suppose $y_t = 5 + 2t + X_t$ where X_t is a zero-mean stationary series with autocorrelation function γ_k.

a) Find the mean function for y_t .

b) Find the autocovariance function for y_t .

c) Is y_t stationary? Why or why not?

d) Consider the process $O_t = y_t - y_{t-1}$.

Is O_t stationary?

2. Please answer the following questions.

a) Describe the difference between the empirical autocorrelation function, r(h), (also known as sample autocorrelation function), and the theoretical autocorrelation function, p(h).

b) How are the empirical autocorrelation function and theoretical autocorrelation function used to identify a time series model?

c) Write stationary conditions for AR(1), MA(1), AR(2), MA(2), ARMA(1,1)

3. For each of the following processes

a) $y_t = 0.3y_{t-1} + Z_t$, where $Z_t \sim WN(0,1)$

b) $y_t = Z_t + 1.3Z_{t-1} + 0.4\,Z_{t-2}$, where $Z_t \sim WN(0,1)$

c) $y_t = 0.5y_{t-1} + Z_t - 1.3Z_{t-1} + 0.4\,Z_{t-2}$, where $Z_t \sim WN(0,1)$

Express each model using backshift operators and determine whether the model is stationary and/or invertible. Also, for each model suggest an appropriate model. (AR(p), MA(q) or ARMA(p,q).

4. Suppose a time series y_t has the following form

$y_t = X_t - \Theta X_{t-2}$ where $X_t \sim$ WN(0,1)

a) Define name of the process for $\{Y_t\}$.

b) Find the autocovariance generating function for the process and calculate autocovariance and autocorrelation functions for this process when $\Theta = 0.6$.

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c) Compute the variance of the sample mean $\bar{Y}$ for a sample size of n=3 when $\Theta = 0.4$.

5. Discuss the stationarity and invertibility of each representation and then write the process in IF and RSF.

a) $y_t = e_t + 0.2e_{t-1} + 1.4\, e_{t-2}$, where $e_t \sim WN(0,1)$

b) $y_t = 1.2 + 0.5\, y_{t-1} + 0.4\, y_{t-2} + e_t$, where $e_t \sim WN(0,1)$

6. Please answer the following multiple choice question. (In your homework sheet, enough to type letter for correct answer.)

- What is the difference between strict stationarity and weak stationarity?

(a) Strict stationarity requires that the mean function and autocovariance function be free of time t. Weak stationarity does not.
(b) Strict stationarity is required to guarantee that MMSE forecasts are unbiased (in ARIMA models). These forecasts may not be unbiased under weak stationarity.
(c) Strict stationarity is a stronger form of stationarity that does not rely on large-sample theory.
(d) Strict stationarity refers to characteristics involving joint probability distributions. Weak stationarity refers only to conditions placed on the mean and autocovariance function.

- Consider an invertible MA(2) process

$$y_t = e_t - \Theta_1 e_{t-1} - \Theta_2 e_{t-2}$$

Which statement is true?

(a) Its PACF can decay exponentially or in a sinusoidal manner depending on the roots of the MA characteristic polynomial.
(b) It is always stationary.
(c) Its ACF is nonzero at lags k = 1 and k = 2 and is equal to zero when k > 2.
(d) All of the above.

- Consider an AR(2) model

$$(1 - \emptyset_1 B - \emptyset_1 B^2) y_t = e_t$$

True or False: If the AR(2) characteristic polynomial $\emptyset(x) = (1 - \emptyset_1 x - \emptyset_1 x^2)$ has imaginary roots, then this model is not stationary.

(a) True
(b) False

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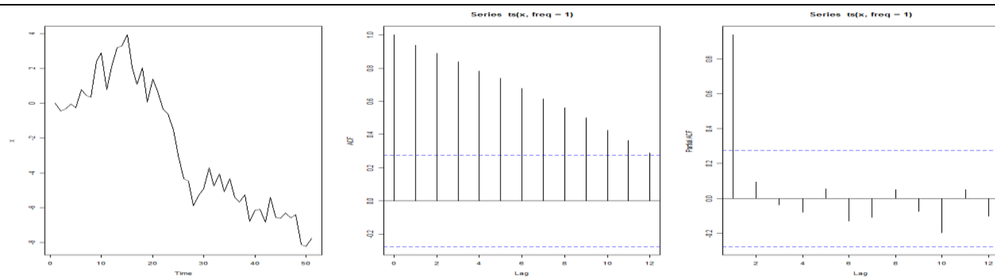
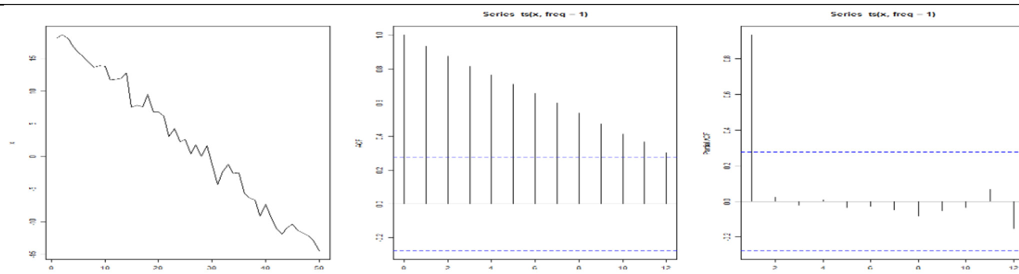
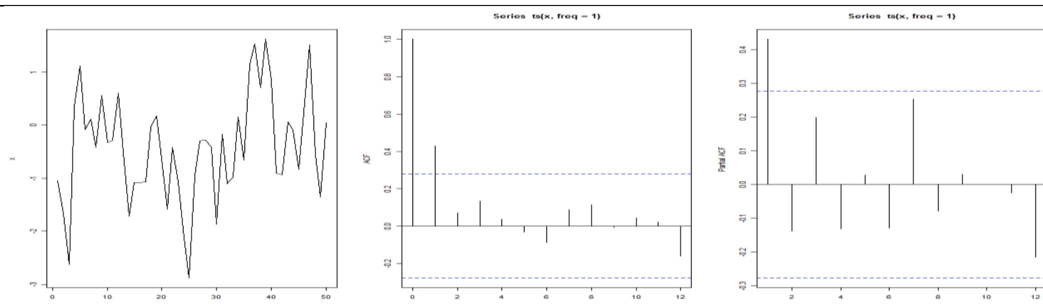
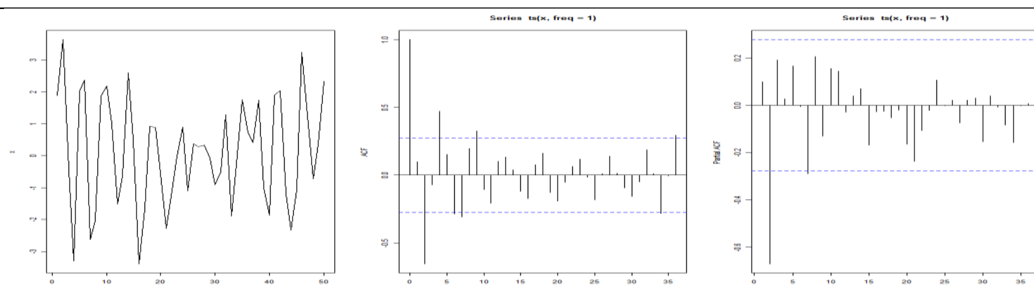
7. Consider the following stochastic processes

i) $Y_t = a_t - 0.5a_{t-1}$

ii) $Y_t - 1.2Y_{t-1} + 0.2Y_{t-2} = a_t$

iii) $Y_t - 0.1Y_{t-1} + 0.7Y_{t-2} = a_t$

iv) $Y_t = 20 - 0.7t + a_t$

A	
B	
C	
D	

Match the presented time series plots A to D with the generating stochastic processes given by i) to vi). Explain the reasons of your identification.

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If you do not understand the solution, please send me an e-mail. Then, I can explain it in details.

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Answer key

1) a) It is given that $E(X_t) = 0$

$$E(y_t) = E(5 + 2t + X_t) \overset{\text{linearity prop of expectation}}{=} \underbrace{E(5) + E(2t)}_{\text{constants}} + \underbrace{E(X_t)}_{0}$$

$$= 5 + 2t$$

b) It is given that $\gamma_{x(k)} = Cov(X_t, X_{t-k})$

$$\gamma_{y(h)} = Cov(y_t, y_{t-h})$$

$$= Cov(5 + 2t + X_t, y_{t-h})$$

$$= Cov(5, y_{t-h}) + 2\,Cov(t, y_{t-h}) + Cov(X_t, y_{t-h})$$

$h=0$

$$\gamma_{y(0)} = 0 + \underbrace{2.Cov(t, y_t)}_{(a)} + \underbrace{Cov(X_t, y_t)}_{(b)} = 0 + 0 + \gamma_{x(0)} = \gamma_{x(0)}$$

$$(a) = Cov(t, 5 + 2t + X_t)$$
$$= \underbrace{Cov(5,t)}_{0} + \underbrace{\underbrace{2Cov(t,t)}_{0}}_{\text{bc constant } t} + \underbrace{Cov(X_t,t)}_{0}$$

$$(b) = Cov(X_t, 5 + 2t + X_t)$$
$$= \underbrace{Cov(X_t, 5)}_{0} + \underbrace{2Cov(X_t, t)}_{0} + \underbrace{Cov(X_t, X_t)}_{\gamma_{x(0)}}$$

$h=1$

$$\gamma_{y(1)} = 0 + \underbrace{2Cov(t, y_{t-1})}_{(a)=0} + \underbrace{Cov(X_t, y_{t-1})}_{(b)} = 0 + \gamma_{x(1)} = \gamma_{x(1)}$$

$$(a) = Cov(t, 5 + 2(t-1) + X_{t-1})$$
$$= \underbrace{Cov(5,t)}_{0} + \underbrace{2Cov(t, t-1)}_{0} + \underbrace{Cov(t, X_{t-1})}_{0} \text{ (not in}$$

$$(b) = Cov(X_t, 5 + 2(t-1) + X_{t-1})$$
$$= \underbrace{Cov(5, X_t)}_{0} + \underbrace{2Cov(X_t, t-1)}_{0} + \underbrace{Cov(X_t, X_{t-1})}_{\gamma_{x(1)}}$$

$h=2$

$$\gamma_{y(2)} = 0 + 2\underbrace{Cov(t, y_{t-2})}_{a} + \underbrace{Cov(x_t, y_{t-2})}_{b}$$

$$= 0 + 2\underbrace{Cov(t, y_{t-2})}_{0} + Cov(x_t, y_{t-2})$$

$$(a) = Cov(t, \eta + 2(t-2) + x_{t-2})$$

$$= \underbrace{Cov(\eta, t)}_{0} + 2\underbrace{Cov(t, (t-2))}_{0} + \underbrace{Cov(t, x_{t-2})}_{0}$$

$$(b) = Cov(x_t, y_{t-2})$$

$$= Cov(x_t, \eta + 2(t-2) + x_{t-2})$$

$$= \underbrace{Cov(x_t, \eta)}_{0} + 2\underbrace{Cov(x_t, (t-2))}_{0} + \underbrace{Cov(x_t, x_{t-2})}_{\gamma_{x(2)}}$$

$\vdots$

$$\gamma_{y(h)} = \begin{cases} \gamma_{x(h)} & h = 0, \pm 1, \ldots \end{cases}$$

c) Since mean function of y_t depends on time, the process is not stationary.

d) Define our new process

$$O_t = y_t - y_{t-1}$$
$$= (\eta + 2t + x_t) - (\eta + 2(t-1) + x_{t-1})$$

$$O_t = 2 + x_t - x_{t-1}$$

To check whether stationary or not, calculate mean, variance and auto cov. of O_t.

$$E(O_t) = E(2 + x_t - x_{t-1}) = \overbrace{E(2)}^{2} + \overbrace{E(x_t)}^{0} - \overbrace{E(x_{t-1})}^{0} = 2$$

$$V(O_t) = V(2 + x_t - x_{t-1}) = V(x_t - x_{t-1})$$
$$= V(x_t) + V(x_{t-1}) - 2Cov(x_t, x_{t-1})$$
$$= \gamma_{x(0)} + \gamma_{x(0)} - 2\gamma_{x(1)}$$
$$= 2(\gamma_{x(0)} - \gamma_{x(1)})$$

$$\gamma_{o(h)} = Cov(o_t, o_{t-h})$$

$$= Cov(2 + x_t - x_{t-1}, o_{t-h})$$

$$= \underbrace{Cov(2, o_{t-h})}_{0} + Cov(x_t, o_{t-h}) - Cov(x_{t-1}, o_{t-h})$$

$$= Cov(x_t, o_{t-h}) - Cov(x_{t-1}, o_{t-h})$$

$\underline{h=0}$

$$\gamma_{o(0)} = \underbrace{Cov(x_t, o_t)}_{(a)} - \underbrace{Cov(x_{t-1}, o_t)}_{(b)} = [\gamma_{x(0)} - \gamma_{x(1)}] - [\gamma_{x(1)} - \gamma_{x(0)}]$$

$$= 2(\gamma_{x(0)} - \gamma_{x(1)}) \; \#$$

$$(a) = Cov(x_t, 2 + x_t - x_{t-1})$$

$$= \underbrace{Cov(x_t, 2)}_{0} + \underbrace{Cov(x_t, x_t)}_{\gamma_{x(0)}} - \underbrace{Cov(x_t, x_{t-1})}_{\gamma_{x(1)}}$$

$$(b) = Cov(x_{t-1}, 2 + x_t - x_{t-1})$$

$$= \underbrace{Cov(x_{t-1}, 2)}_{0} + \underbrace{Cov(x_{t-1}, x_t)}_{\gamma_{x(1)}} - \underbrace{Cov(x_{t-1}, x_{t-1})}_{\gamma_{x(0)}}$$

$\underline{h=1}$

$$\gamma_{o(1)} = \underbrace{Cov(x_t, o_{t-1})}_{(a)} - \underbrace{Cov(x_{t-1}, o_{t-1})}_{(b)} = [\gamma_{x(1)} - \gamma_{x(2)}] - [\gamma_{x(0)} - \gamma_{x(1)}]$$

$$= 2\gamma_{x(1)} - \gamma_{x(2)} - \gamma_{x(0)} \; \#$$

$$(a) = Cov(x_t, 2 + x_{t-1} - x_{t-2})$$

$$= \underbrace{Cov(2, x_t)}_{0} + \underbrace{Cov(x_t, x_{t-1})}_{\gamma_{x(1)}} - \underbrace{Cov(x_t, x_{t-2})}_{\gamma_{x(2)}}$$

$$(b) = Cov(x_{t-1}, 2 + x_{t-1} - x_{t-2})$$

$$= \underbrace{Cov(2, x_{t-1})}_{0} + \underbrace{Cov(x_{t-1}, x_{t-1})}_{\gamma_{x(0)}} - \underbrace{Cov(x_{t-1}, x_{t-2})}_{\gamma_{x(1)}}$$

$\underline{h=2}$

$$\gamma_{o(2)} = \underbrace{Cov(x_t, o_{t-2})}_{(a)} - \underbrace{Cov(x_{t-1}, o_{t-2})}_{(b)} = [\gamma_{x(2)} - \gamma_{x(3)}] - [\gamma_{x(1)} - \gamma_{x(2)}]$$

$$(a) = Cov(x_t, o_{t-2}) = Cov(x_t, 2 + x_{t-2} - x_{t-3}) \quad = 2\gamma_{x(2)} - \gamma_{x(3)} - \gamma_{x(1)}$$

$$= 0 + \gamma_{x(2)} - \gamma_{x(3)}$$

$$(b) = Cov(x_{t-1}, o_{t-2}) = Cov(x_{t-1}, + 2 + x_{t-2} - x_{t-3})$$

$$= 0 + \gamma_{x(1)} - \gamma_{x(2)}$$

$$\gamma_{O}(h) = \begin{cases} 2(\gamma_X(0) - \gamma_X(1)) & h=0 \\ 2\gamma_X(h) - \gamma_X(h+1) - \gamma_X(h-1) & |h| \geq 1 \end{cases}$$

It is revealed that both mean and autocov. fnc of O_t does not depend on t. Therefore, process O_t is stationary.

2) (a) The empirical ACF is calculated from the data, or the sample observed, while the theoretical ACF is mathematically derived form of a process.

(b) Essentially, the empirical ACF is compared to theoretical ACF's, we choose a process for our time series that has a theoretical ACF consistent with the sample ACF observed.

(c) AR(1) is stat if $|\phi| < 1$

MA(1) is already stationary.

AR(2) is stat iff $\phi_1 + \phi_2 < 1$

$\phi_2 - \phi_1 < 1$

$|\phi_2| < 1$

MA(2) is already stationary.

ARMA(1,1) is stat iff $|\phi| < 1$.

3) a) AR(1)

$(1-0.3B)\, y_t = z_t$

- $|\phi_1 = 0.3| < 1 \Rightarrow$ stationary ✓
- It is already in invertible form ✓

b) MA(2)

$y_t = (1 + 1.3B + 0.4B^2) z_t$

- It is already stationary ✓

$\theta_1 = -1.3 \quad \theta_2 = -0.4$

- $\theta_1 + \theta_2 = (\underbrace{-1.3 - 0.4}_{-1.7}) < 1$ ✓

It is invertible

- $\theta_2 - \theta_1 \quad (\underbrace{-0.4 - (+1.3))}_{0.9} < 1$ ✓
- $-1 < -0.4 < 1$ ✓

c) ARMA(1,2)

$(1-0.5B)\, y_t = (1 - 1.3B + 0.4B^2)\, z_t$

- $|\phi_1 = 0.5| < 1$ ✓ : stationary

- $\theta_1 = 1.3$
  $\theta_2 = -0.4$
  - $\theta_1 + \theta_2 = \underbrace{1.3 - 0.4}_{0.9} < 1$ ✓
  - $\theta_2 - \theta_1 = \underbrace{-0.4 - 1.3}_{-1.7} < 1$ ✓
  - $-1 < \theta_2 = -0.4 < 1$ ✓

It is invertible

4) a) It is a MA(2) process.

b) For finding autocoverance function, I can use autogenerating function. That's why, I need to find MA order using Backshift operator.

$\gamma(B) = \sigma_x^2 \psi(B) \psi(B^{-1})$ $\qquad$ $y_t = \underbrace{(1-\theta B^2)}_{\psi(B)} x_t$

$\gamma(B) = \sigma_x^2 (1-\theta B^2)(1-\theta B^{-2})$

$= \sigma_x^2 (1 - \theta B^{-2} - \theta B^2 + \theta^2 B^0)$

$$\gamma_k = \begin{cases} (1+\theta^2)\sigma_x^2 & , k=0 \\ 0 & , k=\pm 1 \\ -\theta\sigma_x^2 & , k=\pm 2 \\ 0 & \text{ow.} \end{cases}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} 1 & k=0 \\ 0 & k=\pm 1 \\ \frac{-\theta}{1+\theta^2} & k=\pm 2 \\ 0 & \text{ow} \end{cases}$$

$\theta = 0.6$

$$\gamma_k = \begin{cases} 1.36 & k=0 \\ -0.6 & k=\pm 2 \\ 0 & \text{ow.} \end{cases}$$ Becase it is given that $\sigma_x^2 = 1$

$$\rho_k = \begin{cases} 1 & k=0 \\ -0.44 & k=\pm 2 \\ 0 & \text{ow} \end{cases}$$

c) $\bar{y} = \frac{y_1+y_2+y_3}{3}$

$V(\bar{y}) = V\left(\frac{y_1+y_2+y_3}{3}\right) = \frac{1}{9}\left[V(y_1)+V(y_2)+V(y_3)+2Cov(y_1,y_2)+2Cov(y_1,y_3)+2Cov(y_2,y_3)\right]$

for $\theta = 0.4$ $\qquad = \frac{1}{9}\left[\gamma_0+\gamma_0+\gamma_0+2\gamma_1+2\gamma_2+2\gamma_1\right]$

$$\gamma_k = \begin{cases} 1.16 & , k=0 \\ -0.4 & k=\pm 2 \end{cases}$$ $= \frac{1}{9}\left[3\gamma_0 + 4\overset{0}{\gamma_1} + 2\gamma_2\right]$

$= \frac{1}{9}\left[3.48+2(-0.4)\right] = 0.298$

6

5) a) $y_t = (1 + 0.2B + 1.4B^2) e_t$ where $e_t \sim WN(0,1)$

It is MA(2). So, it is already stationary
For MA(2), invertibility conditions

$\theta_2 + \theta_1 < 1$ | $-1.4 + (-0.2) = -1.6 < 1$ ✓
$\theta_2 - \theta_1 < 1$ | $-1.4 - (-0.2) = -1.2 < 1$ ✓
$|\theta_2| < 1$ | $|-1.4| \nless 1$ ✗

process is not invertible - we cannot write in IF.

b) $\dot{y}_t = 0.5 \dot{y}_{t-1} + 0.4 \dot{y}_{t-2} + e_t$, $e_t \sim WN(0,1)$ $\dot{y}_t = y_t - 1.2$

It is a AR(2) process. It is already in inverted form. Stationary conditions;

$\phi_2 + \phi_1 < 1$ | $\phi_1 = 0.5$ $\phi_2 = 0.4$
$\phi_2 - \phi_1 < 1$ | $0.4 + 0.5 = 0.9 < 1$ ✓
$|\phi_2| < 1$ | $0.4 - 0.5 = -0.1 < 1$ ✓
| $|0.4| < 1$ ✓

process is stationary
It can be written in MA form (RSF)

To write this process in RSF, we should find $\psi(B)$ by using $\pi(B)$.
To obtain AR polynomial ($\pi(B)$), the process must be rewritten in Backshift.

$$\dot{y}_t = 0.5 \dot{y}_{t-1} + 0.4 \dot{y}_{t-2} + e_t$$
$$\underbrace{(1 - 0.5B - 0.4B^2)}_{\pi(B)} \dot{y}_t = e_t$$

Roots of $\pi(B)$ lie outside the unit circle. Process is stationary.

$\pi(B)\psi(B) = 1$

$$(1 - 0.5B - 0.4B^2)(1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = 1$$
$$(\cancel{1} + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots$$
$$- 0.5B - 0.5\psi_1 B^2 - 0.5\psi_2 B^3 - \dots$$
$$- 0.4B^2 - 0.4\psi_1 B^3 - 0.4\psi_2 B^4 - \dots) = \cancel{1}$$

- $(\psi_1 - 0.5)B = B0 \Rightarrow \psi_1 = 0.5$
- $(\psi_2 - 0.5\psi_1 - 0.4) = 0 \Rightarrow \psi_2 = 0.5\psi_1 + 0.4 = 0.65$
- $(\psi_3 - 0.5\psi_2 - 0.4\psi_1) = 0 \Rightarrow \psi_3 = 0.5\psi_2 + 0.4\psi_1 = 0.525$

$\vdots$

$$\psi_j = 0.5\psi_{j-1} + 0.4\psi_{j-2} \text{ for } j \geq 3$$

$\sum \psi_i^2 < \infty$

writing y_t in RSF

$y_t = \psi(B) e_t = (1 + \psi_1 B + \psi_2 B^2 + \dots) e_t$

$= (1 + 0.75 B + 0.65 B^2 + 0.575 B^3 + \dots) e_t$

$= e_t + 0.75 e_{t-1} + 0.65 e_{t-2} + 0.575 e_{t-3} + \dots$

6) "In this question, If you make a mistake please ask me! we can look it together."

a) (d) Strict stationary refers to characteristics involving joint prob. distributions weak stationary refers only to conditions placed on the mean and autocovariance function.

b) (d) All of the above.

c) (b) False

7) A). The process is not stationary. It has a decreasing trend but not linear.
- ACF has linear trend which is an indication of nonstationary
- PACF no need interpretation.

The app. model is ii) * because it is not stationary and trend not deterministic

B). The process is not stationary. It has a deterministic dec. trend. That means the process depends on time
- ACF has linear trend
- No need interpretation for PACF.

The app Model is iv)

c). The process is stationary
- ACF cuts off after lag 1
- PACF has sinusoidal behaviour.

The app. model is i) $\Rightarrow$ MA(1)

D). The process is stationary around zero
- ACF has sinusoidal behaviour
- PACF cuts off after lag 2.

The app. model is iii) $\Rightarrow$ AR(2)