CME 2001
Data Structures and Algorithms

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Elementary Graph Algorithms

Graphs

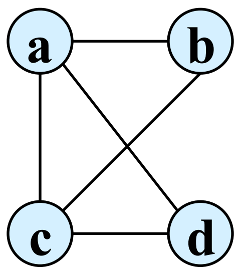
- *Graph* $G = (V, E)$
  - $V =$ set of vertices
  - $E =$ set of edges $\subseteq (V \times V)$
- Types of graphs
  - Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
  - Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
  - Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
- $|E| = O(|V|^2)$

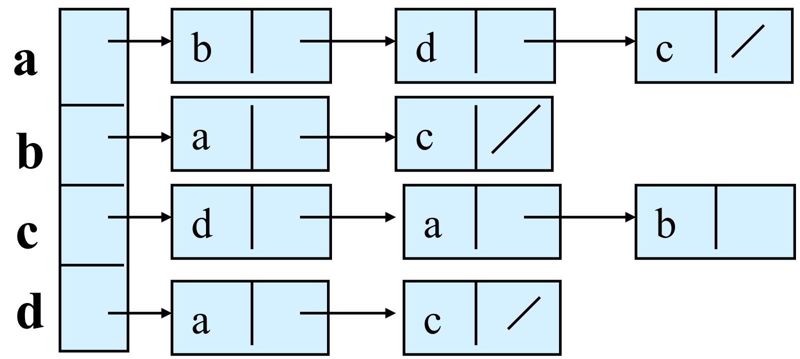
Graphs

- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is:
  - Symmetric if G is undirected.
  - Not necessarily so if G is directed.
- If G is connected:
  - There is a path between every pair of vertices.
  - $|E| \geq |V| - 1$.
  - Furthermore, if $|E| = |V| - 1$, then G is a tree.

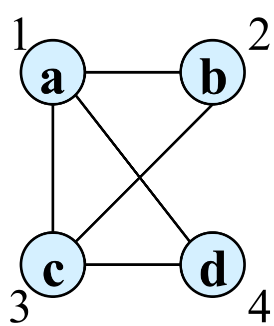
Representation of Graphs

- Two standard ways.
  - Adjacency Lists.





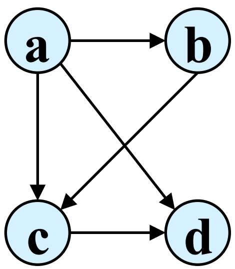
  - Adjacency Matrix.

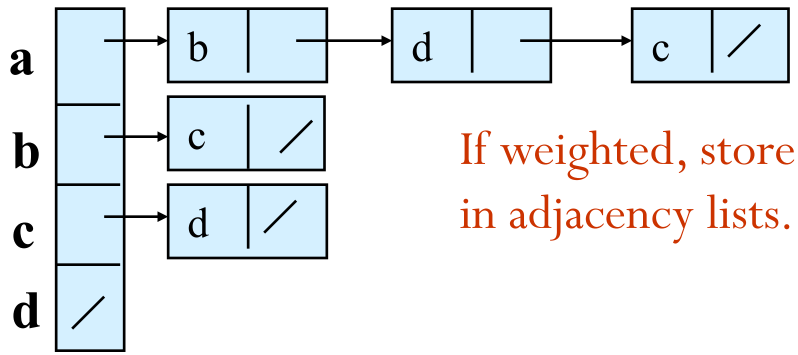


	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

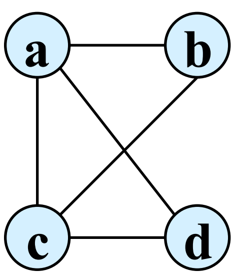
Adjacency Lists

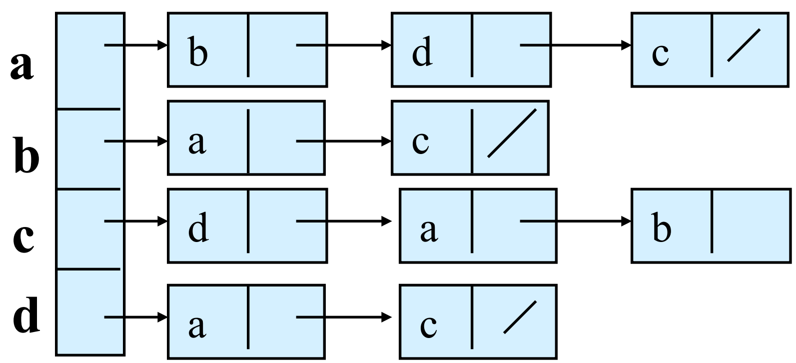
- Consists of an array *Adj* of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.





If weighted, store weights in adjacency lists.





Storage Requirement

- For directed graphs:
  - Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{out-degree}(v) = |E|$$

# of edges leaving v.

  - Total storage: $\Theta(V+E)$
- For undirected graphs:
  - Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

# of edges incident on v.
Edge (u,v) is incident on vertices u and v.

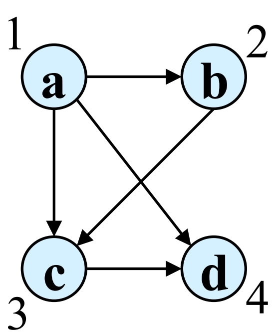
  - Total storage: $\Theta(V+E)$

Pros and Cons: adjacency list

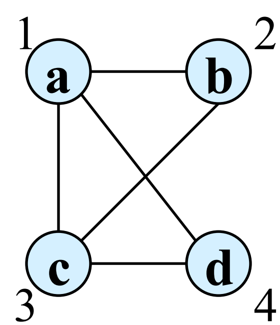
- Pros
  - Space-efficient, when a graph is sparse.
  - Can be modified to support many graph variants.
- Cons
  - Determining if an edge $(u,v) \in G$ is not efficient.
  - Have to search in u's adjacency list. $\Theta(\text{degree}(u))$ time.
  - $\Theta(V)$ in the worst case.

Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- A is then given by: $A[i,j] = a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$



	1	2	3	4
1	0	1	1	1
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



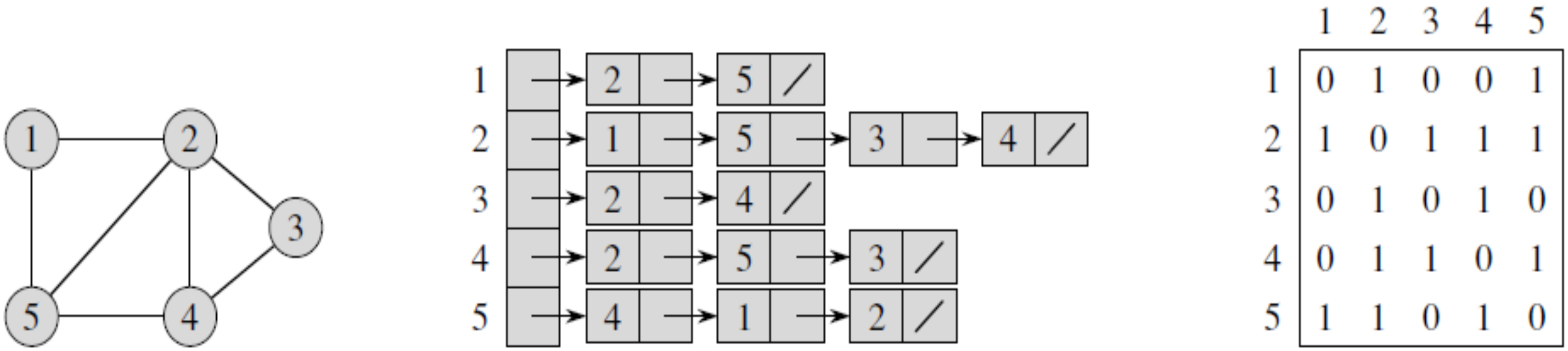
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^{\mathrm{T}}$ for undirected graphs.

Space and Time

- **Space:** $\Theta(V^2)$.
  - Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to u: $\Theta(V)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.

Undirected Graph



(a) (b) (c)

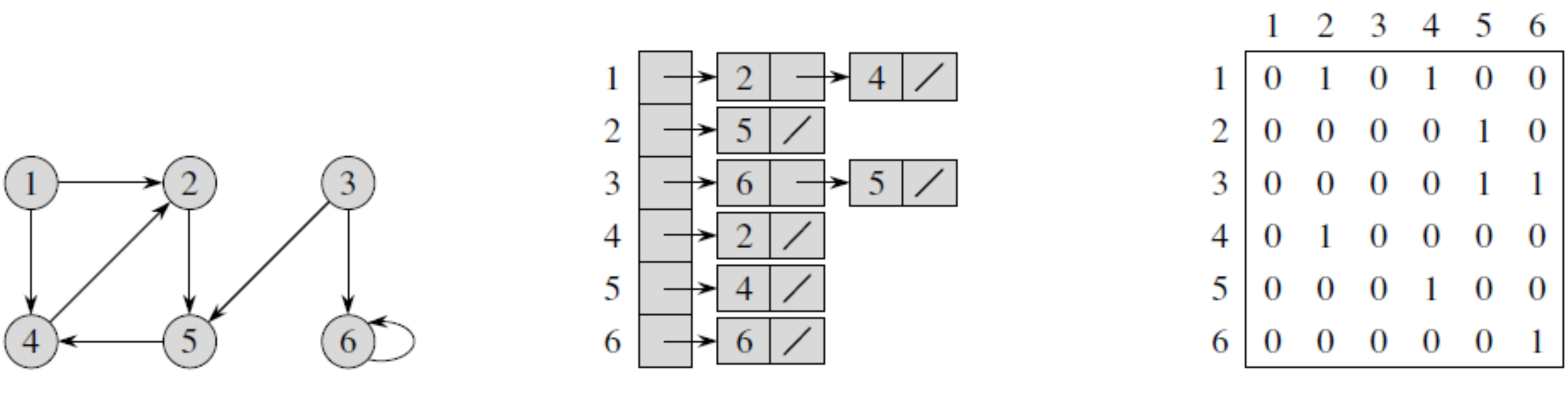
Two representations of an undirected graph.

(a) An undirected graph G *having five* vertices and seven edges.

(b) An adjacency-list representation of G.

(c) The adjacency-matrix representation of G.

Directed Graph



(a) (b) (c)

Two representations of a directed graph.

(a) A directed graph G *having six vertices* and eight edges.

(b) An adjacency-list representation of G.

(c) The adjacency-matrix representation of G.

Graph-searching Algorithms

- Searching a graph:
  - Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - Breadth-first Search (BFS).
  - Depth-first Search (DFS).

Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \leadsto v$.
    - u is v's predecessor.
  - Builds breadth-first tree with root s that contains all reachable vertices.

Definitions:

Path between vertices u and v: Sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $u = v_1$ and $v = v_k$, and $(v_i, v_{i+1}) \in E$, for all $1 \leq i \leq k-1$.

Length of the path: Number of edges in the path.

Path is simple if no vertex is repeated.

Breadth-first Search

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.

  - A vertex is discovered the first time it is found during the search.
  - A vertex is finished if all vertices adjacent to it have been discovered.

Pseudo-Code for Breadth-First Search

- Choose a starting vertex
- Search all adjacent vertices
- Return to each adjacent vertex in turn and visit all of its adjacent vertices

breadth-first-search

```
mark starting vertex as visited; put on queue
while the queue is not empty
  dequeue the next node
  for all unvisited vertices adjacent to this one
    • mark vertex as visited
    • add vertex to queue
```

```
BFS(G, s)                              // G is the graph and s is the starting node
1  for each vertex u ∈ V [G] - {s}
2       do color[u] ← WHITE
3          d[u] ← ∞
4          π[u] ← NIL
5  color[s] ← GRAY
6  d[s] ← 0
7  π[s] ← NIL
8  Q ← Ø
9  ENQUEUE(Q, s)
10  while Q ≠ Ø                          // Lines 10-18 iterates as long as there are gray vertices.
11      do u ← DEQUEUE(Q)
12         for each v ∈ Adj[u]
13             do if color[v] = WHITE             // discover the undiscovered adjacent vertices
14                 then color[v] ← GRAY          // enqueued whenever painted gray
15                      d[v] ← d[u] + 1
16                      π[v] ← u
17                      ENQUEUE(Q, v)
18         color[u] ← BLACK          // painted black whenever dequeued
```

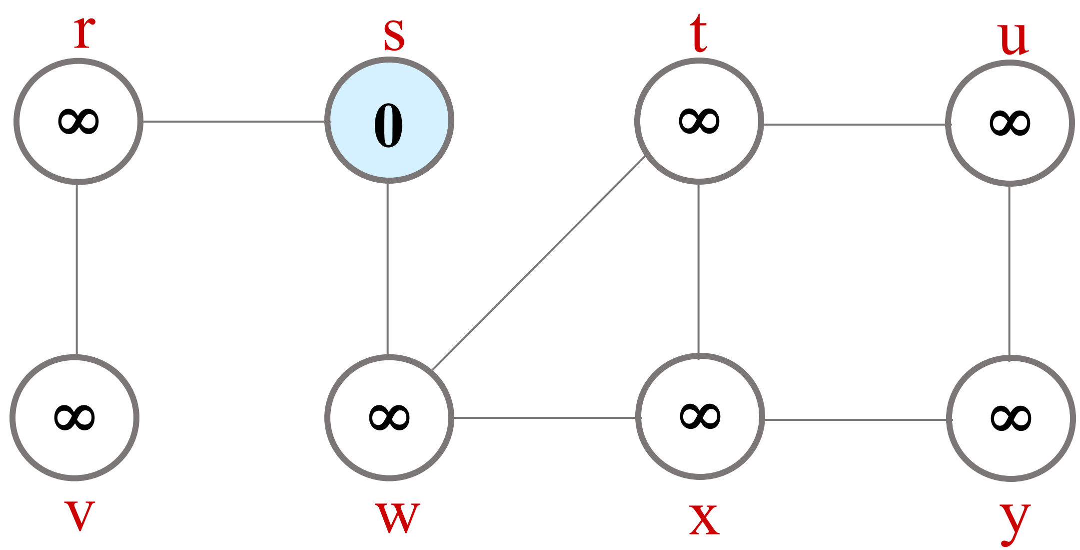
Lines 1–4 paint every vertex white,
Set *d[u]* to be infinity for each vertex *u*,
Set the parent of every vertex to be *NIL*.

Line 5 paints the source vertex *s* ***gray***

Line 6 initializes *d[s]* to 0,
Line 7 sets the predecessor of the source to be *NIL*.
Lines 8–9 initialize *Q* to the queue containing just the vertex *s*.

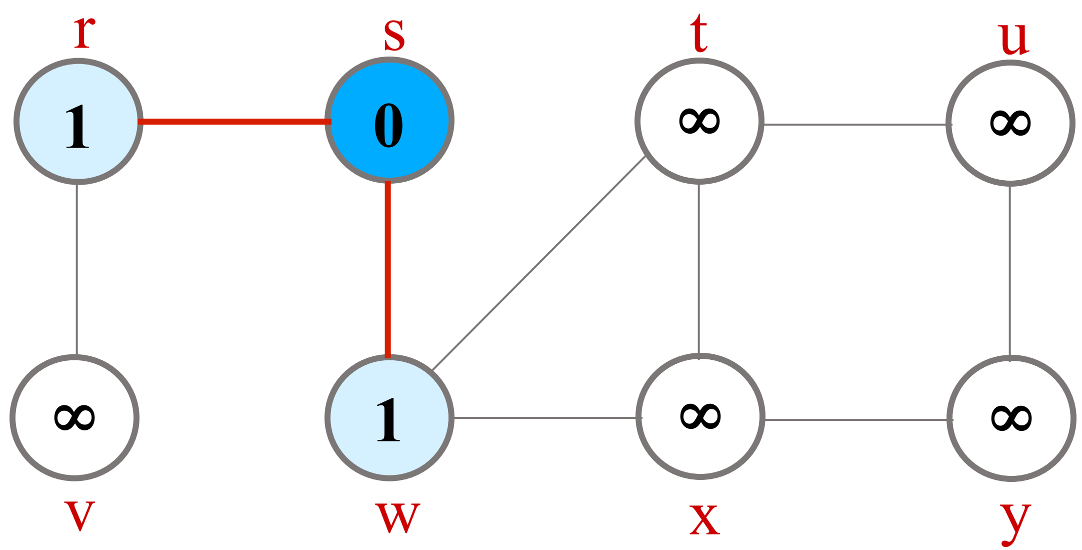
Q: a queue of discovered v
color[*v*]: color of v
d[*v*]: distance from s to v
π[*u*]: predecessor of v
white: undiscovered
gray: discovered
black: finished

Example (BFS)



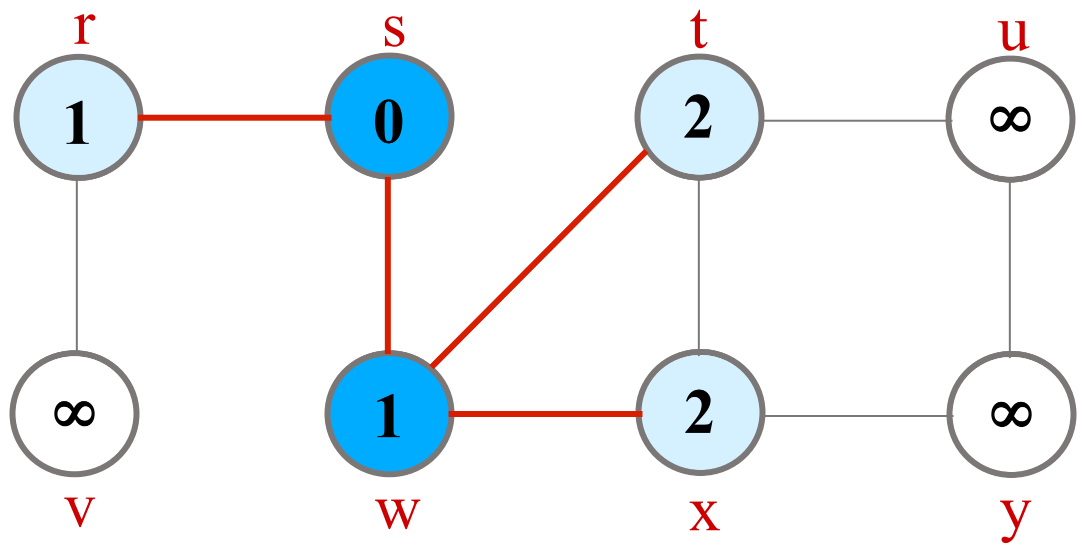
Q:	s
	0

Example (BFS)



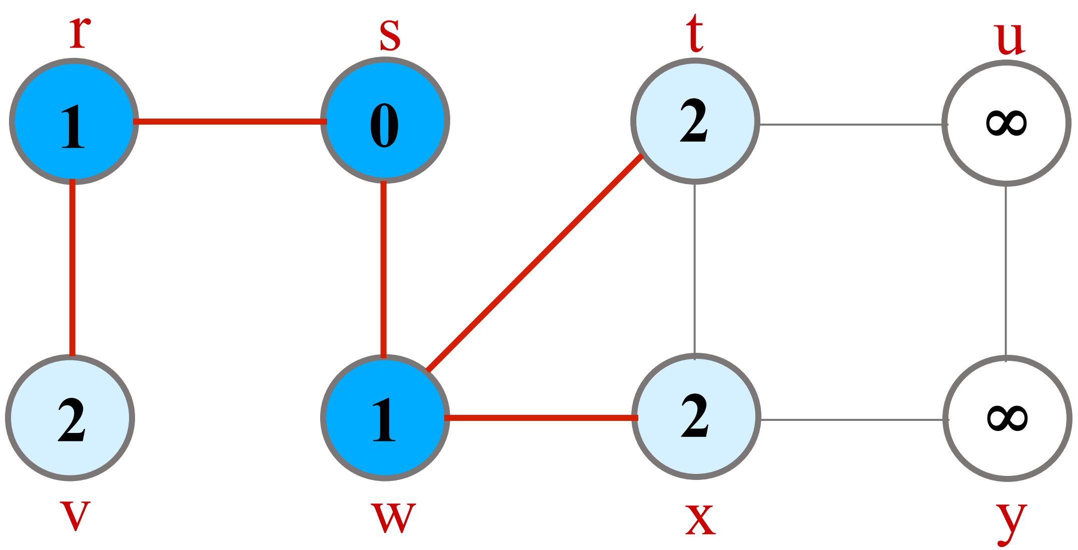
Q: w r
1 1

Example (BFS)



Q:	r	t	x
	1	2	2

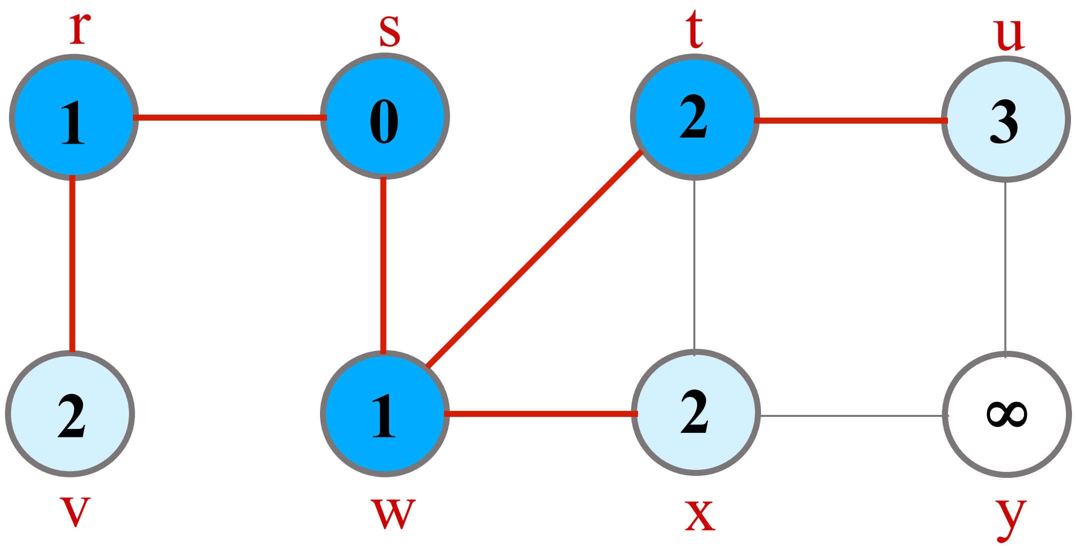
Example (BFS)



Q: t x v

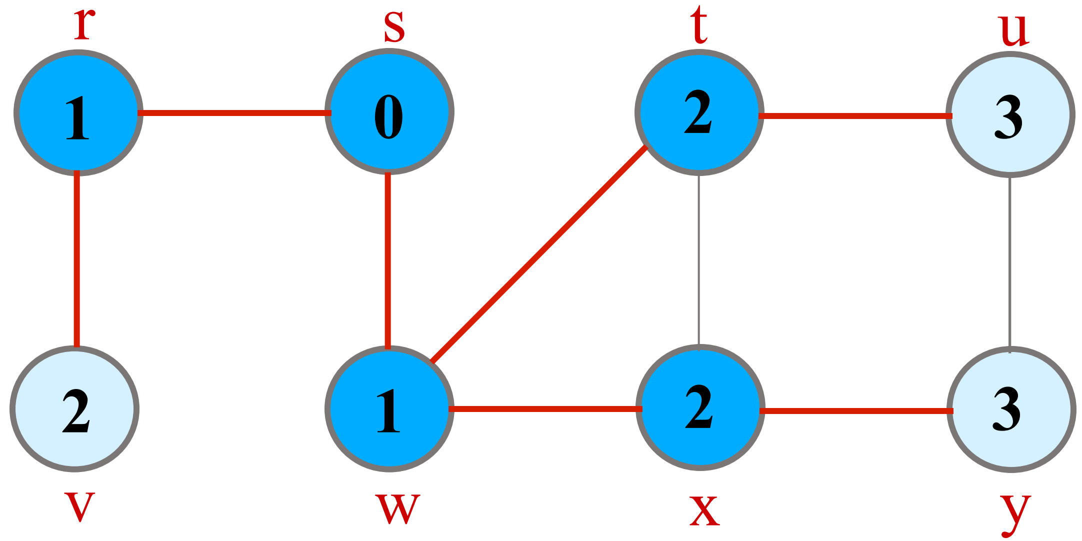
2 2 2

Example (BFS)



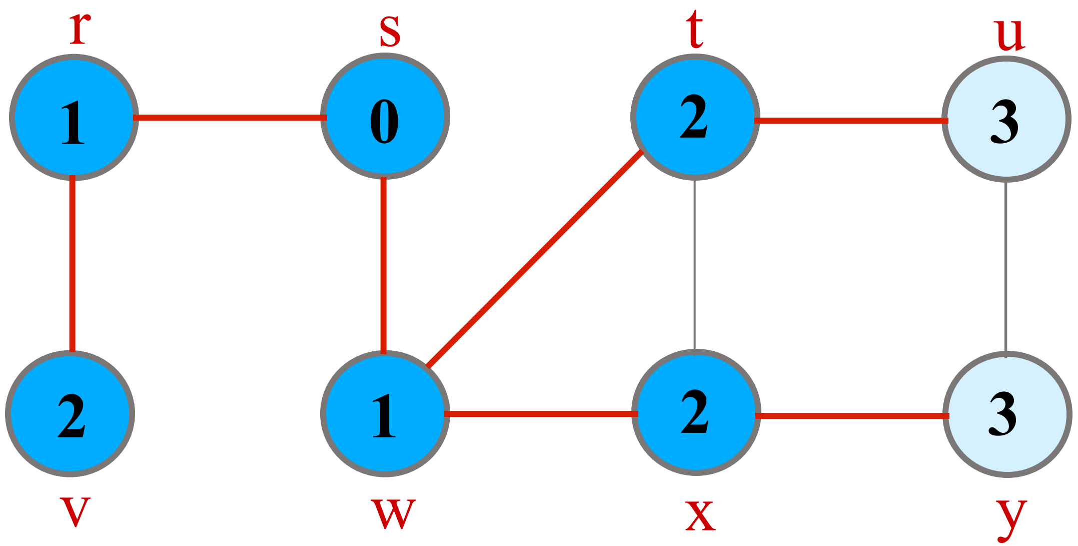
Q:	x	v	u
	2	2	3

Example (BFS)



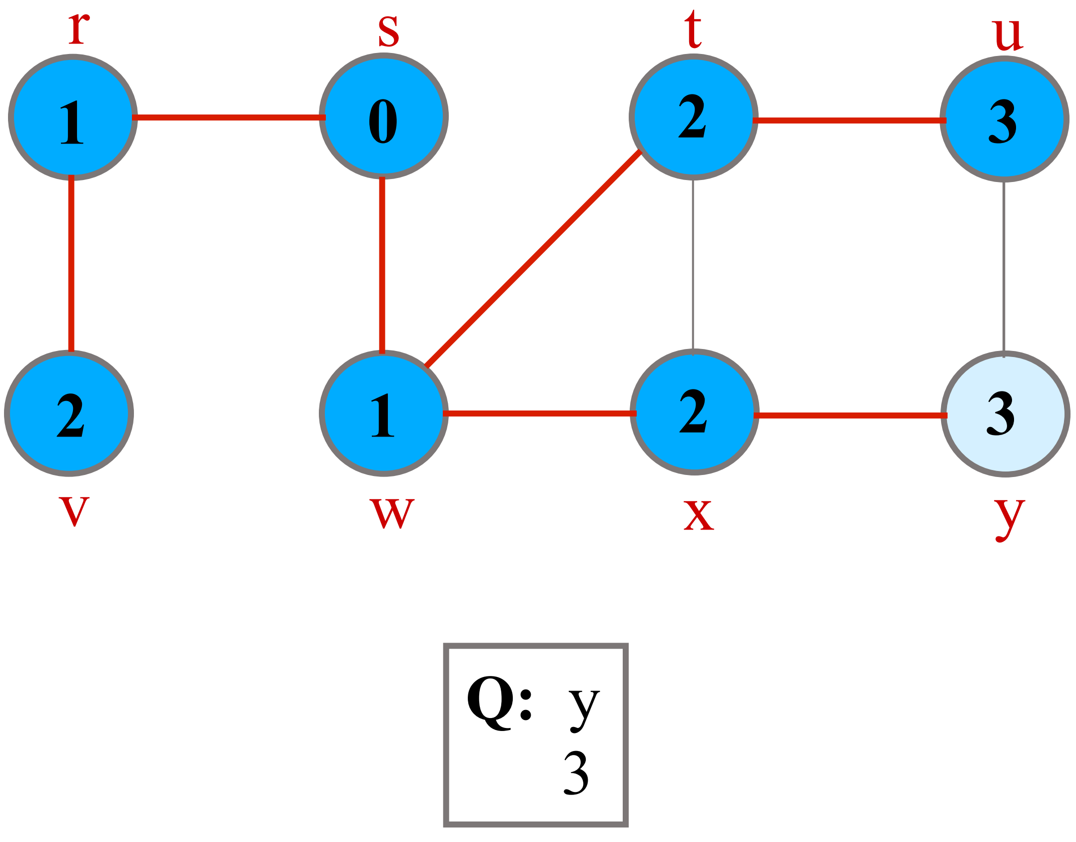
Q:	v	u	y
	2	3	3

Example (BFS)

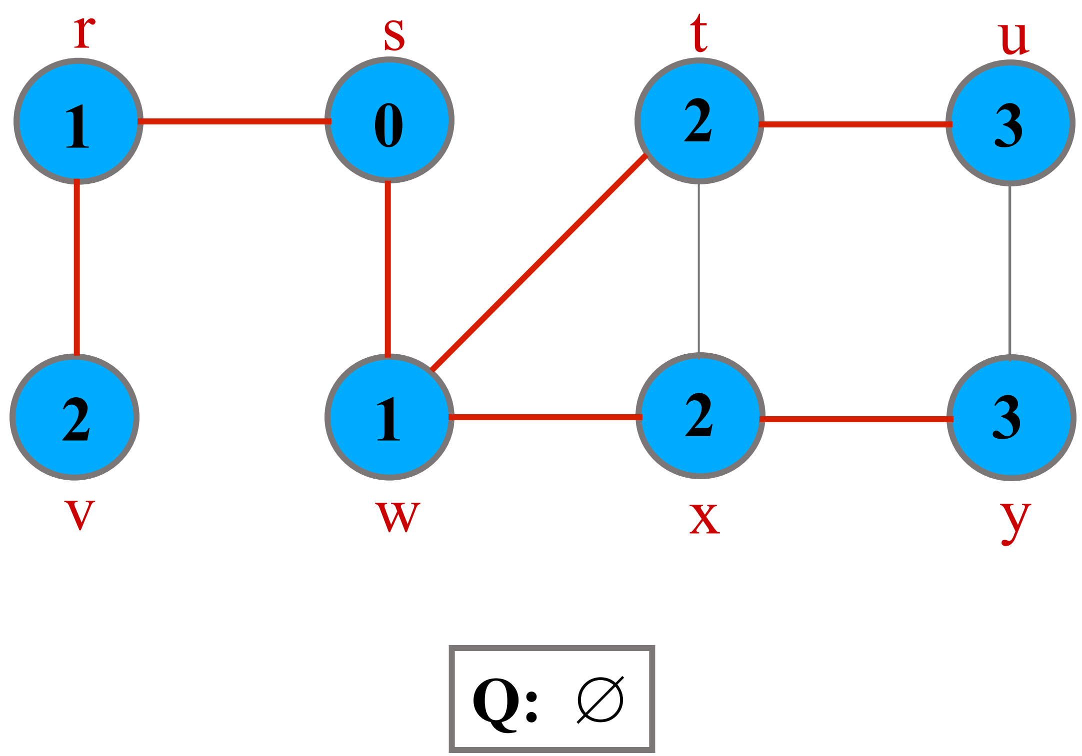


Q: u y
3 3

Example (BFS)

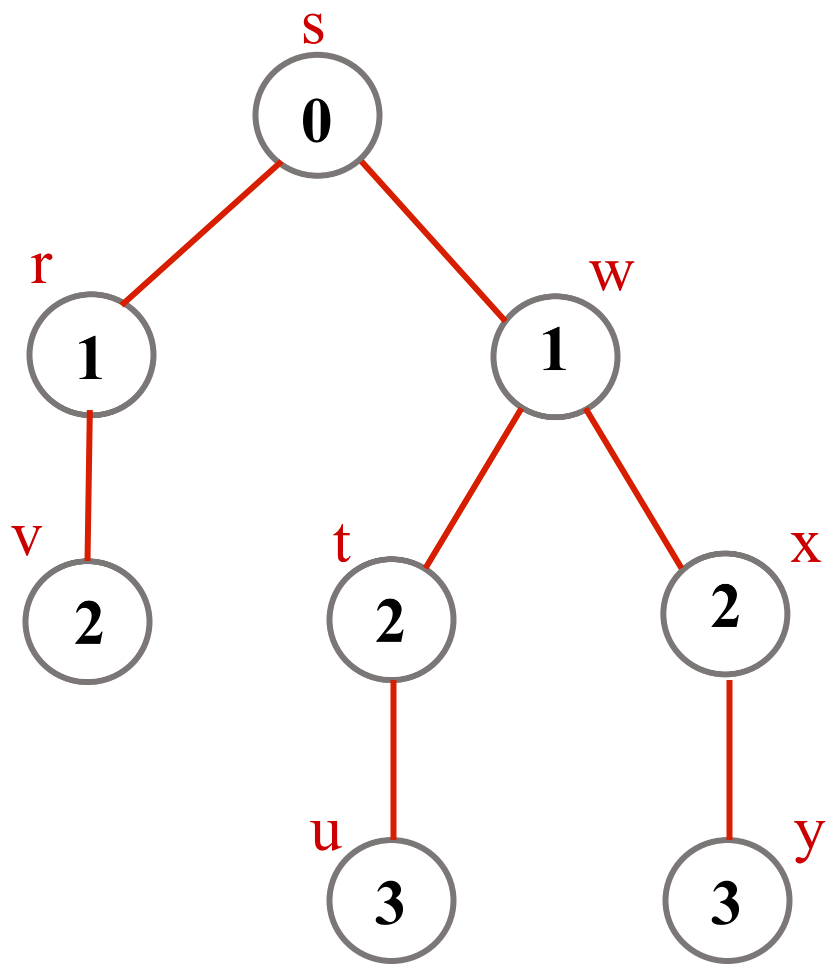


Example (BFS)



Example (BFS)

BF Tree



Analysis of BFS

```
BFS(G, s)
for each vertex u ∈ V [G] - {s}
    do color[u] ← WHITE
       d[u] ← ∞
       π[u] ← NIL
color[s] ← GRAY
d[s] ← 0
π[s] ← NIL
Q ← Ø
ENQUEUE(Q, s)
while Q ≠ Ø
    do u ← DEQUEUE(Q)
       for each v ∈ Adj[u]
           do if color[v] = WHITE
                 then color[v] ← GRAY
                      d[v] ← d[u] + 1
                      π[v] ← u
                      ENQUEUE(Q, v)
       color[u] ← BLACK
```

Lines 1–4: $O(V)$

Lines 9–18: $O(V+E)$

Each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(V)$.

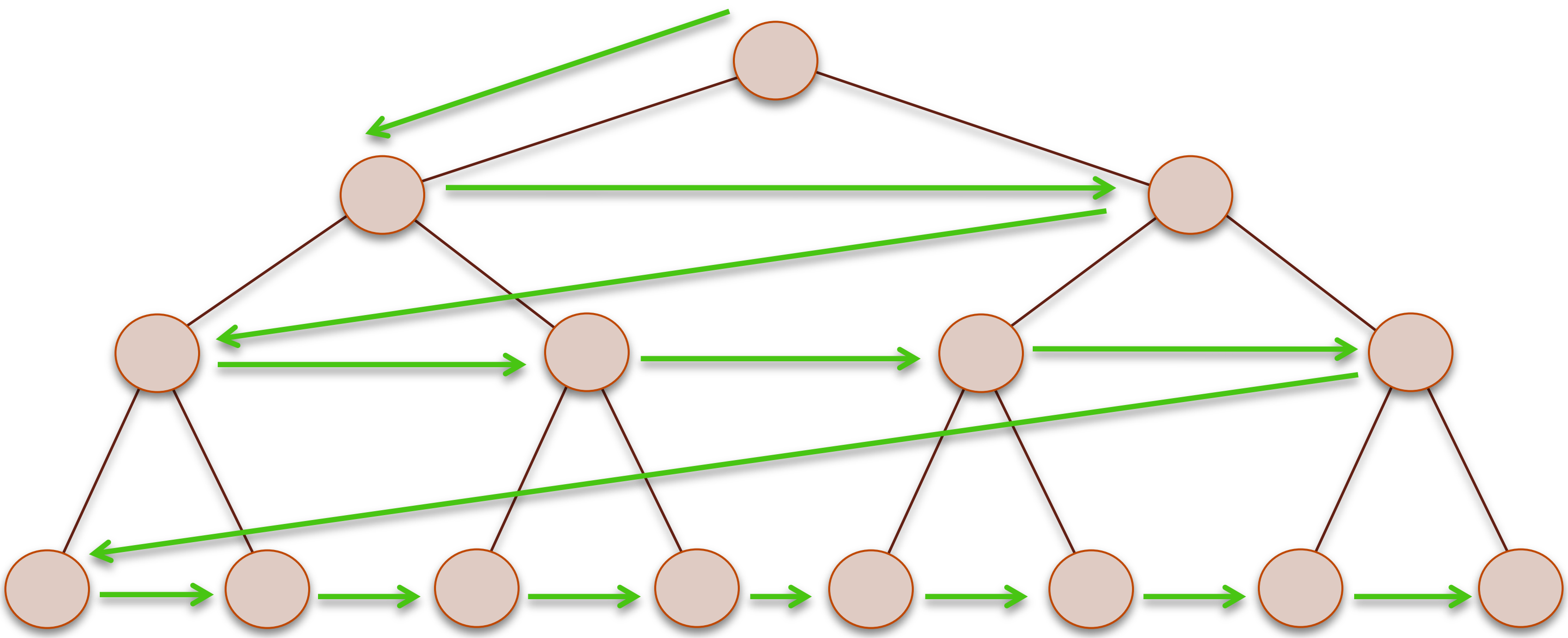
The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $O(E)$.

⇒ Total time complexity : $\boldsymbol{O(V+E)}$

⇒ Space complexity:

- ⇒Linear ($\Theta(V+E)$) in the size of adjacency list representation.
- ⇒Quadratic ($\Theta(V^2)$) in the size of adjacency matrix representation.

Breadth-First Search (BFS)



Applications of BFS

- Shortest paths in graphs which have equal edge weights.
- To compute maximum flow in Ford-Fulkerson method.
- Certain pattern (e.g., triangular) matching in a large graph.
- To test if a graph has the bipartite property or not
- …

Depth-First Search (DFS)

- Explore edges out of the most recently discovered vertex v.
- When all edges of v have been explored, backtrack to explore other edges leaving the vertex from which v was discovered (its *predecessor*).
- "Search as deep as possible first."
- Continue until all vertices reachable from the original source are discovered.
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-First Search

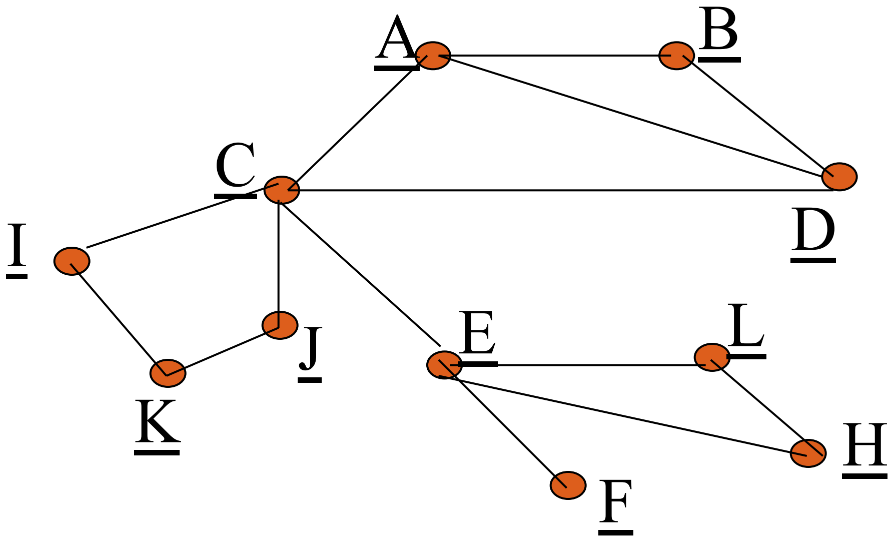
- **Input:** $G = (V, E)$, directed or undirected. No source vertex given!
- **Output:**
  - 2 **timestamps** on each vertex. Integers between 1 and $2|V|$.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f[v]$ = ***finishing time*** (v turns from gray to black)
  - $\pi[v]$: predecessor of v is u, such that v was discovered during the scan of u's adjacency list.
- Uses the same coloring scheme for vertices as BFS.

Depth-First Search

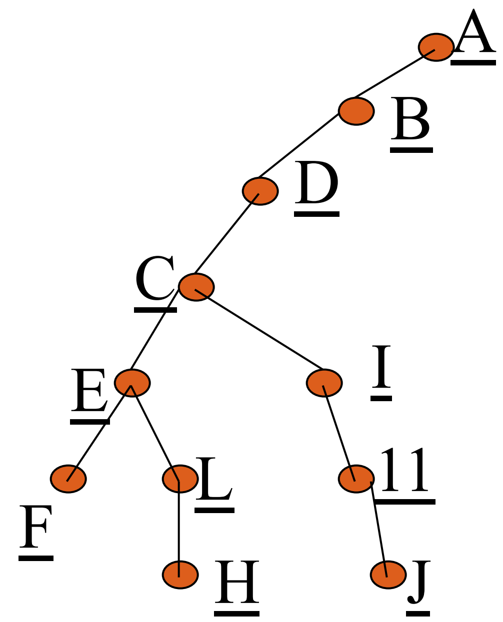
DFS follows the following rules:

1. Select an unvisited node ***x***, visit it, and treat as the current node;
2. Find an unvisited neighbor of the current node, visit it, and make it the new current node;
3. If the current node has no unvisited neighbors, backtrack to the its parent, and make that parent the new current node;
4. Repeat steps 3 and 4 until no more nodes can be visited.
5. If there are still unvisited nodes, repeat from step 1.

Illustration of DFS



Graph G



DFS Tree

DFS(G)

```
1.  for each vertex u ∈ V [G]
2.      do color[u] ← white
3.          π[u] ← NIL
4.  time ← 0
5.  for each vertex u ∈ V [G]
6.      do if color[u] = white
7.          then DFS-Visit(u)
```

Lines 1–3 paint all vertices white and initialize their π fields to *NIL*. Line 4 resets the global time counter. Lines 5–7 check each vertex in ***V*** in turn and, when a white vertex is found, visit it using *DFS-VISIT*. Every time DFS-VISIT(u) is called in line 7, vertex ***u*** becomes the root of a new tree in the depth-first forest. When DFS returns, every vertex ***u*** has been assigned a discovery time ***d[u]*** and a finishing time ***f [u].***

DFS-Visit(u)

```
1.  color[u] ← GRAY   // White vertex u has been
                         discovered
2.  time ← time + 1
3.  d[u] ← time
4.  for each v ∈ Adj[u]
5.      do if color[v] = WHITE
6.          then π[v] ← u
7.               DFS-Visit(v)
8.  color[u] ← BLACK     // it is finished.
9.  f[u] ← (time ← time + 1)
```

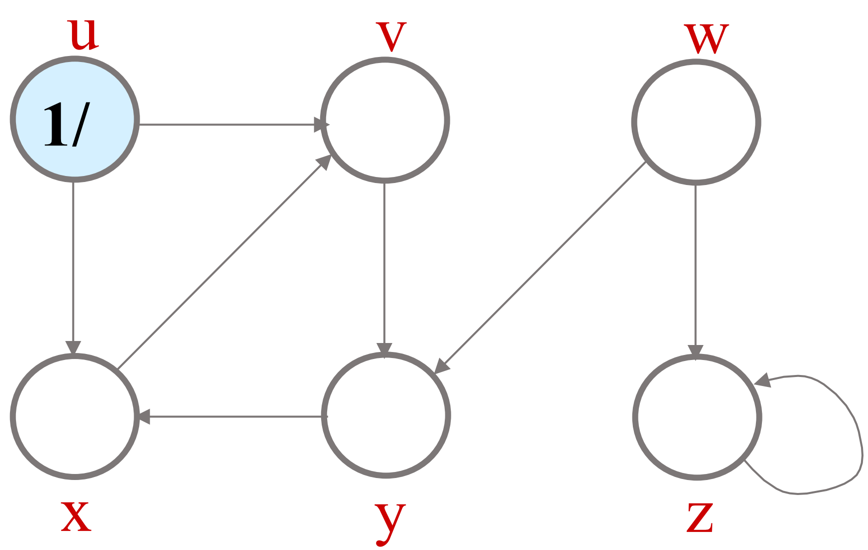
Line 1 paints ***u gray***, line 2 increments the global variable ***time***, and line 3 records the new value of time as the discovery time ***d[u]***. Lines 4–7 examine each vertex ***v*** adjacent to ***u*** and recursively visit ***v*** if it is white. As each vertex ***v*** $\in$ ***Adj[u]*** is considered in line 4, we say that edge ***(u, v)*** is ***explored*** by the depth-first search. Finally, after every edge leaving ***u*** has been explored, lines 8–9 paint ***u*** black and record the finishing time in *f [u]*.

Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

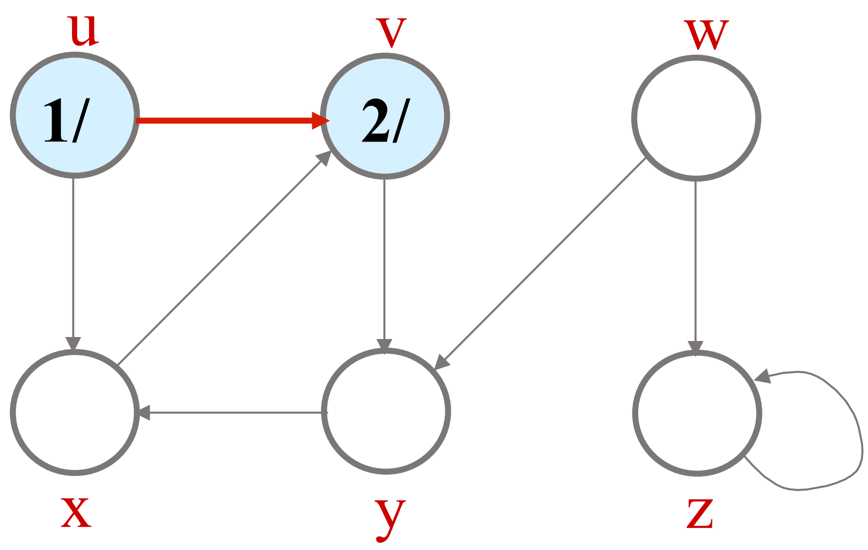
Theorem:
In DFS of an undirected graph, we get only tree and back edges.
No forward or cross edges.

Example (DFS)



discovery t. / finishing t.

Example (DFS)



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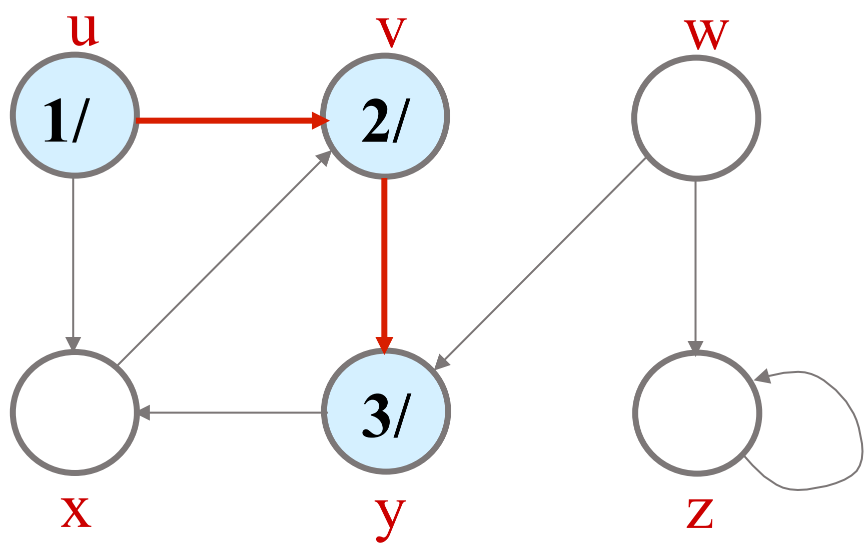

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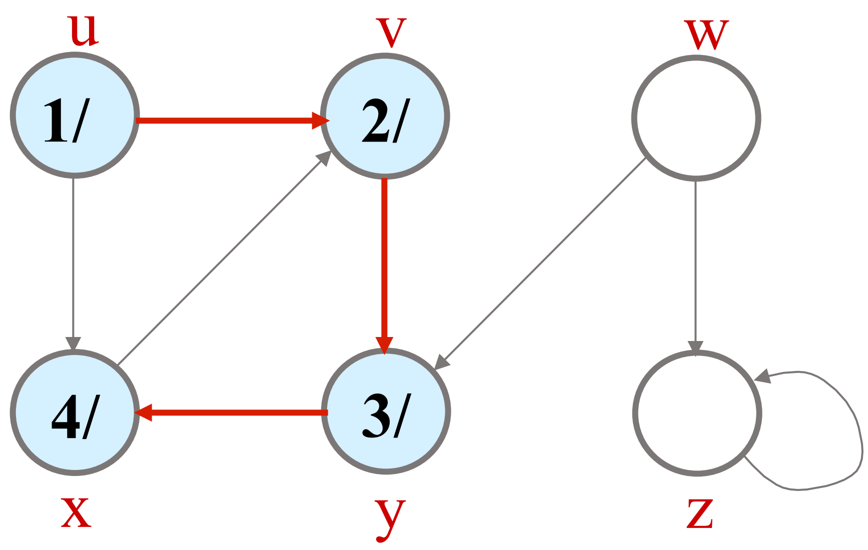

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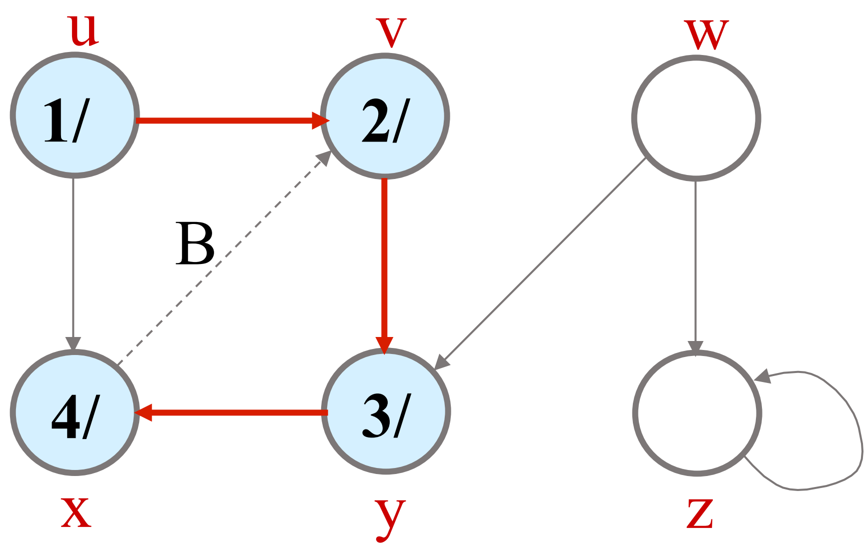

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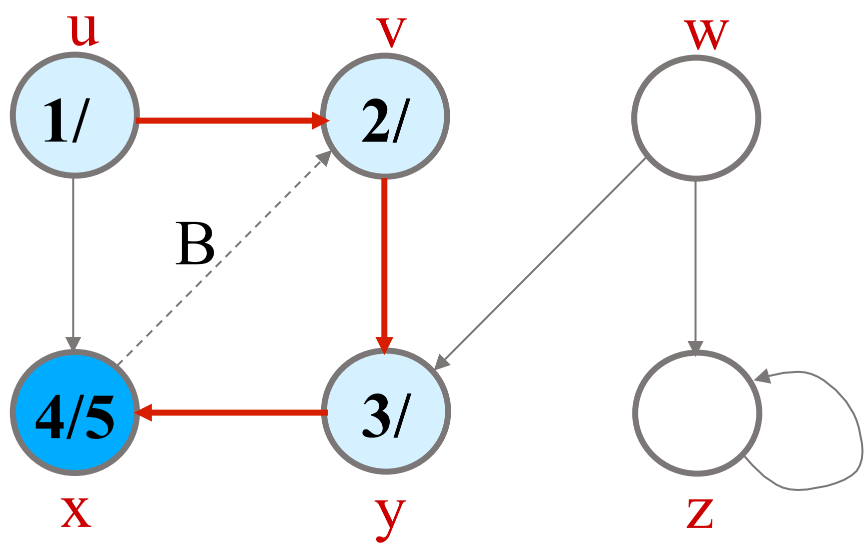

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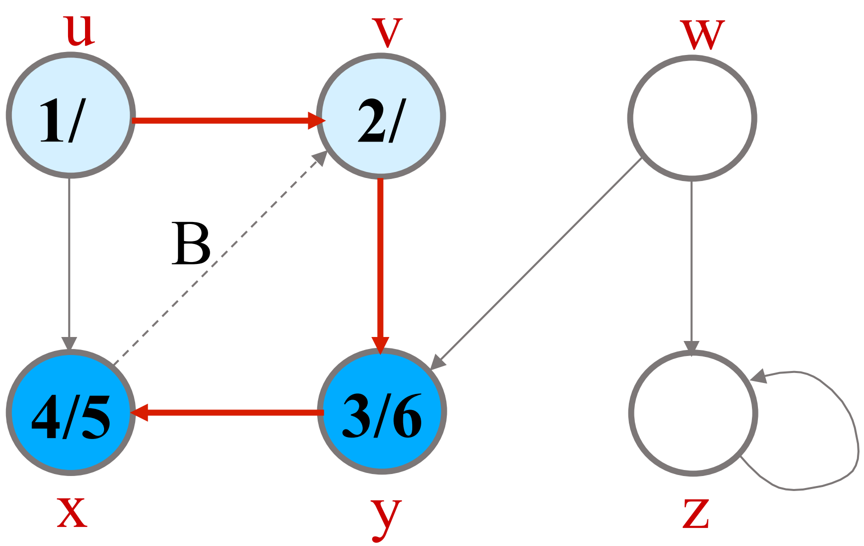

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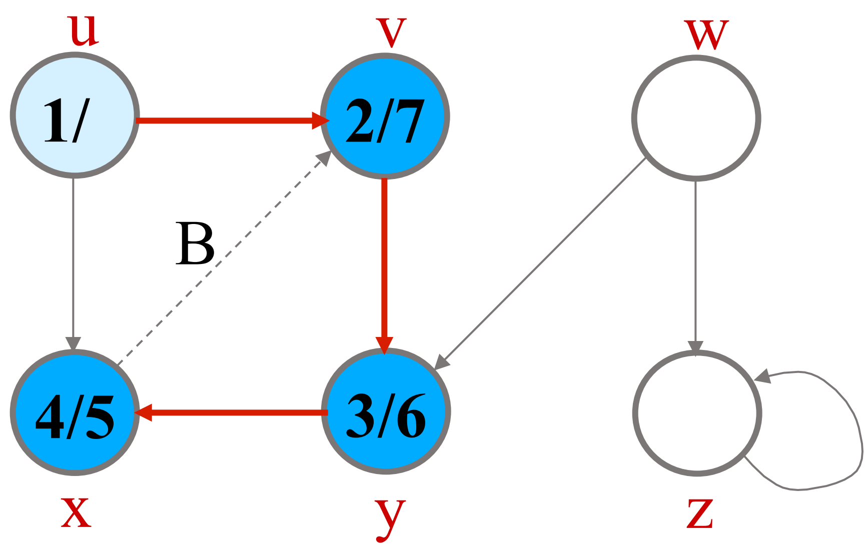

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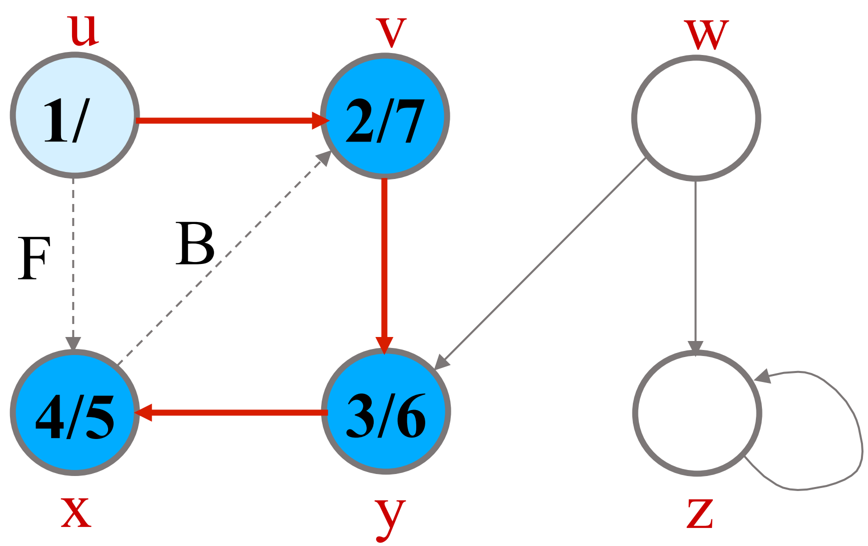

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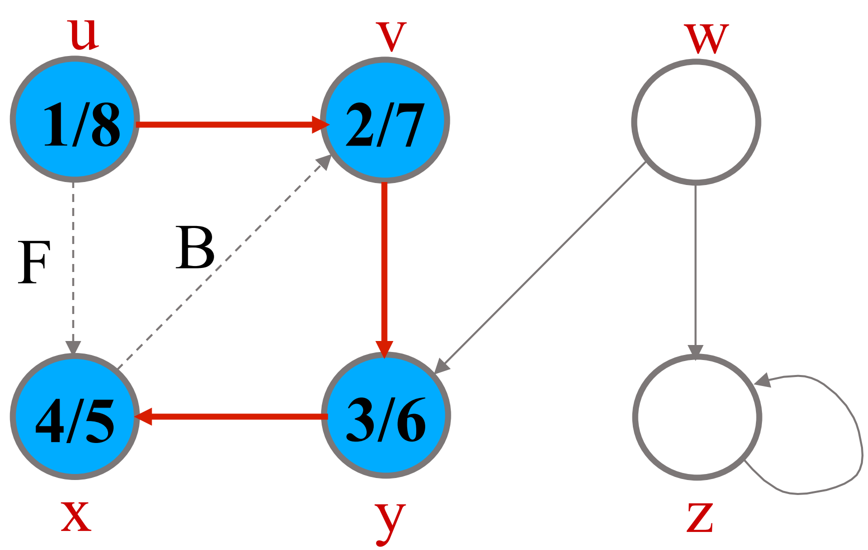

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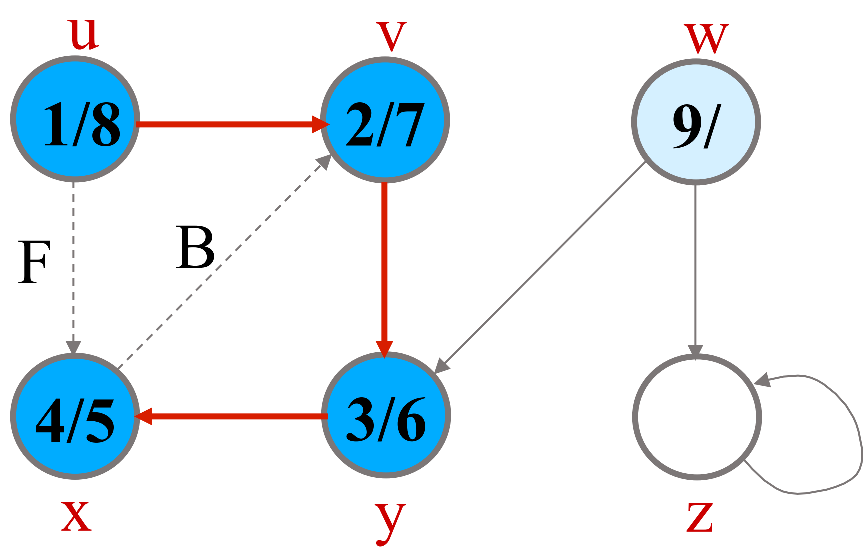

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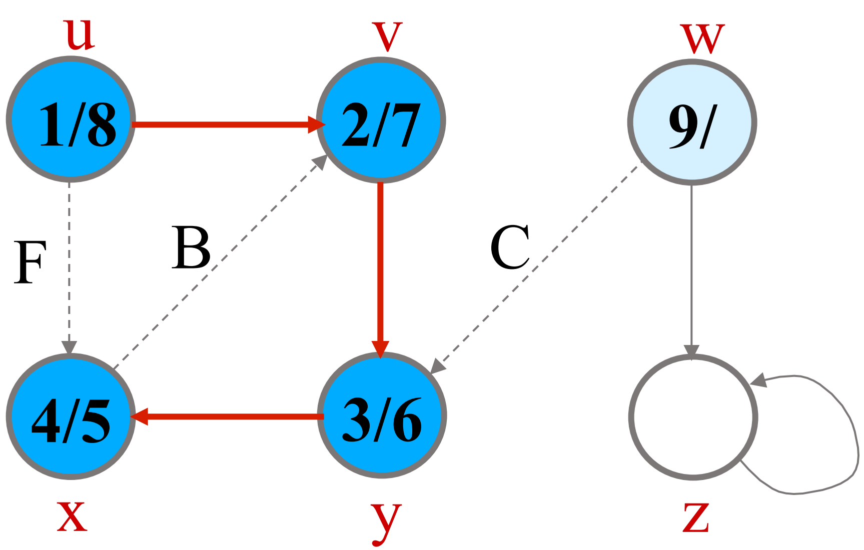

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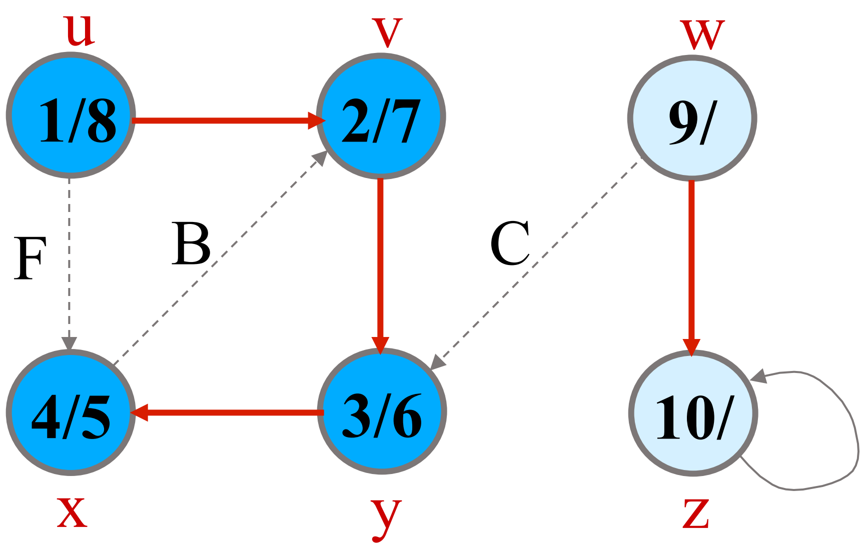

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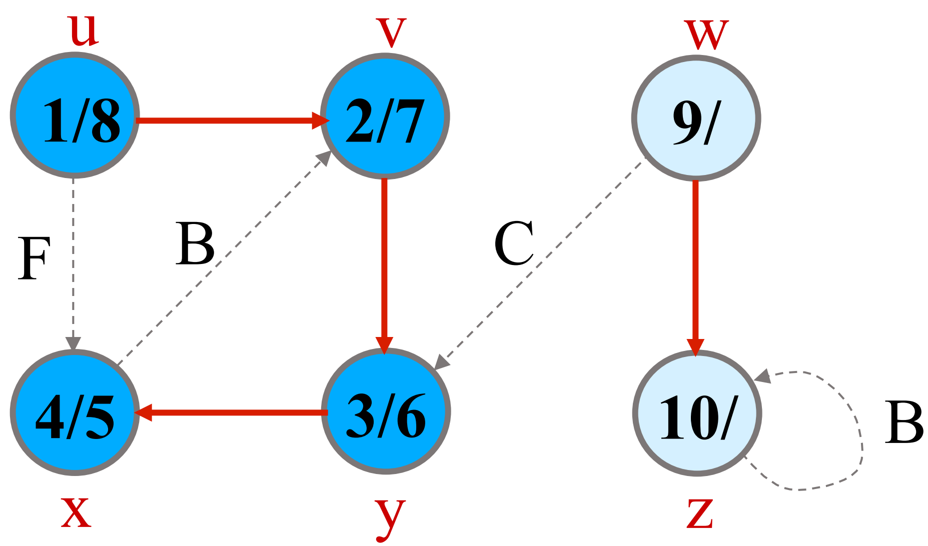

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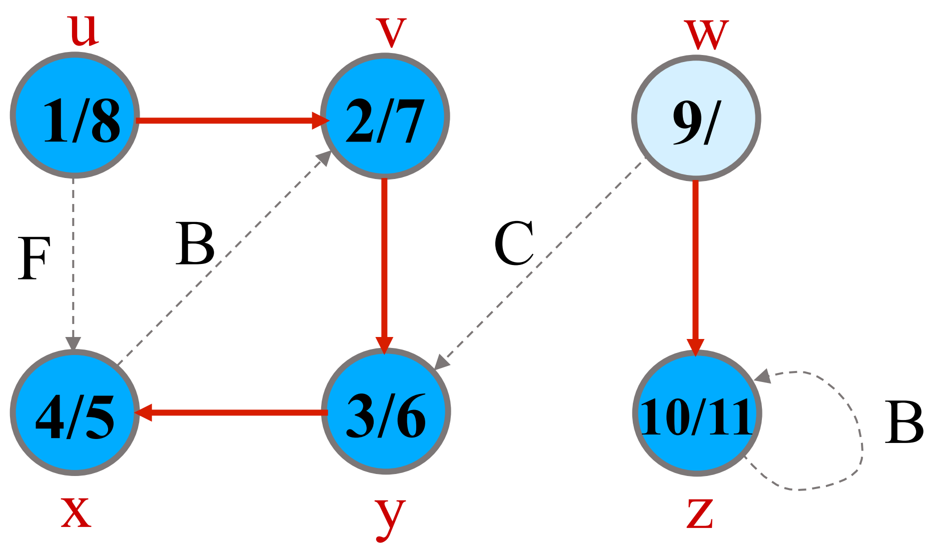
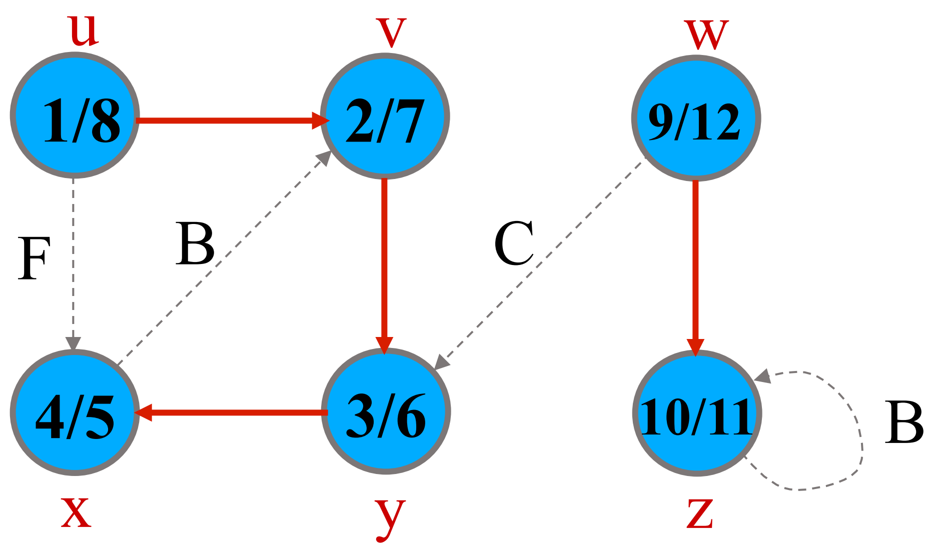
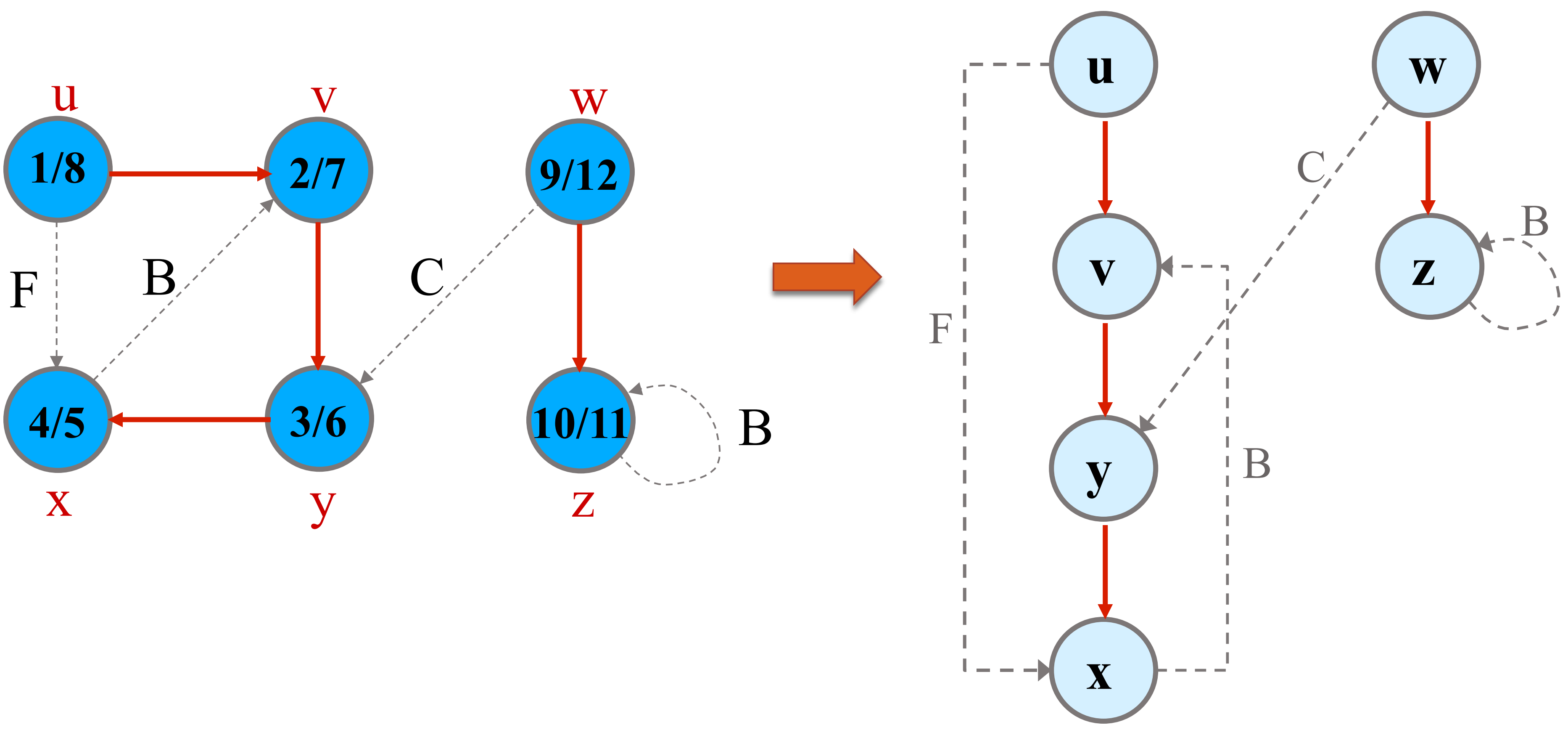


Example (DFS)



DF Tree



Analysis of DFS

DFS(G)

1. **for** each vertex $u \in V[G]$
2. **do** $color[u] \leftarrow$ white
3. $\pi[u] \leftarrow$ NIL
4. $time \leftarrow 0$
5. **for** each vertex $u \in V[G]$
6. **do if** $color[u] =$ white
7. **then** DFS-Visit(u)

Loops on lines 1-2 and 5-7 take $\Theta(V)$ time (excluding time to execute DFS-Visit.)

DFS-Visit(u)

1. $color[u] \leftarrow$ GRAY // White vertex $\boldsymbol{u}$ has been discovered
2. $time \leftarrow time + 1$
3. $d[u] \leftarrow time$
4. **for** each $v \in Adj[u]$
5. **do if** $color[v] =$ WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. $color[u] \leftarrow$ BLACK // it is finished.
9. $f[u] \leftarrow (time \leftarrow time + 1)$

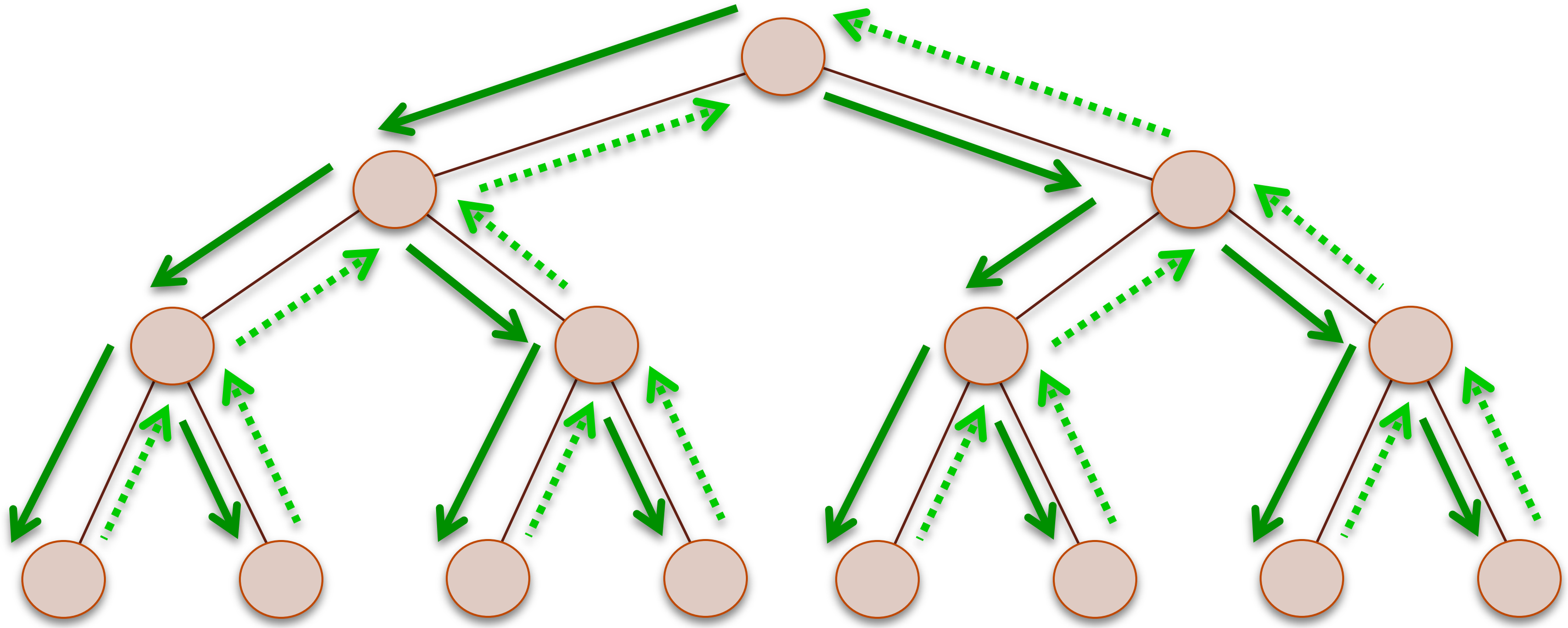
DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time.

Lines 3-6 are executed $|\mathrm{Adj}[v]|$ times.

The total cost of executing DFS-Visit : $\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(E)$

Total running time of DFS : $\Theta(V+E)$

Depth-First Search (DFS)



Applications of DFS

- Topological sorting of vertices
- Find connected components of a large graph
- Find bridges of a graph
- Solve one solution puzzles (e.g. maze)
- Find bi-connectivity in graphs
- …

Next Week Topics

- No Lecture – Exam week
- The week later: Minimum Spanning Trees (Chapter 23)