CME 2001
Data Structures and Algorithms

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Heapsort

Heaps

- A heap is a complete binary tree such that:
  - It is empty, or
  - Its root contains a search key greater than or equal to the search key in each of its children, and each of its children is also a heap.
- The root contains the item with the largest search key
- ***Height*** of node = # of edges on a longest simple path from the node down to a leaf.
- ***Height*** of heap = height of root = $\Theta(\lg n)$.

Heaps

A heap can be stored as an array **A**.

- Root of tree is $A[1]$.
- Parent of $A[i] = A[\lfloor i/2 \rfloor]$.
- Left child of $A[i] = A[2i]$.
- Right child of $A[i] = A[2i + 1]$.

PARENT(i) : return $\lfloor i/2 \rfloor$

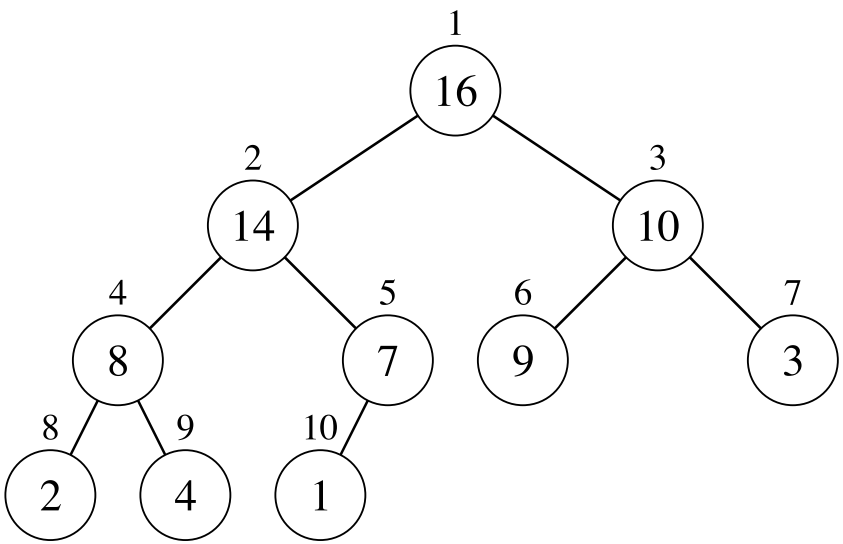
LEFT (i): return $2i$

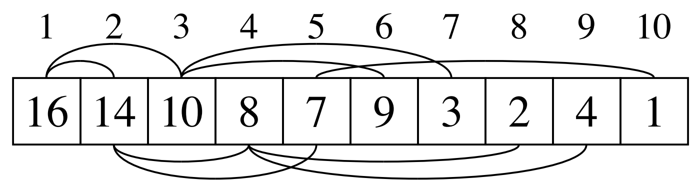
RIGHT (i): return $2i+1$

Heap Property:

- For max-heaps (largest element at root), ***max-heap property:*** for all nodes ***i***, excluding the root, **A**[PARENT(i)] $\geq$ **A**[i].
- For min-heaps (smallest element at root), ***min-heap property:*** for all nodes ***i***, excluding the root, **A**[PARENT(i)] $\leq$ **A**[i].

Max-Heap example





heap-size: # of elements that are already sorted in the heap.

- ➟ heap-size = 10

Max-Heapify

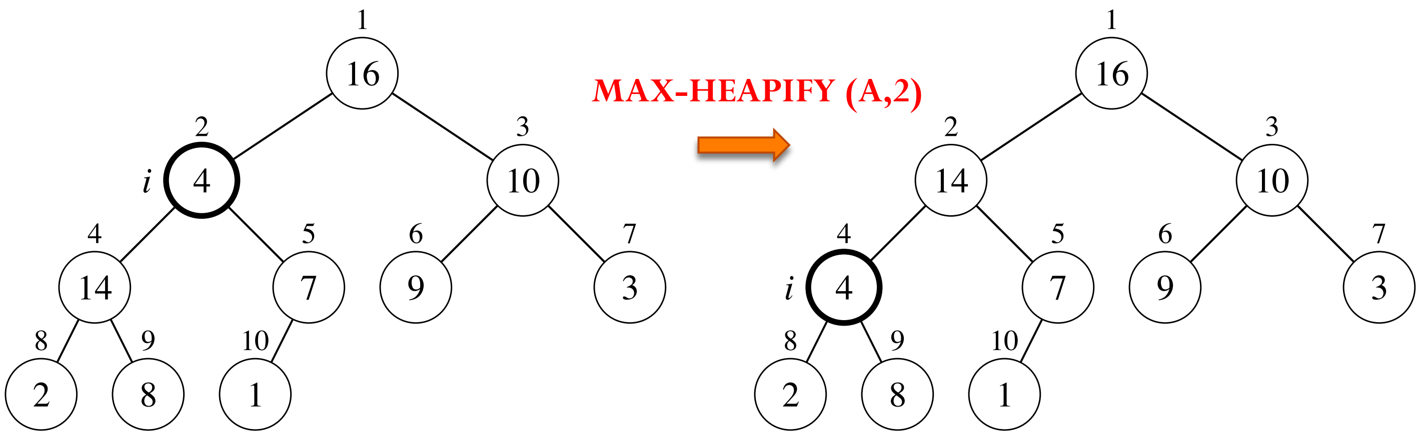
- Used to maintain the max-heap property.

$\text{MAX-HEAPIFY}(A, i, n)$

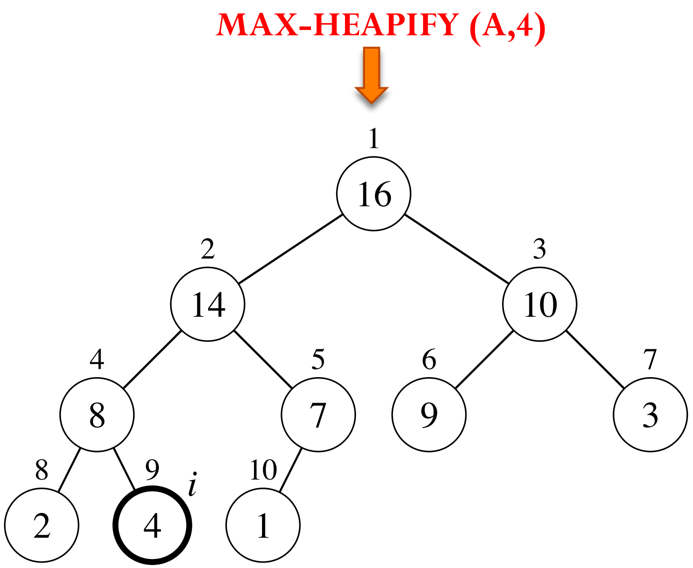
$l = \text{LEFT}(i)$
$r = \text{RIGHT}(i)$
if $l \leq n$ and $A[l] > A[i]$
 $largest = l$
else $largest = i$
if $r \leq n$ and $A[r] > A[largest]$
 $largest = r$
if $largest \neq i$
 exchange $A[i]$ with $A[largest]$
 $\text{MAX-HEAPIFY}(A, largest, n)$

n: heap-size
LEFT (i): return 2i
RIGHT (i): return 2i+1

Max-Heapify …



(a) (b)



(c)

- Node 2 violates max-heap property (a).
- Compare node 2 with its children, and then swap it with the larger of the two children (b).
- Continue swapping until the value is properly placed at the root of a subtree that is a max-heap (c).

Max-Heapify Analysis

MAX-HEAPIFY(A, i, n)
  $l = \text{LEFT}(i)$
  $r = \text{RIGHT}(i)$
  if $l \leq n$ and $A[l] > A[i]$
    $largest = l$
  else $largest = i$
  if $r \leq n$ and $A[r] > A[largest]$
    $largest = r$
  if $largest \neq i$
    exchange $A[i]$ with $A[largest]$
    MAX-HEAPIFY$(A, largest, n)$

Analysis

- $\Theta(1)$: Fix relations among the A[i], A[LEFT(i)], A[RIGHT(i)]
- Children subtreees have size at most 2n/3

$T(n) \leq T(2n/3) + 1$

Apply recurrence ➡ $T(n) = O(\lg n)$

Building a Heap

- Given an unsorted array, build a max-heap

BUILD-MAX-HEAP(A, n)

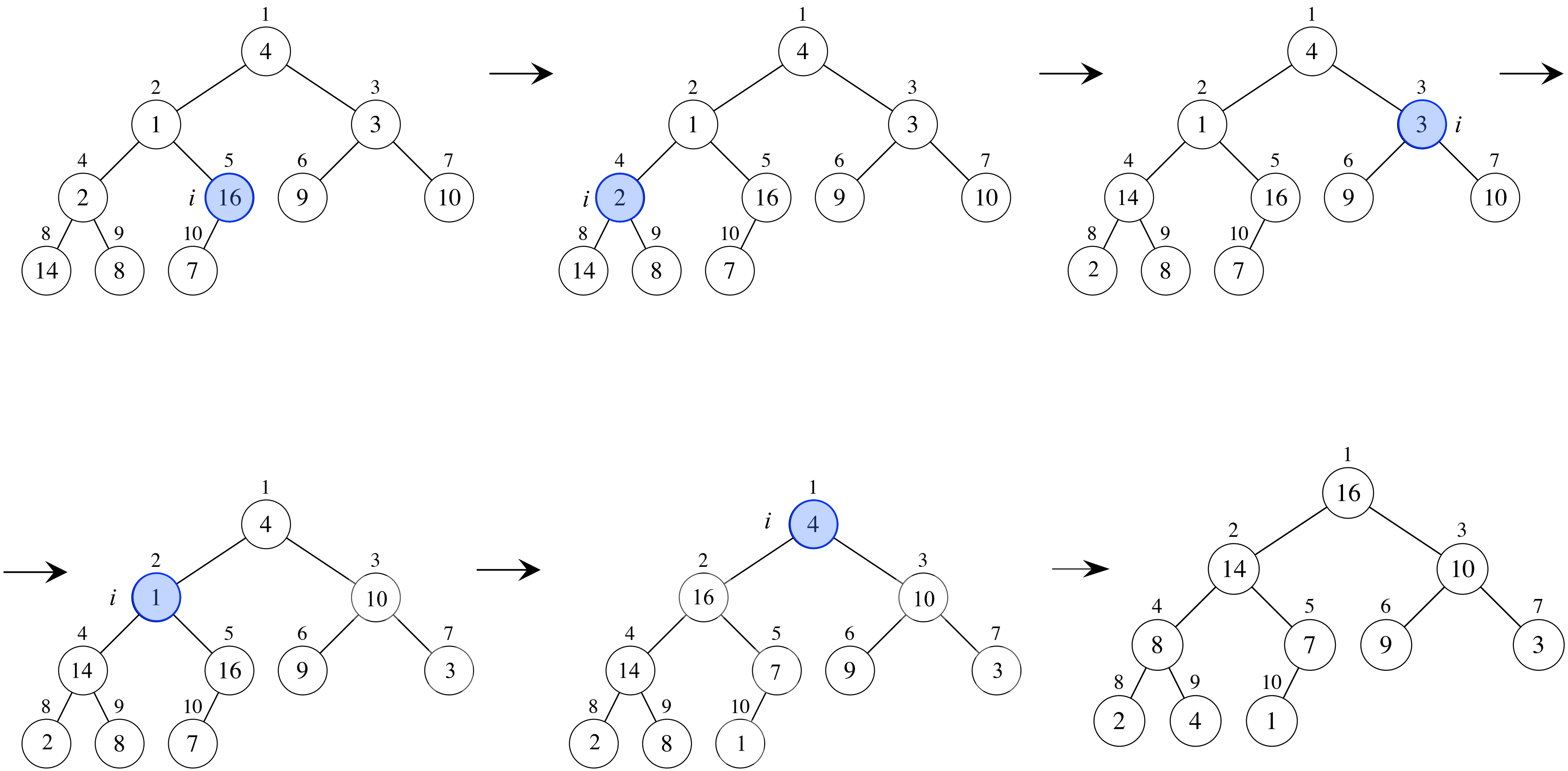
  for $i = \lfloor n/2 \rfloor$ **downto** 1

    MAX-HEAPIFY(A, i, n)

n: unsorted array size

- Why does it start from n/2?
  - All elements A[(n/2+1), …, n] are on the leaves

	1	2	3	4	5	6	7	8	9	10
A	4	1	3	2	16	9	10	14	8	7



Build-Max-Heap Analysis

<u>Analysis</u>

- Each call of Max-Heapify: O(lgn)
- For loop runs n/2 times: O(n)
- Upper bound: O(n. lgn)

- Tight bound : O(n) (check page 159)

BUILD-MAX-HEAP(A, n)

  for $i = \lfloor n/2 \rfloor$ **downto** 1

    MAX-HEAPIFY(A, i, n)

Heap Usage

- Heap sort
  - one of the best sorting methods - not quadratic in the worst-case
- Selection algorithms
  - finding the min, max, median, k^{th} element in sublinear time
- Graph algorithms
  - Prim's minimal spanning tree
  - Dijkstra's shortest path

Heap Sort Algorithm

Given an input array:

- Builds a max-heap from the array.
- Start with the root, place the maximum element into the correct place in the array by swapping it with the element in the last position in the array.
- "Discard" this last node by decreasing the heap size, and calling *MAX-HEAPIFY* on the new root.
- Repeat this "discarding" process until only one node (the smallest element) remains.

HEAPSORT (A,n)
BUILD-MAX-HEAP (A,n)
for i=A.length **downto** 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size -1
    MAX-HEAPIFY(A,1,i-1)

Heap Sort Algorithm

Given an input array:

- Builds a max-heap from the array.
- Start with the root, place the maximum element into the correct place in the array by swapping it with the element in the last position in the array.
- "Discard" this last node by decreasing the heap size, and calling *MAX-HEAPIFY* on the new root.
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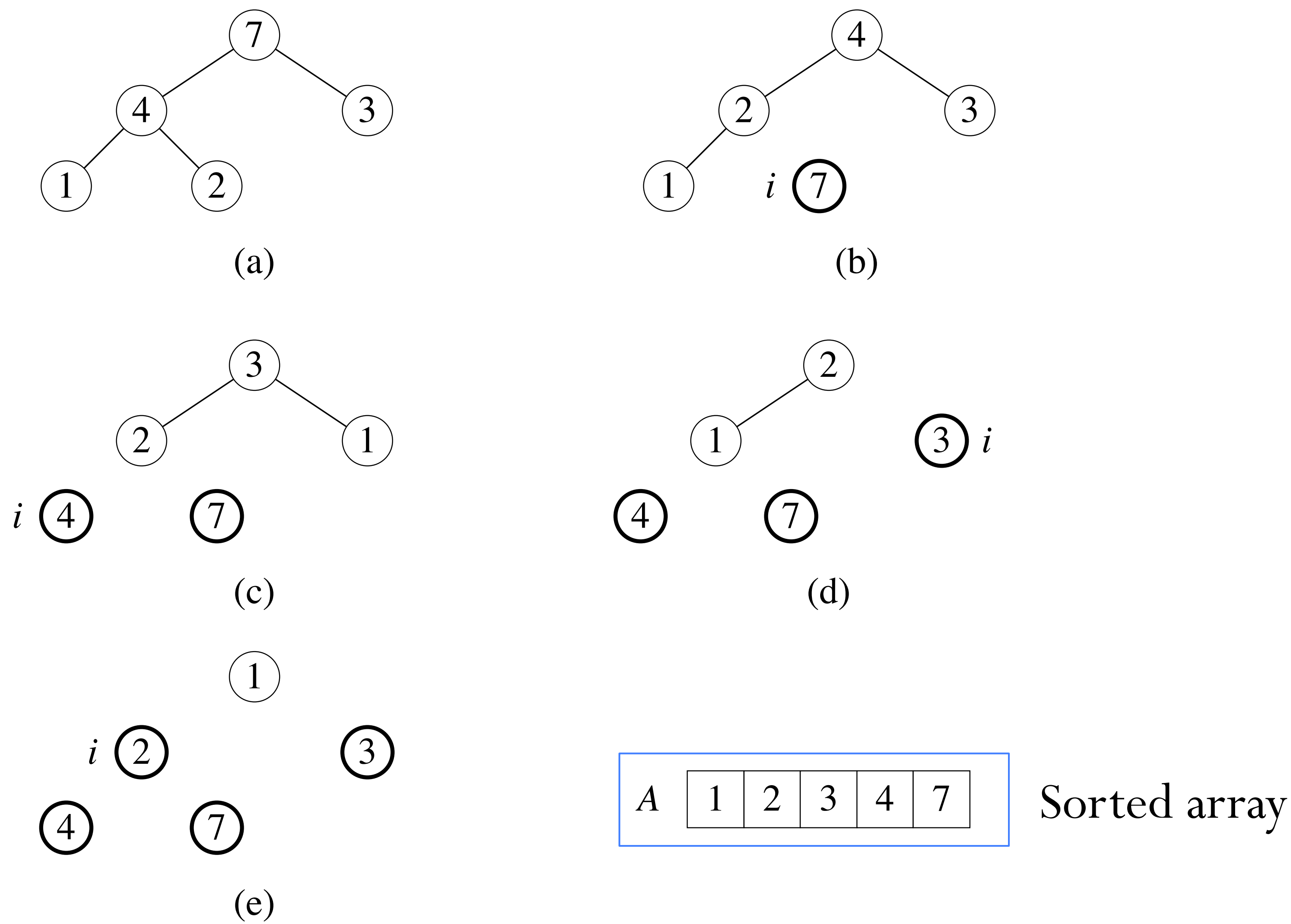
HEAPSORT (A,n)
BUILD-MAX-HEAP (A,n)
for i=A.length **downto** 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size -1
    MAX-HEAPIFY(A,1,i-1)

Analysis

- *BUILD-MAX-HEAP*: O(n)
- for loop runs (n-1) times
- exchange elements: O(1)
- *MAX-HEAPIFY*: O(lgn)
- **Total time:** O(n lgn)

Heap Sort Example



Priority Queues – A Heap Application

- Maintains a dynamic set ***S*** of elements.
- Each set element has a ***key*** - an associated value.
- Max-priority queue supports dynamic-set operations:
  - *INSERT* (S,x): inserts element x into set ***S***.
  - *MAXIMUM* (S): returns element of ***S*** with largest key.
  - *EXTRACT-MAX* (S): removes and returns element of ***S*** with largest key.
  - *INCREASE-KEY* (S,x,k): increases value of element x's key to k. Assume k ≥ x's current key value.
- e.g. Max-priority queue application : schedule jobs on shared computer.

Priority Queue Operations

HEAP-MAXIMUM(A)
  return $A[1]$

- Running Time $= \Theta(1)$.

Priority Queue Operations

HEAP-EXTRACT-MAX(A, n)

```
if n < 1
    error "heap underflow"
max = A[1]
A[1] = A[n]
n = n - 1
MAX-HEAPIFY(A, 1, n)        // remakes heap
return max
```

- Running Time = O(lgn).

Priority Queue Operations

HEAP-INCREASE-KEY(A, i, key)

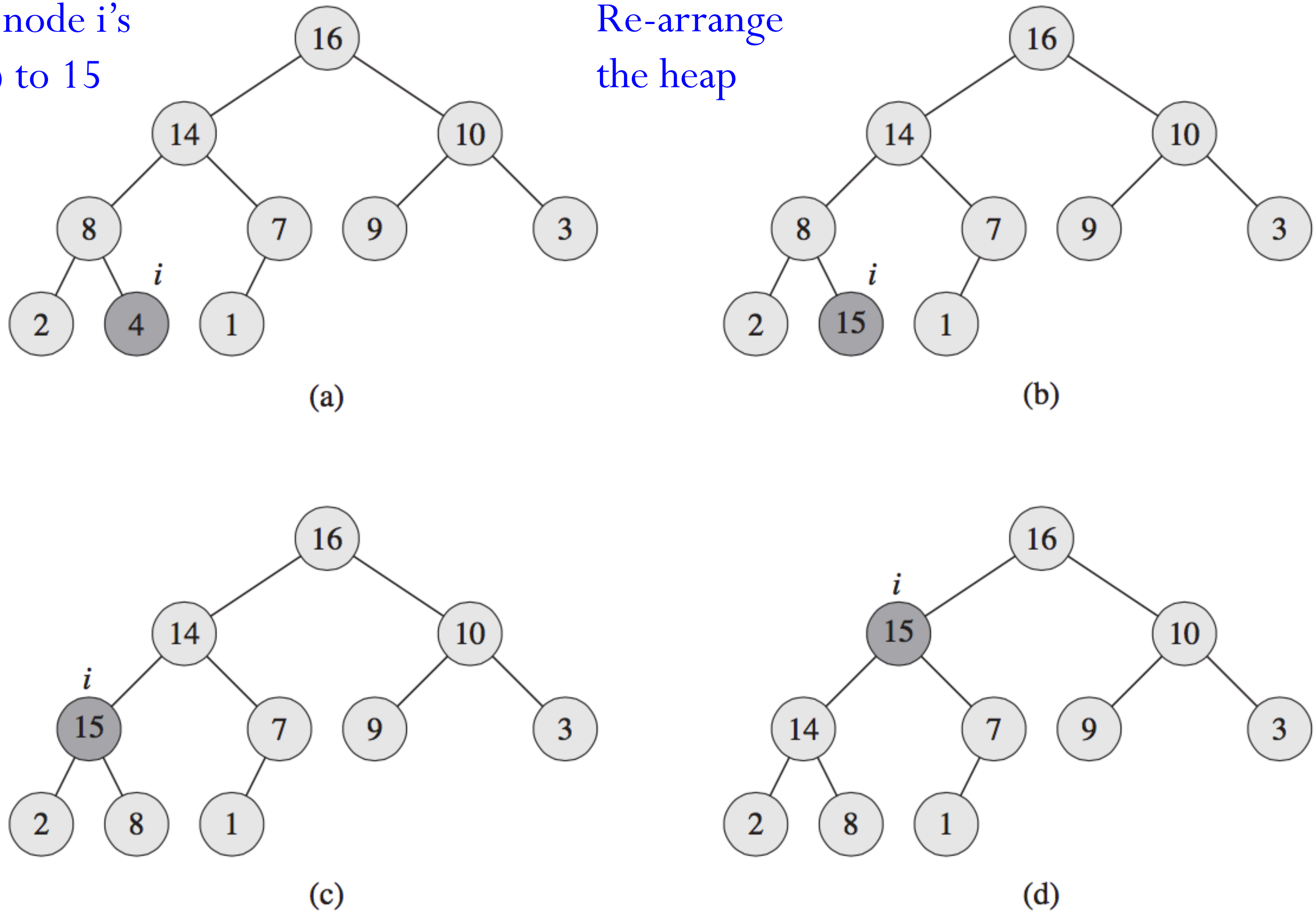
```
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

- Running Time = O(lgn).

Heap-Increase-Key Example

Increase node i's value (4) to 15

Re-arrange the heap



Priority Queue Operations

MAX-HEAP-INSERT(A, key, n)

$n = n + 1$
$A[n] = -\infty$
HEAP-INCREASE-KEY(A, n, key)

- Running Time = O(lgn).

Quicksort

Quicksort

Quicksort is based on *divide-and-conquer* paradigm, similar to Mergesort.

Steps:

1. Partition an array into two subarrays
2. Sort each subarray independently,
3. Combine sorted subarrays.

Quicksort Steps

To sort the subarray $A[p\ldots r]$:

- **Divide:** Partition $A[p\ldots r]$ into two subarrays $A[p\ldots q-1]$ and $A[q+1\ldots r]$, such that each element in the first subarray $A[p\ldots q-1]$ is $\leq A[q]$ and $A[q]$ is $\leq$ each element in the second subarray $A[q+1\ldots r]$. Compute index $\boldsymbol{q}$ as a part of partition procedure
- **Conquer:** Sort the two subarrays by recursive calls to QUICKSORT.
- **Combine:** No work is needed to combine the subarrays, because they are sorted in place.

```
QUICKSORT(A, p, r)
  if p < r
      q = PARTITION(A, p, r)
      QUICKSORT(A, p, q - 1)
      QUICKSORT(A, q + 1, r)
```

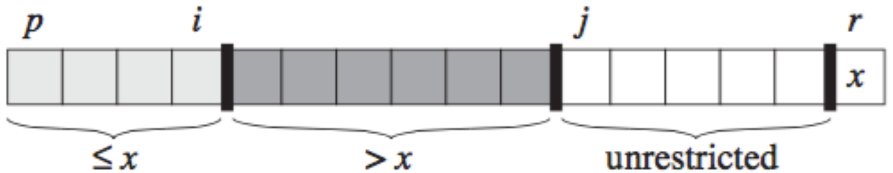
- Initial call is QUICKSORT (A, 1, n).

```
PARTITION(A, p, r)
  x = A[r]
  i = p - 1
  for j = p to r - 1
      if A[j] ≤ x
          i = i + 1
          exchange A[i] with A[j]
  exchange A[i + 1] with A[r]
  return i + 1
```

Runtime of PARTITION : Θ(n)

where n=r-p+1

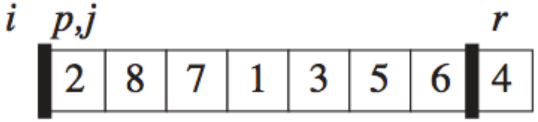
- PARTITION always selects the last element A[r] in the subarray A[p…r] as the ***pivot*** - the element around which to partition.
- As the procedure executes, the array is partitioned into four regions, some of which may be empty.



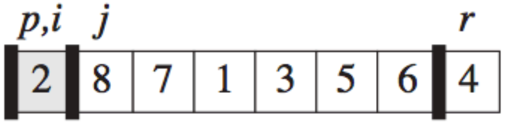
i	p,j							r
	2	8	7	1	3	5	6	4

i	*p,j*							*r*
	2	8	7	1	3	5	6	4

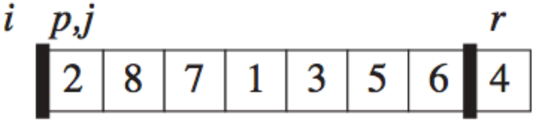
A[1] $\leq$ 4 ✓ i++ Swap A[i] with A[j] ➡ A[1] with A[1]



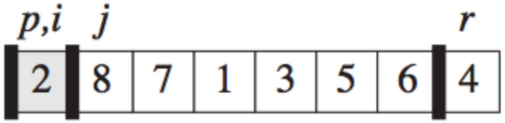
A[1] $\leq$ 4 ✓ i++ Swap A[i] with A[j] ➟A[1] with A[1]



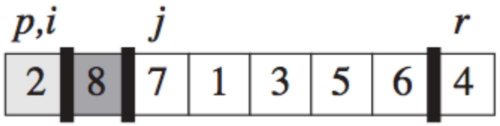
A[2] $\leq$ 4 ✗



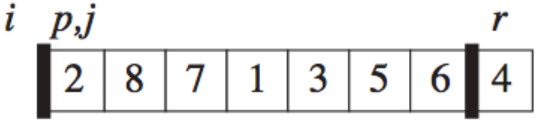
A[1] ≤ 4 ✓ i++ Swap A[i] with A[j] ➡A[1] with A[1]



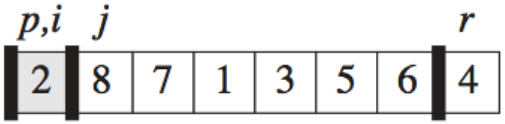
A[2] ≤ 4 ✗



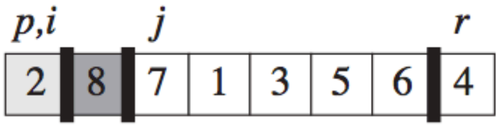
A[3] ≤ 4 ✗



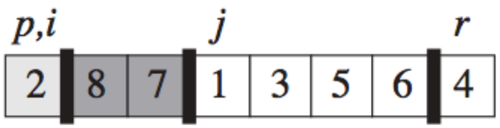
A[1] ≤ 4 ✓ i++ Swap A[i] with A[j] ➟A[1] with A[1]



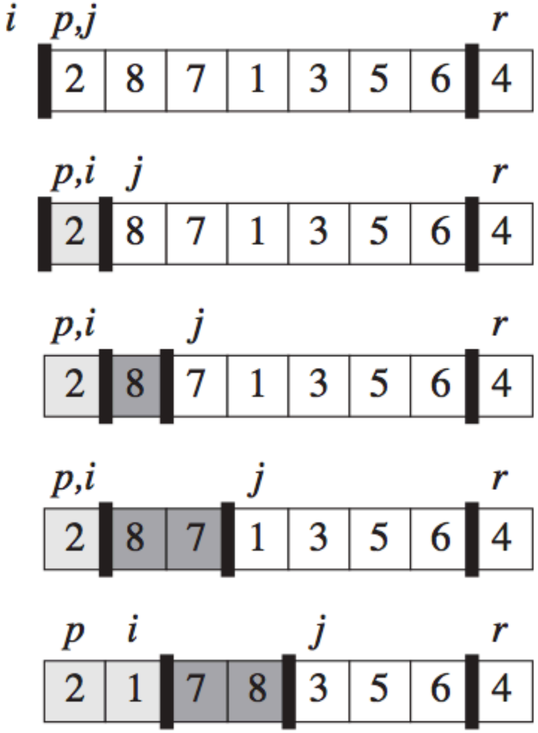
A[2] ≤ 4 ✗



A[3] ≤ 4 ✗



A[4] ≤ 4 ✓ i++ Swap A[2] with A[4]



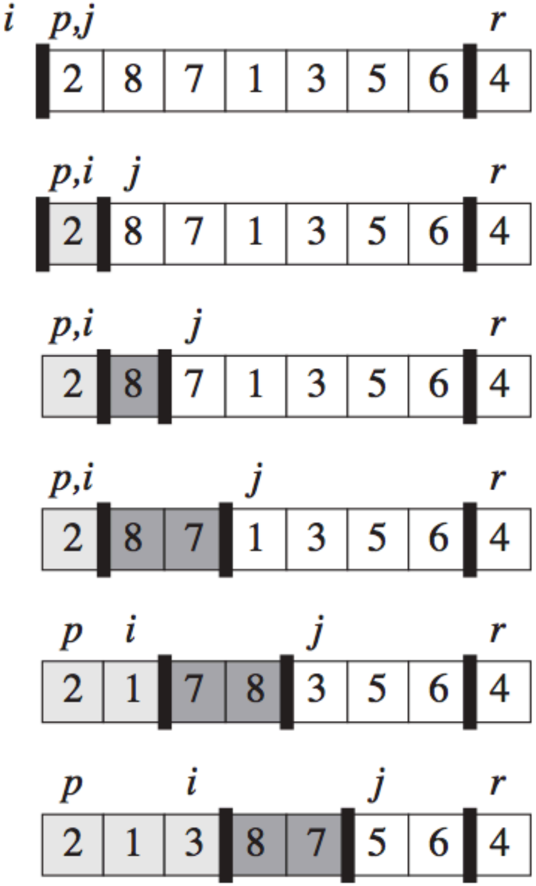
$A[1] \leq 4$ ✓ i++ Swap A[i] with A[j] ➟A[1] with A[1]

$A[2] \leq 4$ ✗

$A[3] \leq 4$ ✗

$A[4] \leq 4$ ✓ i++ Swap A[2] with A[4]

$A[5] \leq 4$ ✓ i++ Swap A[3] with A[5]



A[1] ≤ 4 ✓ i++ Swap A[i] with A[j] ➟A[1] with A[1]

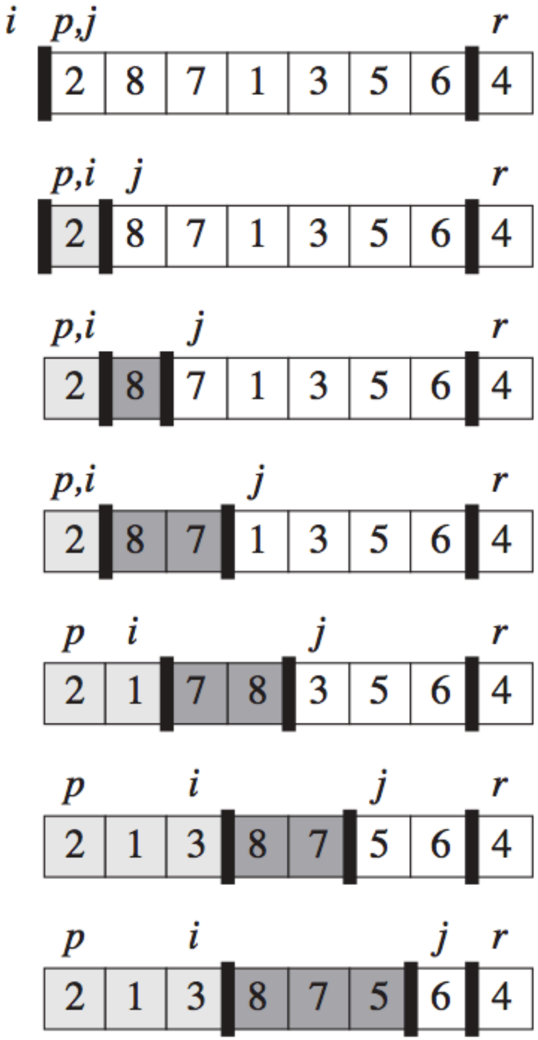
A[2] ≤ 4 ✗

A[3] ≤ 4 ✗

A[4] ≤ 4 ✓ i++ Swap A[2] with A[4]

A[5] ≤ 4 ✓ i++ Swap A[3] with A[5]

A[6] ≤ 4 ✗



$A[1] \leq 4$ ✓ i++ Swap A[i] with A[j] ➟A[1] with A[1]

$A[2] \leq 4$ ✗

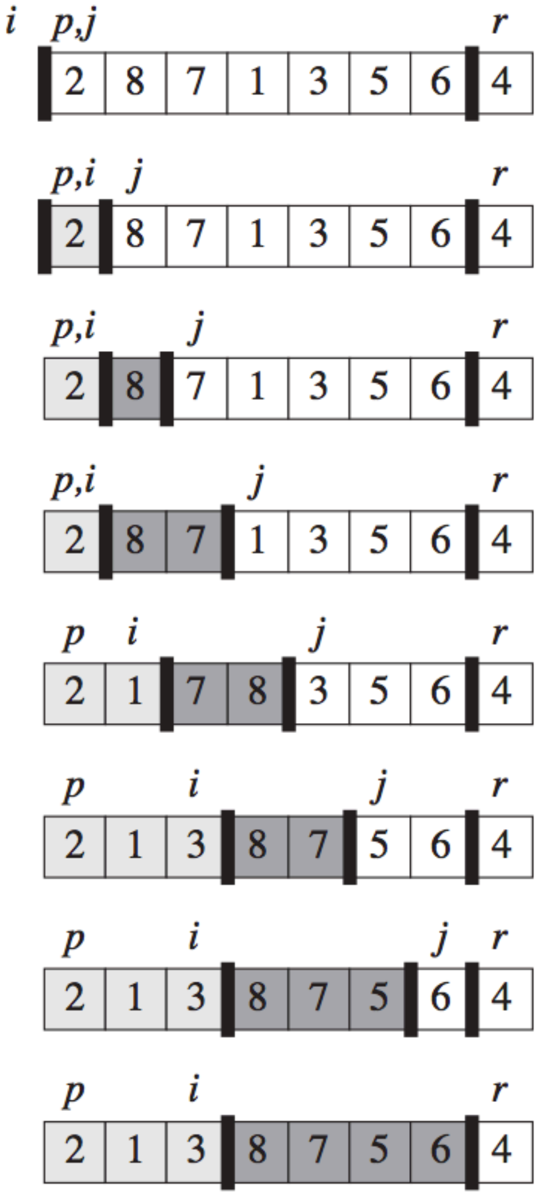
$A[3] \leq 4$ ✗

$A[4] \leq 4$ ✓ i++ Swap A[2] with A[4]

$A[5] \leq 4$ ✓ i++ Swap A[3] with A[5]

$A[6] \leq 4$ ✗

$A[7] \leq 4$ ✗



A[1] ≤ 4 ✓ i++ Swap A[i] with A[j] ➡ A[1] with A[1]

A[2] ≤ 4 ✗

A[3] ≤ 4 ✗

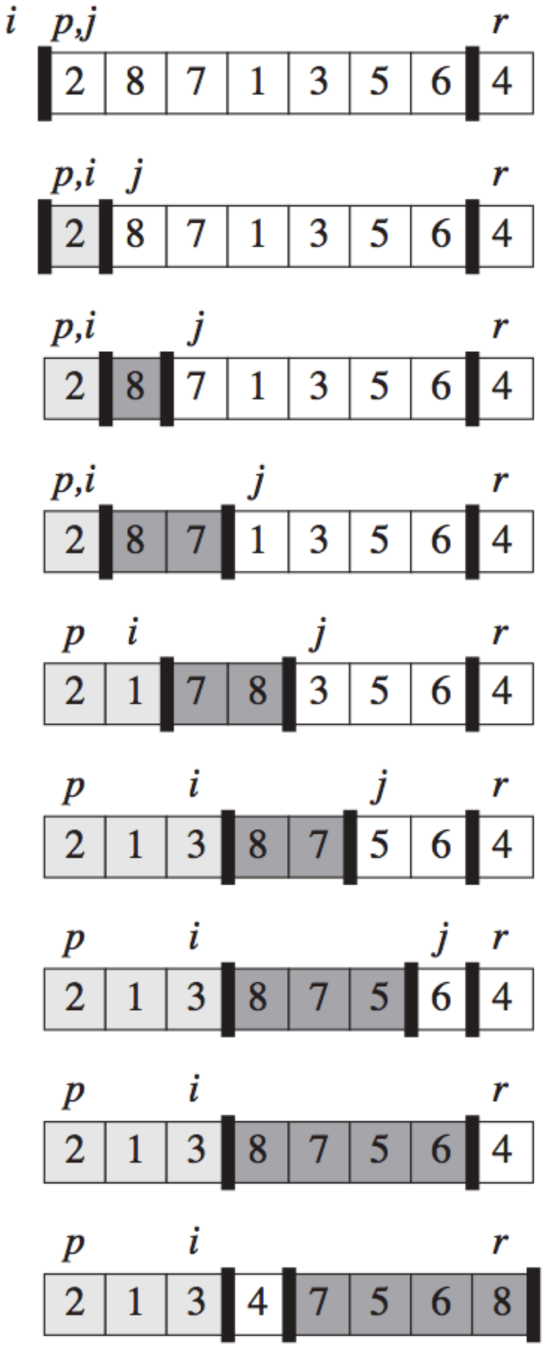
A[4] ≤ 4 ✓ i++ Swap A[2] with A[4]

A[5] ≤ 4 ✓ i++ Swap A[3] with A[5]

A[6] ≤ 4 ✗

A[7] ≤ 4 ✗

End of for (j ≤ r-1)



A[1] ≤ 4 ✓ i++ Swap A[i] with A[j] ➠A[1] with A[1]

A[2] ≤ 4 ✗

A[3] ≤ 4 ✗

A[4] ≤ 4 ✓ i++ Swap A[2] with A[4]

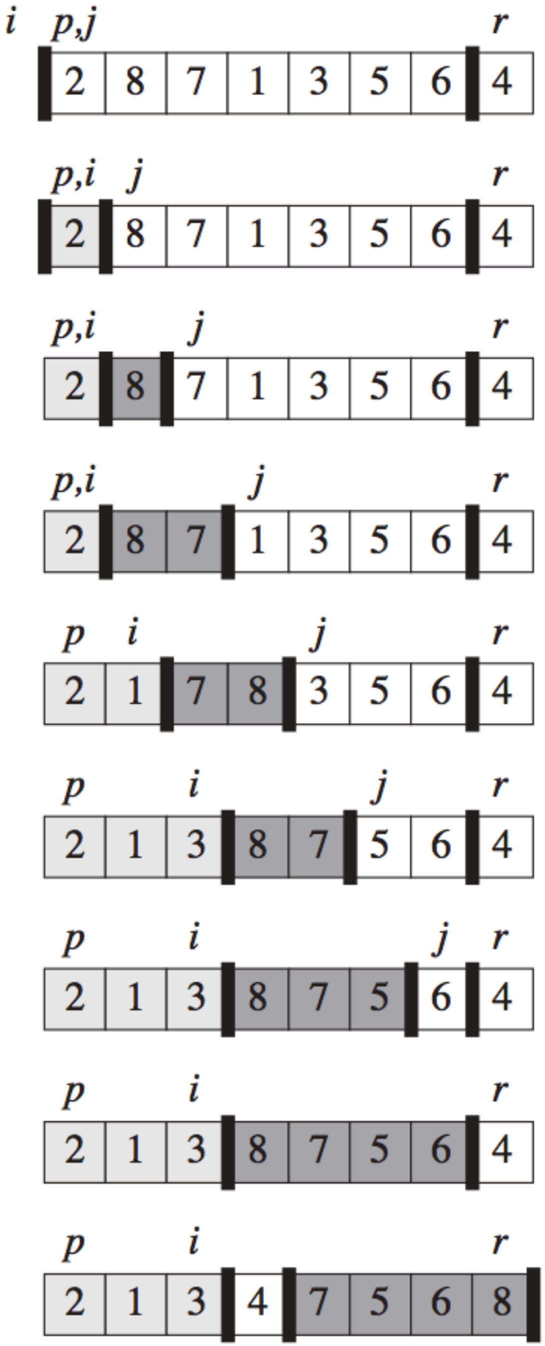
A[5] ≤ 4 ✓ i++ Swap A[3] with A[5]

A[6] ≤ 4 ✗

A[7] ≤ 4 ✗

End of for (j ≤ r-1)

Swap A[i+1] with A[r] ➠ Swap A[4] with A[8]



A[1] ≤ 4 ✓ i++ Swap A[i] with A[j] ➟A[1] with A[1]

A[2] ≤ 4 ✗

A[3] ≤ 4 ✗

A[4] ≤ 4 ✓ i++ Swap A[2] with A[4]

A[5] ≤ 4 ✓ i++ Swap A[3] with A[5]

A[6] ≤ 4 ✗

A[7] ≤ 4 ✗

Finished *PARTITION* (A,1,8) run

q= i+1 ➟ q=4

Now run recursively:

QUICKSORT(A,1,3)

QUICKSORT(A,5,8)

End of for (j ≤ r-1)

Swap A[i+1] with A[r] ➟ Swap A[4] with A[8] ✓

Performance of Quicksort

The running time of quicksort depends on the partitioning of the subarrays:

- If the subarrays are balanced, then quicksort can run as fast as mergesort.
- If they are unbalanced, then quicksort can run as slowly as insertion sort.

Worst-case Partitioning

- Occurs when the sub-arrays are completely unbalanced.
  i.e. one part with "n-1" elements, the other with only 0 element

  $PARTITION : \Theta(n)$

  $$T(n) = T(n-1) + T(0) + \Theta(n) = T(n-1) + \Theta(n) = \sum_{k=1}^{n} \Theta(k) = \Theta(\sum_{k=1}^{n} k)$$

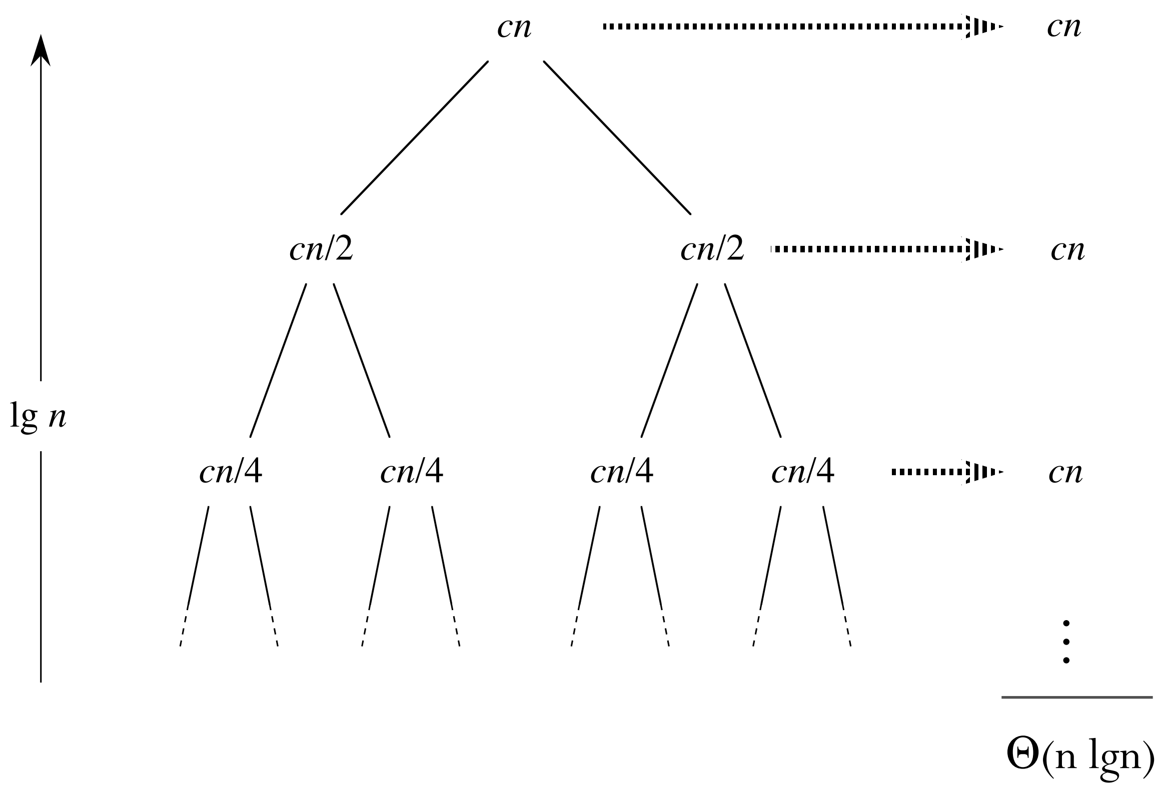
  $$T(n) = \Theta(n^2)$$

- Same running time as insertion sort.
- It happens when input array is already ordered!

Best-case Partitioning

- Assume that PARTITION always produces n/2 splits.

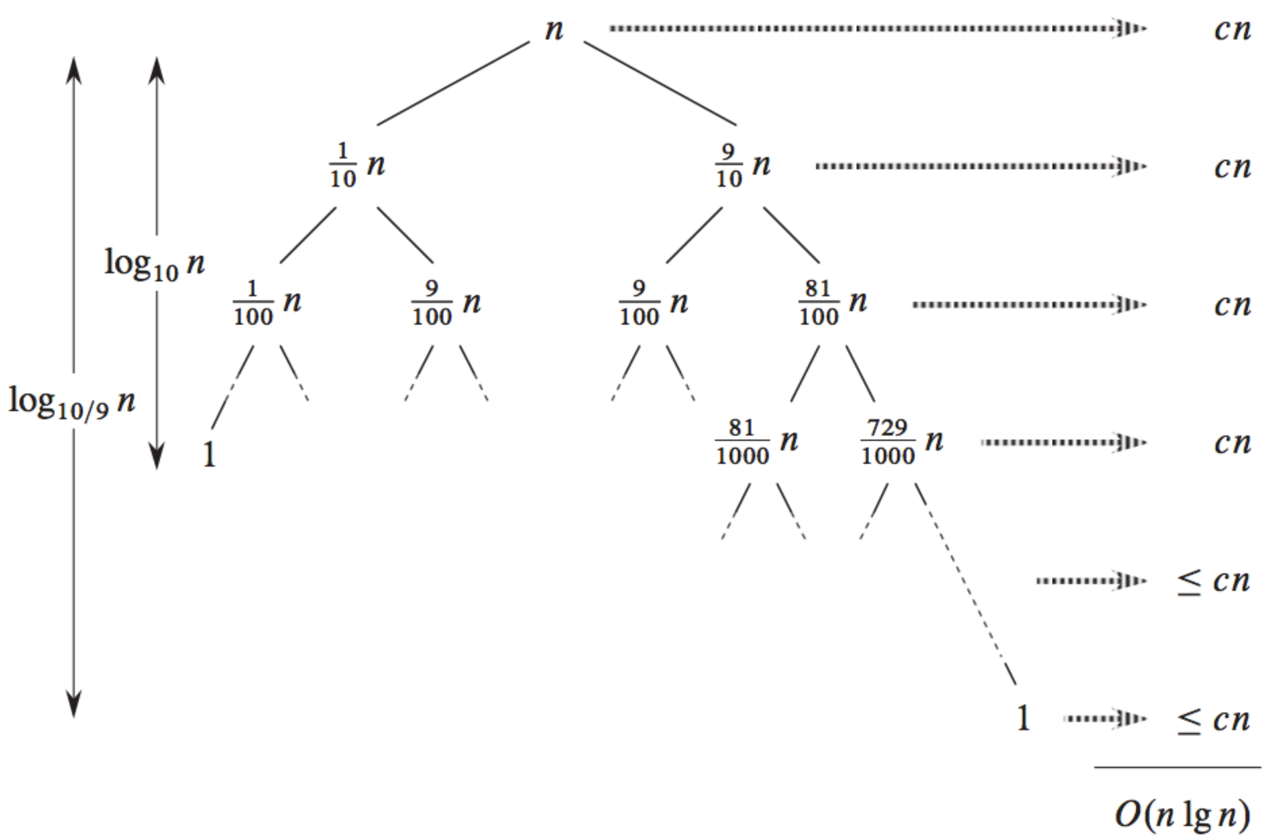
  $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \lg n)$



Balanced Partitioning

- Assume that PARTITION always produces a 9-to-1 split.

  $T(n) = T(9n/10) + T(n/10) + n = \Theta(n \lg n)$



Next Week Topics

- Sorting in Linear Time (Chapter 8)