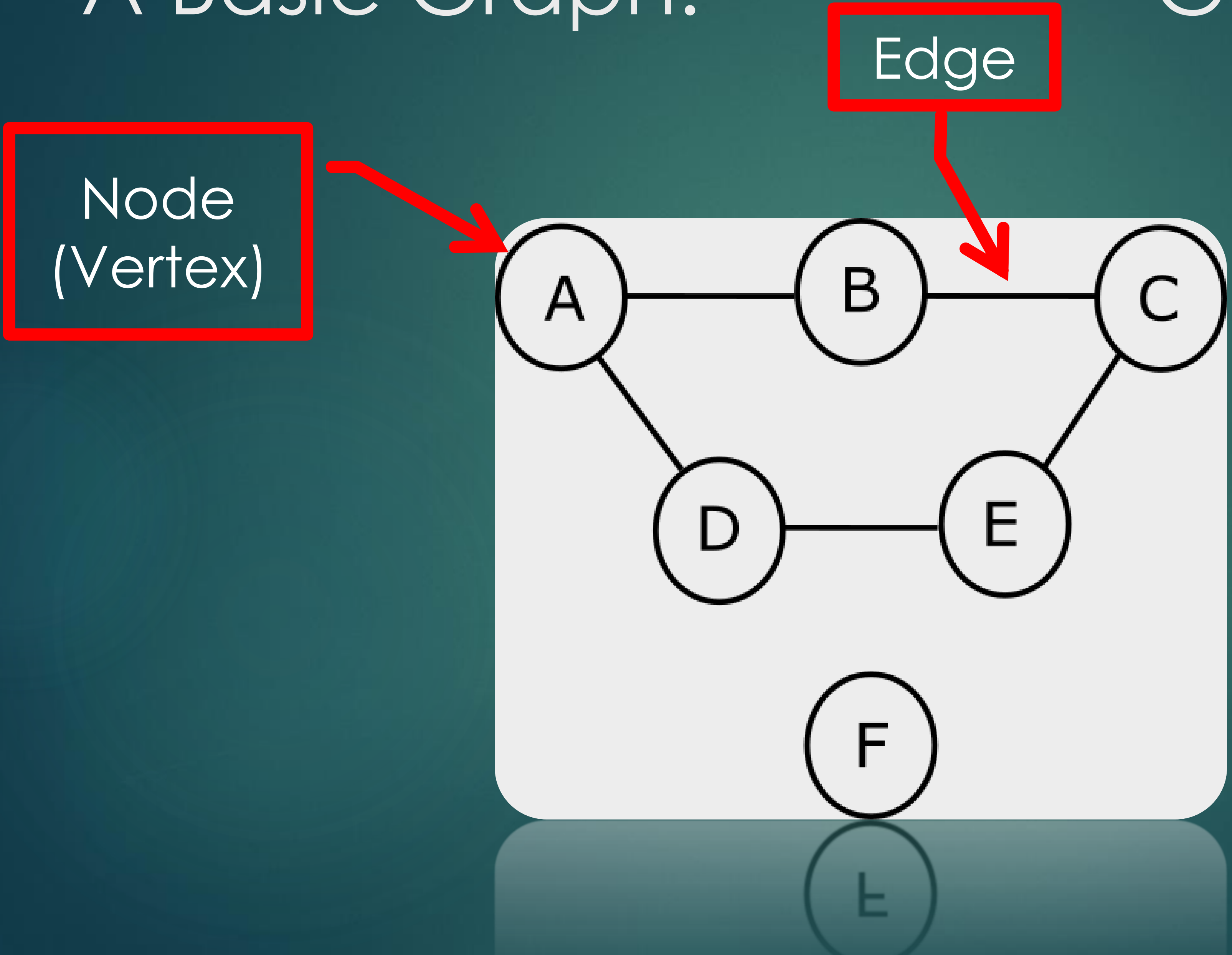
Graph Theory Basics

CME4422, I. ATAKAN KUBILAY, PHD.

Outline

- Definitions I
  - Graph Definition
  - Eccentricity, Diameter, Radius
  - Periphery, Center
- Graph Types
  - Directed\Undirected Graphs
  - Weighted\Non-weighted Graphs
  - Signed Graphs
- Definitions II
  - Path
  - Eulerian Walk
  - Neighborhood
  - Connected/Disconnected Graphs
  - Bridge and Hub
- Graph Representations
  - Adjacency Matrix
  - Adjacency List
  - Edge List
  - Incidence Matrix

A Basic Graph: Graph1

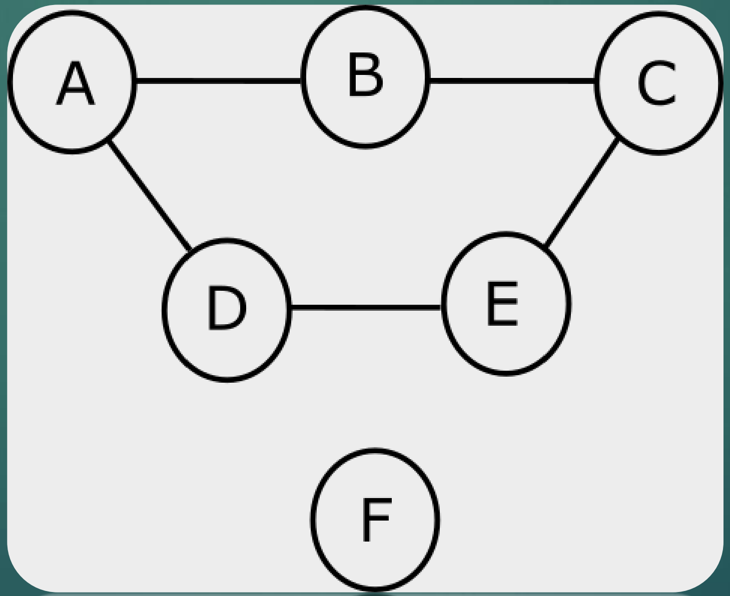


Graph Definition

- The set of Vertices(nodes) in a graph are given as:
$$V = \{v_1, v_2, \cdots v_n\}$$
- For our graph1 above $V = \{A, B, C, D, E, F\}$
- The size n of a graph is denoted by $|V|$ and for graph1 $n = |V| = 6$
- The set of Edges in a graph are given as: $E = \{e_1, e_2, \dots, e_n\}$
- The size m of a graph is denoted by $|E|$ and for graph1 $m = |E| = 5$

Simple Graph

- A simple graph has no loops.
- At most one edge between the nodes.



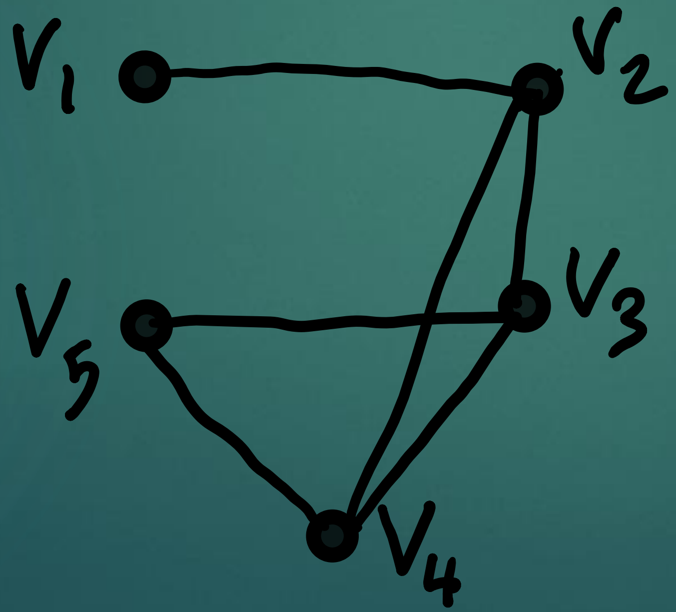
Mathematical Notation

- The mathematical notation for an undirected, unweighted graph is given by $G(V, E)$.
- For graph1:
- $V = \{A, B, C, D, E, F\}$
- $E = \{AB, BC, CE, ED, DA\}$

Eccentricity, Diameter and Radius

Eccentricity

- Eccentricity is defined individually for every node in the graph.
- It's the maximum distance from the node to any other node.



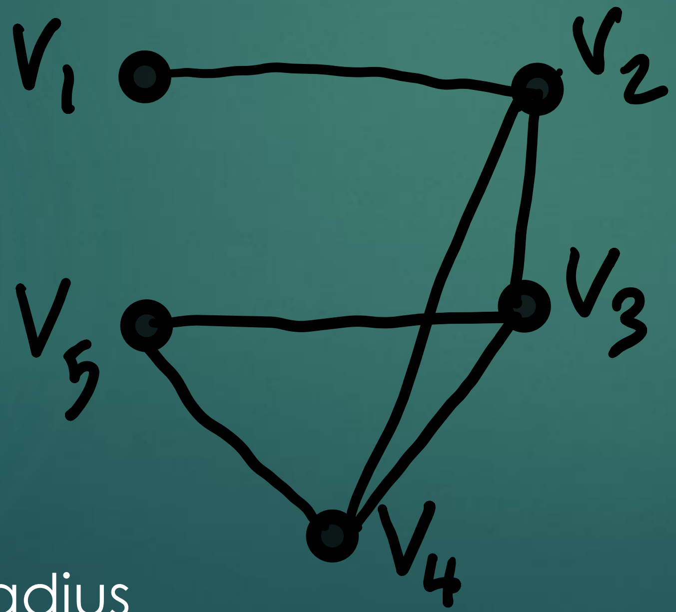
$$e(V_1) = 3$$
$$e(V_2) = 2$$
$$e(V_3) = 2$$
$$e(V_4) = 2$$
$$e(V_5) = 3$$

Diameter and Radius

- Diameter and Radius are defined for the entire graph.
- Diameter is the maximum eccentricity of the graph.
- Radius is the minimum eccentricity of the graph.

Diameter: 3

Radius: 2



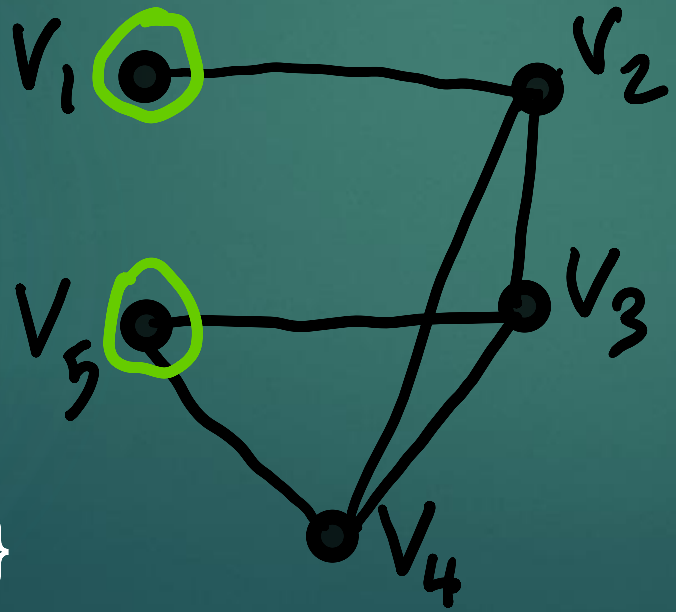
$$e(V_1) = 3$$
$$e(V_2) = 2$$
$$e(V_3) = 2$$
$$e(V_4) = 2$$
$$e(V_5) = 3$$

Radius ≤ Diameter ≤ 2Radius

Periphery and Center

Periphery

- If a node's eccentricity is equal to the diameter of the graph, then it's a peripheral node.
- The set of all such nodes is the periphery of the graph.



$$e(V_1) = 3$$
$$e(V_2) = 2$$
$$e(V_3) = 2$$
$$e(V_4) = 2$$
$$e(V_5) = 3$$

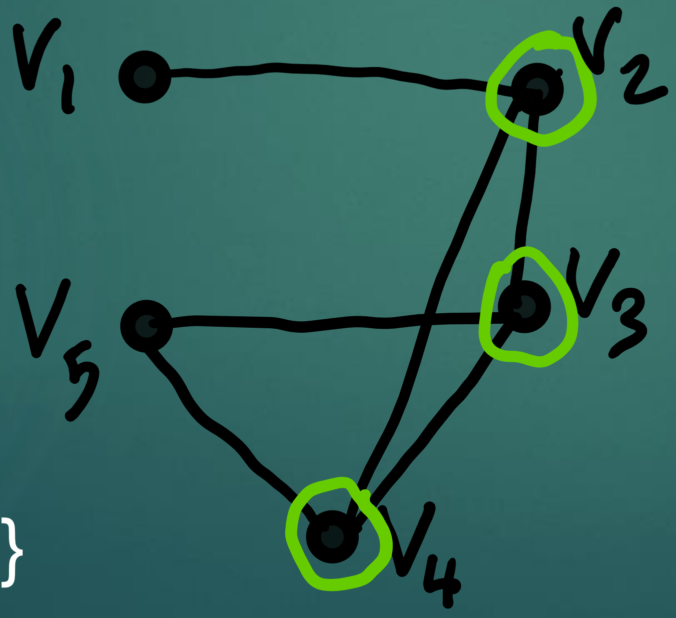
Peripheral Node

Peripheral Node

Periphery=$\{V_1, V_5\}$

Center

- If a node's eccentricity is equal to the radius of the graph, then it's a central node.
- The set of all such nodes is the center of the graph.



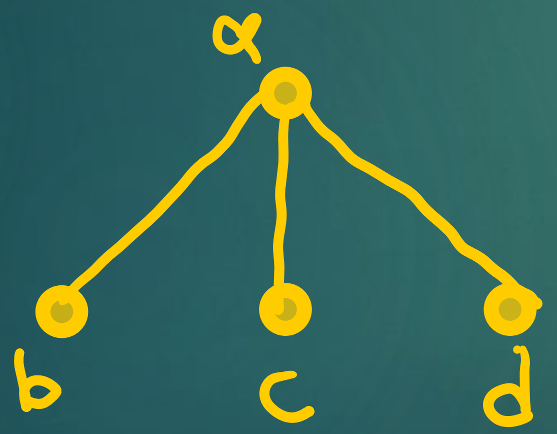
$e(V_1) = 3$
$e(V_2) = 2$
$e(V_3) = 2$
$e(V_4) = 2$
$e(V_5) = 3$

Central Nodes

Center={V_2,V_3,V_4}

Example

- Find the eccentricities of all nodes.
- Find the diameter and radius for the graph.
- Find the periphery and center for the graph.



$e(a) = 1 \longrightarrow$ radius

$e(b) = e(c) = e(d) = 2 \rightarrow$ diameter

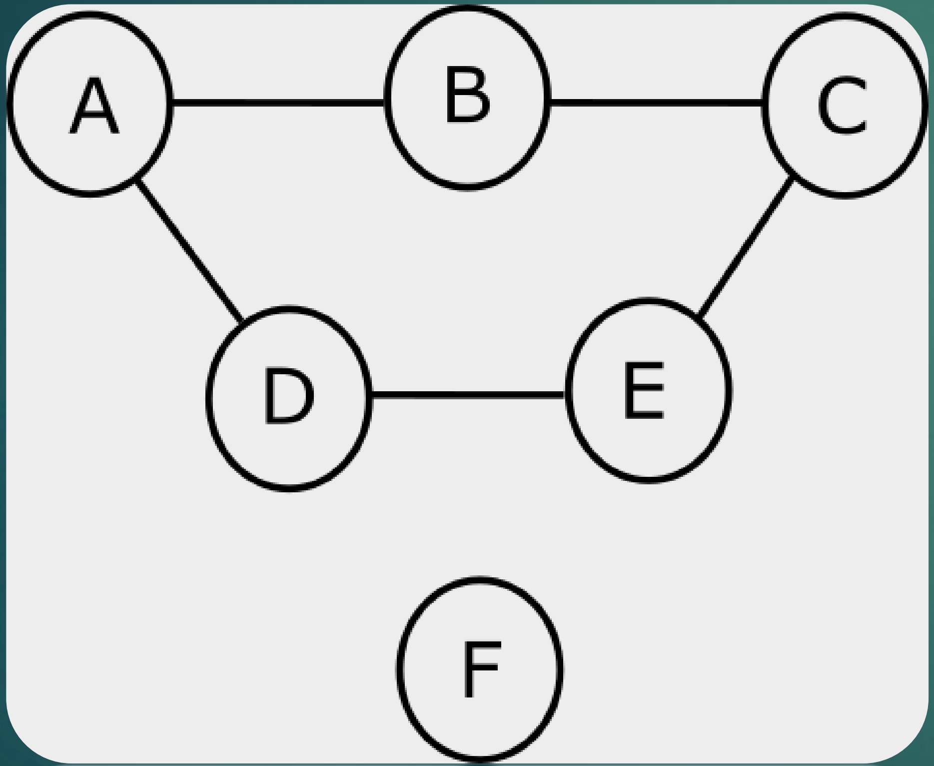
For a node e(V)=1
iff V is adjacent to all other nodes.

Graph Types

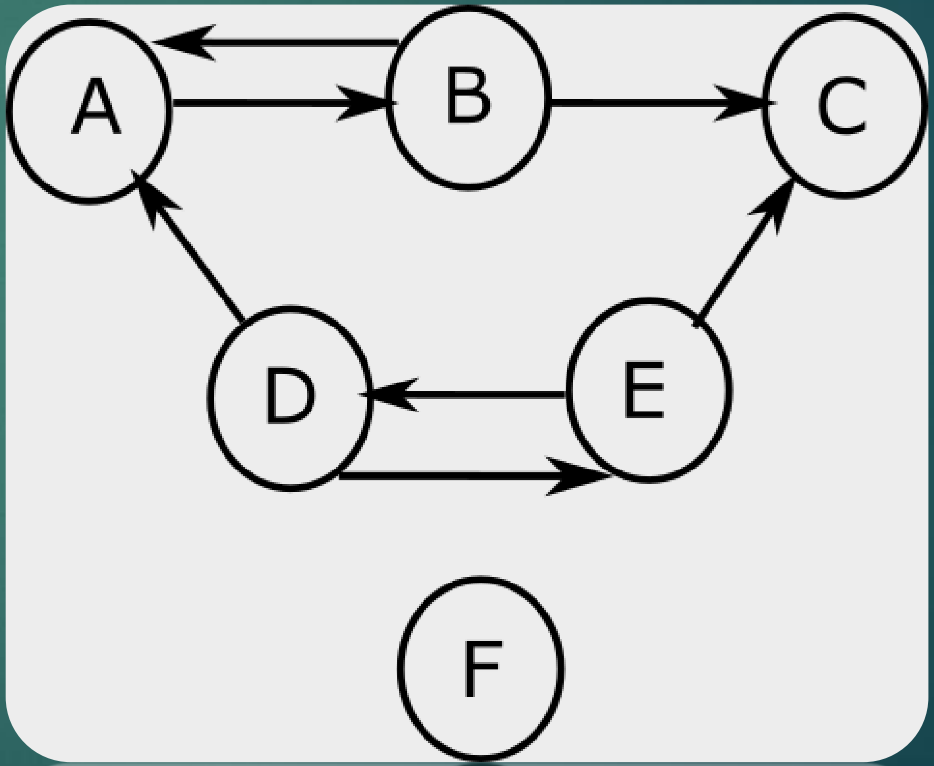
DIRECTED/UNDIRECTED GRAPHS, WEIGHTED/UNWEIGHTED GRAPHS

Directed/Undirected Graphs

Ex. Friendship.

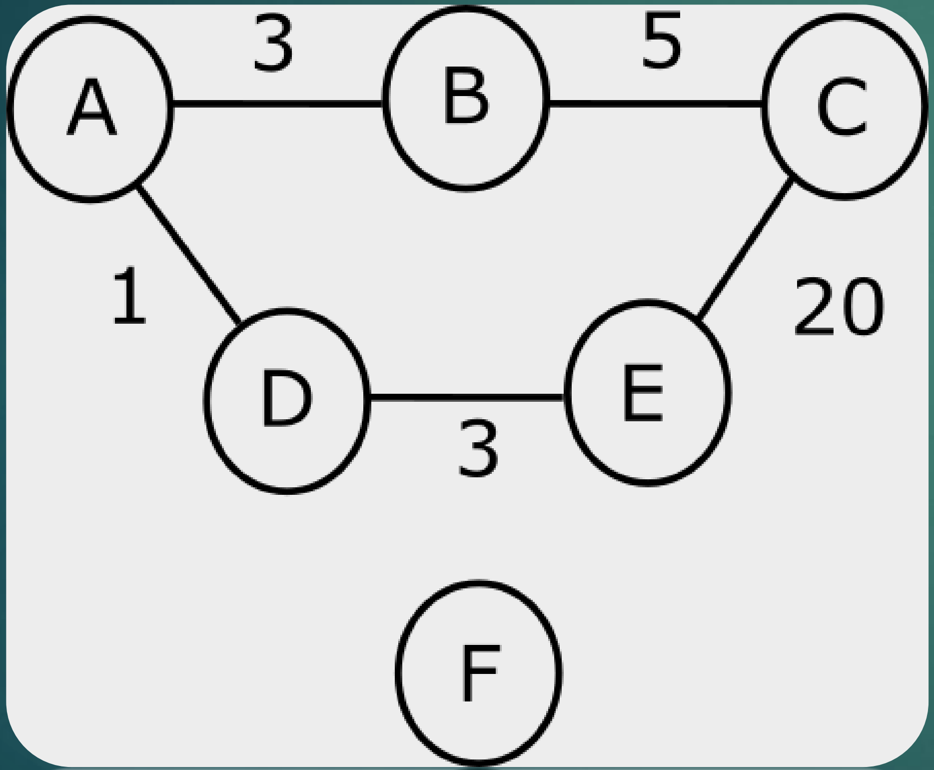


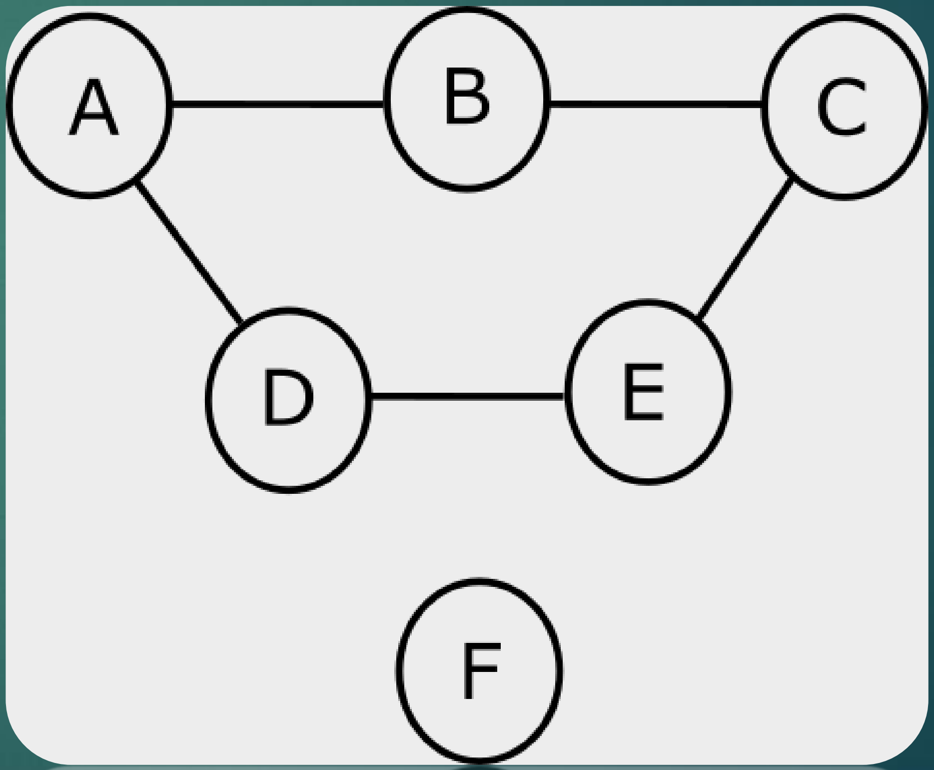
Ex. Following.



Weighted/Unweighted Graphs

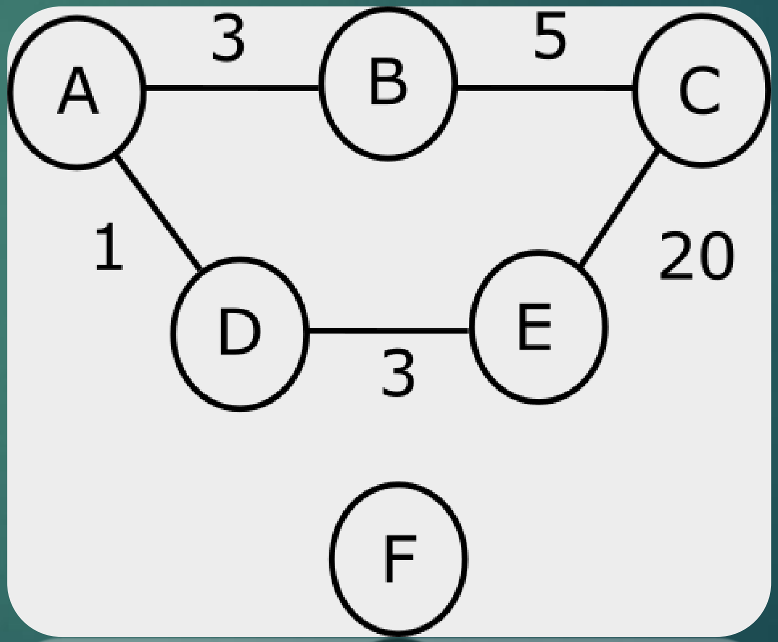
Ex. Crow's Flight Distance.





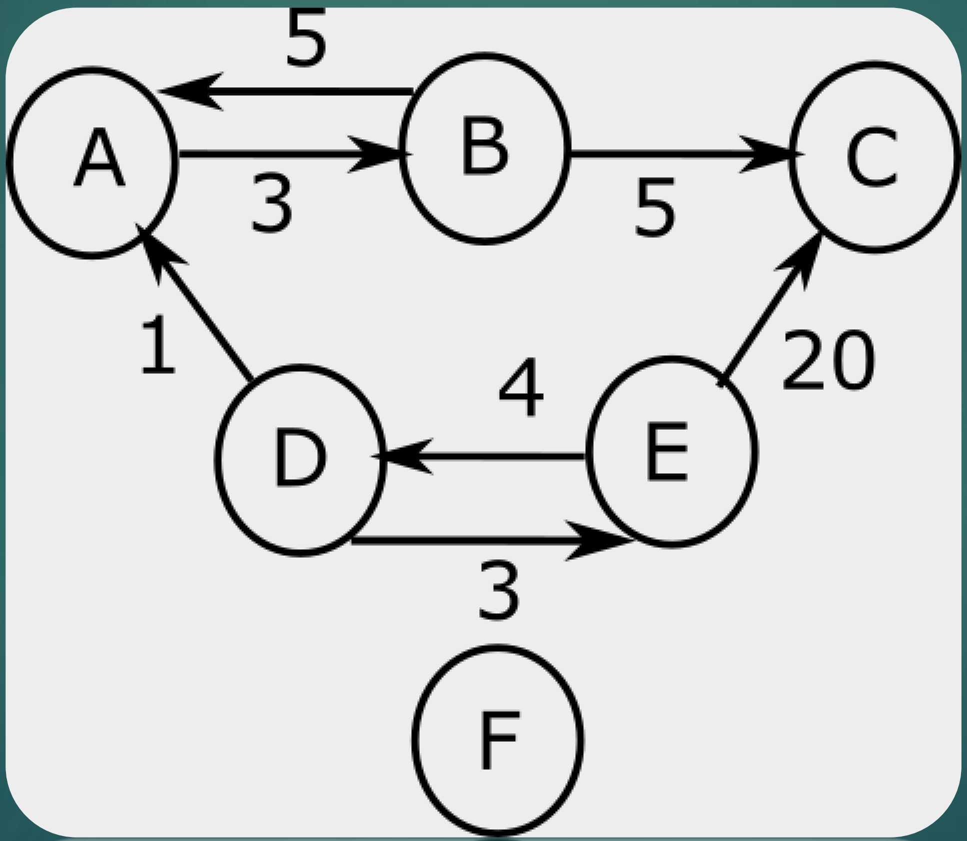
Weighted Graphs

- Weighted Graphs are mathematically denoted by $G(V, E, W)$ such that $|W| = |E|$.
- *In our example:*
- $W=\{3,5,20,3,1\}$
- $E=\{AB,BC,CE,ED,DA\}$
- $V = \{A,B,C,D,E,F\}$



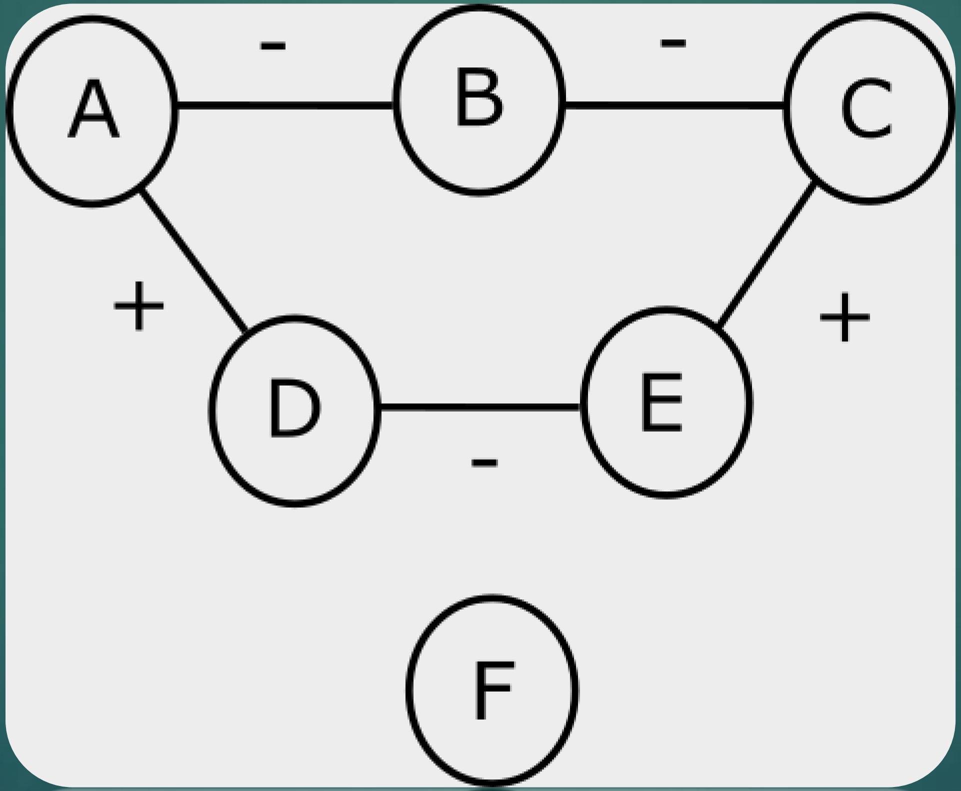
Directed and Weighted

Ex. Highway distance between cities.



Signed Graphs

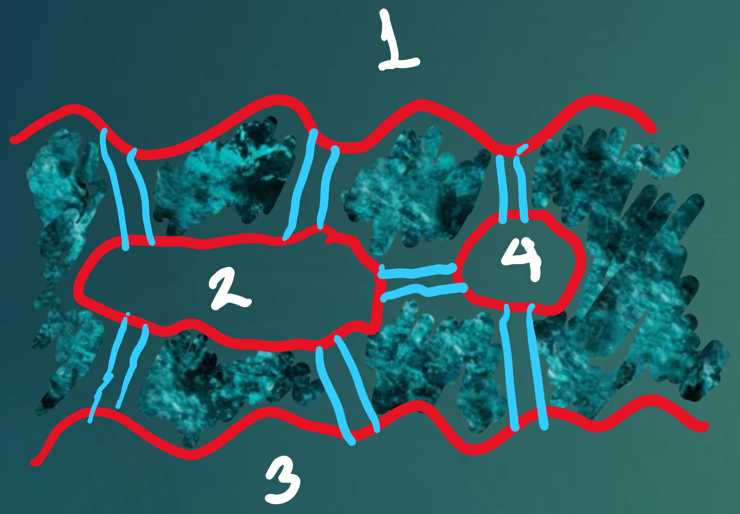
Ex. Friendship(+) and Animosity(-).

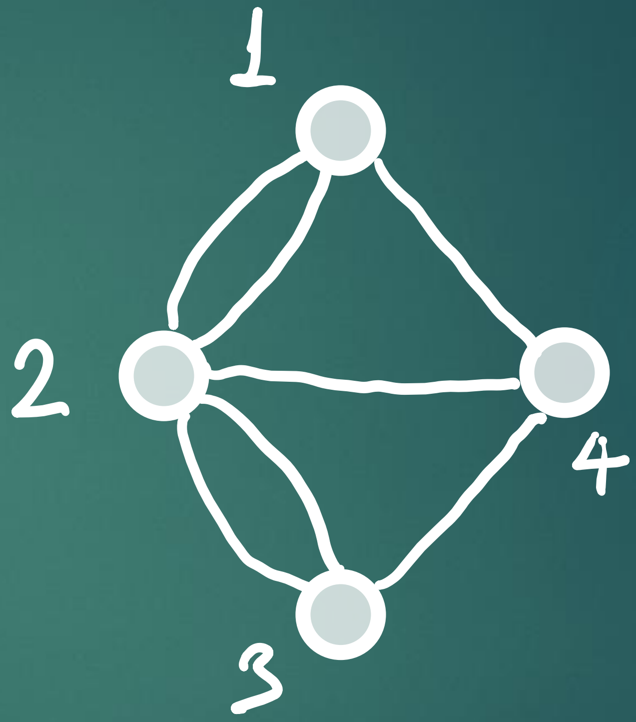


Path

- A path is the sequence of nodes that travel from one node to another without a loop or repetition.
- There can be several paths from one node to another.

Bridges of Königsberg

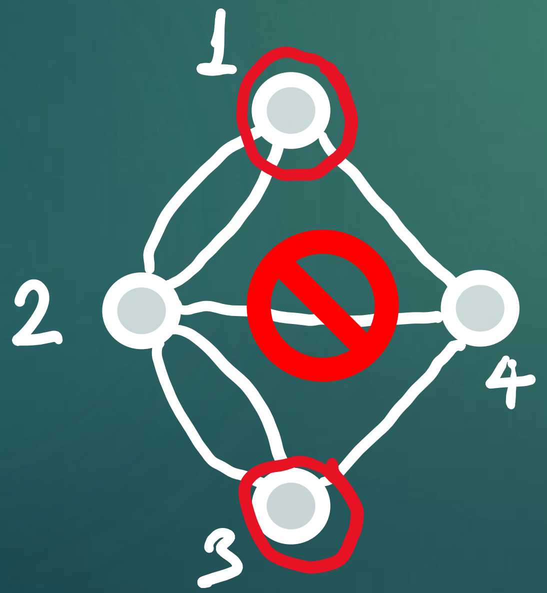




$d(1) = 3$
$d(2) = 5$
$d(3) = 3$
$d(4) = 3$

Eulerian Walk

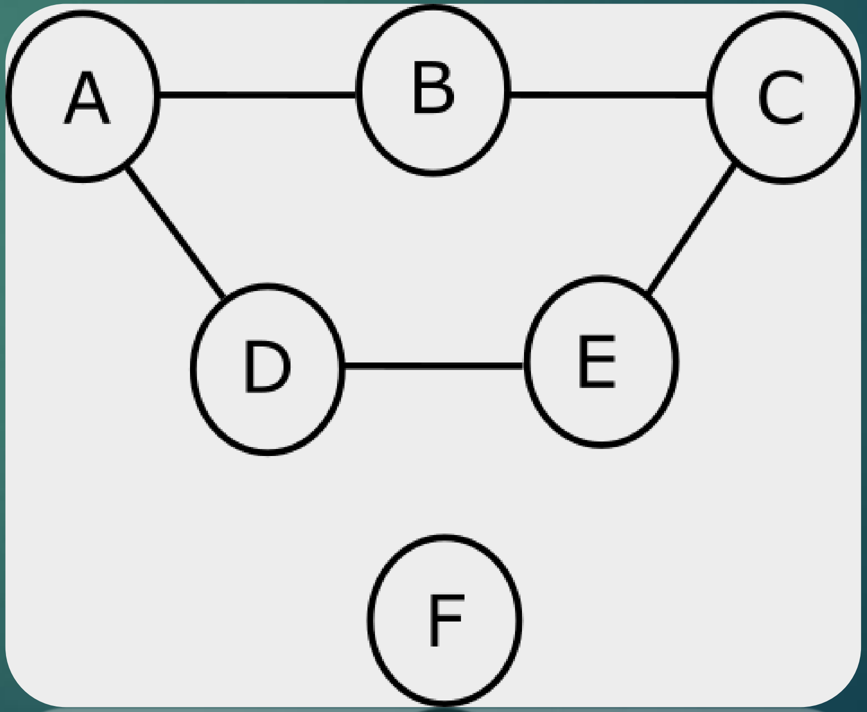
- In order to pass each edge precisely once, we need one of the following conditions:
- All nodes have even degree. OR
- Only the start and end nodes have odd degree.



$d(1)=3$
$d(2)=4$
$d(3)=3$
$d(4)=2$

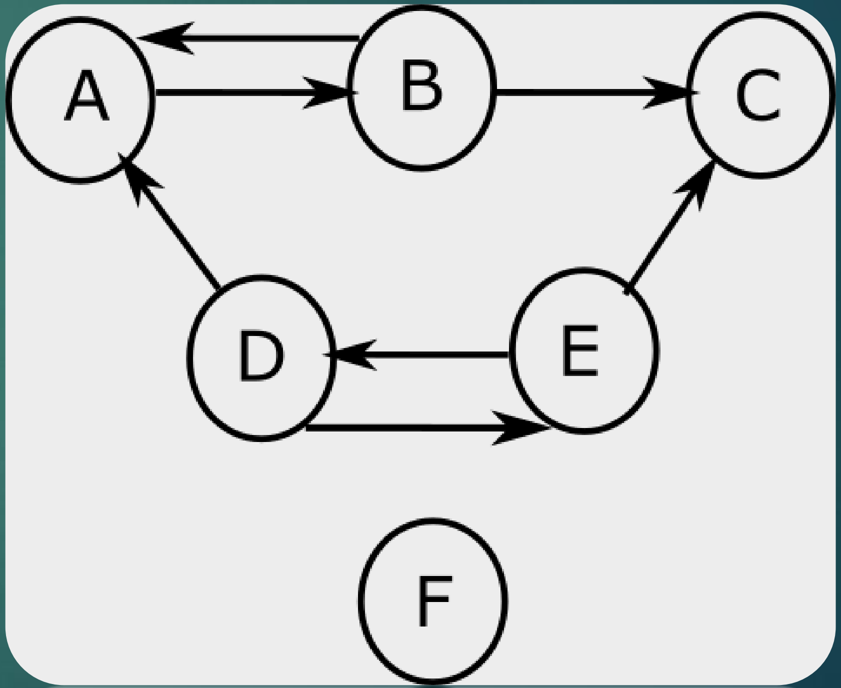
Neighborhood

- For each vertex v_i the set of all vertices that are connected to v_i are called the neighborhood $N(v_i)$ of v_i.
- For graph1:
- $N(A) = \{B,D\}$
- $N(B) = \{A,C\}$
- $N(C) = \{B,E\}$
- $N(D) = \{A,E\}$
- $N(E) = \{D,C\}$
- $N(F) = \{\}$



Neighborhood for Directed Graphs

- Now we have to consider both incoming neighbors $N_{in}(v_i)$ and outgoing neighbors $N_{out}(v_i)$.
- $N_{in}(A) = \{B,D\}$ and $N_{out}(A) = \{B\}$
- $N_{in}(B) = \{A\}$ and $N_{out}(B) = \{A,C\}$
- $N_{in}(C) = \{B,E\}$ and $N_{out}(C) = \{\}$
- $N_{in}(D) = \{E\}$ and $N_{out}(D) = \{A,E\}$
- $N_{in}(E) = \{D\}$ and $N_{out}(E) = \{D,C\}$
- $N_{in}(F) = \{\}$ and $N_{out}(F) = \{\}$

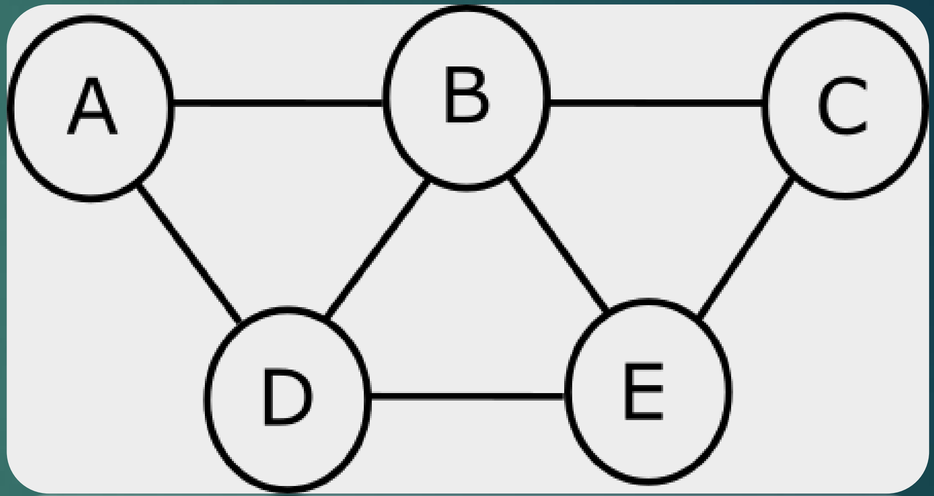


Example

- Paths between A and E:
- ABE,ADE,ABCE,ADBE,ABDE,

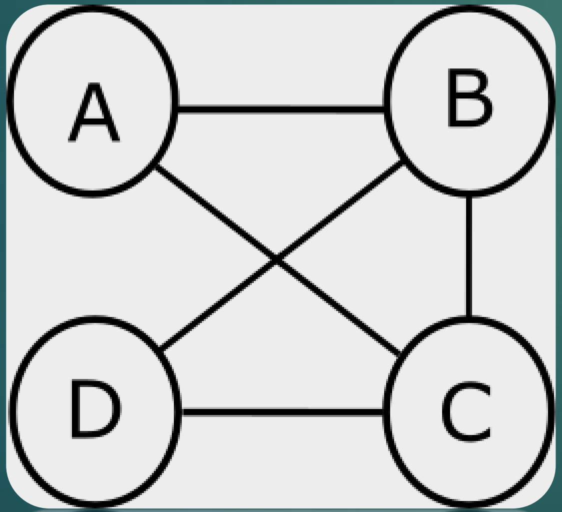
ADBCE

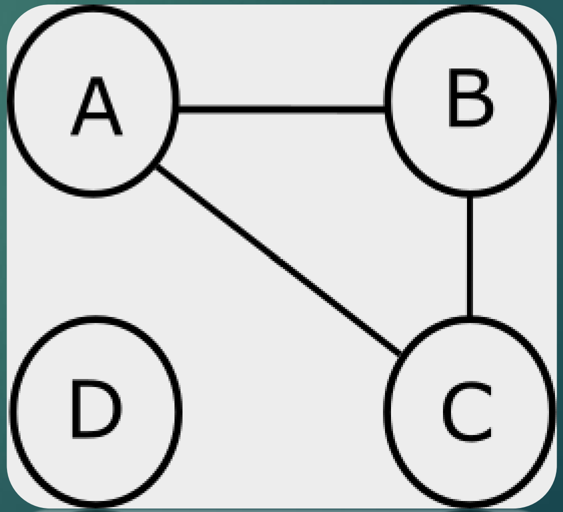
- Shortest Paths:
- ABE,ADE
- All edges are taken as having length 1 in unweighted graphs.



Undirected Graphs: Connected vs. Disconnected

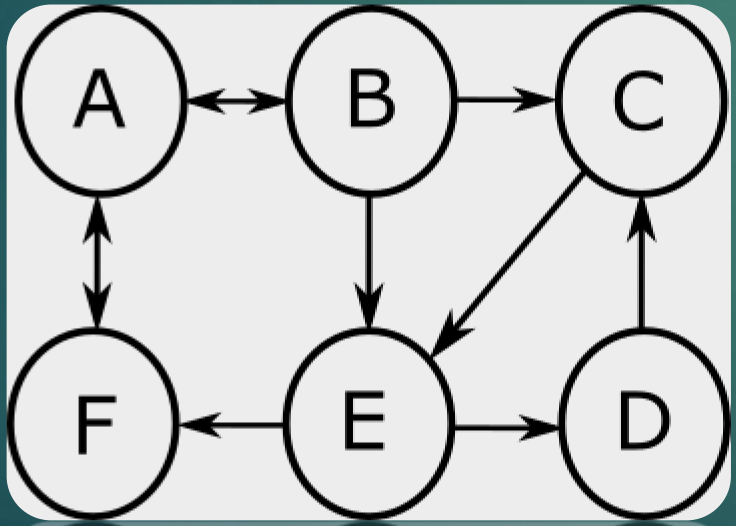
- Connected Graph: There is a path from every node to every other node.
- Disconnected Graph: Some nodes can’t be reached from all other nodes.

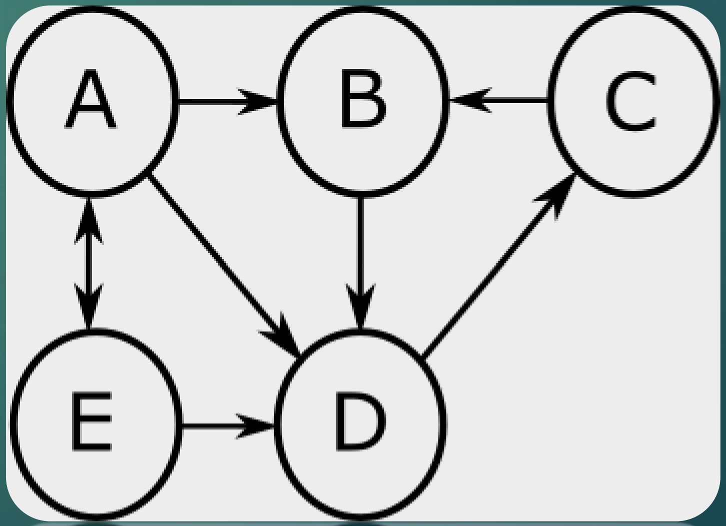


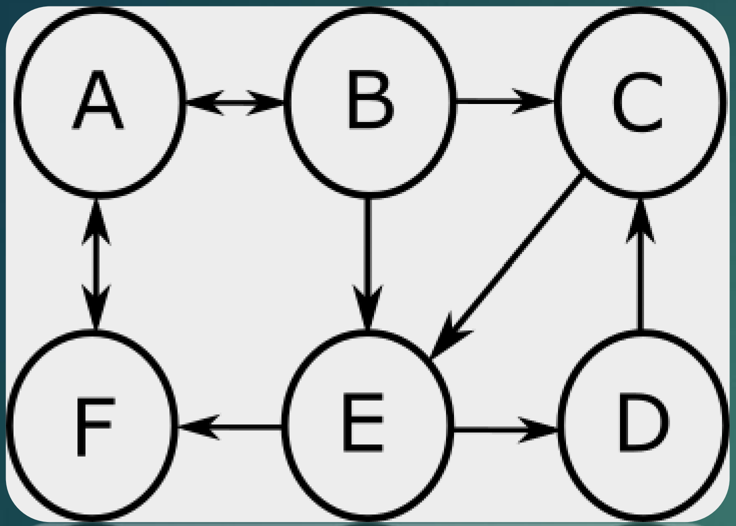


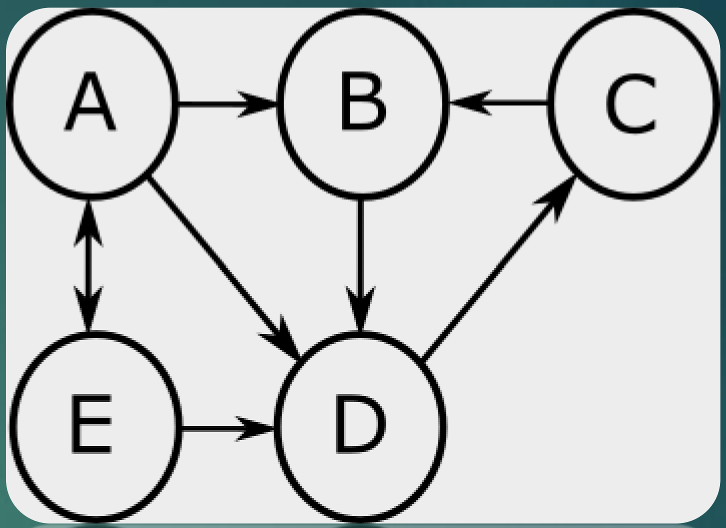
Directed Graphs:
Disconnected, Weakly Connected, Strongly Connected

- If a path from any node to any other node may be found considering the directions, this is a <u>strongly connected</u> graph.
- If a path from any node to any other node may be found only if we ignore the directions, this is a <u>weakly connected</u> graph.





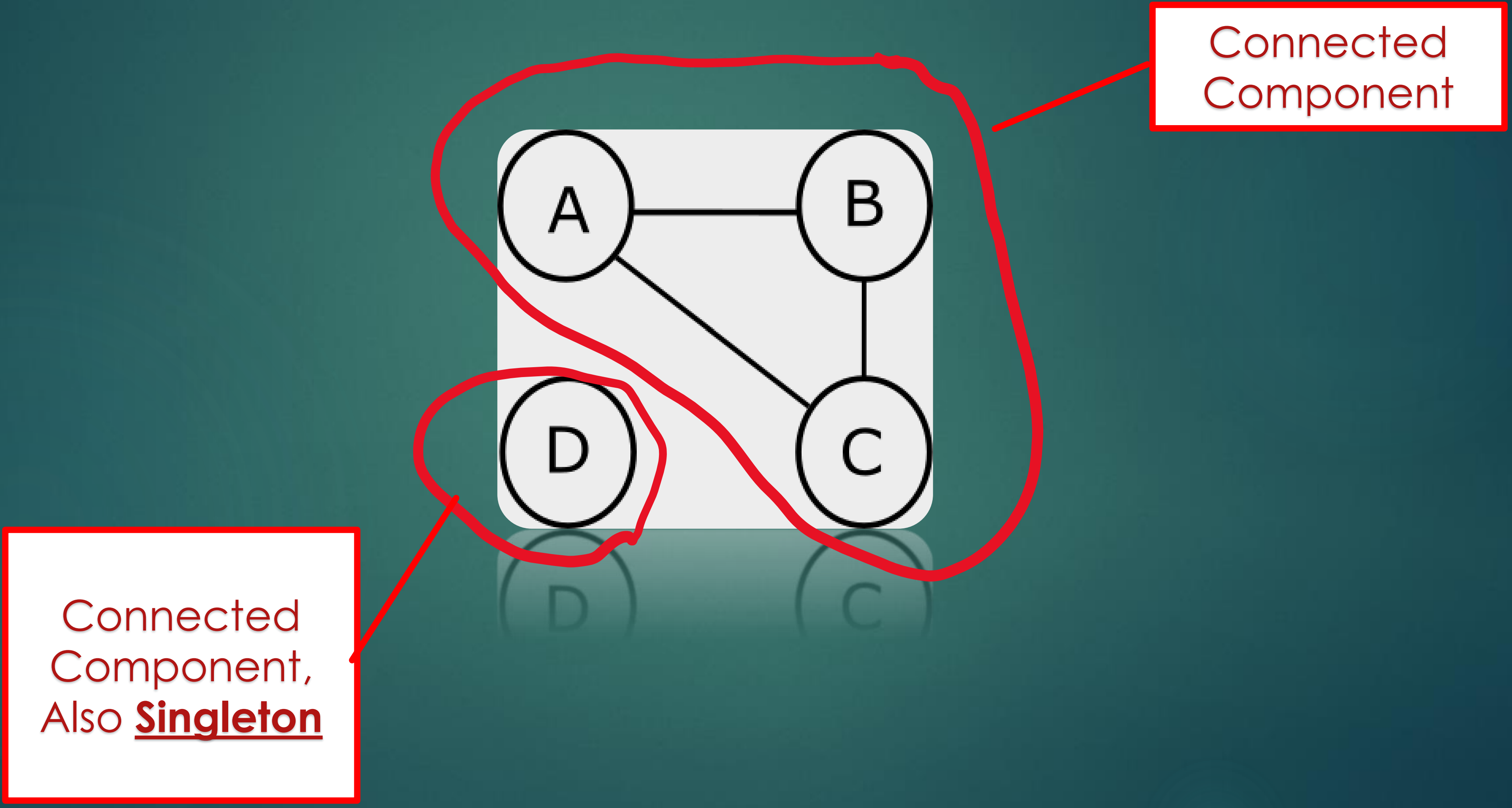




	A	B	C	D	E	F
A		✓	✓	✓	✓	✓
B	✓		✓	✓	✓	✓
C	✓	✓		✓	✓	✓
D	✓	✓	✓		✓	✓
E	✓	✓	✓	✓		✓
F	✓	✓	✓	✓	✓	

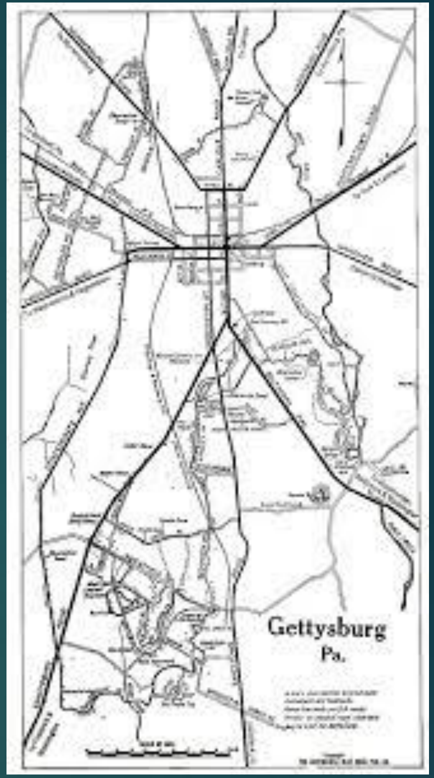
	A	B	C	D	E
A		✓	✓	✓	✓
B	☹		✓	✓	☹
C	☹	✓		✓	☹
D	☹	✓	✓		☹
E	✓	✓	✓	✓	

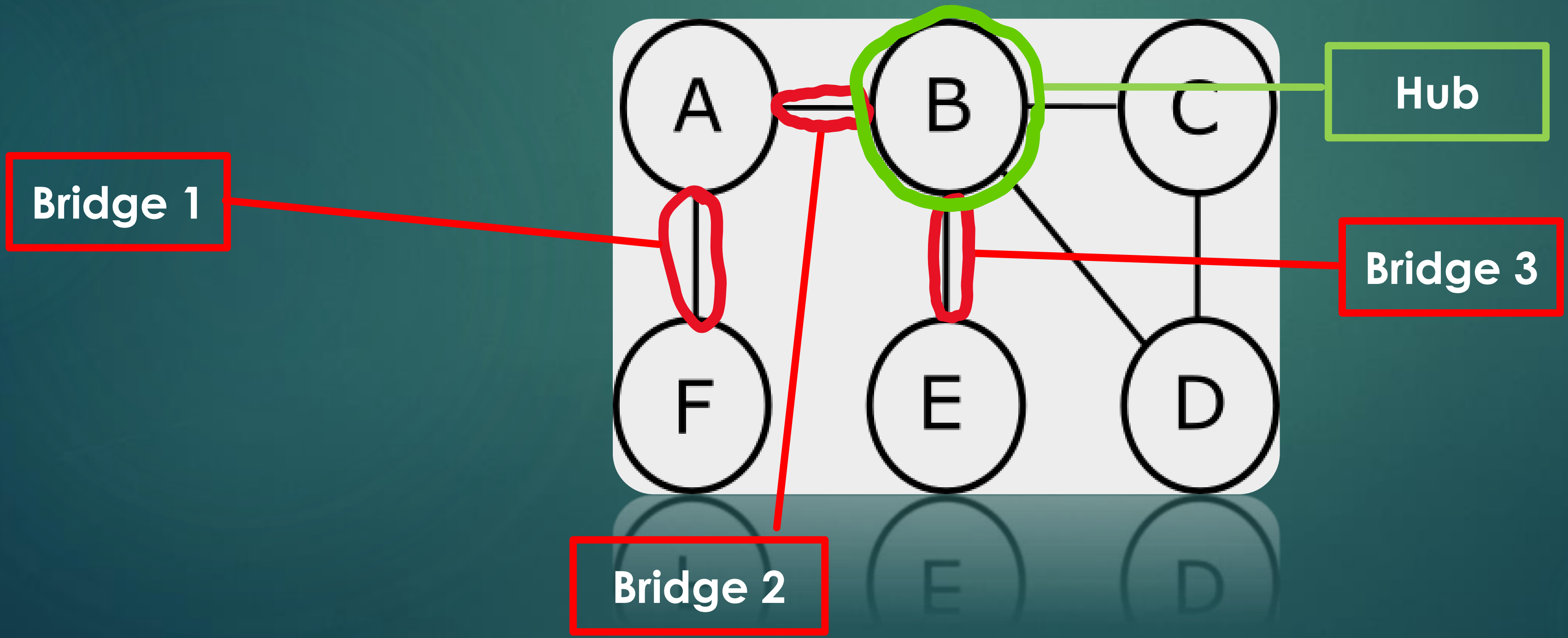
Connected Components



Bridge and Hub

- An edge that would separate the graph into connected components if removed is called a <u>bridge</u>.
- The node with most connections is called a <u>hub</u>.



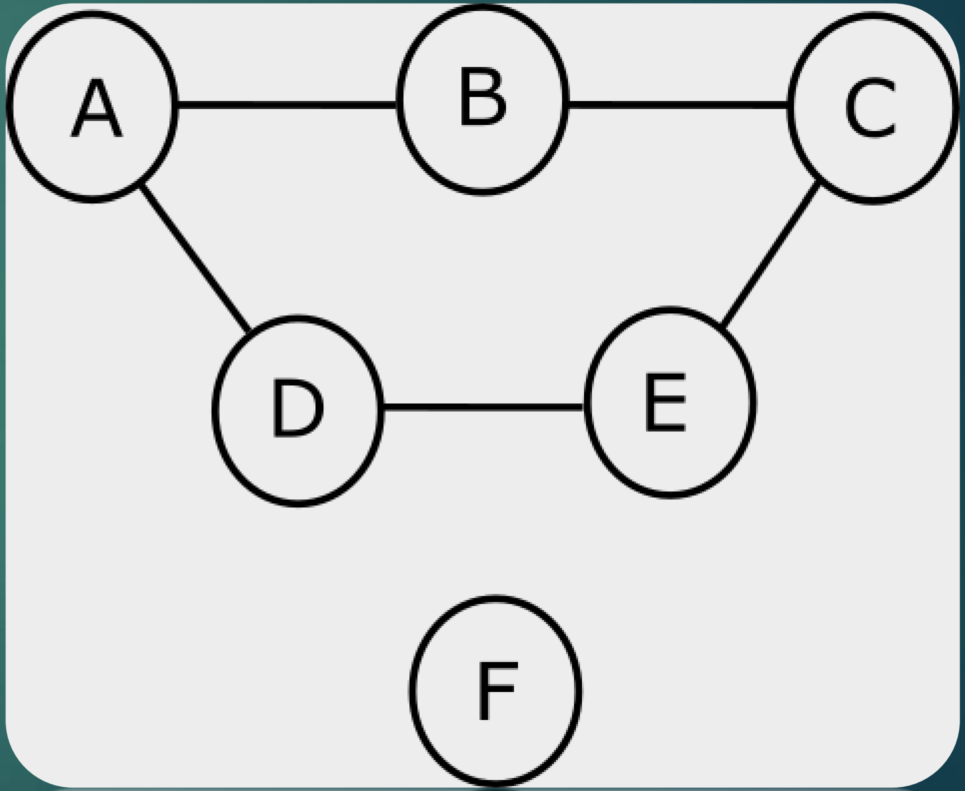


Graph Representation Types

Adjacency Matrix

There can be a 1 at the diagonal only if there is a loop connection from a vertex to itself.

	A	B	C	D	E	F
A	0	1	0	1	0	0
B	1	0	1	0	0	0
C	0	1	0	0	1	0
D	1	0	0	0	1	0
E	0	0	1	1	0	0
F	0	0	0	0	0	0



For an undirected graph, the adjacency matrix is symmetric along the diagonal.

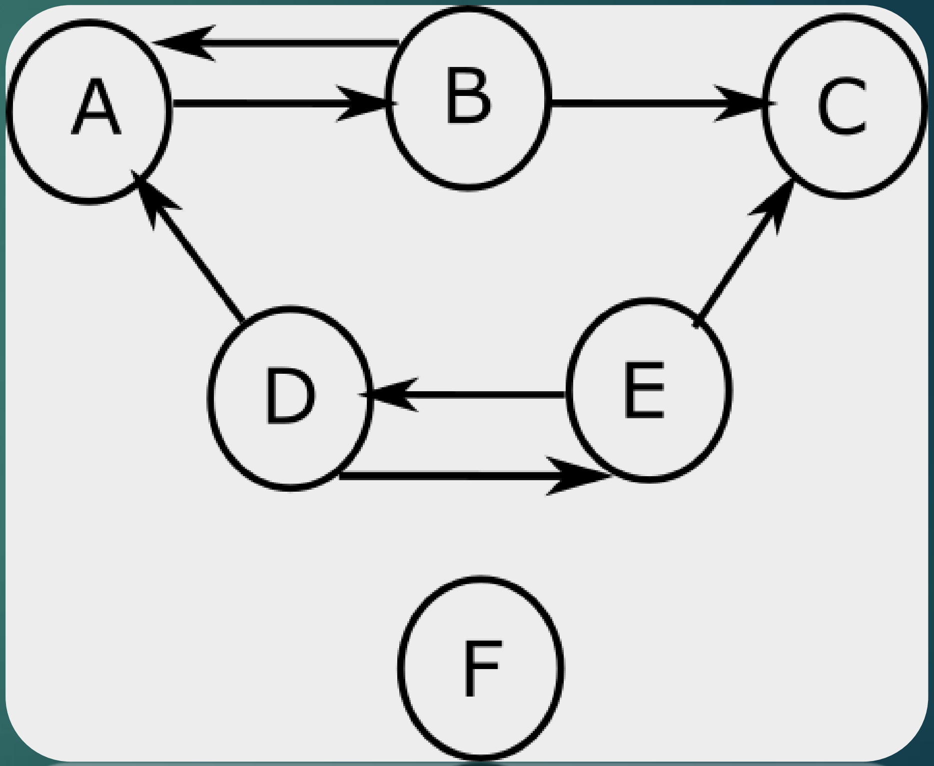
Adjacency Matrix

OUT

	A	B	C	D	E	F
A	0	1	0	0	0	0
B	1	0	1	0	0	0
C	0	0	0	0	0	0
D	1	0	0	0	1	0
E	0	0	1	1	0	0
F	0	0	0	0	0	0

IN

For a directed graph, the adjacency matrix may not be symmetric along the diagonal.

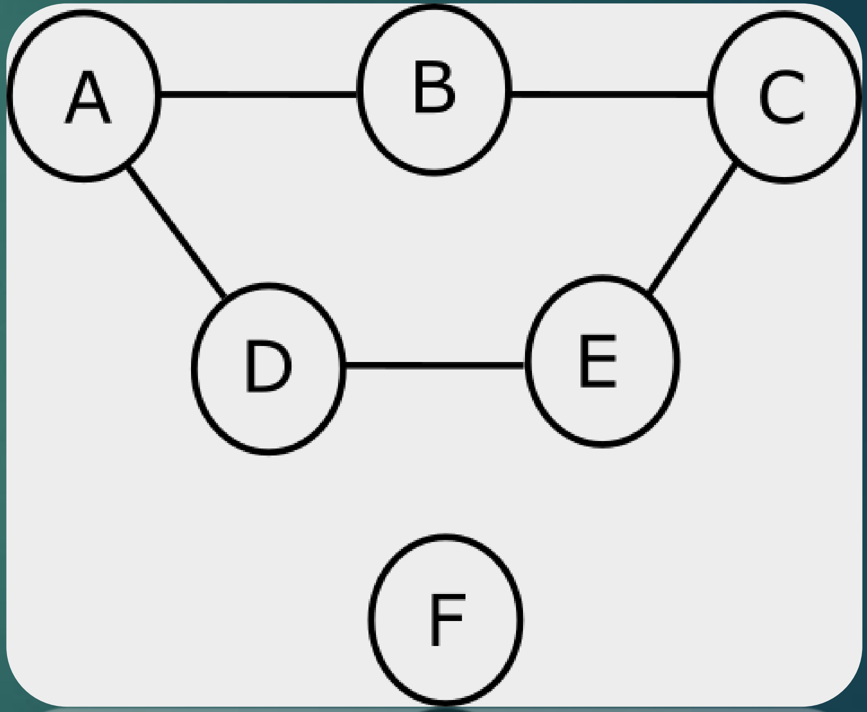


Sparse Matrix

- Sparse matrix has most of its elements valued 0.
- If there is a large number of vertices(nodes) but few connections, the adjacency matrix associated with that graph will be a sparse matrix.
- To avoid keeping all values of a sparse matrix most of which are 0, we can use an adjacency list.

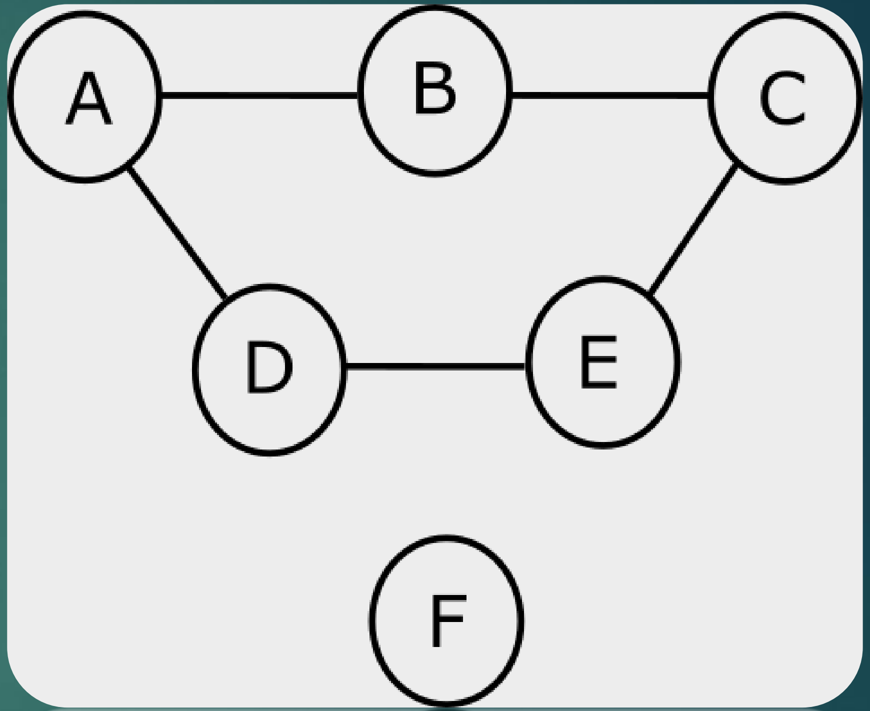
Adjacency List

Vertex(node)	Neighbors
A	B,D
B	A,C
C	B,E
D	A,E
E	D,C
F	-



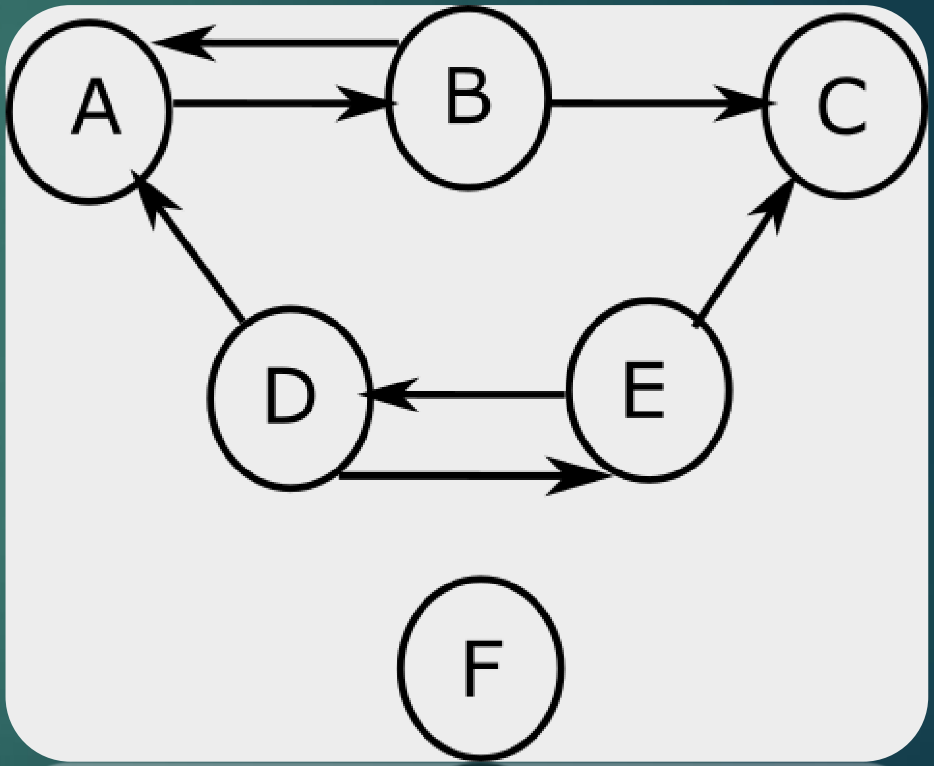
Edge List

Edge
A,B
A,D
B,C
C,E
D,E



Edge List (Directed Graph)

Edge
A,B
B,A
B,C
E,C
D,A
D,E
E,D



Incidence Matrix

	e1	e2	e3	e4	e5
A	1	0	0	0	1
B	1	1	0	0	0
C	0	1	1	0	0
D	0	0	0	1	1
E	0	0	1	1	0
F	0	0	0	0	0

Not suitable for large graphs.