CS180 Winter 2011

Homework 3

The following homework is due Wednesday, January 26 at the beginning of lecture.

When submitting your homework, please include your name at the top of each page. If you submit multiple pages, please staple them together. We also ask that you do something to indicate which name is your last name on the first page, such as underlining it.

Please provide complete arguments and time complexity analysis for all solutions, unless otherwise stated.

1. Given a graph G, the Line Graph G' of G is a graph where all edges in G are vertices in G'. Two vertices in G' (which were edges in G) have an edge between them iff their edges in G are both incident to a common vertex.
   Prove or disprove: The line graph of a bipartite graph is a bipartite graph.

2. For an undirected connected graph $G = (V, E)$, an Euler[1] Tour is an ordering $t_1, t_2, ...t_m$ of edges such that any two consecutive edges in the ordering share a vertex and each edge appears once in the ordering. That is, it is a path that travels across each *edge* exactly once. An Euler Tour does *not* need to begin and end at the same vertex. Furthermore, one does not exist for every graph, and a graph may have multiple valid Euler tours.

   (a) An Euler Tour always exists for a graph where every vertex has even degree. Give an efficient algorithm to find an Euler Tour in a graph where every vertex has even degree.

   (b) What if there are some vertices with odd degree? There is an Euler Tour for some, but not all, such graphs. Under what conditions will a graph have an Euler Tour? Give a brief justification for your answer (you do not need to give a formal proof).

   (c) Give an algorithm that, given a graph G for which there exists an Euler Tour, will find an Euler Tour for that graph. If your solution requires only a small change from (a), you may simply state the difference.

3. In a graph, a Hamiltonian Path is a path that visits each vertex exactly once. Not every graph has a Hamiltonian Path. Give a linear-time algorithm to determine if a given DAG G has a Hamiltonian Path.

4. In class, we discussed the Couples Matching problem, which is as follows: we have n women, numbered $w_1 \dots w_n$ and n men, $m_1 \dots m_n$. We are going to create some number of (woman, man) pairs, subject to the following. Each woman w_i has a preference for some man in the list, and can only be matched up with that man or no one. However, m_i can be involved in a match if and only if w_i is involved in a (possibly different) matching. Our goal is to make as many matches as possible, while respecting these constraints.

   The solution proposed was as follows. Remove from consideration any man m_i who is not preferred by any women under consideration. For each such man removed, we must also remove the woman w_i (with the same number). We repeat this as long as there is such a man. We showed that, when we are done with this process, we have a matching.

[1] Euler is pronounced "Oiler" and not "U-ler"

Prove that, if at any step, we have multiple men who are not preferred, that the order in which we remove them does not matter; we will get the same solution.