CME 2001
Data Structures and Algorithms

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B-Trees

B-Trees

- B-trees are balanced search trees and designed to work well on disks or other direct access secondary storage devices.
- Many database systems use B-trees, or variations to store information.
- The idea of the B-tree also motivates the design of many other disk-based index structures.

B-Trees

- When data volume is large and does not fit in memory, a B-tree is used.
- The B-tree is always balanced (all leafs appear at the same level)
- Since each disk access exchanges a whole block of information between memory and disk, a node of the B-tree is expanded to hold more than two child pointers, up to the block capacity.
- The B-tree requires that every node (except the root) has to be at least *half full*.
- An exact match query, insertion, deletion need to access O(log*B n)* nodes, where
  - ***B****: the page capacity* in number of child pointers
  - **n***: is the number of objects*

Disk –Based Environment

- The computer CPU deals directly with the primary storage (main memory).
- We can access data stored in main memory quickly, but cannot store everything in memory:
  - because memory is expensive.
  - memory is volatile, i.e. if there is a power failure, information stored in memory gets lost.
- The secondary storage stands for magnetic disks. Although it has slower access, it is less expensive and non-volatile.

Disk –Based Environment

- The CPU does not deal with disk directly, any data has to be read from disk to memory first.
- Data is stored on disk in units called ***blocks*** or ***pages***.

- If a disk page is 8KB (8192 bytes), while a node in the BST is 16 bytes (four integers: key, value, two child pointers)

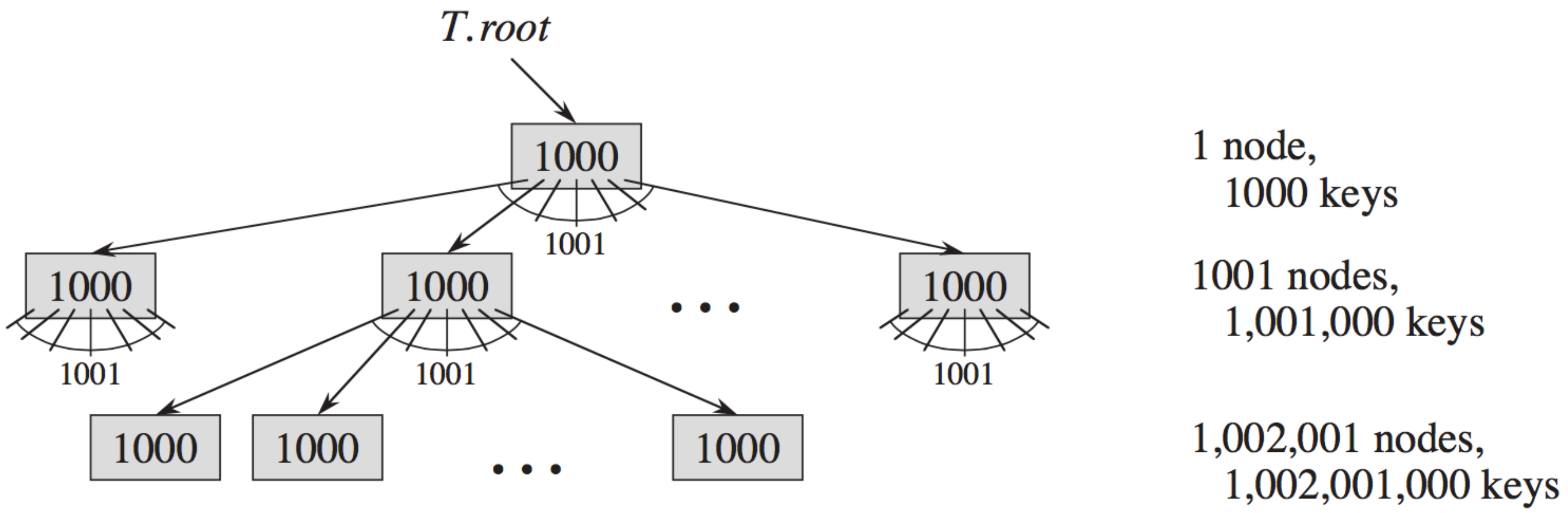
  => every page is only 8192/16 = 0.2% full.
- To improve space efficiency, we should store multiple tree nodes in one disk page.

Disk –Based Environment

- The running time of a B-tree operations highly depends on the number of *DISK-READ* and *DISK-WRITE* operations.
- ***Branching factor***: total # of children of a parent node
- For a large B-tree stored on a disk, ***branching factors*** are between 50 and 2000, depending on the size of a key relative to the size of a page.
- A large *branching factor* reduces :
  - the height of the tree
  - the number of disk accesses required to find any key

B-tree example

- B-tree :
  - branching factor = 1001
  - height = 2 (that can store over ***one billion keys***)
- can keep the root node permanently in main memory,
- can find any key in this tree by making at most only **2** disk accesses.



Definition of B-trees

1. Every node ***x*** has the following attributes:
    a) ***x.n***: the number of keys currently kept in node ***x***
    b) the ***x.n*** keys themselves stored in an increasing order, so that $x.key_1 \leq x.key_2 \leq \ldots \leq x.key_{x.n}$
    c) ***x.leaf***: boolean value, which is *TRUE* if ***x*** is a leaf, *FALSE* if ***x*** is an internal node.

2. Each internal node ***x*** contains ***x.n+1*** pointers $x.c_1, x.c_2, \ldots, x.c_{n+1}$ to indicate its children. Leaf nodes have no children, no c_i attributes.

3. The keys $x.key_i$ separate the ranges of keys stored in each subtree. If k_i is any key stored in the subtree with root $x.c_i$ then:

$$k_1 \leq x.key_1 \leq k_2 \leq x.key_2 \leq \ldots \leq x.key_{x.n} \leq k_{x.n+1}$$

Definition of B-trees

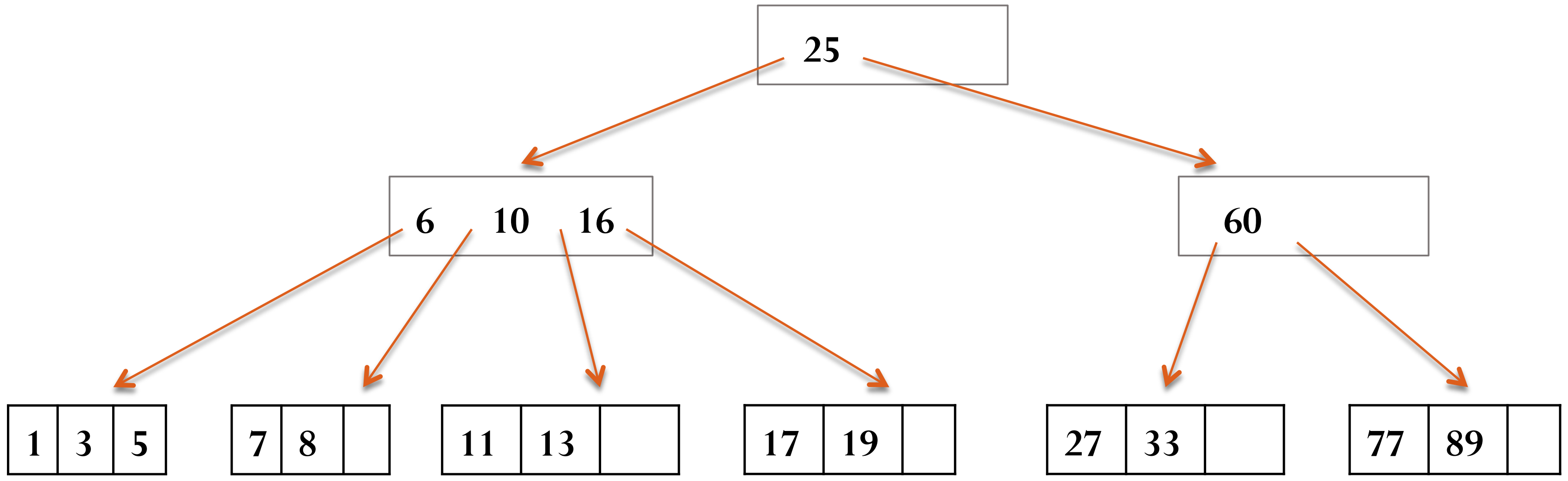
4. All leaves have the same depth, which is the tree's height ***h*.**

5. Nodes have lower and upper bounds on the number of keys they can contain. These bounds are represented by a fixed integer ***t*** $\geq$ *2,* called the ***minimum degree*** of the B-tree:

   a) Every node (except root) must have at least ***t***-1 keys. Every internal node has at least ***t*** children. If tree in non-empty, root must have at least one key.

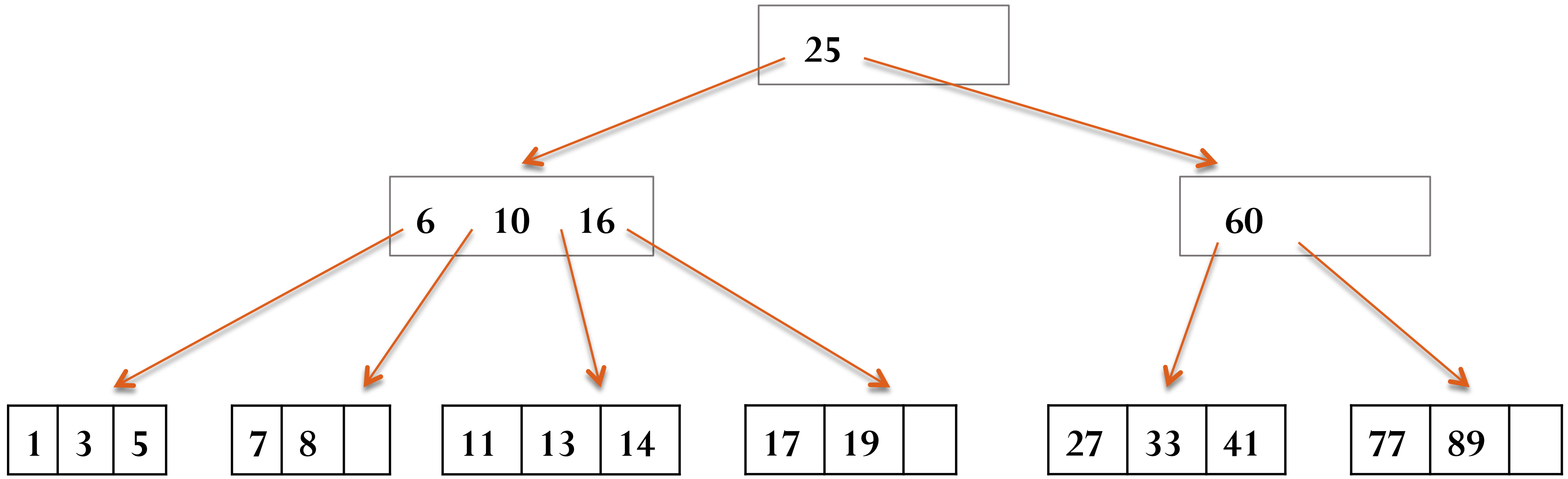
   b) Every node may contain at most 2***t***-1 keys. An internal node may have at most 2***t*** children. A node its contains exactly 2***t***-1 keys, it becomes ***full.***

B-tree example



Insert : 14, 41

B-tree example



B-tree Operations: Search

```
B-TREE-SEARCH$(x, k)$
$i = 1$
while $i \leq x.n$ and $k > x.key_i$
    $i = i + 1$
if $i \leq x.n$ and $k == x.key_i$
    return $(x, i)$
elseif $x.leaf$
    return NIL
else DISK-READ$(x.c_i)$
    return B-TREE-SEARCH$(x.c_i, k)$
```

- B-TREE-SEARCH applies a similar method with BST-SEARCH.
- Starts with B-TREE-SEARCH (*T.root*, *k*)
- If *k* is found, it returns *(x,i)* node *x* and index *i* such that $x.key_i = k$
- Otherwise returns *NIL*.
- DISK-READ reads the requested object from secondary memory and puts into main memory.

Running time: $O(t.h) = O(t \lg_t n)$

- ***while*** loop runs ***t*** times

- recursion runs at most the depth of the tree = $O(h)$

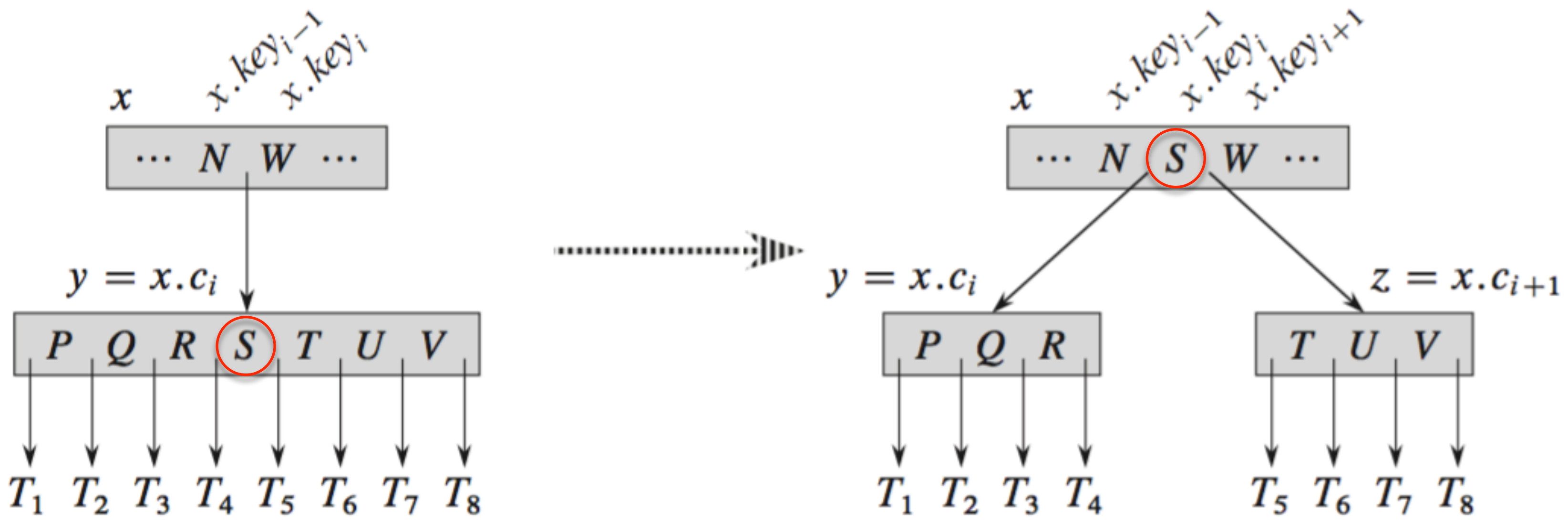
Create an empty B-tree

```
B-TREE-CREATE($T$)
1  $x$ = ALLOCATE-NODE()
2  $x.leaf$ = TRUE
3  $x.n = 0$
4  DISK-WRITE($x$)
5  $T.root = x$
```

- ALLOCATE-NODE allocates one disk page to be used as a new node in O(1) time.
- DISK-WRITE saves attributes of new node to the disk.
- Running time: O(1)

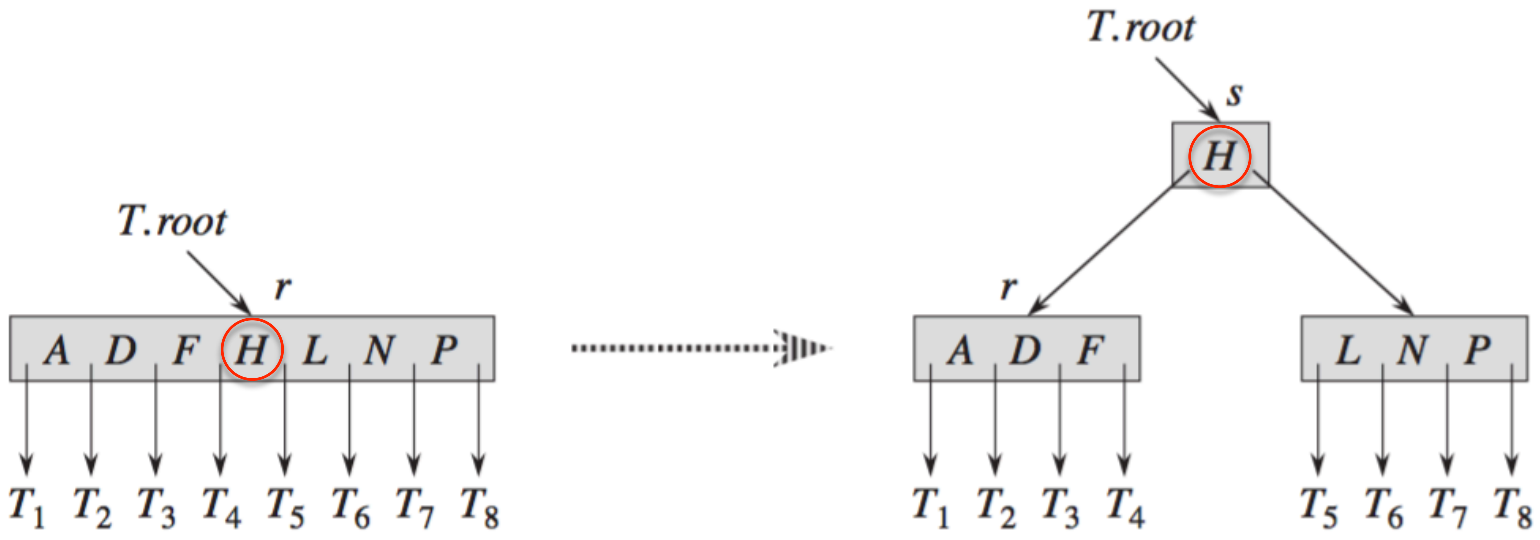
Insert a key into a B-tree

- To insert new key ***X***, find a suitable leaf node:
  - If leaf node is not full, put ***X*** into empty slot (easy case!)
  - If leaf node is full, split leaf node and adjust parents up to root.
- Insert "S" into below B-tree in which **t**=4



Insert a key into a B-tree …

- If the key should be put into a full root node, which should be divided into *two new nodes* and *a new root*.
- Insert "H" into the root node of below B-tree in which **t**=4



Insert a key into a B-tree

```
B-TREE-INSERT($T, k$)
1   $r = T.root$
2   if $r.n == 2t - 1$
3       $s$ = ALLOCATE-NODE()
4       $T.root = s$
5       $s.leaf$ = FALSE
6       $s.n = 0$
7       $s.c_1 = r$
8       B-TREE-SPLIT-CHILD($s$, 1)
9       B-TREE-INSERT-NONFULL($s, k$)
10  else B-TREE-INSERT-NONFULL($r, k$)
```

$\longrightarrow$ Is Root node is full?

If "yes", then run lines 3-9

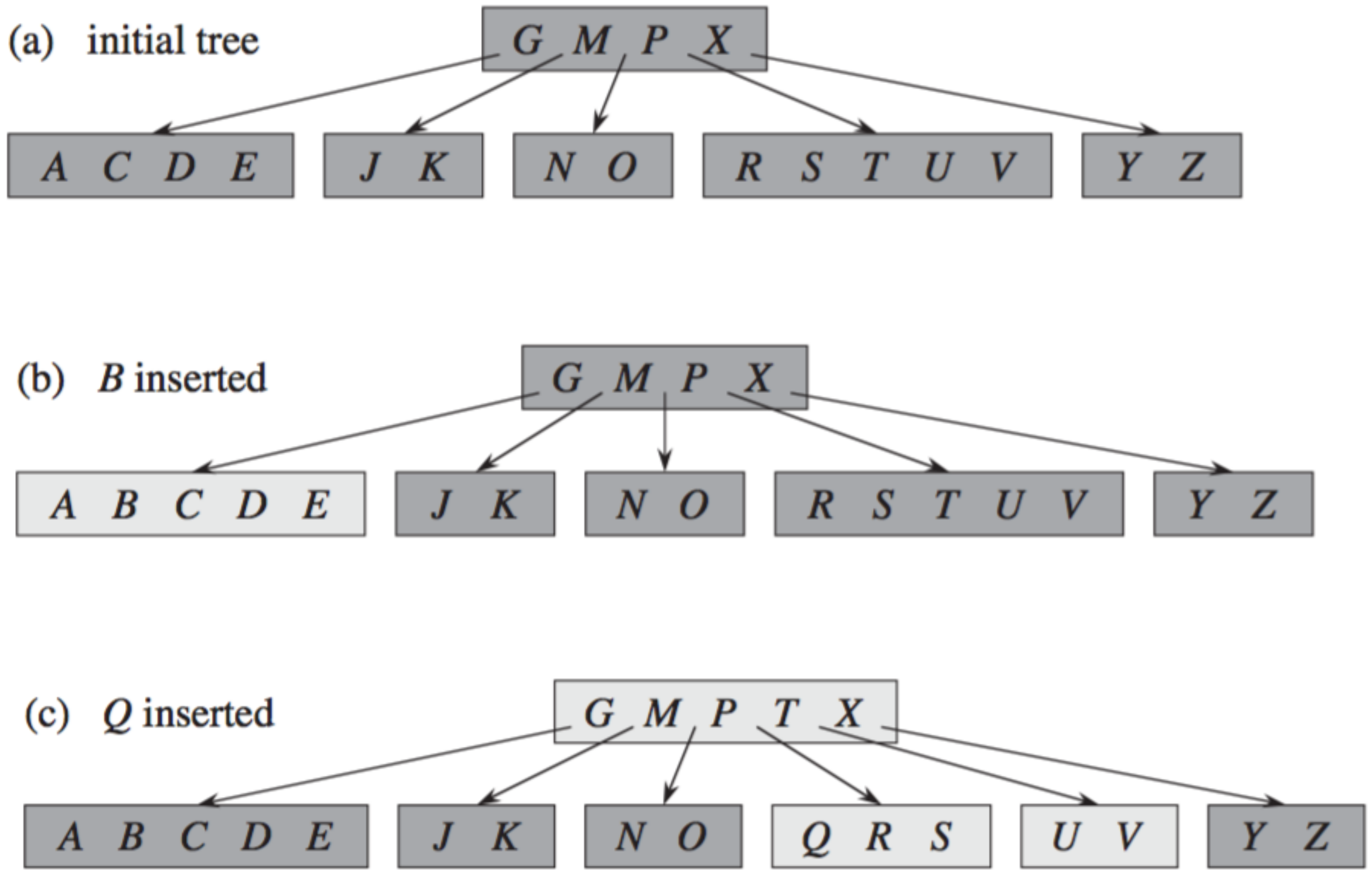
Running time: $O(t \lg_t n)$

```
B-TREE-SPLIT-CHILD$(x, i)$
 1  $z$ = ALLOCATE-NODE()
 2  $y = x.c_i$
 3  $z.leaf = y.leaf$
 4  $z.n = t - 1$
 5  for $j = 1$ to $t - 1$
 6      $z.key_j = y.key_{j+t}$
 7  if not $y.leaf$
 8      for $j = 1$ to $t$
 9          $z.c_j = y.c_{j+t}$
10  $y.n = t - 1$
11  for $j = x.n + 1$ downto $i + 1$
12      $x.c_{j+1} = x.c_j$
13  $x.c_{i+1} = z$
14  for $j = x.n$ downto $i$
15      $x.key_{j+1} = x.key_j$
16  $x.key_i = y.key_t$
17  $x.n = x.n + 1$
18  DISK-WRITE$(y)$
19  DISK-WRITE$(z)$
20  DISK-WRITE$(x)$
```

```
B-TREE-INSERT-NONFULL$(x, k)$
 1  $i = x.n$
 2  if $x.leaf$
 3      while $i \geq 1$ and $k < x.key_i$
 4          $x.key_{i+1} = x.key_i$
 5          $i = i - 1$
 6      $x.key_{i+1} = k$
 7      $x.n = x.n + 1$
 8      DISK-WRITE$(x)$
 9  else while $i \geq 1$ and $k < x.key_i$
10          $i = i - 1$
11      $i = i + 1$
12      DISK-READ$(x.c_i)$
13      if $x.c_i.n == 2t - 1$
14          B-TREE-SPLIT-CHILD$(x, i)$
15          if $k > x.key_i$
16              $i = i + 1$
17      B-TREE-INSERT-NONFULL$(x.c_i, k)$
```

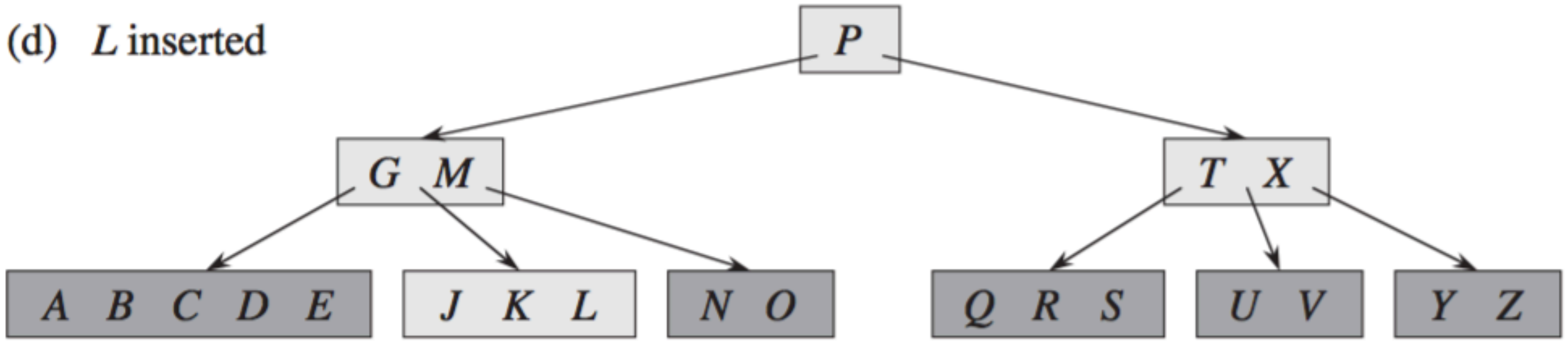
Insertion example

$t = 3$

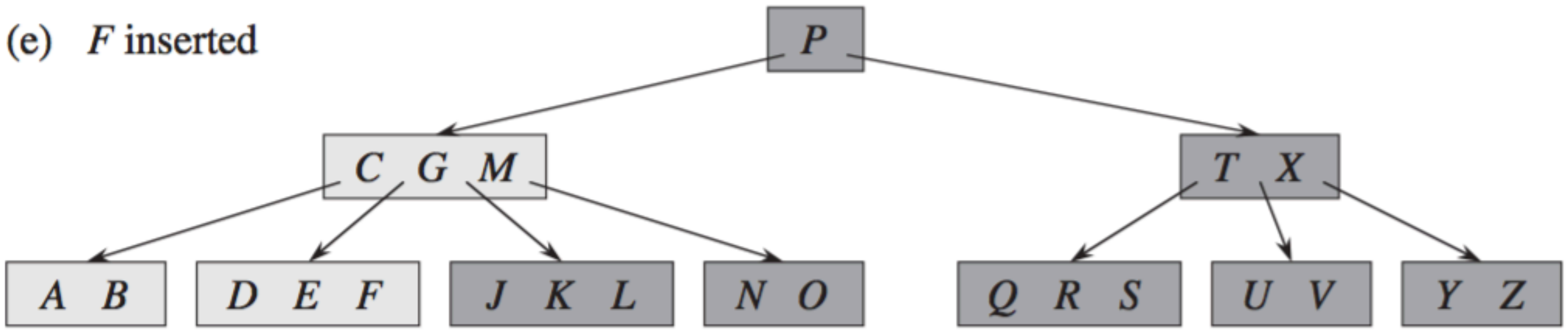


Insertion example

(d) *L* inserted



(e) *F* inserted

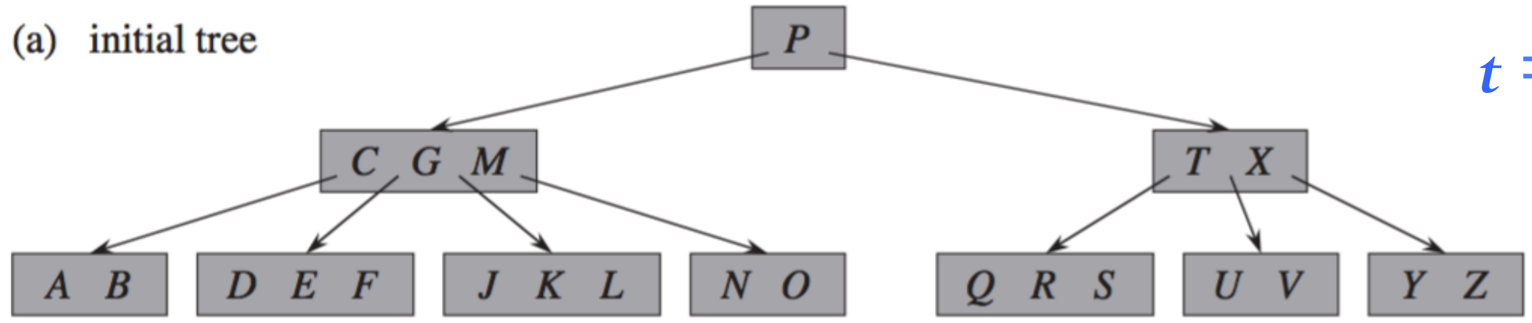


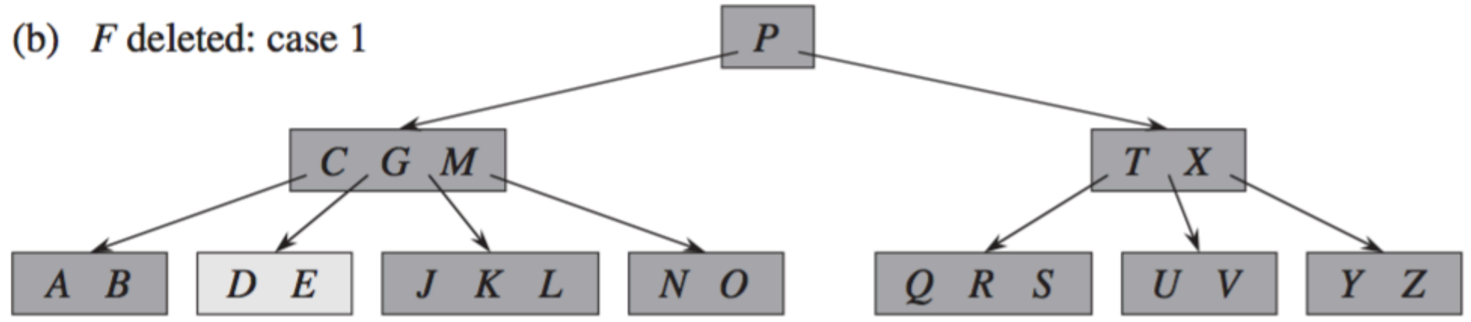
Delete a key

There could be several cases while delete a key ***k*** from B-tree

Case 1: If ***k*** is in node *x* and *x* is a leaf

$t = 3$

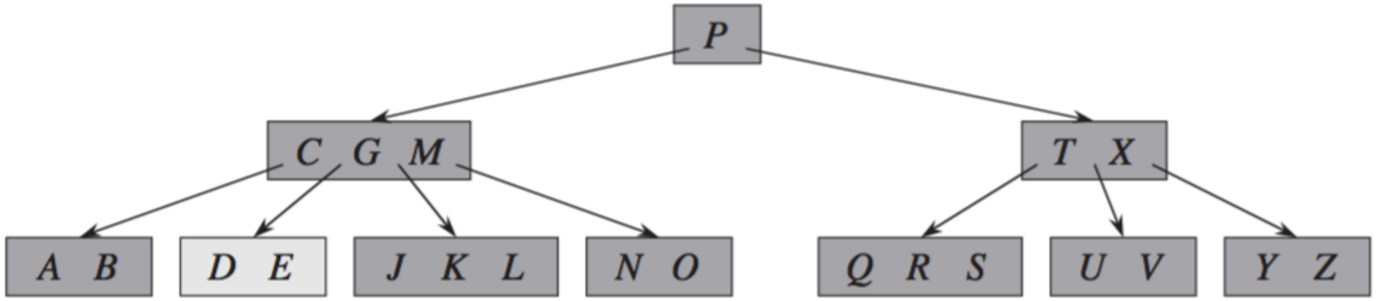




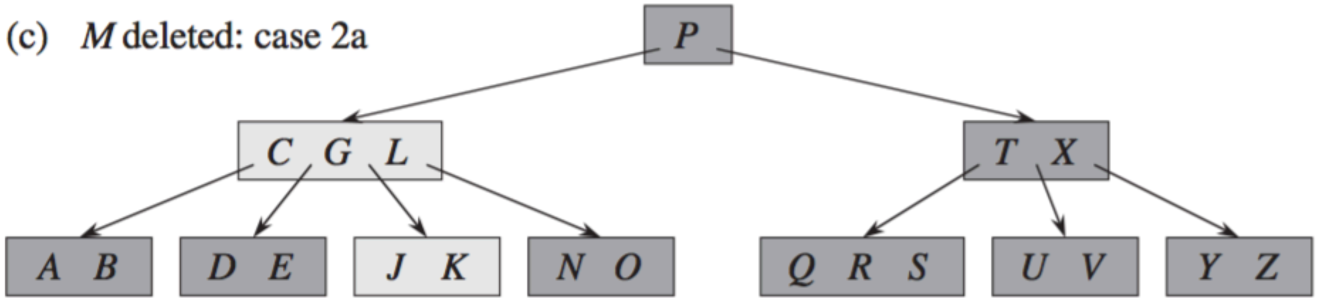
Delete a key …

Case 2: If ***k*** is in node *x* and *x* is an internal node:

2a) If the child ***y*** that precedes ***k*** in node ***x*** has at least ***t*** keys, then find the predecessor of ***k*** in the subtree rooted by ***y***.



(c) *M* deleted: case 2a

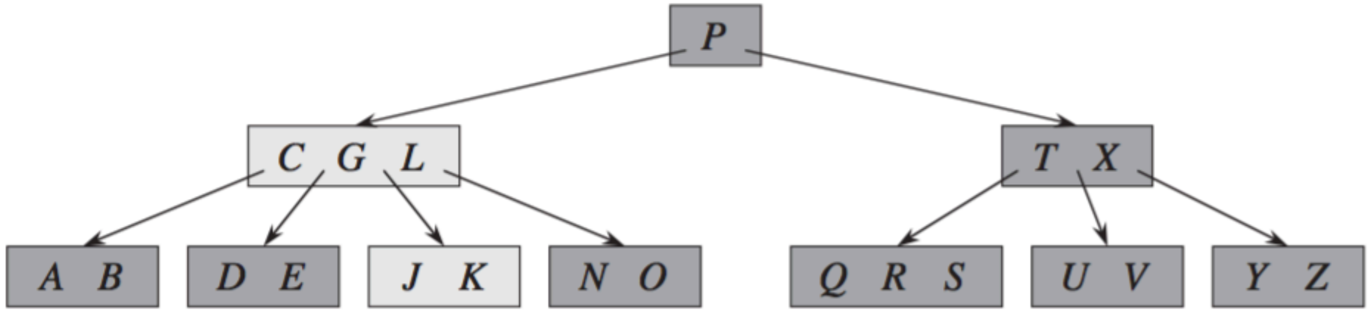


Delete a key …

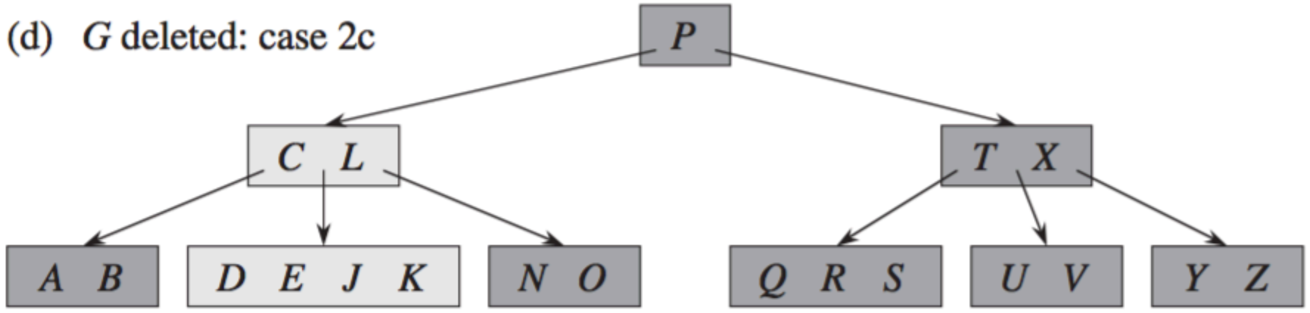
Case 2b) If child ***y*** has fewer than ***t*** keys, then, symmetrically, examine the child ***z*** that follows ***k*** in node ***x***. If ***z*** has at least ***t*** keys, then find the successor of ***k*** in the subtree rooted at ***z***.

Delete a key …

Case 2c) if both *y* and ***z*** have only ***t*** -1 keys, merge ***k*** and all of ***z*** into *y*, so that *x* loses both ***k*** and the pointer to ***z***, and y now contains 2***t*** -1 keys. Then free ***z*** and recursively delete ***k*** from *y*.



(d) *G* deleted: case 2c



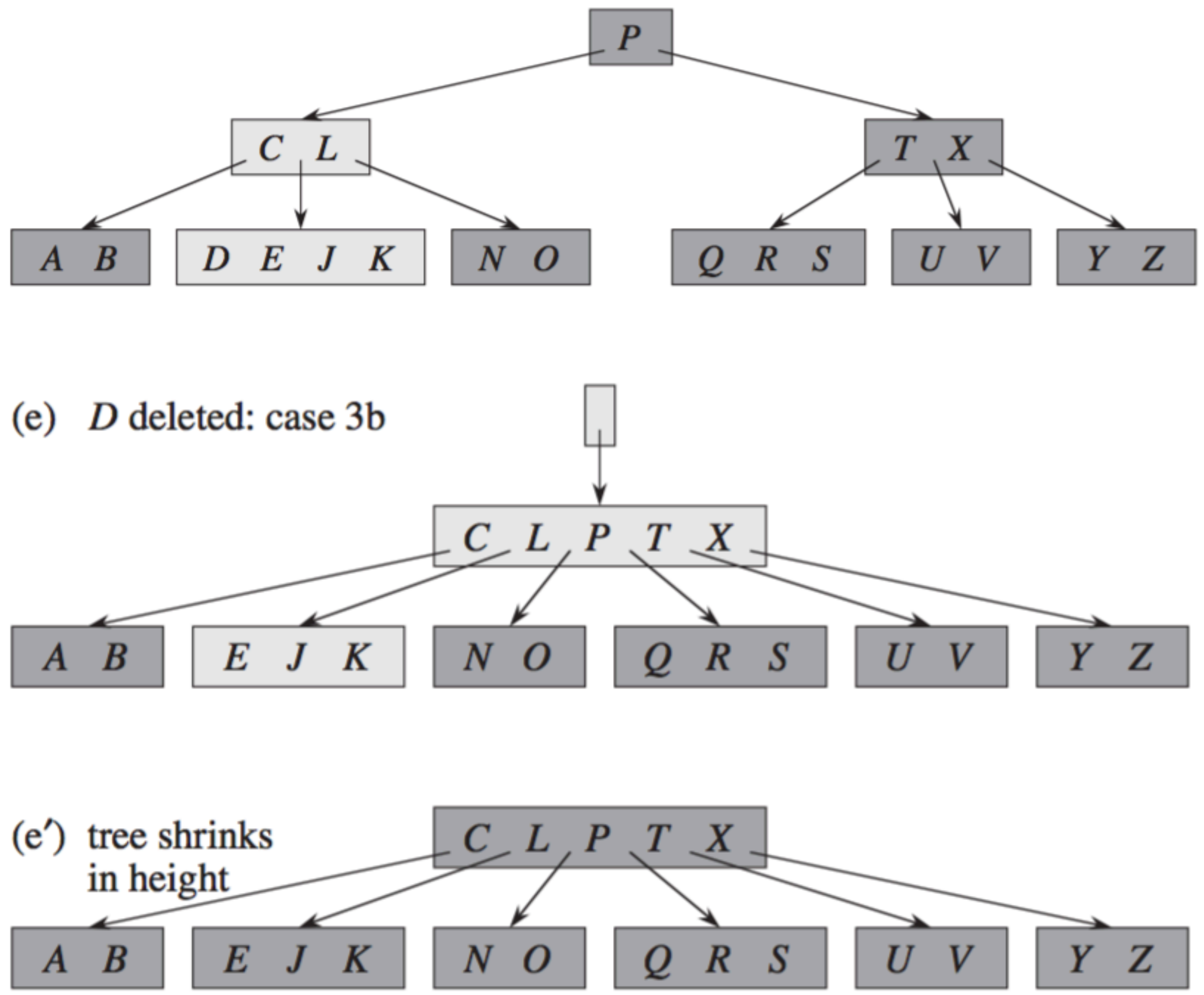
Delete a key …

Case 3: If $\boldsymbol{k}$ does not present in internal node $\boldsymbol{x}$, determine the root $\boldsymbol{x.c_i}$ of the appropriate subtree that must contain $\boldsymbol{k}$, if $\boldsymbol{k}$ is in the tree at all. If $\boldsymbol{x.c_i}$ has only $\boldsymbol{t}$-1 keys, execute step 3a or 3b as necessary to guarantee that we descend to a node containing at least $\boldsymbol{t}$ keys. Then finish by recursing on the appropriate child of $\boldsymbol{x}$.

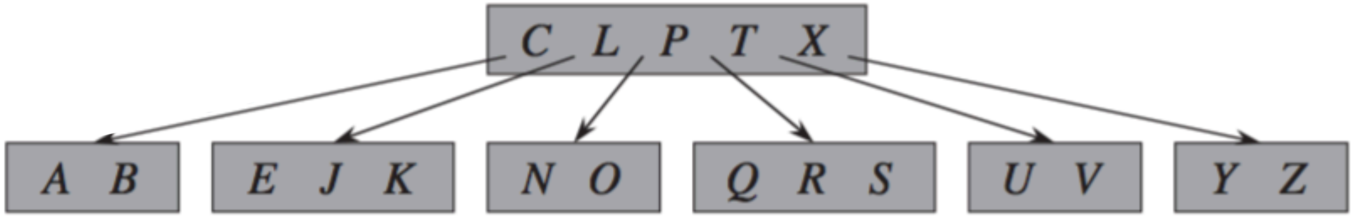
3a) If $\boldsymbol{x.c_i}$ has only t -1 keys but has an immediate sibling with at least $\boldsymbol{t}$ keys, give $\boldsymbol{x.c_i}$ an extra key by moving a key from $\boldsymbol{x}$ down into $\boldsymbol{x.c_i}$, moving a key from $\boldsymbol{x.c_i}$'s immediate left or right sibling up into $\boldsymbol{x}$, and moving the appropriate child pointer from the sibling into $\boldsymbol{x.c_i}$.

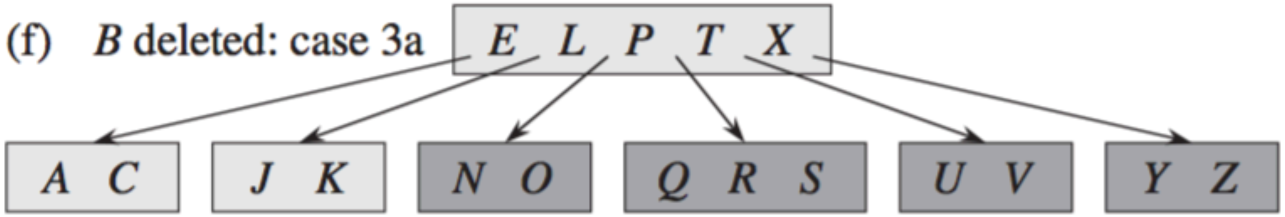
3b) If $\boldsymbol{x.c_i}$ and both of $\boldsymbol{x.c_i}$'s immediate siblings have $\boldsymbol{t}$ -1 keys, merge $\boldsymbol{x.c_i}$ with one sibling, which involves moving a key from $\boldsymbol{x}$ down into the new merged node to become the median key for that node.

Delete a key …


P
C L
T X
A B
D E J K
N O
Q R S
U V
Y Z
(e) D deleted: case 3b
C L P T X
A B
E J K
N O
Q R S
U V
Y Z
(e′) tree shrinks in height
C L P T X
A B
E J K
N O
Q R S
U V
Y Z

Delete a key …




(f) *B* deleted: case 3a