Linear Algebra:
The Algebra of Vectors and Matrices
(and Scalars)

Vector spaces

Matrix algebra

Coordinate systems

Affine transformations

Vectors

N-tuple of scalar elements

$$\mathbf{v} = (x_1, x_2, \ldots, x_n), \quad x_i \in \Re$$

Vector:
Bold lower-case

Scalar:
Italic lower-case

Vectors

N-tuple:

$$\mathbf{v} = (x_1, x_2, \ldots, x_n), \quad x_i \in \Re$$

Magnitude:

$$|\mathbf{v}| = \sqrt{x_1^2 + \ldots + x_n^2}$$

Unit vectors

$$\mathbf{v} : |\mathbf{v}| = 1$$

Normalizing a vector

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Operations with Vectors

Addition

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$$

Multiplication with scalar (scaling)

$$a\mathbf{x} = (ax_1, \ldots, ax_n), \quad a \in \Re$$

Properties

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

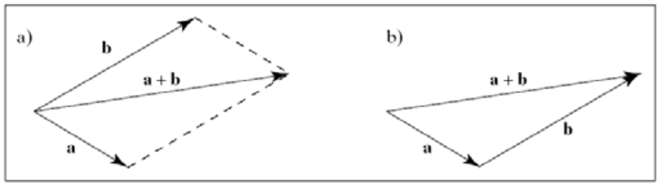
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}, \quad a \in \Re$$

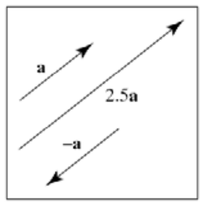
$$\mathbf{u} - \mathbf{u} = \mathbf{0}$$

Visualization of 2D and 3D Vectors

Addition

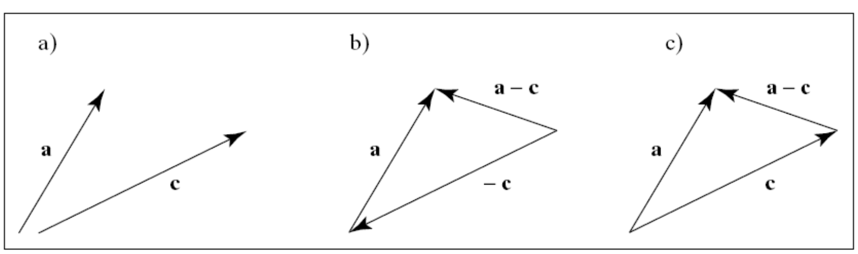


Scaling



Subtraction

Adding the negatively scaled vector



Linear Combination of Vectors

Definition

A linear combination of the *m* vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m$ is a vector of the form:

$$\mathbf{w} = a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m, \quad a_1,\ldots,a_m \text{ in R}$$

Special Cases

Linear combination

$$\mathbf{w} = a_1\mathbf{v}_1 + \ldots a_m\mathbf{v}_m, \quad a_1,\ldots,a_m \text{ in R}$$

Affine combination:

A linear combination for which $a_1+\ldots+a_m = 1$

Convex combination

An affine combination for which $a_i \geq 0$ for i = 1,…,m

Linear Independence

For vectors $v_1, \ldots, v_m$

If $a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m = \mathbf{0}$ iff $a_1 = a_2 = \ldots = a_m = 0$

then the vectors are linearly independent

Generators and Base Vectors

How many vectors are needed to generate a vector space?

- Any set of vectors that generate a vector space is called a generator set
- Given a vector space $\mathbf{R}^n$ we can prove that we need minimum n vectors to generate all vectors $\mathbf{v}$ in $\mathbf{R}^n$
- A generator set with minimum size is called a basis for the given vector space

Standard Unit Vectors

$$\mathbf{v} = (x_1, \ldots, x_n), \quad x_i \in \Re$$

$$\begin{aligned}(x_1, x_2, ..., x_n) &= x_1(1, 0, 0, \ldots, 0, 0) \\ &\quad +x_2(0, 1, 0, \ldots, 0, 0) \\ &\quad \ldots \\ &\quad +x_n(0, 0, 0, \ldots, 0, 1)\end{aligned}$$

Standard Unit Vectors

For any vector space R^n:

$$\mathbf{i}_1 = (1, 0, 0, \ldots, 0, 0)$$
$$\mathbf{i}_2 = (0, 1, 0, \ldots, 0, 0)$$
$$\ldots$$
$$\mathbf{i}_n = (0, 0, 0, \ldots, 0, 1)$$

The elements of a vector v in R^n are the scalar coefficients of the linear combination of the basis vectors

Standard Unit Vectors in 2D & 3D

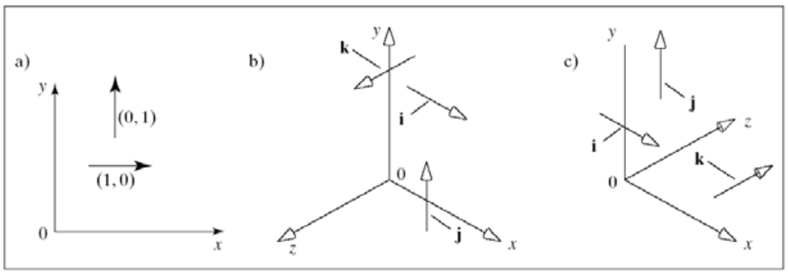
i = (1,0)

j = (0,1)

i = (1,0,0)

j = (0,1,0)

k = (0,0,1)



Right handed

Left handed

Representation of Vectors Through Basis Vectors

Given a vector space R^n, a set of basis vectors B {b_i in R^n, i=1,…n} and a vector v in R^n we can always find scalar coefficients such that:

$\mathbf{v} = a_1\mathbf{b}_1 + \ldots + a_n\mathbf{b}_n$

So, vector **v** expressed with respect to *B* is:

$\mathbf{v}_B = (a_1, \ldots, a_n)$

Dot Product

Definition:

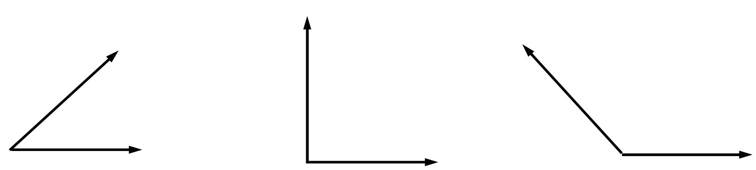
$$\mathbf{w}, \mathbf{v} \in \Re^n$$
$$\mathbf{w} \cdot \mathbf{v} = \sum_{i=1}^{n} w_i v_i$$

Properties

1. Symmetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$
5. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$

Dot Product and Perpendicularity

From Property 5:



a · b > 0   **a · b =** 0   **a · b <** 0

Perpendicular Vectors

Definition

Vectors **a** and **b** are perpendicular iff **a** · **b** = 0

Also called normal or orthogonal vectors

It is easy to see that the standard unit vectors form an orthogonal basis:

I · **j** = 0, **j** · **k** = 0, **I** · **k** = 0

Cross Product

Defined only for 3D vectors and with respect to the standard unit vectors

Definition

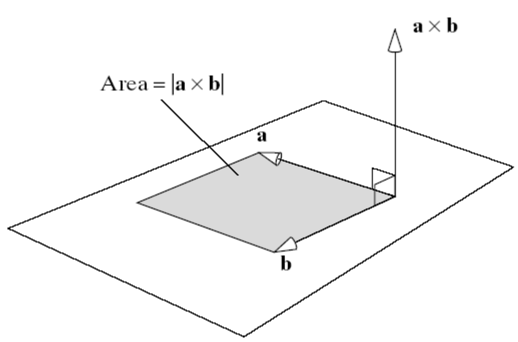
$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Properties of the Cross Product

1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$
2. Antisymmetry: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
3. Linearity: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. Homogeneity: $(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b})$
5. The cross product is normal to both vectors: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$
6. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$

Geometric Interpretation of the Cross Product



Matrices

Rectangular arrangement of scalar elements

Matrix:
Bold upper-case

$$\mathbf{A}_{3\times 3} = \begin{pmatrix} -1 & 2.0 & 0.5 \\ 0.2 & -4.0 & 2.1 \\ 3 & 0.4 & 8.2 \end{pmatrix}$$

$$\mathbf{A} = (\mathbf{A}_{ij})$$

Special Square Matrices

Zero: $A_{ij} = 0$, *for all* i,j

Identity: $I_n = \begin{cases} I_{ii} = 1, & \text{for all } i \\ I_{ij} = 0 & \text{for } i \neq j \end{cases}$

Symmetric: $(A_{ij})_{n \times n} = (A_{ji})_{n \times n}$

Operations with Matrices

Addition:

$$\mathbf{A}_{m\times n} + \mathbf{B}_{m\times n} = (a_{ij} + b_{ij})$$

Properties:

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $f(\mathbf{A} + \mathbf{B}) = f\mathbf{A} + f\mathbf{B}$
4. Transpose: $\mathbf{A}^T = (a_{ij})^T = (a_{ji})$

Multiplication

Definition:

$$\mathbf{C}_{m\times r} = \mathbf{A}_{m\times n}\mathbf{B}_{n\times r}$$

$$(\mathbf{C}_{ij}) = (\sum_{k=1}^{n} a_{ik}b_{kj})$$

Properties:

1. $\mathbf{AB} \neq \mathbf{BA}$
2. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
3. $f(\mathbf{AB}) = (f\mathbf{A})\mathbf{B}$
4. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$,
   $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
5. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Inverse of a Square Matrix

Definition

$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$

Important property

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$

Dot Product as a Matrix Multiplication

A vector is a column matrix

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \mathbf{a}^T \mathbf{b} \\ &= (a_1, a_2, a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

Convention

Vectors and Points are represented as column matrices

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix} \qquad P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}$$

Lines and Planes

In addition to vectors and points, lines and planes are fundamental geometric entities in computer graphics

- Recall how we represent them mathematically…

Lines

Representations of a line (in 2D)

- Explicit $$y = \frac{dy}{dx}(x - x_0) + y_0$$
- Implicit $$F(x, y) = (x - x_0)dy - (y - y_0)dx$$

$$\text{if} \quad \begin{array}{l} F(x,y) = 0 \\ F(x,y) > 0 \\ F(x,y) < 0 \end{array} \quad \text{then} \quad \begin{array}{l} (x,y) \text{ is on line} \\ (x,y) \text{ is below line} \\ (x,y) \text{ is above line} \end{array}$$

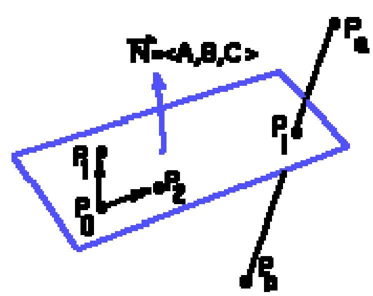
- Parametric

$$\begin{aligned} x(t) &= x_0 + t(x_1 - x_0) \\ y(t) &= y_0 + t(y_1 - y_0) \\ & t \in [0, 1] \end{aligned}$$

$$\begin{aligned} P(t) &= P_0 + t(P_1 - P_0), \text{ or} \\ P(t) &= (1 - t)P_0 + tP_1 \end{aligned}$$

Planes

Plane equations



Implicit

$F(x, y, z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$

Points on Plane $F(x, y, z) = 0$

Explicit

$z = -(A/C)x - (B/C)y - D/C, \; C \neq 0$

Parametric

$Plane(s,t) = P_0 + s(P_1 - P_0) + t(P_2 - P_0)$

P_0, P_1, P_2 not collinear

or

$Plane(s,t) = P_0 + sV_1 + tV_2$ where V_1, V_2 are basis vectors

convex combination defines a triangle :

$Plane(s,t) = (1 - s - t)P_0 + sP_1 + tP_2$