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I Preliminaries

1.1 Review of Elementary Math

Sequence $s_1, s_2, \ldots, s_n, \ldots$

Series is a sequence S_n where $S_n = s_1 + s_2 + \ldots + s_n = \sum_{i=1}^{n} s_i$

Arithmetic Progression

A sequence $\{s_i\}$ is an arithmetic progression if there is d such that $s_{n+1} - s_n = d \ \forall n$

Thus, we obtain

$s_2 = s_1 + d$

$s_3 = s_2 + d = s_1 + 2d$

$\ldots$

$s_n = s_{n-1} + d = s_1 + (n-1)d$

In order to calculate the series $S_n = \sum_{i=1}^{n} s_i$ we proceed as follows:

$S_n = s_1 + s_2 + \ldots + s_n$

Reversing it,

$S_n = s_n + s_{n-1} + \ldots + s_1$

By adding the above two expressions, we get

$$\begin{aligned} 2S_n &= (s_1 + s_n) + (s_2 + s_{n-1}) + \ldots + (s_n + s_1) \\ &= [s_1 + s_1 + (n-1)d] + [(s_1+d) + s_1 + (n-2)d] + \ldots + [s_1 + (n-1)d + s_1] \\ &= n[2s_1 + (n-1)d] \end{aligned}$$

or
$$S_n = \frac{n(s_1 + s_n)}{2}$$

Geometric Progression

A geometric progression is a sequence of numbers $s_1, s_2, \ldots, s_n, \ldots$ such that $s_n = r\, s_{n-1}$; r is called the *common ratio*. Thus, we obtain

$s_2 = rs_1$

$s_3 = rs_2 = r^2 s_1$

...

$s_n = r^{n-1} s_1$

In order to derive a closed formula for series $S_n = \sum_{i=1}^{n} s_i$ we proceed as follows:

$S_n = s_1 + s_2 + \dots + s_n$

$rS_n = rs_1 + rs_2 + \dots + rs_n = s_2 + s_3 + \dots + s_{n+1}$

Taking the difference, we get

$$S_n - rS_n = s_1 - s_{n+1}$$

$$S_n = \frac{s_1 - s_{n+1}}{1-r} = \frac{s_1(1-r^n)}{1-r}$$

For $|r| < 1$, as $n \to \infty$, the sum converges to $S_\infty = \dfrac{s_1}{1-r}$

Also assuming $s_1 = 1$, we get $S_\infty = \dfrac{s_1}{1-r} \Rightarrow S_\infty = \dfrac{1}{1-r}$

Power and Binomial Series

Taylor series

A Taylor series is a series expansion of a function about a point $x = x_0$ (sometimes written instead $x = \alpha$).

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)(x-\alpha)^2}{2!} + \dots + \frac{f^{(n-1)}(\alpha)(x-\alpha)^{n-1}}{(n-1)!} + R_n$$

where R_n is a remainder term known as the Lagrange remainder, which is given by

$R_n = \dfrac{f^{(n)}(x^*)(x-\alpha)^n}{n!}$ for some $x^* \in (\alpha, x)$.

If $\alpha = 0$, the expansion is known as a *Maclaurin series*.

Example 1.1

$f(x)=\frac{1}{1-x}$; using the Taylor series above, we get

$$\frac{1}{1-x}=1+x+x^2+...+x^n+... \text{ (geometric series)}$$

For $|x|<1$, the series converges as $n\to\infty$.

For $|x|>1$, the series diverges.

Similarly, we can derive:

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...=\sum_{i=0}^{\infty}\frac{x^i}{i!}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...$$

The last three series (e^x, $\sin x$, $\cos x$) converge for all values of x.

Binomial Theorem

$$(\alpha+b)^p=\alpha^p+\frac{p}{1!}\alpha^{p-1}b+\frac{p(p-1)}{2!}\alpha^{p-2}b^2+...+b^p$$

$$=\sum_{n=0}^{p}\frac{p!}{(p-n)!n!}\alpha^{p-n}b^n \qquad \text{for every positive integer } p$$

where the numbers $\binom{p}{n}=\frac{p!}{n!(p-n)!}$, $n=0,1,\ldots,p$ are referred to as binomial coefficients.

Example 1.2 B*inomial series*

$$(1+x)^p=1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+...+x^p$$

$$=\sum_{n=0}^{p}\frac{p!}{(p-n)!n!}x^n=\sum_{n=0}^{p}\binom{p}{n}x^n$$

Multinomial Theorem

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r): \\ n_1+n_2+\ldots+n_r=n}} \binom{n}{n_1\, n_2 \ldots n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

where the numbers $\binom{n}{n_1\, n_2 \ldots n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$ are referred to as multinomial coefficients.

Leibnitz's Formula

$$\frac{\partial^n}{\partial x^n} f(x) g(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

1.2 Permutations and Combinations

The number of ways to arrange N objects in N positions, is

$$N \times N-1 \times N-2 \times \ldots \times 2 \times 1 \equiv N!$$

The number of ways of arranging $K \leq N$ objects chosen (from N objects) in K positions, without replacement, is

$$N \times N-1 \times N-2 \times \ldots \times (N-(K-1))$$

$$= N(N-1)(N-2)\ldots(N-K+1) = \frac{N!}{(N-K)!}$$

With replacement allowed, the number of permutations is N^k

Combination is the number of ways of selecting k objects out of N objects.

It is the same as number of ways of placing k out of N in k positions, divided by the number of ways k objects can be arranged in k positions, thus:

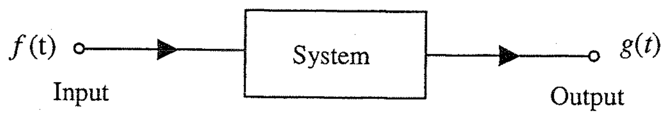
$$\binom{N}{k} = \frac{\frac{N!}{(N-k)!}}{k!} = \frac{N!}{(N-k)!k!}$$

1.3 Transforms

Why

1) Naturally arise in formulation
2) Simplify calculations

1.3.1 Linear Time-Invariant Systems (LTI)



Without loss of generality, let f and g be functions of time, that is, $f(t)$ and $g(t)$

Notation: $f(t) \to g(t)$ represents an input/output relationship.

Linearity

If $f_1(t) \to g_1(t)$, and $f_2(t) \to g_2(t)$, and $a\, f_1(t) + b\, f_2(t) \to a\, g_1(t) + b\, g_2(t)$, for a, b constants with respect to time, the system is said to be linear.

Time Invariance

If $f(t) \to g(t)$ and $f(t+\tau) \to g(t+\tau)$, the system is said to be time-invariant.

Question

Which forms of $f(t)$ passes through with no change in form except for scalar (time-invariant) multiplier, i.e. $f(t) \to H f(t)$

Functions satisfying the above relation are called **characteristic functions**, **invariants**, or **eigenfunctions** for the LTI systems.

Let $f_e(t)$ be the eigenfunction of a linear, time-invariant system. We can prove that $f_e(t) = e^{st}$, where s, in general, is a complex variable.

Proof

Let $f_e(t) = e^{st} \rightarrow g_e(t)$

By the linearity property, $\alpha f_e(t) \rightarrow \alpha g_e(t)$

Let τ be a constant time interval; thus $\alpha = e^{s\tau}$ is also a constant. Then,

$$e^{s\tau} f_e(t) \rightarrow e^{s\tau} g_e(t)$$

$$e^{s\tau} e^{st} \rightarrow e^{s\tau} g_e(t)$$

$$e^{s(\tau+t)} \rightarrow e^{s\tau} g_e(t) \tag{1.3.1}$$

By the time-invariant property,

$$f_e(t+\tau) \rightarrow g_e(t+\tau)$$

$$e^{s(t+\tau)} \rightarrow g_e(t+\tau) \tag{1.3.2}$$

The (unique) solution to (1.3.1) and (1.3.2) is

$$g_e(t) = He^{st}$$

where H is independent of t.

Thus, we have proved that the eigenfunction of a linear, time-invariant system is of the form $f_e(t) = e^{st}$. Thus,

$$e^{st} \rightarrow H(s)e^{st} \tag{1.3.3}$$

$H(s)$ is called the **system** or **transfer function**.

1.3.2 Transform Method of Analysis

If we were able to decompose a function $f(t)$ into complex exponentials whose sum or integral contributes to $g(t)$ as above, we have a (simpler) method of calculating $g(t)$ given $f(t)$:

1. Decompose *input* into sum of exponentials
2. Compute response of each exponential as in (1.3.3)
3. Reconstitute *output* from sum of exponentials

1.3.3 Discrete Functions of Time

$f(t) \equiv f(t = nT)$, where n is in $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and T is a constant interval.

Notation: $f(nT) \equiv f_n$

The input/output relationship can be expressed as $f_n \to g_n$

Linearity

$$\alpha f_n^{(1)} + b f_n^{(2)} \to \alpha g_n^{(1)} + b g_n^{(2)}$$

Time Invariance

$$f_{n+m} \to g_{n+m}$$

Eigenfunctions

$$f_n^{(e)} \equiv e^{st} \equiv e^{snT}$$

Let $z \equiv e^{-sT}$, where z is a complex variable, then $f_n^{(e)} = z^{-n}$

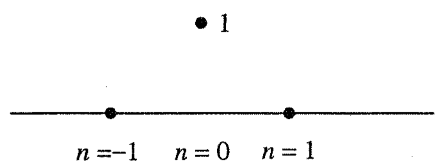
Exercise

Show that $z^{-n} \to H(z) z^{-n}$ (where $H(z)$ is independent of n), thus proving that z^{-n} are indeed eigenfunctions for discrete, linear, time-invariant systems.

Again $H(z)$ is called a *system* or *transfer function* of the LTI system. It gives the output we get when the input is one unit of the exponential. It defines the impact of the system on the exponential function.

Unit Function (Kronecker Delta Function)

$$u_n = \begin{cases} 1, & n = 0 \\ 0, & \text{elsewhere} \end{cases}$$



The input/output relationship is $u_n \to h_n$, where h_n is called the *unit response*.

Using time invariance: $u_{n+m} \to h_{n+m}$

And using linearity: $z^m u_{n+m} \to z^m h_{n+m}$

Multiplying both sides of the relation above by $z^{-n}z^n$ we get:

$$z^{-n}z^n z^m u_{n+m} \to z^{-n}z^n z^m h_{n+m}$$

$$z^{-n}\sum_m z^{n+m}u_{n+m} \to z^{-n}\sum_m z^{n+m}h_{n+m}$$

But, $\sum_m z^{n+m}u_{n+m} = 1$ by definition of u_n, so

$$z^{-n} \to z^{-n}\sum_m z^{n+m}h_{n+m}$$

or

$$z^{-n} \to z^{-n}\sum_k z^k h_k$$

Thus, the transfer function is

$$H(z) = \sum_k z^k h_k$$

$H(z)$ is called the **z-transform** of h_k.

As we see from the above derivation, the system or transfer function can be determined from the unit response function h_n.

It is worth noting that only a single experiment (applying the unit function as input and measuring the resulting output h_n) is sufficient to compute all responses of the system.

1.3.4 z-Transforms

It is also called **characteristic function** or **generating function**.

The z-transform maps a series f_n indexed by an integer value n into a function of a complex variable z.

Consider a sequence f_n, $n = 0, 1, 2, \ldots$

The z-transform of f_n is defined by:

$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

For $z = 1$, we get

$$F(1) = \sum_{n=0}^{\infty} f_n$$

Notation: We denote the relationship between the transform pair as $f_n \Leftrightarrow F(z)$.

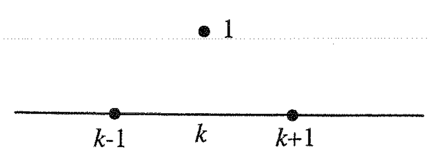
Example 1.3

Let us consider the unit function $u_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

Then, the z-transform of u_n is $F(z) = \sum_{n=0}^{\infty} u_n z^n = 1$

Using the notation we defined earlier, we can write $u_n \Leftrightarrow 1$

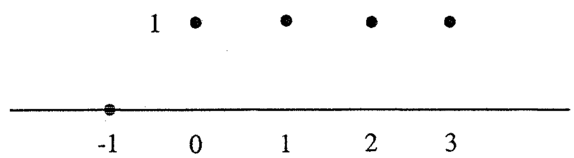
We can show that for a unit function shifted to the right k units, that is, $u_{n-k} = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$



we get $u_{n-k} \Leftrightarrow z^k$

Example 1.4 *The (discrete) unit step function* δ_n

$$\delta_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Then, $\delta_n \Leftrightarrow \sum_{n=0}^{\infty} z^n \delta_n$

$$\delta_n \Leftrightarrow \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots$$

$$\delta_n \Leftrightarrow \frac{1}{1-z}$$

Example 1.5 *Geometric sequence*

$f_n = A\alpha^n$, $n = 0, 1, 2, \ldots$

Then, $A\alpha^n \Leftrightarrow A\sum_{n=0}^{\infty} \alpha^n z^n$

$$A\alpha^n \Leftrightarrow \frac{A}{1-\alpha z}$$

Convolution: Important z -Transform Property

Consider the sequences f_n and g_n for $n = 0, 1, 2, \ldots$

Denoting the convolution operator by $\otimes$ the *convolution* is defined as

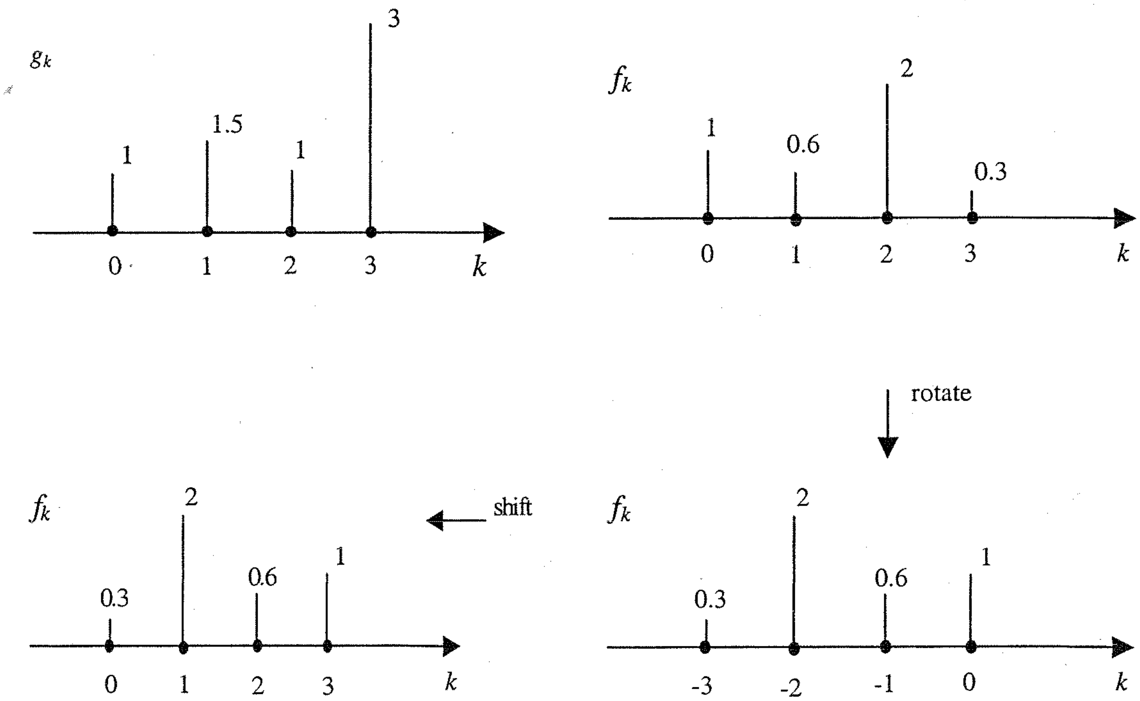
$$f_n \otimes g_n = \sum_{k=0}^{n} f_{n-k} g_k$$

z-transforms are useful in simplifying convolution calculations. Procedure:

1. Calculate $F(z)$, $G(z)$
2. Perform inverse transformation of the product $F(z)G(z)$

Example 1.6

$f_3 \otimes g_3 = [(0.3 \times 1) + (2 \times 1.5) + (0.6 \times 1) + (1 \times 3)] = 0.3 + 3.0 + 0.6 + 3.0 = 6.9$

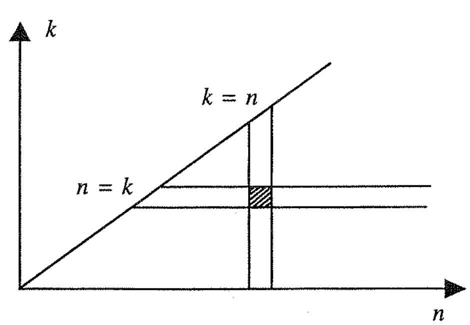


We can show that convolution operation corresponds to a product operation in the transfer domain.

$$f_n \otimes g_n \Leftrightarrow \sum_{n=0}^{\infty} (f_n \otimes g_n) z^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_{n-k} g_k z^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} f_{n-k} g_k z^n = \sum_{k=0}^{\infty} g_k z^k \sum_{n=k}^{\infty} f_{n-k} z^{n-k}$$

$$f_n \otimes g_n \Leftrightarrow G(z)F(z)$$



Other z-Transform Properties

$$F(z)=\sum_{k=0}^{\infty} f_k z^k = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \ldots$$

1. $F(0)=f_0$

2. $F'(1)=\sum_{k=0}^{\infty} k f_k$

3. To show that $f_n = \frac{1}{n!}\frac{\partial^n F(z)}{\partial z^n}\Big|_{z=0}$, we proceed as follows:

$$\frac{\partial F(z)}{\partial z} = f_1 + 2 f_2 z + 3 f_3 z^2 + \ldots$$

$$\frac{\partial F(z)}{\partial z}\Big|_{z=0} = f_1$$

$$\frac{\partial^2 F(z)}{\partial z^2} = 2 f_2 + 6 f_3 z + \ldots$$

$$\frac{1}{2}\frac{\partial^2 F(z)}{\partial z^2}\Big|_{z=0} = f_2$$

$$\frac{\partial^3 F(z)}{\partial z^3} = 6 f_3 + \ldots$$

$$\frac{1}{6}\frac{\partial^3 F(z)}{\partial z^3}\Big|_{z=0} = f_3$$

4. $n f_n \Leftrightarrow \sum_{n=0}^{\infty} n f_n z^n = z\sum_{n=0}^{\infty} n f_n z^{n-1}$

$$= z\frac{\partial}{\partial z}\sum_{n=0}^{\infty} f_n z^n = z\frac{\partial F(z)}{\partial z}, \text{ where } f_n \Leftrightarrow F(z)$$

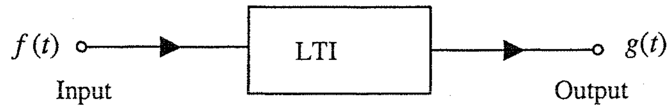
5. $\frac{1}{n!} \Leftrightarrow \sum_{n=0}^{\infty}\frac{1}{n!} z^n = e^z$

6. $n \Leftrightarrow \sum_{n=1}^{\infty} n z^n = z\sum_{n=1}^{\infty} n z^{n-1}$

$$= z\frac{\partial}{\partial z}\sum_{n=1}^{\infty} z^n = z\frac{\partial}{\partial z}\frac{1}{1-z}$$

$$n \Leftrightarrow \frac{z}{(1-z)^2}$$

1.3.5 Continuous, Linear, Time-Invariant Systems



For continuous time, we have shown that the eigenfunctions are of the form e^{st} and $e^{st} \to H(s)e^{st}$

Consider now $u_0(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$ where $\int_{-\infty}^{\infty} u_0(t)dt = 1$

This is the **Dirac delta function** or the **unit impulse function** (we shall see it in more detail later).

Consider applying a unit impulse function as input to the linear, time-invariant system. Let the output be the function $h(t)$ usually called the **unit impulse response function**. Thus, $u_0(t) \to h(t)$.

Using time invariance, we can write
$u_0(t+\tau) \to h(t+\tau)$

And using linearity, we get

$e^{-s\tau}u_0(t+\tau) \to e^{-s\tau}h(t+\tau)$

also,

$e^{-st}e^{st}e^{-s\tau}u_0(t+\tau) \to e^{-st}e^{st}e^{-s\tau}h(t+\tau)$

$e^{st}\int_{-\infty}^{\infty} e^{-s(t+\tau)}u_0(t+\tau)d\tau \to e^{st}\int_{-\infty}^{\infty} e^{-s(t+\tau)}h(t+\tau)d\tau$

by $\int_{-\infty}^{\infty} e^{-s(t+\tau)}u_0(t+\tau)d\tau \to e^{-s(t+\tau)}\Big|_{t+\tau=0} = 1$

Thus,

$$e^{st} \to e^{st}\int_{-\infty}^{\infty} e^{-s(t+\tau)}h(t+\tau)d\tau$$

or by changing variables in the integral

$$e^{st} \to e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau$$

Thus

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

This relates the *system transfer function* to the unit *impulse response function.*

The above is also known as the Laplace transform. It transforms the time domain function $h(\tau)$ to the complex domain function $H(s)$.

1.3.6 Laplace Transforms

The Laplace transform maps a continuous function $f(t)$, where $f(t) = 0$ for $t < 0$, into a function of a complex variable s. It is defined as follows:

$$F^*(s) \equiv \int_{0-}^{\infty} f(t) e^{-st} dt$$

0^- means any accumulation at 0 will be included in the integration.

$$F^*(0) = \int_{0^-}^{\infty} f(t) dt$$

Many properties of z-transforms at $z = 1$ are equivalent to those of Laplace transforms at $s = 0$.

Notation: $f(t) \Leftrightarrow F^*(s)$

Laplace transforms have a number of useful properties that make them a very practical tool for calculations that would otherwise be difficult in the time domain!

Example 1.7

Consider the function $f(t) = \begin{cases} Ae^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$

Then,

$$F^*(s) = A\int_0^{\infty} e^{-\alpha t} e^{-st} dt = A\int_0^{\infty} e^{-(\alpha+s)t} dt = A\left[\frac{e^{-(\alpha+s)t}}{-(\alpha+s)}\right]_0^{\infty} = \frac{A}{\alpha+s}$$

Thus,

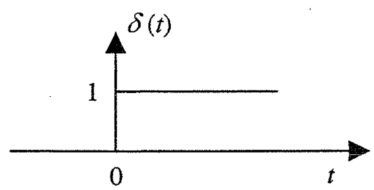
$$Ae^{-at}\delta(t) \Leftrightarrow \frac{A}{\alpha+s}$$

where $\delta(t)$ is the unit step function.

Example 1.8 *Continuous unit step function*

Consider the function $f(t) = \delta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

The Laplace transform of the unit step function is given by $F^*(s) = \int_0^\infty \delta(t) e^{-st} dt$



Using the function of the previous example we view the $\delta(t)$ as $\delta(t) = Ae^{-\alpha t}$ with $A = 1$ and $\alpha = 0$. Then the Laplace transform is $F^*(s) = \frac{1}{s}$

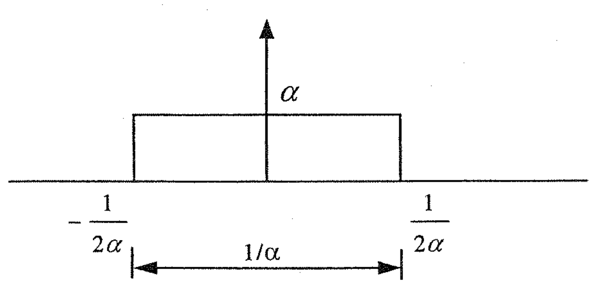
Thus, $\delta(t) \Leftrightarrow \frac{1}{s}$

Example 1.9 *Unit impulse function*

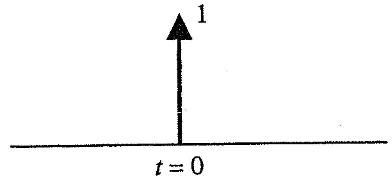
The *Dirac delta function* provides a way for handling discontinuities

Consider $f_\alpha(t) = \begin{cases} \alpha & |t| \leq \frac{1}{2\alpha} \\ 0 & |t| > \frac{1}{2\alpha} \end{cases}$ and $\int_{-\infty}^{\infty} f_\alpha(t) dt = 1$

The unit impulse function $u_0(t)$ is obtained from $f_\alpha(t)$ through $\lim_{\alpha \to \infty} f_\alpha(t)$. Note that, $\int_{-\infty}^{\infty} u_0(t) dt = 1$.

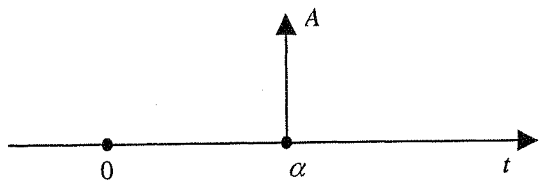


A graphical notation for $u_0(t)$ is shown below:



Time shift impact and amplitude change impact

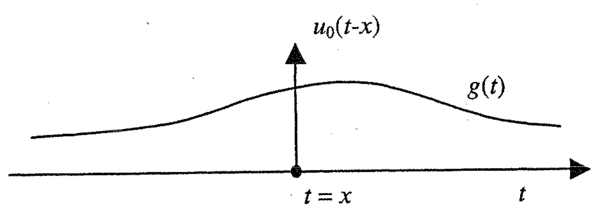
$f(t) = Au_0(t-\alpha)$



Sampling or sifting property

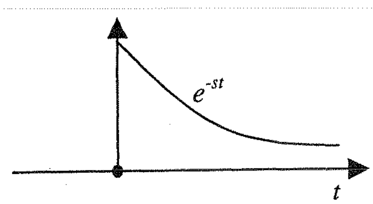
For an arbitrary differentiable function $g(t)$

$$\int_{-\infty}^{\infty} u_0(t-x)g(t)dt = g(x)$$



We can use the sifting property to derive the Laplace transform for $u_0(t)$ as follows:

$$u_0(t) \Leftrightarrow \int_{0-}^{\infty} u_0(t)e^{-st}dt = 1$$



Thus, $u_0(t) \Leftrightarrow 1$

1.3.6.1 Properties of Laplace Transforms

We can show

1. $f(t-\alpha) \Leftrightarrow e^{-\alpha s} F^*(s)$

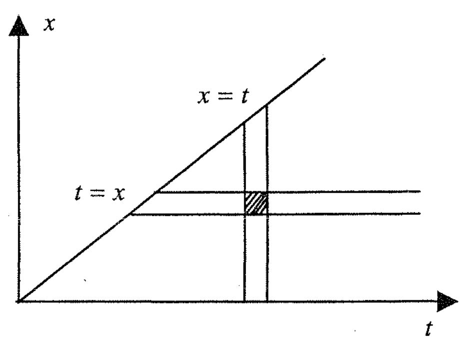
2. $tf(t) \Leftrightarrow -\dfrac{dF^*(s)}{ds}$

3. $t^n f(t) \Leftrightarrow (-1)^n \dfrac{d^n F^*(s)}{ds^n}$

4. $f(t) \otimes g(t) \equiv \int_{0^-}^{t} f(t-x) g(x) dx \Leftrightarrow F^*(s) G^*(s)$ (Convolution)

Proof

$$f(t) \otimes g(t) \Leftrightarrow \int_{t=0^-}^{\infty} \int_{x=0^-}^{t} f(t-x) g(x) dx \, e^{-st} dt$$



By changing variables

$$f(t) \otimes g(t) \Leftrightarrow \int_{x=0^-}^{\infty} \int_{t=x}^{\infty} e^{-st} f(t-x) g(x) \, dt dx$$

$$= \int_{x=0^-}^{\infty} \int_{t=x}^{\infty} e^{-s(t-x)} f(t-x) dt \;\; g(x) e^{-sx} dx$$

$$= \int_{x=0^-}^{\infty} g(x) e^{-sx} dx \int_{t=x}^{\infty} e^{-s(t-x)} f(t-x) dt$$

$$= G^*(s) \int_{\tau=0^-}^{\infty} e^{-s\tau} f(\tau) d\tau$$

$$= G^*(s) F^*(s)$$

1.3.7 Using Transforms to Solve Difference and Differential Equations

Example 1.10

Consider the difference equation

$$2f_n = 3f_{n-1} - f_{n-2} \qquad \text{for } n = 2, 3, \ldots$$

with initial conditions: $f_0 = 1$ and $f_1 = 2$. We want to find f_n that satisfies the above difference equation. We will use z-transform to determine f_n.

Solution

Multiplying both sides by z^n and summing we get

$$2\sum_{n=2}^{\infty} f_n z^n = 3\sum_{n=2}^{\infty} f_{n-1} z^n - \sum_{n=2}^{\infty} f_{n-2} z^n$$

$$2\,[F(z) - f_0 - f_1 z] = 3z\,[F(z) - f_0] - z^2 F(z)$$

$$F(z)\,(2 - 3z + z^2) = f_0\,(2 - 3z) + 2f_1 z$$

Using initial conditions we get

$$F(z) = \frac{2-3z+4z}{z^2-3z+2} = \frac{2+z}{(2-z)(1-z)}$$

But $\quad F(z) = \dfrac{2+z}{(2-z)(1-z)} = \dfrac{A}{2-z} + \dfrac{B}{1-z}$

$$A = [(2-z)F(z)]_{z=2} = \left.\frac{2+z}{1-z}\right|_{z=2} = -4$$

$$B = [(1-z)F(z)]_{z=1} = \left.\frac{2+z}{2-z}\right|_{z=1} = 3$$

Thus

$$F(z) = \frac{3}{1-z} - \frac{4}{2-z}$$

By inspection we can do the inverse z-transform to obtain:

$$f_n = 3(1)^n - 2\left(\frac{1}{2}\right)^n \qquad \text{for } n = 0, 1, 2, \ldots$$

or

$$f_n = 3 - 2\left(\frac{1}{2}\right)^n$$

II Introduction to Probability Theory

Provides mathematical models to analyze random phenomena and random behavior.

Random: Unpredictable or irregular

Example 2.1

Tossing a coin once is a random event because we cannot predict the outcome of a trial.

Statistical Regularity: Applies to behavior about which one can make accurate statements of prediction for a large collection of trials.

Example 2.2

Tossing a coin a large number of times will yield about one-half of the times heads and the other half tails, if the coin is fair. When we say a coin is fair, we mean that when it is flipped, it is equally likely to come up heads or tails.

2.1 Experiments and Their Corresponding Probability Model

Experiment	Model	Example
Set of Outcomes	**Sample Space** Let $S = \{\omega: \omega \text{ corresponds to an outcome}\}$	Toss a die 1 2 3 4 5 6
Set of Results	Set of **Events**, $E = \{A_1, A_2, \ldots\}$ where $A_i \subseteq S$ subset of S	1 2 3 4 5 6 A = [even number]
Relative frequency	**Probability Measure** Mapping from E into real line, with the following properties: (1) $P[A] \geq 0$ (2) $P[S] = 1$ (3) $P[A \cup B] = P[A] + P[B] - P[A \cap B]$	$P[A] = \frac{1}{2}$

2.2 Sample Spaces

Example 2.3

Toss a coin twice;

$S = \{(H, H), (H, T), (T, H), (T, T)\}$

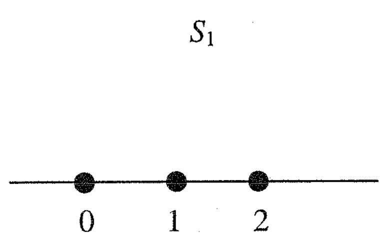
Example 2.4

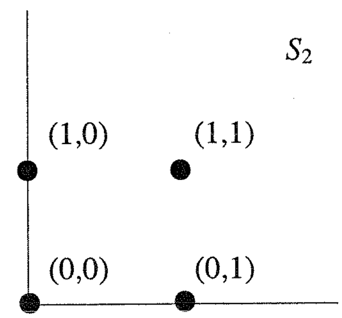
Observe 2 devices at $t = 120$ minutes

Two possible sample spaces:

$S_1 = \{0, 1, 2\}$ (where 0 = both down, 1 = one up, and 2 = both up)

$S_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ ($0 \Rightarrow$ device down, $1 \Rightarrow$ device up)





Example 2.5

Observe n devices at $t = \tau$;

$S = \{(0, 0,\ldots, 0), (0, 0,\ldots, 1), (0, 0,\ldots 1, 0),\ldots\}$

This is an n–dimensional space with 2^n sample points.

Example 2.6

Lifetime of a car;

$S = [0, \infty)$

Example 2.7

Check devices coming out of an assembly line; continue until a defective device is found

$S = \{1, 2, 3,\ldots, \infty\}$

Sample Space Classification

(a) *Finite* (Examples 2.3, 2.4, 2.5)

Outcomes are countable (one-to-one correspondence with natural numbers) and finite.

(b) *Discrete* (Example 2.7)

Outcomes are countable but infinite.

(c) *Continuous* (Example 2.6)

Uncountable or non-denumerable; sample space constitutes a continuum on some interval of the real line.

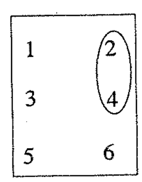
2.3 Events and Their Algebra

An **event** A is a subset of the sample space S, $A \subseteq S$

If $s \in S$, $s \in A$, and s is the outcome of the trial, we say event A has occurred. That is, it is sufficient that the outcome of the trial corresponds to one element of A to say that A has occurred.

Example 2.8

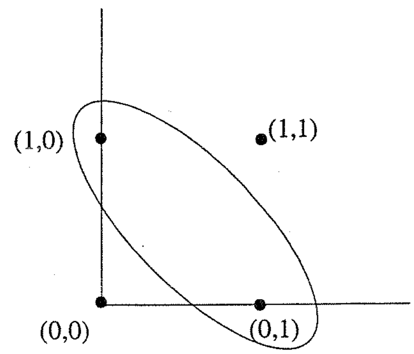
$A \equiv$ [outcome of tossing a die is a power of 2]



Example 2.9

Observe two devices at $t = \tau$

Let 0 means device is down and 1 means device is up.



$S = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is the sample space.

Let us define the event

A = [exactly one device is up]

Then,

A^C = [0 or both devices are up]

and

$A \cup A^C = S$ (S which is also called the **universal** event)

Let

$B = \{(0,0)\} \equiv$ [no devices up];

B is called an **elementary** event since it is made up of exactly one sample element.

$S^C = \varnothing$ where $\varnothing$ is called the **null** event

In this example, there are 16 subsets of S, including $\varnothing$. Each subset corresponds to an event. Note that, here, the cardinality of the set S and its power set is $|S| = 4$ and $|E| = 2^4 = 16$ respectively.

Example 2.10

Let $S = [0, \infty)$ represents the lifetime of a car (Jaguar?)

Let $A \equiv$ [car lasts at least 4 years]$=\{x: x \geq 4\}= [4, \infty)$

Example 2.11 *System with five tape drives is observed at time $t = \tau$*

Let 1 means device is available and 0 means device is in use (occupied).

$S = \{(n_1, n_2, n_3, n_4, n_5): n_i = 0$ or 1 depending on the status of the drive$\}$

$|S| = 2^5 = 32$; and $|E| = 2^{32}$.

Let $E_1 \equiv$ [at least 4 drives are available] = {(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0 1, 1), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1)}

If we write each bit string (event) as binary integers between 0 and 31 (i.e. (0, 0, 0, 0, 0) is S_0 (0, 0, 0, 0, 1) is S_1, … (1, 1, 1, 1, 1) is S_{31}), we can rewrite E_1 as follows:

$E_1 = \{S_{15}, S_{23}, S_{27}, S_{29}, S_{30}, S_{31}\}$

$E_1^C \equiv$ [at most 3 drives are available] = $\{S_0 \rightarrow S_{14}, S_{16} \rightarrow S_{22}, S_{24} \rightarrow S_{26}, S_{28}\}$.

2.3.1 Event Intersection

$A_1 \cap A_2 = \{\omega : \omega \in A_1 \text{ and } \omega \in A_2\}$

$$\bigcap_{i=1}^{n} A_i \equiv A_1 \cap A_2 \cap \ldots \cap A_n$$

In example 2.11 above, set $E_2 \equiv$ [at most 4 drives are available].

Then, $E_2 = \{S_0 \rightarrow S_{30}\}$ and $E_2^C = \{S_{31}\}$.

Recall, $E_1 \equiv$ [at least 4 drives are available]. Then, $E_1 \cap E_2 \equiv$ [exactly 4 drives are available].

We can also define a set $E_3 = E_1 \cap E_2 = \{S_{15}, S_{23}, S_{27}, S_{29}, S_{30}\}$.

2.3.2 Event Union

$A_1 \cup A_2 = \{\omega : \omega \in A_1 \text{ or } \omega \in A_2\}$

$$\bigcup_{i=1}^{n} A_i \equiv A_1 \cup A_2 \cup \ldots \cup A_n$$

In example 2.11 above, set $E_4 \equiv$ [tape drive 1 is available]. Then, $E_4 = \{S_{16} \rightarrow S_{31}\}$.

Recall, $E_1 = \{S_{15}, S_{23}, S_{27}, S_{29}, S_{30}, S_{31}\}$. We can define set $E_5 = E_1 \cup E_4 = \{S_{15} \rightarrow S_{31}\}$

Then, $E_5 \equiv$ [drive 1 or at least 4 drives are available].

2.3.3 Cardinality

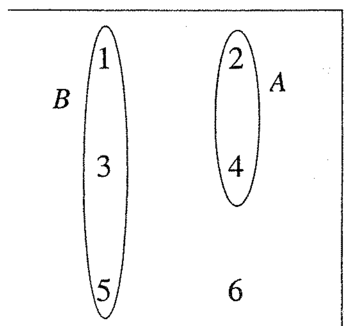
It is the number of elements in an event.

Note that $|E_5| \leq |E_1| + |E_4|$ in the above case.

2.3.4 Mutual Exclusion

Events A and B are **mutually exclusive** if they cannot occur in the same trial. That is, for such events $A \cap B = \varnothing$.

Example 2.12 *Toss a die*



$A \equiv$ [power of $2 > 1$]

$B \equiv$ [odd]

$A \cap B = \varnothing$.

Extension of Definition

Events $A_1, A_2, \ldots, A_n$ are mutually exclusive if and only if (iff) $A_i \cap A_j = \varnothing, \forall\ i \neq j$.

2.3.5 Exhaustive Events

Events A and B are **exhaustive** iff $A \cup B = S$.

Also, $\bigcup_{i=1}^{n} A_i = S \Leftrightarrow \{A_i\}$ is a set of exhaustive events.

2.3.6 Sample Space Partition

Let a set of events A_i, $i = 1, 2, \ldots, n$ be a set of mutually exclusive and exhaustive (M.E events. We say that A_i, $i = 1, 2, \ldots, n$ define a **partition** of the sample space.

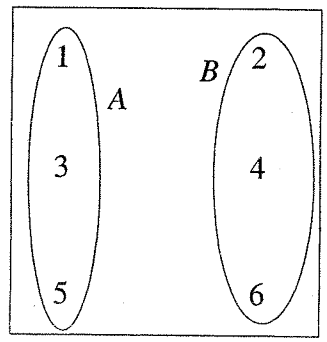
Example 2.13

Consider once again tossing a die.

$A \equiv$ [odd]

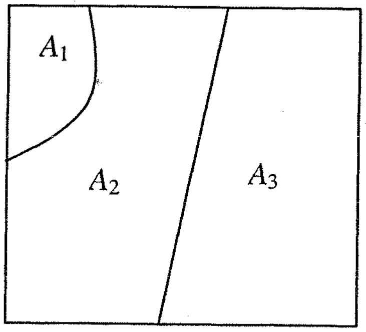
$B \equiv$ [even]

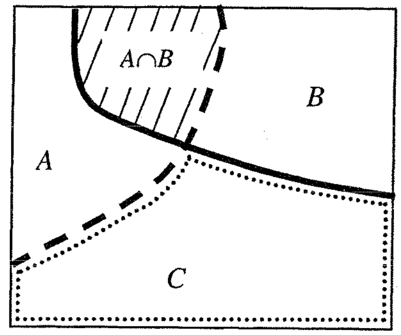
A, B partition the sample space S



Venn Diagrams

A sometimes useful representation of (S, E), the sample space and the set of events defir on it.

<table>
<tr>
<td>

Partition</td>
<td>
</td>
</tr>
<tr>
<td>$A_1 \cup A_2 \cup A_3 = S$
$A_1 \cap A_2 = \varnothing$
$A_1 \cap A_3 = \varnothing$
$A_2 \cap A_3 = \varnothing$</td>
<td>$A \cap S = A$
$A \cup S = S$
$A \cup A^C = S$
$(A^C)^C = A$
$A \cup B \cup C = S$
$A \cap C = \varnothing \rightarrow A, C$ are M.E.
$B \cap C = \varnothing \rightarrow B, C$ are M.E.
$A \cap B \neq \varnothing$</td>
</tr>
</table>

Tree Representation
Useful for sequential sample spaces.

Example 2.14

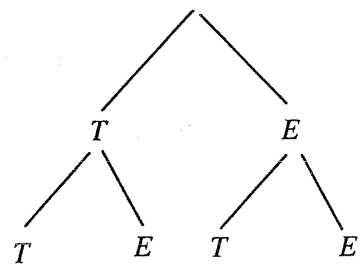
If C, then T else E

Let the trial be two executions of the if statement above.

The outcome of each trial is a pair of the following sample space:

$S = \{(T, T), (T, E), (E, T), (E, E)\}$

Each *event* is a path from the root to some leaf in the tree below:



Coordinate System
Useful in a 2-dimensional system representation

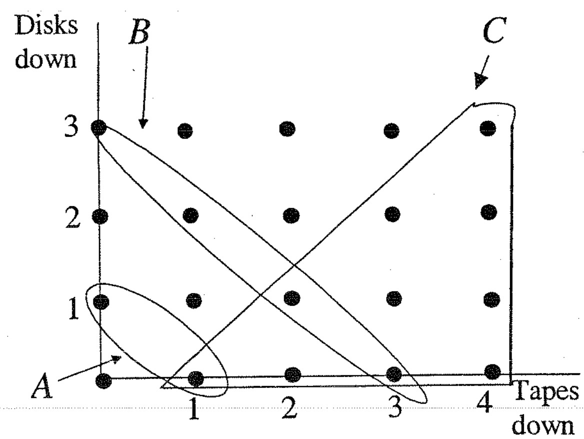
Example 2.15 *4 tape drives, 3 disk drives*

At time $t = \tau$, how many of each type of drive are down?

$A \equiv$ [one device is down]

$B \equiv$ [3 devices are down]

$C \equiv$ [more tape drives are down than disk drives].



2.4 Probability Measure

Mapping from set of events E into [0,1].

$P[A]$ represents the relative likelihood that A would occur. $P[A]$ is assigned to A based on intuition, assumed, or based on experiments.

Axioms (or Conditions)

(i) $P[\mathrm{A}] \geq 0$: nonnegative real number

(ii) $P[\mathrm{S}] = 1$: normalization effect $\Rightarrow P[\mathrm{A}] \leq 1$

(iii) $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ in general

$P[A \cup B] = P[A] + P[B]$ when A and B are mutually exclusive.

Corollaries

a) $P\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} P[A_i]$ for $\{A_i\}$ mutually exclusive events.

b) $P[S] = 1 = P[A \cup A^C] = P[A] + P[A^C]$

$\Rightarrow P[A] = 1 - P[A^C]$

Can similarly show that $P[\varnothing] = 0$.

Method to Calculate $P[A]$

$S = \{s_i\text{: } s_i \text{ is an outcome}\}$

Recall that s_i are called elementary events. $P[s_i]$ is assigned by intuition or experimentation and $\{s_i\}$ is a set of mutually exclusive events.

$$P[A] = \sum_{i=1}^{n} P[s_i] \text{ for } s_i \in A.$$

In many cases $P[s_i] = p$, i.e. all outcomes have same probability p. The remaining task is to count the number of outcomes that belong to some event A_j. This leads to the need for combinatorics.

Example 2.16

A box contains 75 good Integrated Circuits (IC) and 25 bad integrated circuits. Twelve ar pulled at random. The condition of any chip is independent from the condition of all othe chips. All chips in the box are equally likely to be pulled. What is P [at least one IC in the 1: pulled is defective]?

Solution

Sample Space $S = \{$all possible combinations of 12 chips$\}$

$$|S| = \binom{100}{12}$$

Let $E \equiv$ [at least one IC of those pulled is defective]

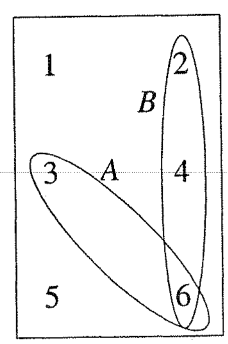
Then, $E^C \equiv$ [no IC pulled is defective]

$$|E^C| = \binom{75}{12}$$

$$P[E^C] = \frac{\binom{75}{12}}{\binom{100}{12}}$$

$P\,[E] = 1 - P\,[E^C]$

Example 2.17 *Event union and intersection*



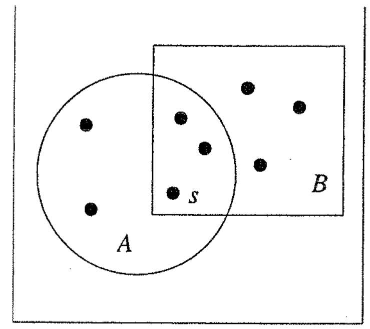
Toss a die.

Let $A \equiv$ [divisible by 3], and

$B \equiv$ [even].

$P\,[A \cup B] = P\,[A] + P\,[B] - P\,[A \cap B] = 1/3 + 1/2 - 1/6 = 2/3.$

2.5 Conditional Probabilities



Let $P\,[s \mid B] \equiv P$ [outcome corresponding to s occurs given that event B has occurred]

$$P[s \mid B] = \begin{cases} \dfrac{P[s]}{P[B]}, & s \in B \\ 0, & s \notin B \end{cases}$$

$$P[A \mid B] = \sum_{s \in A \cap B} P[s \mid B]$$

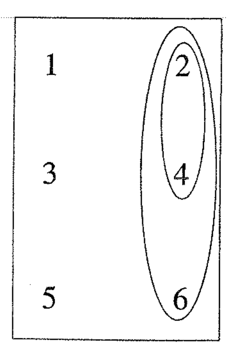
or

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]}$$

Knowledge that B occurred changes the effective sample space.

Division by $P\,[B]$ in $P[s \mid B] = \dfrac{P[s]}{P[B]}$ is necessary for normalization, i.e. $\sum_{s \in B} P[s \mid B] = 1$.

Example 2.18 *Toss a die once*



$$P[\text{power of two} > 1 \mid \text{even}] = \frac{P[\text{power of two} > 1 \text{ and even}]}{P[\text{even}]} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Example 2.19 *Box of 5000 IC's*

1000 from manufacturer X

4000 from manufacturer Y

10% of chips from X are defective

5% of chips from Y are defective

What is P [a chip picked at random was from X | defective]?

Solution

Let A be the event [picked chip was from X]

and B be the event [defective chip]. Then,

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{\frac{100}{5000}}{\frac{300}{5000}} = \frac{1}{3}$$

2.6 Independent Events

Given that B occurred, P [A] may increase, decrease, or remain the same.

Definition

A and B are independent iff $P[A|B] = P[A]$.

From the definition of conditional probability, it follows that:

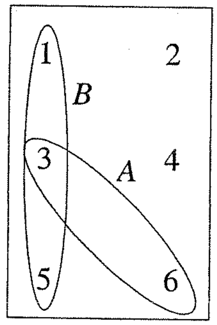
$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = P[A]$$

$\Rightarrow P[A \cap B] = P[A]\, P[B]$ for independent events A, B.

Notes

1. Independence is *not* transitive; i.e. if A and B are independent events and B and C are independent, it does *not* follow that A and C are independent. They may or may not be.
2. $A_1, A_2, \ldots, A_n$ are mutually independent events **iff all subsets of all sizes** are mutually independent.

Example 2.20 *Toss a fair die*



Let

$A \equiv$ [outcome is divisible by 3], and

$B \equiv$ [outcome is odd]

$$P[\text{divisible by } 3 \mid \text{odd}] = \frac{P[A \cap B]}{P[B]} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

By $P[A] = \frac{1}{3} \Rightarrow A$, B are independent.

Example 2.21

Toss 2 fair dice. Each outcome of the possible 36 outcomes is equally likely.

1,1	2,1	3,1	4,1	5,1	6,1
1,2	2,2	3,2	4,2	5,2	6,2
1,3	2,3	3,3	4,3	5,3	6,3
1,4	2,4	3,4	4,4	5,4	6,4
1,5	2,5	3,5	4,5	5,5	6,5
1,6	2,6	3,6	4,6	5,6	6,6

Note

[first = 4] reduces the effective sample space to 6 possible outcomes that are all equally likely.

Consider the following events:

$A \equiv$ [sum of two dice results = 6]

$B \equiv$ [first dice result = 4]

$C \equiv$ [sum of two dice results = 7]

Determine whether A, B are independent events

Determine whether B, C are independent events

$$P[\text{sum} = 6 \mid \text{first} = 4] = \frac{P[\text{sum} = 6 \text{ and first} = 4]}{P[\text{first} = 4]} = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$$

P [sum = 6] = 5/36.

Thus, [sum = 6], [first = 4] are not independent.

$$P[\text{sum} = 7 \mid \text{first} = 4] = \frac{P[\text{sum} = 7 \text{ and first} = 4]}{P[\text{first} = 4]} = \frac{1}{6}\left(= \frac{\frac{1}{36}}{\frac{1}{6}}\right)$$

P [sum = 7] = 1/6.

Thus, [sum = 7], [first = 4] are independent events!

What is the difference? When [sum = 6] is the event of interest, the first die result has an impact. For example P [sum = 6 | first = 6] = 0.

This is not the case for [sum = 7]; P [sum = 7 | first outcome = x] is the same for all x.

Example 2.22

Given an urn with 4 distinct balls. Pick 1 ball at random; balls are equally likely to be picked.

Let $E = \{1, 2\}$, $F = \{1, 3\}$, $G = \{1, 4\}$.

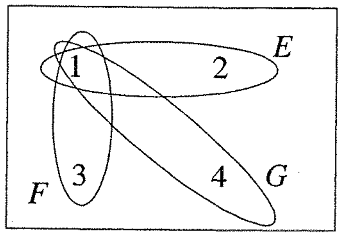
Consider the probabilities of the intersections below:

$P[E \cap F] = P[\{1\}] = 1/4 = P[E]\,P[F]$

$P[F \cap G] = P[\{1\}] = 1/4 = P[F]\,P[G]$

$P[E \cap G] = P[\{1\}] = 1/4 = P[E]\,P[G]$

$P[E \cap F \cap G] = P[\{1\}] = 1/4 \neq P[E]\,P[F]\,P[G]$



The events E, F, G are called **pairwise** independent. They are **not** a set of **mutually** independent events.

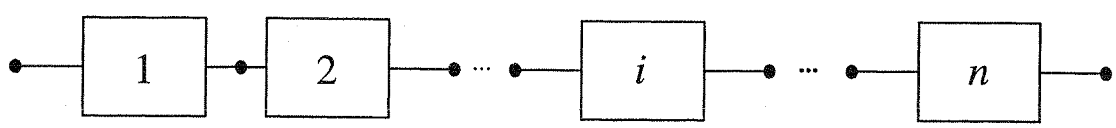
Example 2.23 *Reliability of series/parallel system structures*

Series $\rightarrow$ if any component fails, the system fails

Parallel $\rightarrow$ all components must fail for the system to fail

Assume component failures are mutually independent events.

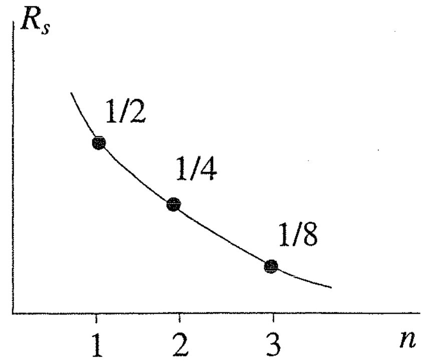
Series System



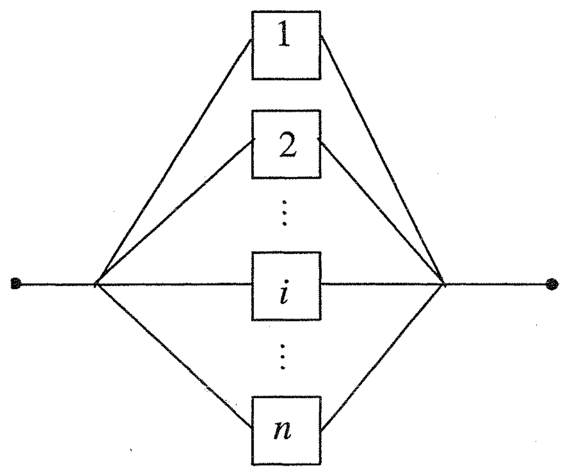
Let $P[A_i] = P$ [component i is functioning]

Define Series system Reliability as: $R_S = \prod_{i=1}^{n} P[A_i] \equiv P$ [series system is functional at some instance in time] since $R_S = P[A_1 \cap A_2 \cap \ldots \cap A_n]$.

R_s can reduce fast as n grows as shown:



Parallel System



$A_i \equiv$ [component i is functioning]

$A_i^C \equiv$ [component i fails]

Define Parallel system Reliability as: P [parallel system is functioning at instance in time] $\equiv$

R_p

$R_p = 1 - P$ [parallel system is not functioning] $= 1 - P\left[A_1^C \cap A_2^C \cap \ldots \cap A_n^C\right] = 1 - \prod_{i=1}^{n} P\left[A_i^C\right]$

As n increases, R_p approaches 1 as shown: