All problems are provided with solution during the discussion session. Thus there should be no problem on this assignment. Just to let you know to make your life easier. Thanks.

Problem 1:

With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \ \lambda_i = 0, \quad i \geq 4$$
$$\mu_1 = 2, \ \mu_2 = \mu_3 = 4$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, \ P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, \ P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0 \ P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0 \ P_3 = \left[\frac{3}{4}\right]^3 P_0$$

and $\quad P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}$$

Problem 2

Let the state be the idle server. The balance equations are

Rate Leave = Rate Enter,

$$(\mu_2 + \mu_3)P_1 = \frac{\mu_1}{\mu_1 + \mu_2}P_3 + \frac{\mu_1}{\mu_1 + \mu_3}P_2,$$

$$(\mu_1 + \mu_3)P_2 = \frac{\mu_2}{\mu_2 + \mu_3}P_1 + \frac{\mu_2}{\mu_2 + \mu_1}P_3,$$

$$\mu_1 + \mu_2 + \mu_3 = 1.$$

These are to be solved and the quantity P_i represents the proportion of time that server i is idle.

Problem 3:

Let the state be the number of taxis waiting. Then we get a birth-death process with $\lambda = 1$ and $\mu = 2$. Also, this can be thought of as an M/M/1 system where being serviced is equivalent to waiting for a customer. Therefore:

a) Average number of taxis waiting $= \frac{1}{\mu - \lambda} = 1$.

b) The proportion of arriving customers tht get a taxi is the proportion of arriving customers that find at least one taxi waiting. This is equivalent to the proportion of time the system is **not** in state 0. This is equal to $1 - P_0 = 1 - (1 - \frac{\lambda}{\mu}) = \frac{1}{2}$.

Problem 4:

This is a 5 state birth death process, each state being the number of customers in service.
$\lambda = 40$ customers/hour, and $\mu = 30$ customer/hour.
The state transition diagram for the system is shown below: (Note that this is **not** *M*/*M*/2 system)

$$
\begin{tikzcd}
0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 4 \arrow[l, bend left, "2\mu"]
\end{tikzcd}
$$

Writing balance equations we get
$\lambda p_0 = \mu p_1$
$(\lambda+\mu)p_1 = \lambda p_0 + \mu p_2$
$(\lambda+\mu)p_2 = \lambda p_1 + 2\mu p_3$
$(\lambda+2\mu)p_3 = \lambda p_2 + 2\mu p_4$
$\lambda p_3 = 2\mu p_4$

Solving we get

$$p_1 = \frac{\lambda}{\mu} p_0$$

$$p_2 = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

$$p_3 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^3 p_0$$

$$p_4 = \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^4 p_0$$

Taking into account the conservation of probability, $p_0 + p_1 + p_2 + p_3 + p_4 = 1$ and solving we get

$$(p_0, p_1, p_2, p_3, p_4) = \left(\frac{81}{493}, \frac{108}{493}, \frac{144}{493}, \frac{96}{493}, \frac{64}{493}\right)$$

a) Fraction of time both attendants are free is simply p_0

b) The attendant works for $1 - p_0$ of the time and the assistant works for $p_3 + p_4$ of the time. So the assistant works for $\frac{p_3 + p_4}{1 - p_0} = 0.388$ of the time the attendant works. If x is the amount of the attendant, $0.388x + x = 100 \Rightarrow x = \$72.05$. Therefore, the amount of the assistant is $100 - x = \$27.95$

c) $\overline{N} = \sum_{k=0}^{4} k p_k = 1.906$

Then by Little's result, $T = \overline{N} / \lambda = 0.047$ hours $= 2.86$ minutes

d) $\overline{N_q} = \sum_{k=1}^{4} (k-1) p_k = \frac{528}{493} = 1.071$

Waiting time, $W = \overline{N_q} / \lambda = 1.606$ minutes

Problem 5:

There are four states, 0, A, B, AB signifying which of the servers are free. The balance equations are:
$2p_0 = 2p_B, 4p_A = 2p_0 + 2p_{AB}, 4p_B = 4p_A + 4p_{AB}, 6p_{AB} = 2p_B$. Solving, we get, $p_0 = 3/9, p_A = 2/9, p_B = 3/9, p_{AB} = 1/9$.
a) $p_0 + p_B = 2/3$.

c) Average number in system $= p_A + p_B + 2p_{AB} = 7/9$.
d) Effective arrival rate $= \lambda(p_0 + p_B) = 4/3$ per hour. Using Little's formula, we get T $=$

$(7/9)/(4/3) = 7/12$.We could also solve this by conditioning on when the customer arrived. We'd get $1/2(1/4 + 1/2) + 1/2(1/4 + (2/6)(1/2)) = 7/12$.

Problem 6:

The waiting time for an arrival is simply the time it spends in the queue waiting for the customers in front of it to be served. If an arrival finds k customers in the system, it will have to wait for all k of them to be served. The distribution of this time is the sum of k iid Random Variables each with exponential distribution with parameter μ. Instead of a lengthy convolution we take the Laplace transform of waiting time W*(s) by multiplying the LTs of the k RVs.
W*(s| arrival finds k customers in system) $= [B*(s)]^k$
Unconditioning, we get,

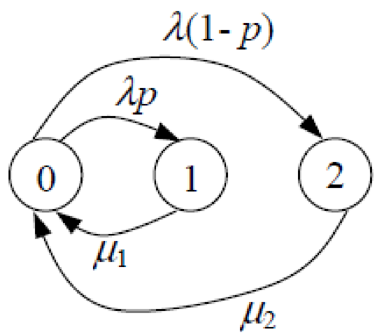
$$\begin{aligned}
W*(s) &= \sum_{k=0}^{\infty}[B*(s)]^k p_k \\
&= \sum_{k=0}^{\infty}[B*(s)]^k(1-\rho)\rho^k \\
&= (1-\rho) + \sum_{k=1}^{\infty}[B*(s)]^k(1-\rho)\rho^k \\
&= (1-\rho) + (1-\rho)\frac{B*(s)\rho}{1-B*(s)\rho} \\
&= (1-\rho) + (1-\rho)\frac{\frac{\mu\rho}{s+\mu}}{\frac{s+\mu-\rho\mu}{s+\mu}} \\
&= (1-\rho) + \rho\frac{\mu(1-\rho)}{s+\mu(1-\rho)} \\
\Leftrightarrow w(y) &= (1-\rho)u_0(t) + \rho(1-\rho)\mu e^{-\mu(1-\rho)t}
\end{aligned}$$

W(y) is a mixed distribution as there is an impulse at t = 0 and a continuous distribution from 0 to ∞.

Problem 7:

Define the states of the Continuous Time Markov chain as:
0: machine is operational
1: machine has failure of type 1
2: machine has failure of type 2

Then, we obtain the following state-transition rate diagram:



The equivalent transition rate matrix is:

$$\mathbf{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} -2 & 0.6 & 1.4 \\ 10 & -10 & 0 \\ 15 & 0 & -15 \end{bmatrix} \end{array}$$

To calculate the steady state probabilities, we solve the system:

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$$
$$\sum_i \pi_i = 1$$

where we get the following equations:

$-2\pi_0 + 0.6\pi_1 + 15\pi_2 = 0$ (1)
$0.6\pi_0 - 10\pi_1 = 0$ (2)
$1.4\pi_0 - 15\pi_2 = 0$ (3)
$\pi_0 + \pi_1 + \pi_2 = 1$ (4)

Using (4) and any two of the (1)-(3) we get

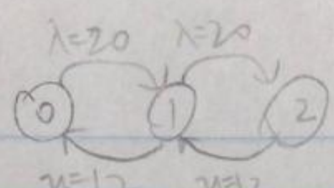
$$\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2) = \left(\frac{150}{173}, \frac{9}{173}, \frac{14}{173}\right)$$

a) The proportion of time the machine is down due to a type 2 failure is given by $\pi_2 = \frac{14}{173}$

b) The proportion of time the machine is up is given by $\pi_0 = \frac{150}{173}$

Problem 8 & Problem 9:

Problem 8



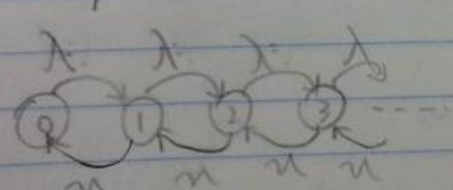
$$\begin{cases} P_0+P_1+P_2=1 \\ \lambda P_0 = \mu P_1 \\ \mu P_2 + \lambda P_0 = \mu P_1 + \lambda P_1 \end{cases}$$

$$\begin{cases} P_0 = \frac{9}{49} \\ P_1 = \frac{15}{49} \\ P_2 = \frac{25}{49} \end{cases}$$

(a) Proportion computer is busy $= P_1 + P_2 = \frac{40}{49}$

(b) Proportion routed to other places: $P_2 = \frac{25}{49}$

Problem 9.



Little's result: $T = \frac{N}{\lambda} = \frac{\lambda}{\lambda(\mu-\lambda)} \leq \frac{1}{20}$

$\mu \geq 20$.

Problem 10:

This problem can be modeled by an M/M/1 queue in which $\lambda=6$, $\mu=8$. The average cost rate will be

\$10 per hour per machine x average number of broken machines.

The average number of broken machines is just L, which can be computed from the equation provided in lectures:

$$L = \lambda / (\mu - \lambda) = 6/2 = 3.$$

Hence, the average cost rate = \$30/hour.

Problem 11:

The state is the number of customers in the system, and the balance equations are

$$m\theta P_0 = \mu P_1$$

$$((m-j)\theta + \mu)P_j = (m-j+1)\theta P_{j-1} + \mu P_{j+1}, \quad 0 < j < m$$

$$\mu P_m = \theta P_{m-1}$$

$$1 = \sum_{j=0}^{m} P_j$$

(a) $\lambda_\alpha = \sum_{j=0}^{m} (m-j)\theta P_j$

(b) $L/\lambda_\alpha = \sum_{j=0}^{m} jP_j / \sum_{j=0}^{m} (m-j)\theta P_j$

Problem 12:

(a) The states are 0, 1, 2, 3 where the state is i when there are i in the system.

(b) The balance equations are

$$\lambda P_0 = \mu P_1$$
$$(\lambda + \mu)P_1 = \lambda P_0 + 2\mu P_2$$
$$(\lambda + 2\mu)P_2 = \lambda P_1 + 2\mu P_3$$
$$2\mu P_3 = \lambda P_2$$

$$P_0 + P_1 + P_2 + P_3 = 1$$

The solution of these equations is

$$P_1 = (\lambda/\mu)P_0, P_2 = (\lambda^2/2\mu^2)P_0, P_3 = (\lambda^3/4\mu^3)P_0$$
$$P_0 = [1 + \lambda/\mu + \lambda^2/(2\mu^2) + \lambda^3/(4\mu^3)]^{-1}$$

(c) $$\begin{aligned} E[\text{Time}] &= E[\text{Time in queue}] \\ &\quad + E[\text{time in service}] \\ &= 1/(2\mu) + 1/\mu. \end{aligned}$$

(d) $1 - P_3$.

(e) Conditioning on the state as seen by the arrival

$$W = [(1/\mu)(P_0 + P_1) + (2/\mu)P_2]/(1 - P_3).$$

Could also use $W = L/\lambda_a$.