# CS112 DISCUSSION 1 OCT 4 FALL 2013

1. Elementary Math

(1) If $a_1 = \frac{1}{2}$, $a_{n+1} = a_n + \frac{1}{n^2+n}$, what is $a_n$?

$a_{n+1} - a_n = \frac{1}{n} - \frac{1}{n+1}$

$a_n - a_{n-1} = \frac{1}{n-1} - \frac{1}{n}$

$a_{n-1} - a_{n-2} = \frac{1}{n-2} - \frac{1}{n-1}$

$\dots$

$a_3 - a_2 = \frac{1}{2} - \frac{1}{3}$

$a_2 - a_1 = 1 - \frac{1}{2}$

So: $a_n - a_1 = 1 - \frac{1}{n}$, $a_n = \frac{3}{2} - \frac{1}{n}$

(2) If $a_1 = 1$, $a_{n+1} = 2a_n + 3$, what is $a_n$?

$a_{n+1} = 2a_n + 3$

$a_{n+1} + 3 = 2(a_n + 3)$

$\frac{a_{n+1}+3}{a_n+3} = 2$

$\frac{a_n+3}{a_{n-1}+3} = 2$

$\dots$

$\frac{a_2+3}{a_1+3} = 2$

So $a_n + 3 = (a_1 + 3) \cdot 2^{n-1}$, $a_n = 2^{n+1} - 3$

(3) Given a sequence $\frac{1}{2}, \frac{3}{4}, \frac{5}{7}, ..., \frac{2n-1}{n^2}$, calculate the sum of the first $n$ terms.

$S_n = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \cdots + \frac{2n-1}{2^n}$

$\frac{1}{2}S_n = \frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \cdots + \frac{2n-3}{2^n} + \frac{2n-1}{2^{n+1}}$

$S_n - \frac{1}{2}S_n = \frac{1}{2}S_n = \frac{1}{2} + (\frac{2}{4} + \frac{2}{8} + \frac{2}{16} + \cdots + \frac{2}{2^2}) - \frac{2n-1}{2^{n+1}}$

$\frac{1}{2}S_n = \frac{1}{2} + (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}) - \frac{2n-1}{2^{n+1}}$

$\frac{1}{2}S_n = \frac{1}{2} + \frac{\frac{1}{2}[1-(\frac{1}{2})^{n-1}]}{1-\frac{1}{2}} - \frac{2n-1}{2^{n+1}}$

$\frac{1}{2}S_n = \frac{3}{2} - (\frac{1}{2})^{n-1} - \frac{2n-1}{2^{n+1}}$

So: $S_n = 3 - \frac{2n+3}{2^n}$

(4) Given a sequence $a, 2a^2, 3a^3, ..., na^n, ...$, if $a > 0$ and $a \neq 1$, calculate the sum of the first $n$ terms.

$S_n = a + 2a^2 + 3a^3 + \cdots + na^n$

$aS_n = a^2 + 2a^3 + 2a^4 + \cdots + na^{n+1}$

$(1-a)S_n = a + a^2 + a^3 + \cdots + a^n - na^{n+1}$

$S_N = \frac{a(1-a^n)}{1-a} - na^{n+1}$

2. Permutation and Combination

(1) There are 43 students. We choose 5 from them. If at least one from student A, B, and C has to be chosen, then how many ways are there to chooose the 5 students?

$\binom{43}{5} - \binom{40}{5}$

(2) There are 8 students and 4 teachers standing in a line. If the students can sit next to each other but none of the teachers can sit next to each other, how many ways are there to line them up?

$8!\binom{7}{4}4!$

(3) There are 9 people standing in a line. If people A has to stand in front of B and they don't need necessarily stand next to each other, how many ways are there to line them up?

$\frac{9!}{2}$

3. Transform

(1) Properties of Laplace transforms

(i) $a_1 f_1(t) + a_2 f_2(t) \iff a_1 F_1(s) + a_2 F_2(s)$

(ii) $f(\alpha t) \iff \frac{1}{\alpha} F(\frac{s}{\alpha}), a > 0$

(iii) $f(t-\alpha) \iff e^{-\alpha s} F(s)$

(iv) $f^{'}(t) \iff sF(s) - f(0)$

$f^{"}(t) \iff s^2 F(s) - sf(0) - f^{'}(0)$

$\dots$

$f^{(n)}(t) \iff s^n F(s) - \sum_{m=0}^{n-1} s^{n-1-m} f^{(m)}(0)$

(v) $f(t) \otimes g(t) \equiv \int_t^0 f(t-x) g(x) dx \iff F(s)G(s)$

(2) Properties of z transforms

(i) $a_1 f_1(k) + a_2 f_2(k) \iff a_1 F_1(z) + a_2 F_2(z)$

(ii) $f(k-m) \iff z^{-m} F(z) + \sum_{k=0}^{m-1} f(k-m) z^{-k}$

(iii) $f(\alpha^k) \iff F(\frac{z}{\alpha}), a \neq 0$

(iv) $f_n \otimes g_n \equiv \sum_{k=0}^{n} f_{n-k} g_k \iff G(z)F(z)$

(3) Practice of z transform

(i) $x(n) = \frac{1}{n}$

(4) using laplace transform to solve the differential equations

$f" - 3f^{'} + 2f(t) = 2e^{3x}$, $f(0) = 0$, $f^{'} = 0$

if $f(t) \iff F(s)$, $e^{3x} \iff \frac{1}{s-3}$ then

$s^2F(s) - sf(0) - f^{'}(0) - 3(sF(s) - f(0)) + 2F(s) = \frac{2}{s-3}$

$s^2F(s) - 3sF(s) + 2F(s) = \frac{2}{s-3}$

$F(s) = \frac{2}{(s-1)(s-2)(s-3)} = \frac{1}{s-1} - \frac{2}{s-2} + \frac{1}{s-3}$

So, $f(x) = e^x - 2e^{2x} + e^{3x}$