Digraphs

CME4422 Graph Theory

Directed Graph

- A directed graph or <u>digraph</u> has directed edges.
- A directed edge is called an <u>arc</u>.

PERT

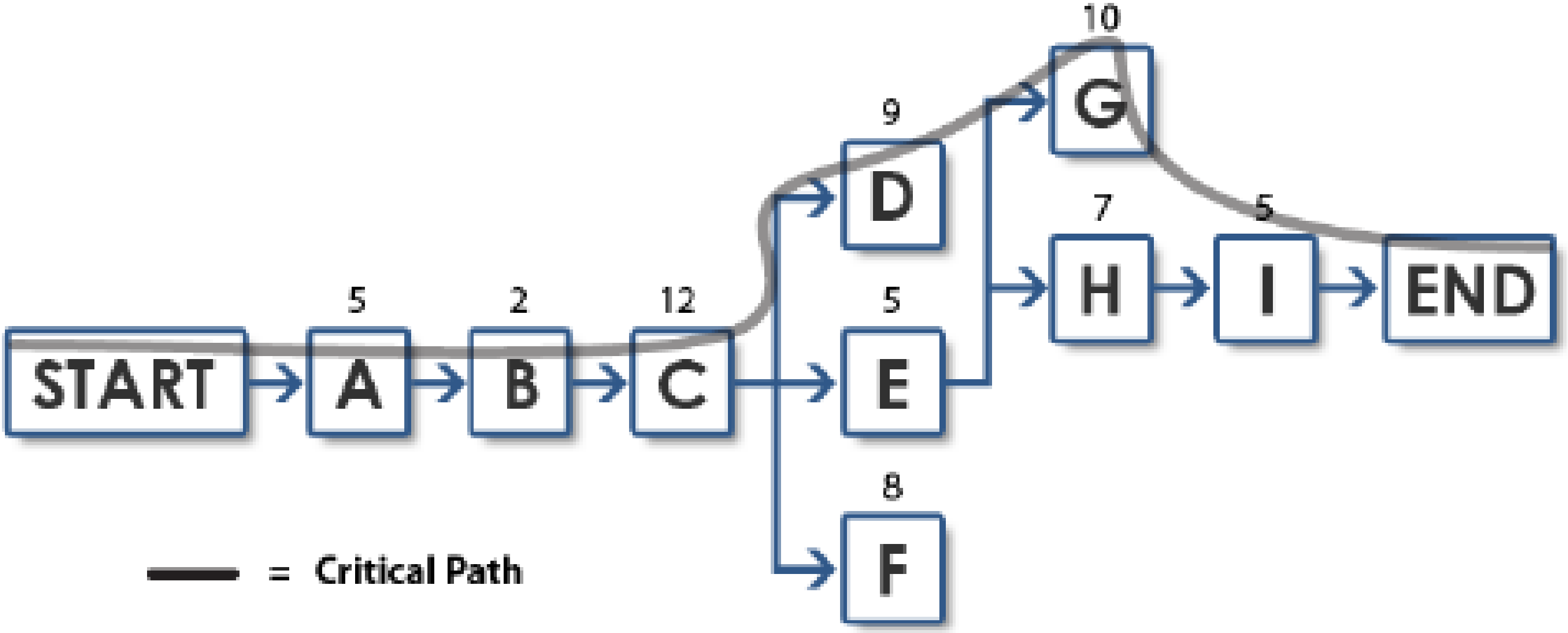
- Activity Network is a weighted digraph used to represent durations of tasks for a project.
- All tasks(arcs) must be traversed to complete the project.
- The longest path from A to L is called the critical path, because any delay on this path delays the entire project.
- Programme Evaluation and Review Technique is used to find the longest path.

Critical Path Finding Algorithm

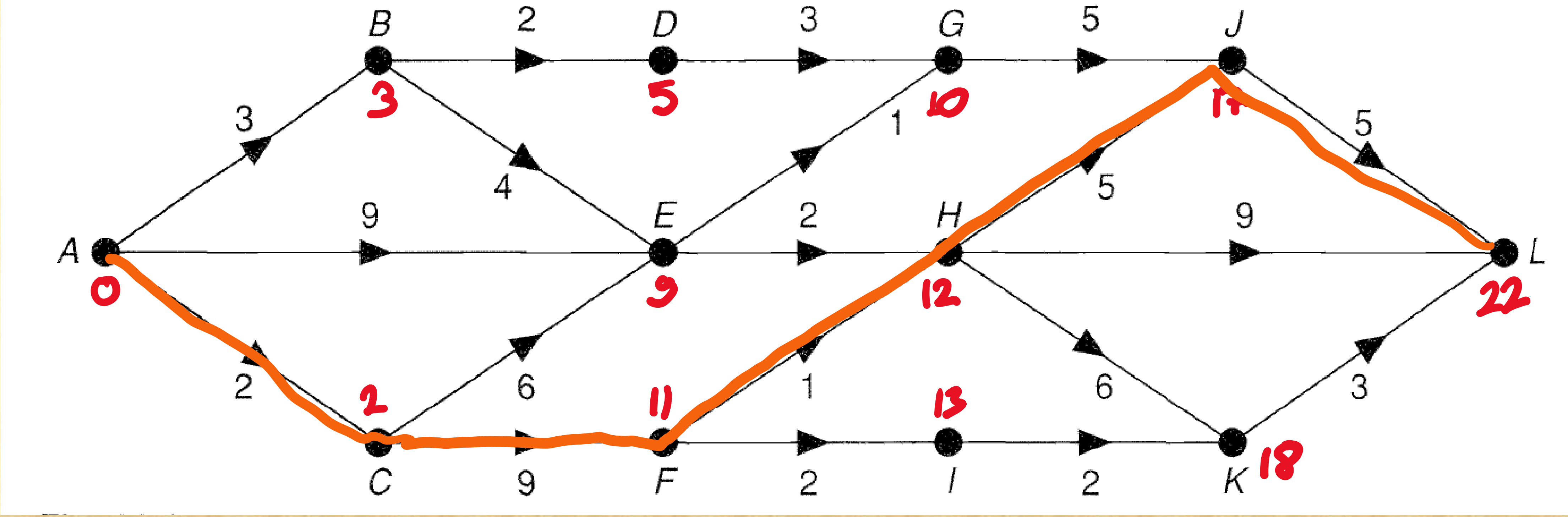
- The edges are tasks and vertices are milestones.
- A is labeled 0.
- Label all nodes by the max. time it takes to reach that node from node A.
- In this way, we find the longest path; the shortest time needed to complete the project.

Critical Path Example

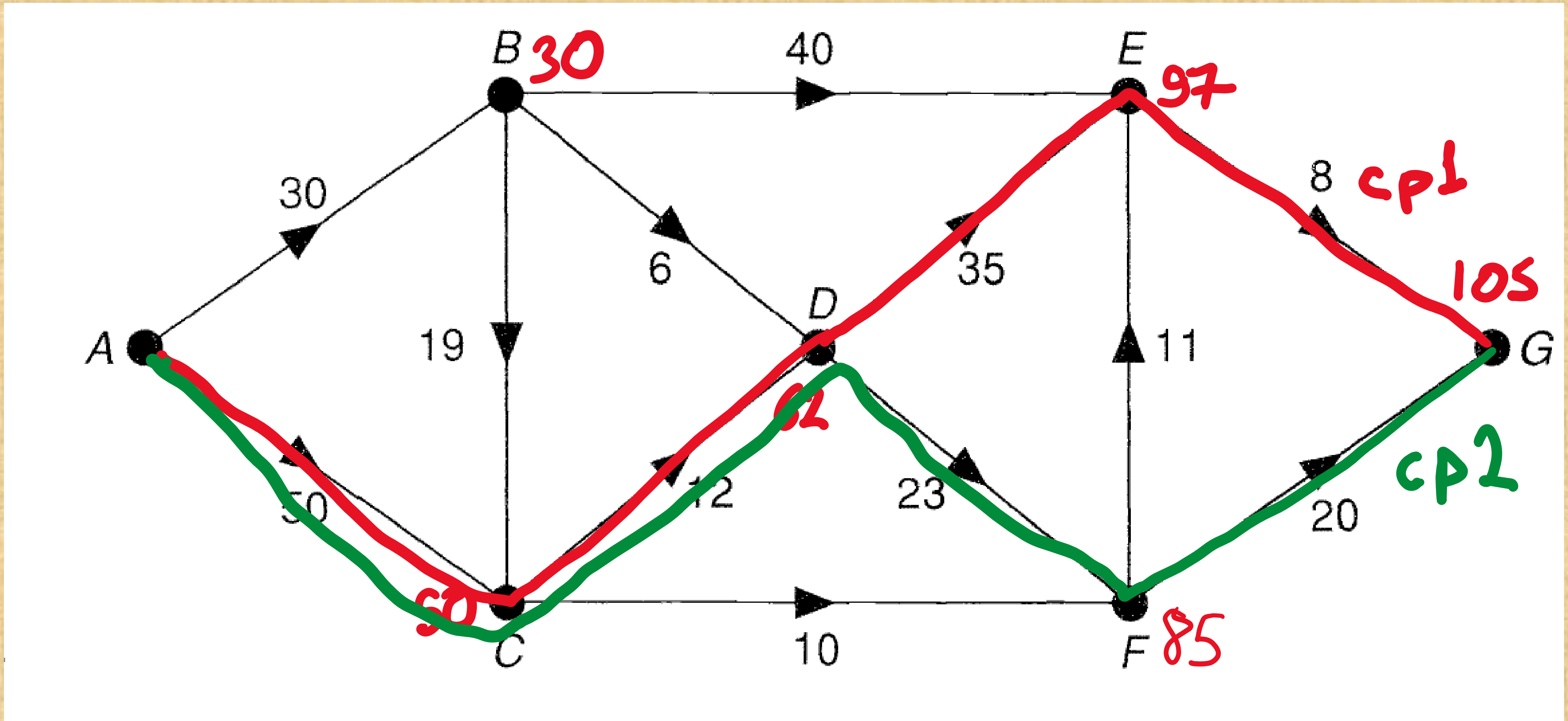
A. Excavate	5 days
B. Foundation	2 days
C. Frame	12 days
D. Electrical	9 days
E. Roof	5 days
F. Masonry	8 days
G. Interior	10 days
H. Exterior	7 days
I. Landscape	5 days



PERT Chart

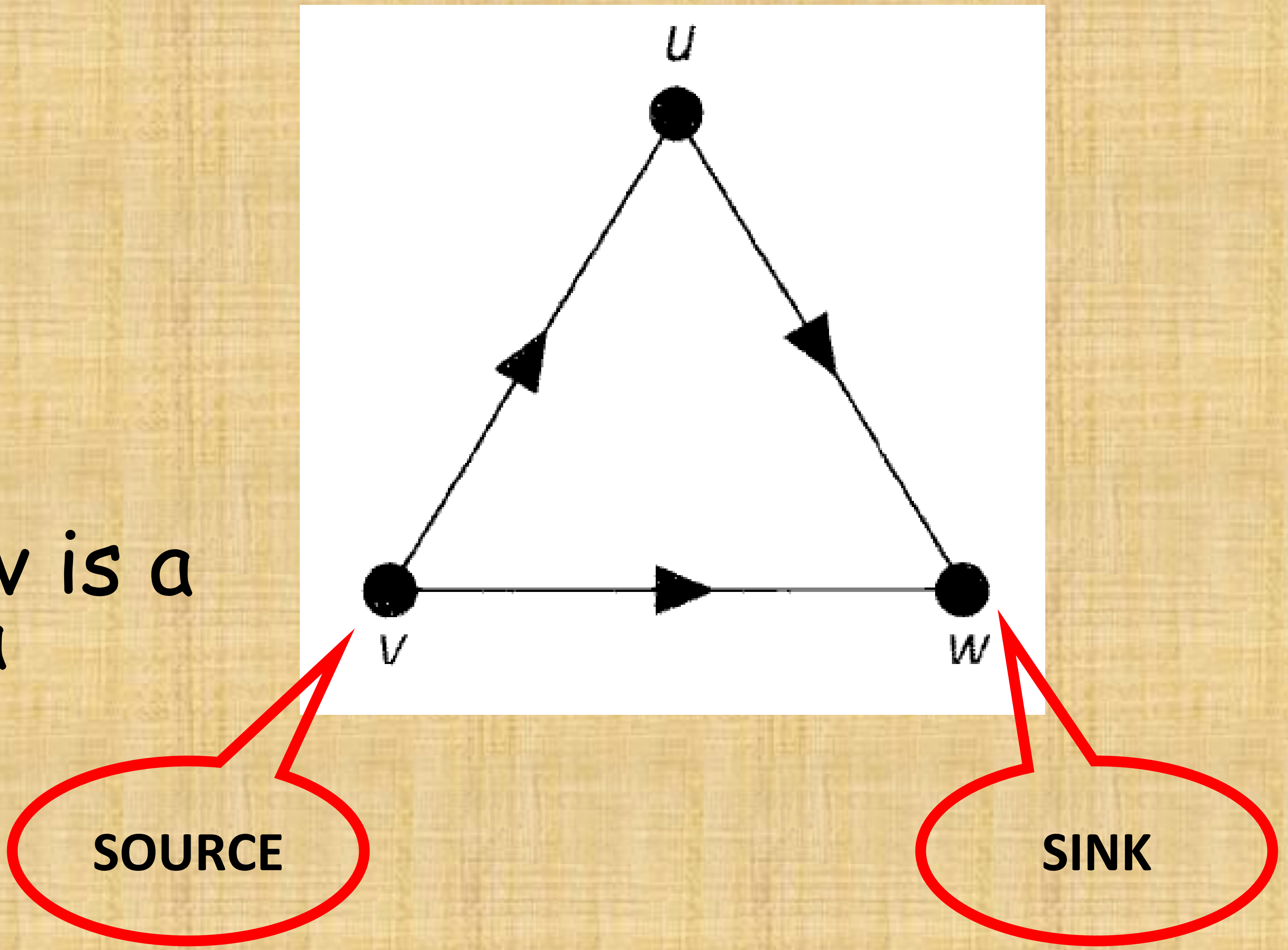


Another Example: Two Critical Paths



Sinks and Sources

- A vertex with no outgoing arcs is called a sink.
- A vertex with no incoming arcs is called a source.
- In the example, v is a source and w is a sink.



Handshaking Dilemma

- Sum of in and out degrees of all vertices are equal.
- Every outgoing arc of one vertex adds to the ingoing arc total of another vertex.

Example

- Verify the handshaking dilemma for the tournament below:

$\Sigma = 10$

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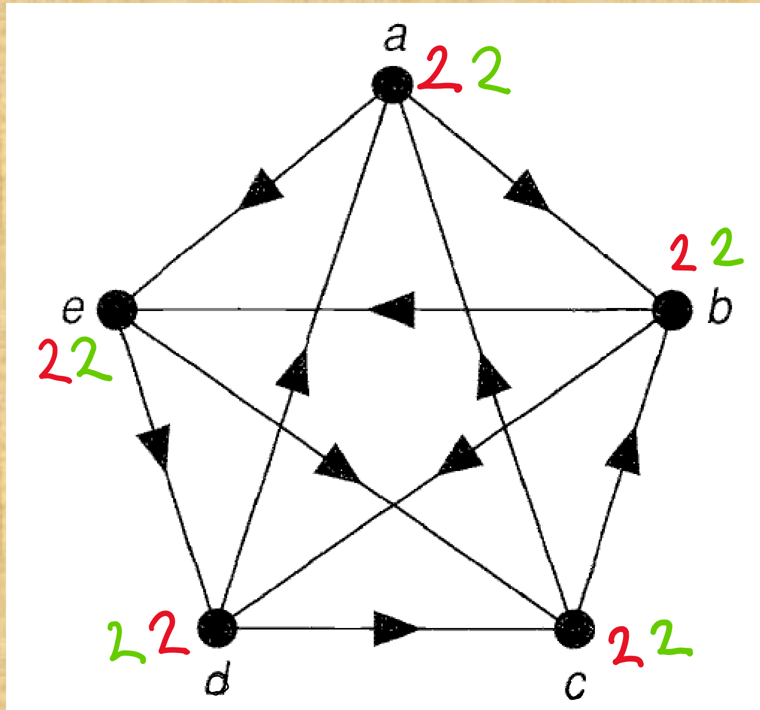
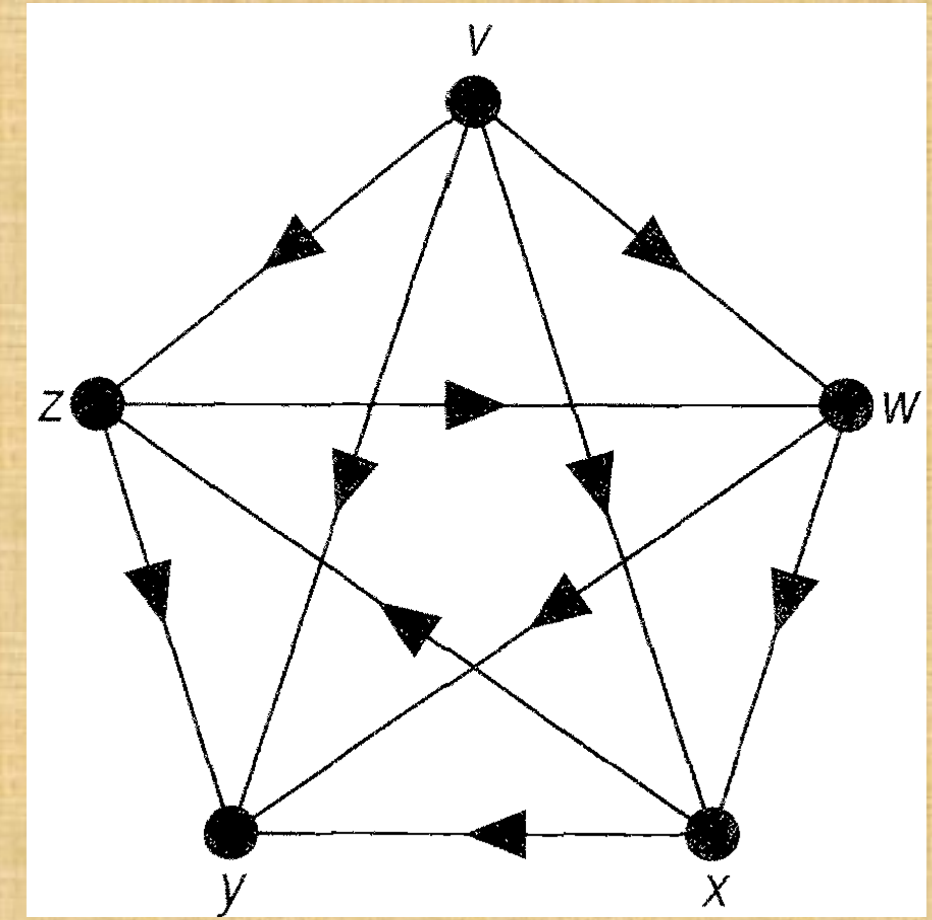


Tournament

- It's a digraph where any two vertices are connected by exactly one arc.
- Outgoing arcs represent a victory for the vertex, incoming arcs mean defeat. No ties.

Example: The winner is ?

- v has 4 wins and no losses. It's also a source.
- w,x and z have 2 wins and 2 losses.
- y has no wins and 4 losses. It's a sink.

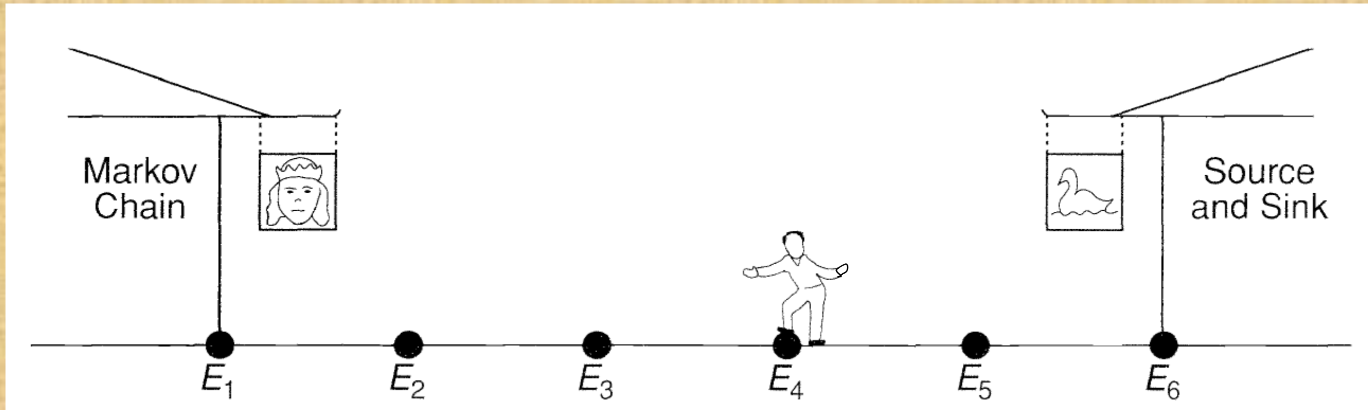


Example

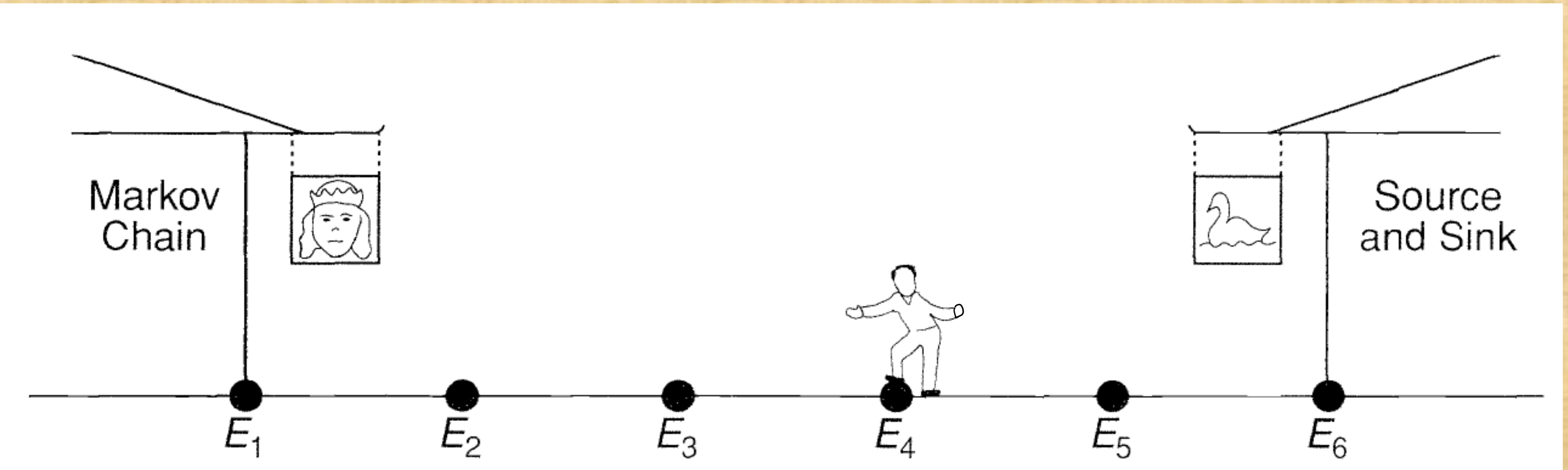
- Prove that in a tournament there can be no more than one source.
- If there are n competitors and two sources then each source must have an outgoing node to every other vertex (n-1). But one source can't have an outgoing vertex to the other source because both of them can't win.

Markov Chains

- Consider this example. A child is initially at position E4. We describe this by the vector x=[0,0,0,1,0,0]. If s/he enters a candy store, s/he stays there.



- The child can:
  - Move left with probability 1/2
  - Move right with probability 1/3
  - Stay at where s/he is with probability 1/6.



- These probability of the child's location after one minute may be represented by a transition vector [0,0,1/2,1/6,1/3,0].
- After two minutes, we have:

->[0,1/4,1/6,13/36,1/9,1/9]

- It's difficult to find the child's location after k minutes.
- We can build a transition matrix P whose rows are the probability vectors.

Transition Matrix P

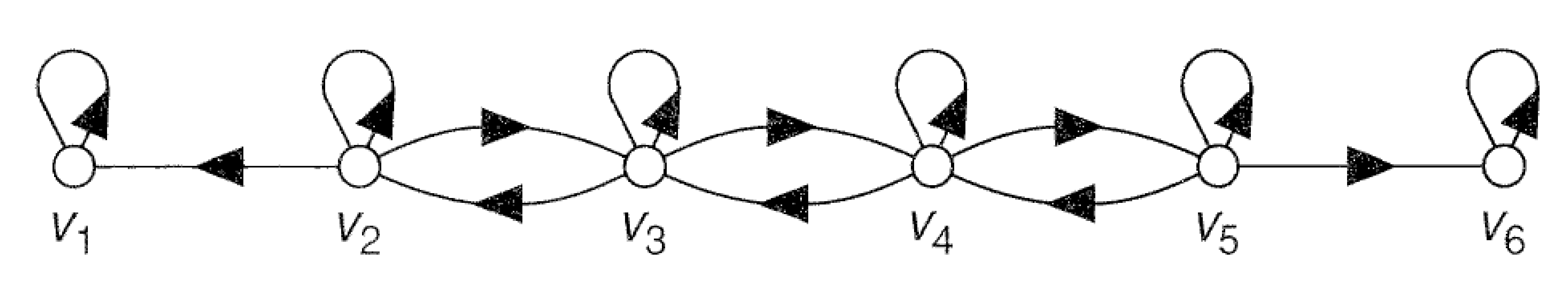
- To find the probability vector after k minutes multiply x by P^k.
- A Markov Chain is composed of an initial state vector x and a transition matrix P.

States

$$
\begin{array}{c}
\begin{array}{cccccc} E1 & E2 & E3 & E4 & E5 & E6 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Markov Chains as Digraphs

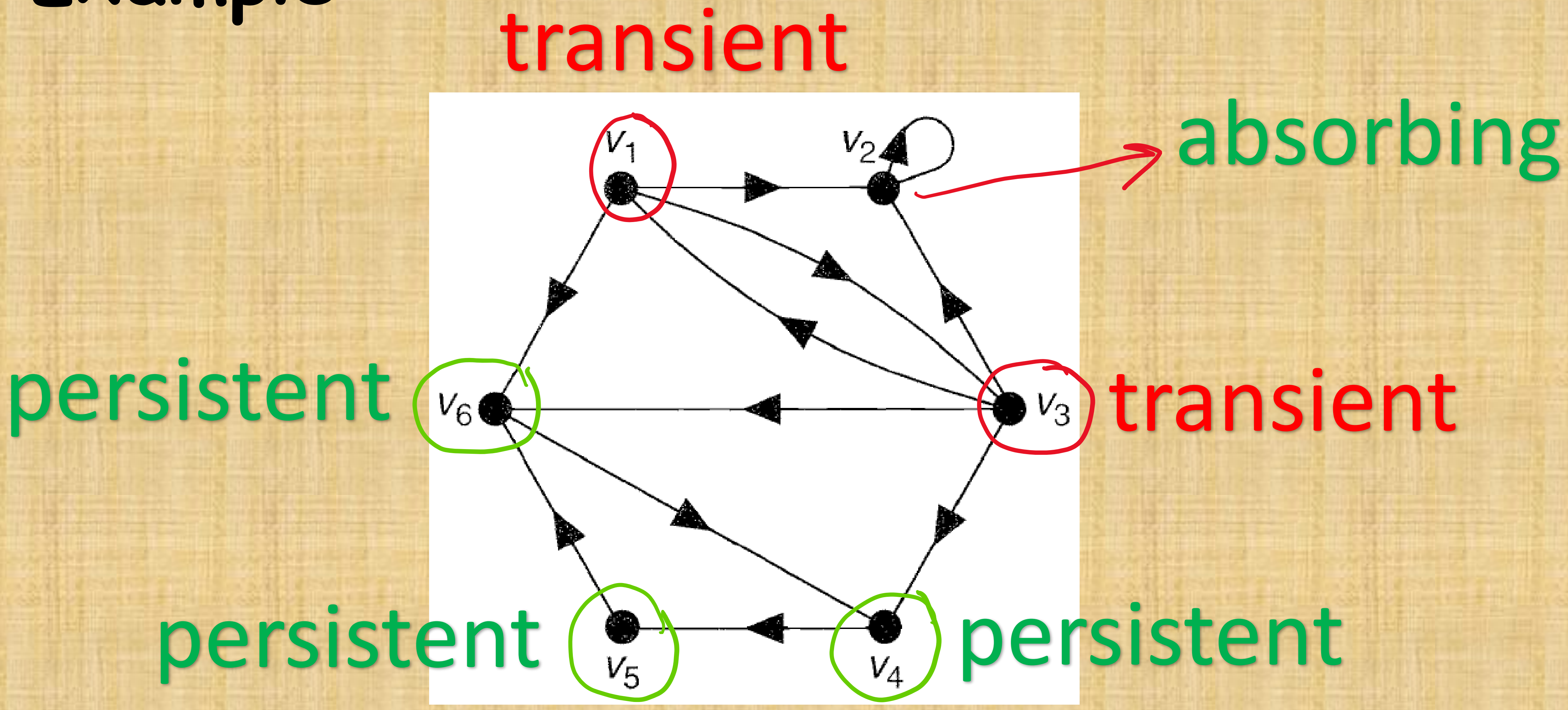
- We can represent a Markov Chain by a digraph:



Some definitions

- Irreducible chain: The digraph is strongly connected, there is a path drom any node to any other node. Our example is NOT an irreducible chain.
- Persistent State: If there is a path v_i to v_j then there is a path v_j to v_i. The probability of returning back to v_i is 1.
- Absorbing state: No arc to any other state.

Example



transient
absorbing
v_1
v_2
persistent
v_6
v_3
transient
persistent
v_5
v_4
persistent