Problem 1

(a)

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1-p & p \\ 1 & 0 & 0 \\ 0 & q & 1-q \end{bmatrix} \end{array}$$

(b) if $\beta=0$, state 3 is unreachable from state 1 and 2. If $\alpha=0$, state 1 and 2 are unreachable from state 3. In both cases, the Markov chain is reducible. If $\alpha=1$, $\beta=1$ the Markov Chain has a period of 3. In other cases, the Markov Chain is irreducible and aperiodic

(c) In the case of equilibrium, $\pi = \pi P$.

$\pi_1 = \pi_2$

$\pi_2 = (1-p)\pi_1 + q\pi_3$

$\pi_3 = p\pi_1 + (1-q)\pi_3$

with the constraint that $\pi_1+\pi_2+\pi_3=1$

$$\pi = [\pi_1\ \pi_2\ \pi_3] = \left[\frac{q}{p+2q}, \frac{q}{p+2q}, \frac{p}{p+2q}\right]$$

(d) Mean recurrence time State 2 $= \frac{1}{\pi_2} = \frac{p}{q} + 2$

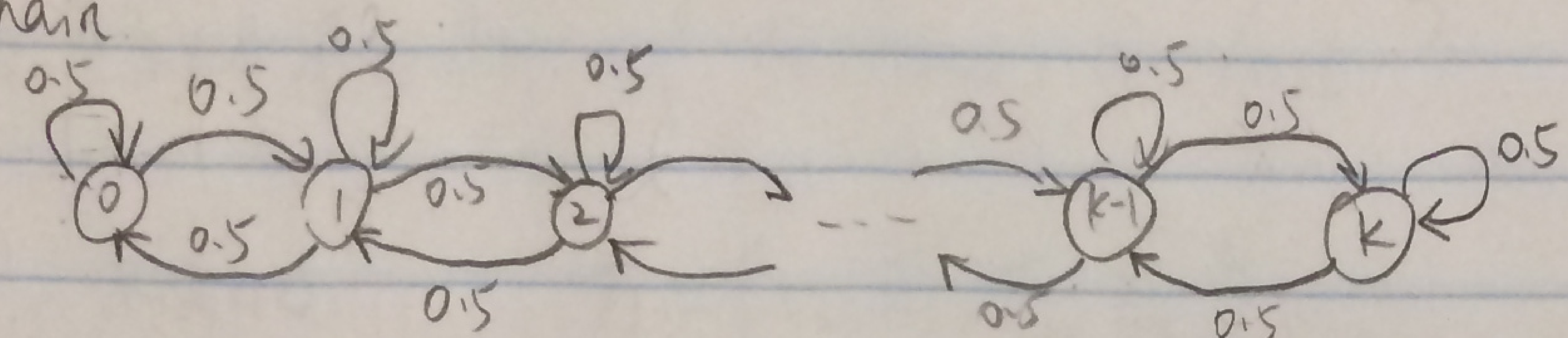
Problem 2

	(A)	(B)	(C)	(D)
(a) transient state:	1	3	1	1 2
absorbing state:	3			
recurrent state:	3 2.	2	2 1 3	4 3

(c) Since a Markov Chain would have stationary probability distribution if all of its states are recurrent, thus (C) has stationary probability distribution.

Problem 3

Let X_n be the numbers of shoes at the front door. Then $X_n \in \{0,1,2,\cdots k\}$ is a Markov chain



Writing the balance equation

$$\tfrac{1}{2}\pi_0 = \tfrac{1}{2}\pi_1 \Rightarrow \pi_0 = \pi_1$$

$$\tfrac{1}{2}\pi_1 = \tfrac{1}{2}\pi_2 \Rightarrow \pi_1 = \pi_2.$$

$$\vdots \qquad\qquad \vdots$$

$$\tfrac{1}{2}\pi_{k-1} = \tfrac{1}{2}\pi_k \Rightarrow \pi_{k-1} = \pi_k$$

$$\pi_0 + \pi_1 \cdots \pi_{k-1} + \pi_k = 1 \qquad \pi_0 = \pi_1 = \pi_2 = \cdots = \pi_k = \frac{1}{k+1}$$

Thus the proportion he runs barefoot is: $\pi_0 = \frac{1}{k+1}$

Problem 4.

$$k=0 \quad \frac{dP_0(t)}{dt} = -\lambda P_0(t) + uP_1(t)$$

$$k \geq 1 \quad \frac{dP_k(t)}{dt} = -(\lambda + ku)P_k(t) + \lambda P_{k-1}(t) + (k+1)uP_{k+1}(t)$$

Problem 5.

For $k=0$:

$$\frac{dP_0(t)}{dt} = uP_1(t) - K\lambda P_0(t)$$

For $0<k<K$

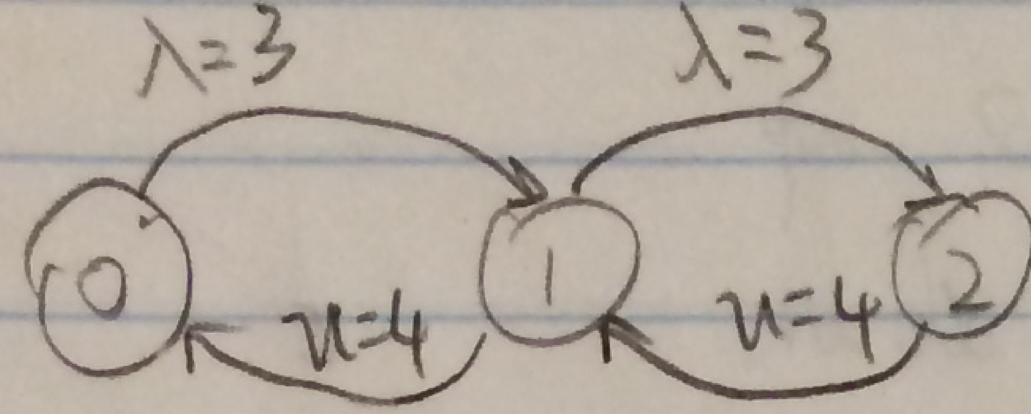
$$\frac{dP_k(t)}{dt} = \lambda(K-k+1)P_{k-1}(t) + (k+1)uP_{k+1}(t) - [(K-k)\lambda + ku]P_k(t).$$

For $k=K$

$$\frac{dP_K(t)}{dt} = \lambda P_{K-1}(t) - KuP_K(t)$$

Problem 6

(a) Let state be the number of customers in the shop.



we have $3\pi_0 = 4\pi_1$

$$\begin{aligned} 7\pi_1 &= 3\pi_0 + 4\pi_2 \\ 4\pi_2 &= 3\pi_1 \\ \pi_0+\pi_1+\pi_2 &= 1 \end{aligned} \Rightarrow \begin{cases} \pi_0 = \frac{16}{37} \\ \pi_1 = \frac{12}{37} \\ \pi_2 = \frac{9}{37} \end{cases}$$

(a) average # of customers $= 0\times\pi_0 + 1\times\pi_1 + 2\times\pi_2 = \frac{30}{37}$

(b) Proportion of customers that enters:

$$\pi_0+\pi_1 = \frac{28}{37}$$

(c) Replacing $\mu=8$ in the transition diagram and we reach the desired condition. In that case we have

$$\begin{aligned} 3\pi_0 &= 8\pi_1 \\ 11\pi_1 &= 3\pi_0 + 8\pi_2 \\ 8\pi_2 &= 3\pi_1 \\ \pi_0+\pi_1+\pi_2 &= 1 \end{aligned} \Rightarrow \begin{cases} \pi_0 = \frac{64}{97} \\ \pi_1 = \frac{24}{97} \\ \pi_2 = \frac{9}{97} \end{cases}$$

Proportion of customers that can enter: $\pi_0+\pi_1 = \frac{88}{97}$ increased 20%.

Problem 7

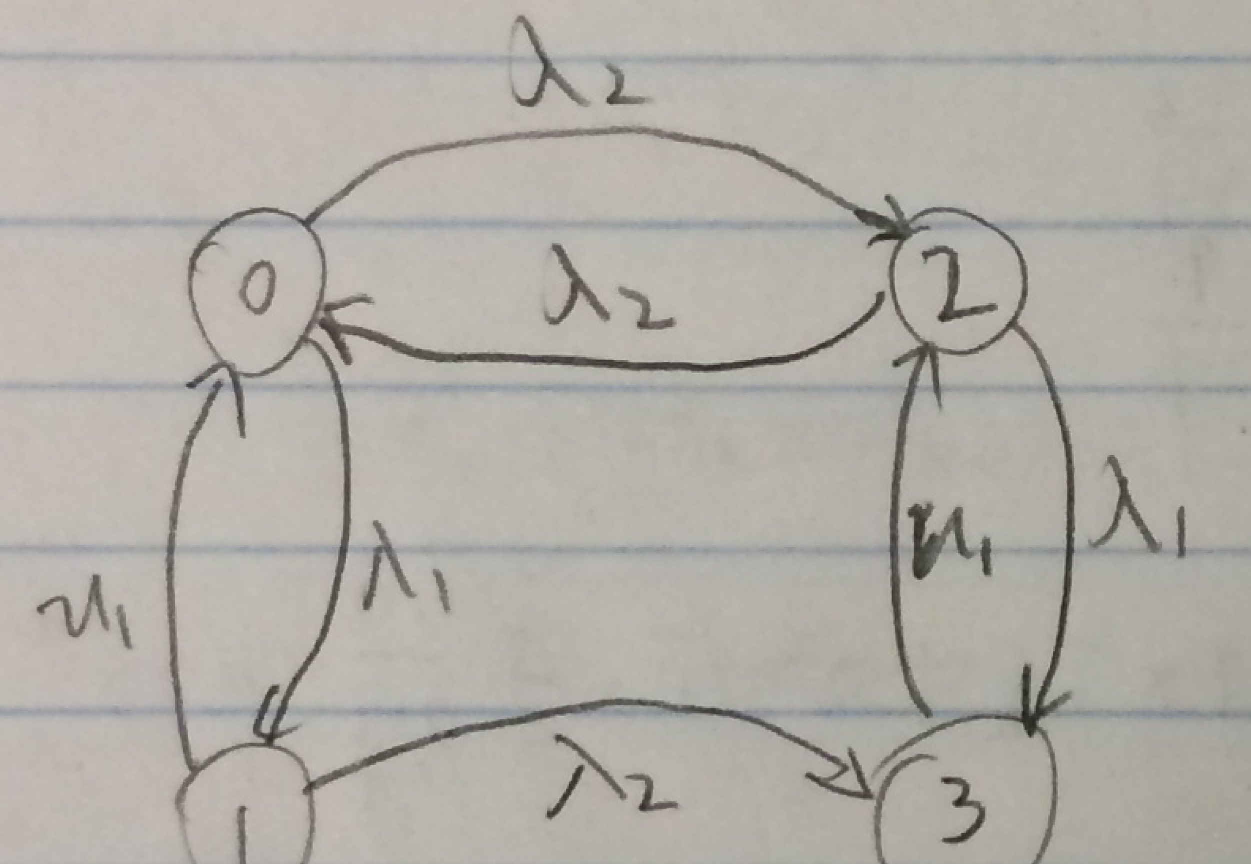
There are 4 States:

state 0: no machines are down.

state 1: machine 1 down, machine 2 up.

state 2: machine 1 up, machine 2 down.

state 3: both down.



The balance equation are:

$$(\lambda_1+\lambda_2)P_0 = \mu_1 P_1 + \mu_2 P_2$$

$$(\mu_1+\lambda_2)P_1 = \lambda_1 P_0$$

$$(\lambda_1+\mu_2)P_2 = \lambda_2 P_0 + \mu_1 P_3$$

$$\mu_1 P_3 = \lambda_2 P_1 + \lambda_1 P_2$$

$$P_0+P_1+P_2+P_3 = 1$$

The proportion of time machine 2 is down is $\pi_2+\pi_3$