Introduction to Linear Algebra International Edition (2019)

Solutions to Selected Exercises

Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

6 The components of every $c\boldsymbol{v} + d\boldsymbol{w}$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$.

9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$.

11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum $=$ zero vector. Sum $= -2{:}00$ vector $= 8{:}00$ vector. 2:00 is 30° from horizontal $= (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

16 All combinations with $c + d = 1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V} = -\boldsymbol{v} + 2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v} + c\boldsymbol{w}$ are on the line passing through $(0, 0)$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v} + \boldsymbol{w}$ and back beyond $(0, 0)$. With $c \geq 0$, half of this line is removed, leaving a *ray* that starts at $(0, 0)$.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} =$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

Problem Set 1.2, page 19

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{24}{25}$. The vectors $\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

4 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = -1$ (b) $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = 0$ so $\theta = 90°$ (notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$) (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = 1 - 4 = -3$.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. All vectors (x, y, z) with $x + y + z = 0$ lie on a *plane*. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.

9 If $v_2 w_2 / v_1 w_1 = -1$ then $v_2 w_2 = -v_1 w_1$ or $v_1 w_1 + v_2 w_2 = \boldsymbol{v} \cdot \boldsymbol{w} = 0$: perpendicular!

11 $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ means angle $> 90°$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $\boldsymbol{v} \cdot (\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v} \cdot \boldsymbol{w} / \boldsymbol{v} \cdot \boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to perpendicular vectors.

15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

21 $2\boldsymbol{v} \cdot \boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

22 $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \le v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2$ which is $(v_1 w_2 - v_2 w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1 w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2 w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

28 Three vectors in the plane could make angles $> 90°$ with each other: $(1, 0), (-1, 4), (-1, -4)$. Four vectors could not do this ($360°$ total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$?

29 Try $\boldsymbol{v} = (1, 2, -3)$ and $\boldsymbol{w} = (-3, 1, 2)$ with $\cos\theta = \frac{-7}{14}$ and $\theta = 120°$. Write $\boldsymbol{v} \cdot \boldsymbol{w} = xz + yz + xy$ as $\frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. If $x + y + z = 0$ this is $-\frac{1}{2}(x^2 + y^2 + z^2) = -\frac{1}{2}\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -\frac{1}{2}$.

Problem Set 1.3, page 29

1 $2\boldsymbol{s}_1 + 3\boldsymbol{s}_2 + 4\boldsymbol{s}_3 = (2, 5, 9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (2, 3, 4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side = column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{ any constant vector.}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \qquad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ -x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

6 Equation $1 +$ equation $2 -$ equation 3 is now $0 = -4$. Line misses plane; *no solution*.

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3,3,3,2)$ is $\boldsymbol{x} = (0,0,1,2)$ if A has columns $(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)$. The equations are $x+y+z+t=3, y+z+t=3, z+t=3, t=2$.

11 $A\boldsymbol{x}$ equals $(14,22)$ and $(0,0)$ and $(9,7)$.

14 $2x+3y+z+5t=8$ is $A\boldsymbol{x}=\boldsymbol{b}$ with the 1 by 4 matrix $A=[\,2\ \ 3\ \ 1\ \ 5\,]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

16 $90°$ rotation from $R=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $180°$ rotation from $R^2=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=-I$.

18 $E=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$ and $E=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}$ subtract the first component from the second.

22 The dot product $A\boldsymbol{x}=[\,1\ \ 4\ \ 5\,]\begin{bmatrix}x\\y\\z\end{bmatrix}=(1\text{ by }3)(3\text{ by }1)$ is zero for points (x,y,z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A=[\,1\ \ 2\ \ ;\ \ 3\ \ 4\,]$ and $\boldsymbol{x}=[\,5\ \ -2\,]'$ and $\boldsymbol{b}=[\,1\ \ 7\,]'$. $\boldsymbol{r}=\boldsymbol{b}-A*\boldsymbol{x}$ prints as zero.

25 **ones**$(4,4)*$**ones**$(4,1)=[\,4\ \ 4\ \ 4\ \ 4\,]'$; $B*\boldsymbol{w}=[\,10\ \ 10\ \ 10\ \ 10\,]'$.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_7,\boldsymbol{v}_7,\boldsymbol{w}_7$ are all close to $(.6,.4)$. Their components still add to 1.

30 $\begin{bmatrix}.8&.3\\.2&.7\end{bmatrix}\begin{bmatrix}.6\\.4\end{bmatrix}=\begin{bmatrix}.6\\.4\end{bmatrix}=$ *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix}.8&.3\\.2&.7\end{bmatrix}$.

31 $M=\begin{bmatrix}8&3&4\\1&5&9\\6&7&2\end{bmatrix}=\begin{bmatrix}5+u&5-u+v&5-v\\5-u-v&5&5+u+v\\5+v&5+u-v&5-u\end{bmatrix}$; $M_3(1,1,1)=(15,15,15)$; $M_4(1,1,1,1)=(34,34,34,34)$ because $1+2+\cdots+16=136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u}+d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution*.

33 $\boldsymbol{w}=(5,7)$ is $5\boldsymbol{u}+7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ has the solution $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}4\\7\\8\\6\end{bmatrix}$.

35 $\boldsymbol{x}=(1,\ldots,1)$ gives $S\boldsymbol{x}=$ sum of each row $=1+\cdots+9=45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4=1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. A solution is $(1, 1, -1)$.

17 If row $1 =$ row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column $2 =$ column 1, then column 2 has no pivot.

19 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular — no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2 =$ row 3 on the left side but not the right side: $x + y + z = 0$, $x - 2y - z = 1$, $2x - y = 4$. No parallel planes but still no solution.

25 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

28 $A(2,:) = A(2,:) - 3 * A(1,:)$ will subtract 3 times row 1 from row 2.

29 Pivots 2 and 3 can be arbitrarily large. I believe their averages are infinite! *With row exchanges* in MATLAB's lu code, the averages are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

30 If $A(5, 5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1, \ldots, j$ of A. If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U is the diagonal of A when A is *lower triangular*.

Problem Set 2.3, page 63

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

12 The first product is $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

14 E_{21} has $-\ell_{21} = \frac{1}{2}$, E_{32} has $-\ell_{32} = \frac{2}{3}$, E_{43} has $-\ell_{43} = \frac{3}{4}$. Otherwise the E's match I.

18 $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21} - a_{11}$ (c) $a_{21} - 2a_{11}$ (d) $(E_{21}A\boldsymbol{x})_1 = (A\boldsymbol{x})_1 = \sum a_{1j}x_j$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3.

27 (a) No solution if $d = 0$ and $c \neq 0$ (b) Many solutions if $d = 0 = c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

30 $EM = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ then $FEM = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ then $EFEM = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ then $EEFEM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$. So after inverting with $E^{-1} = A$ and $F^{-1} = B$ this is $M = ABAAB$.

Problem Set 2.4, page 75

2 (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF) = (EA)F$: Matrix multiplication is *associative*.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

15 (a) mn (use every entry of A) (b) $mnp = p \times$part (a) (c) n^3 (n^2 dot products).

16 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

18 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

22 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;
$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

24 $(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

27 (a) (row 3 of A) $\cdot$ (column 1 of B) and (row 3 of A) $\cdot$ (column 2 of B) are both zero.

(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$: **both upper**.

28 A times B with cuts $\quad A\begin{bmatrix} | & | & | & | \end{bmatrix}$, $\begin{bmatrix} \text{———} \end{bmatrix} B$, $\begin{bmatrix} \text{———} \end{bmatrix}\begin{bmatrix} | & | & | & | \end{bmatrix}$, $\begin{bmatrix} | & | & | \end{bmatrix}\begin{bmatrix} \text{———} \\ \text{———} \end{bmatrix}$

30 In **29**, $\boldsymbol{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - \boldsymbol{cb}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in the lower corner of EA.

32 A times $X = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,]$ will be the identity matrix $I = [\,A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3\,]$.

33 $\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $\boldsymbol{x}_1 = (1,1,1), \boldsymbol{x}_2 = (0,1,1), \boldsymbol{x}_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

35 $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$,

aba, ada	**cba, cda**	These show
bab, bcb	**dab, dcb**	16 2-step
abc, adc	**cbc, cdc**	paths in
bad, bcd	**dad, dcd**	the graph

Problem Set 2.5, page 89

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

7 (a) In $A\boldsymbol{x} = (1,0,0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $\boldsymbol{x} = (1,1,-1)$ solves $A\boldsymbol{x} = \boldsymbol{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

12 Multiply $C = AB$ on the left by A^{-1} and on the right by C^{-1}. Then $A^{-1} = BC^{-1}$.

14 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

18 $A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

21 Six of the sixteen $0 - 1$ matrices are invertible, including all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = [\,I \;\; A^{-1}\,]$;

$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = [\,I \;\; A^{-1}\,]$.

24 $\begin{bmatrix}1&a&b&1&0&0\\0&1&c&0&1&0\\0&0&1&0&0&1\end{bmatrix}\to\begin{bmatrix}1&a&0&1&0&-b\\0&1&0&0&1&-c\\0&0&1&0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0&1&-a&ac-b\\0&1&0&0&1&-c\\0&0&1&0&0&1\end{bmatrix}.$

27 $A^{-1}=\begin{bmatrix}1&0&0\\-2&1&-3\\0&0&1\end{bmatrix}$ (notice the pattern); $A^{-1}=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}.$

31 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1}=\dfrac{1}{a(a-b)}\begin{bmatrix}a&0&-b\\-a&a&0\\0&-a&a\end{bmatrix}.$

33 $\boldsymbol{x}=(1,1,\ldots,1)$ has $P\boldsymbol{x}=Q\boldsymbol{x}$ so $(P-Q)\boldsymbol{x}=\mathbf{0}$.

34 $\begin{bmatrix}I&0\\-C&I\end{bmatrix}$ and $\begin{bmatrix}A^{-1}&0\\-D^{-1}CA^{-1}&D^{-1}\end{bmatrix}$ and $\begin{bmatrix}-D&I\\I&0\end{bmatrix}.$

35 A can be invertible with diagonal zeros. B is singular because each row adds to zero.

38 The three Pascal matrices have $P=LU=LL^{\mathrm{T}}$ and then $\mathsf{inv}(P)=\mathsf{inv}(L^{\mathrm{T}})\mathsf{inv}(L)$.

42 $MM^{-1}=(I_n-UV)\,(I_n+U(I_m-VU)^{-1}V)$ (this is testing formula **3**)
$=I_n-UV+U(I_m-VU)^{-1}V-UVU(I_m-VU)^{-1}V$ (keep simplifying)
$=I_n-UV+U(I_m-VU)(I_m-VU)^{-1}V=I_n$ (formulas **1**, **2**, **4** are similar)

43 4 by 4 still with $T_{11}=1$ has pivots $1,1,1,1$; reversing to $T^*=UL$ makes $T^*_{44}=1$.

44 Add the equations $C\boldsymbol{x}=\boldsymbol{b}$ to find $0=b_1+b_2+b_3+b_4$. Same for $F\boldsymbol{x}=\boldsymbol{b}$.

Problem Set 2.6, page 102

3 $\ell_{31}=1$ and $\ell_{32}=2$ (and $\ell_{33}=1$): reverse steps to get $A\boldsymbol{u}=\boldsymbol{b}$ from $U\boldsymbol{x}=\boldsymbol{c}$: 1 times $(x+y+z=5)+2$ times $(y+2z=2)+1$ times $(z=2)$ gives $x+3y+6z=11$.

4 $L\boldsymbol{c}=\begin{bmatrix}1&&\\1&1&\\1&2&1\end{bmatrix}\begin{bmatrix}5\\2\\2\end{bmatrix}=\begin{bmatrix}5\\7\\11\end{bmatrix};\quad U\boldsymbol{x}=\begin{bmatrix}1&1&1\\&1&2\\&&1\end{bmatrix}\begin{bmatrix}\\ \boldsymbol{x}\\ \\ \end{bmatrix}=\begin{bmatrix}5\\2\\2\end{bmatrix};\ \boldsymbol{x}=\begin{bmatrix}5\\-2\\2\end{bmatrix}.$

6 $\begin{bmatrix}1&&\\0&1&\\0&-2&1\end{bmatrix}\begin{bmatrix}1&&\\-2&1&\\0&0&1\end{bmatrix}A=\begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}=U.$ Then $A=\begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}U$ is the same as $E_{21}^{-1}E_{32}^{-1}U=LU$. The multipliers $\ell_{21},\ell_{32}=2$ fall into place in L.

10 $c=2$ leads to zero in the second pivot position: exchange rows and not singular. $c=1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

12 $A=\begin{bmatrix}2&4\\4&11\end{bmatrix}=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}2&4\\0&3\end{bmatrix}=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}=LDU;$ $\boldsymbol{U}$ **is** $\boldsymbol{L}^{\mathrm{T}}$

$\begin{bmatrix}1&&\\4&1&\\0&-1&1\end{bmatrix}\begin{bmatrix}1&4&0\\0&-4&4\\0&0&4\end{bmatrix}=\begin{bmatrix}1&&\\4&1&\\0&-1&1\end{bmatrix}\begin{bmatrix}1&&\\&-4&\\&&4\end{bmatrix}\begin{bmatrix}1&4&0\\0&1&-1\\0&0&1\end{bmatrix}=LDL^{\mathrm{T}}.$

14 $\begin{bmatrix}a&r&r&r\\a&b&s&s\\a&b&c&t\\a&b&c&d\end{bmatrix}=\begin{bmatrix}1&&&\\1&1&&\\1&1&1&\\1&1&1&1\end{bmatrix}\begin{bmatrix}a&r&r&r\\&b-r&s-r&s-r\\&&c-s&t-s\\&&&d-t\end{bmatrix}.$ Need $\begin{matrix}a\neq 0\\b\neq r\\c\neq s\\d\neq t\end{matrix}$

15 $\begin{bmatrix}1&0\\4&1\end{bmatrix}\boldsymbol{c}=\begin{bmatrix}2\\11\end{bmatrix}$ gives $\boldsymbol{c}=\begin{bmatrix}2\\3\end{bmatrix}$. Then $\begin{bmatrix}2&4\\0&1\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}2\\3\end{bmatrix}$ gives $\boldsymbol{x}=\begin{bmatrix}-5\\3\end{bmatrix}$.
$A\boldsymbol{x}=\boldsymbol{b}$ is $LU\boldsymbol{x}=\begin{bmatrix}2&4\\8&17\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}2\\11\end{bmatrix}$. Forward to $\begin{bmatrix}2&4\\0&1\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}2\\3\end{bmatrix}=\boldsymbol{c}$.

18 (a) Multiply $LDU=L_1D_1U_1$ by inverses to get $L_1^{-1}LD=D_1U_1U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.
(b) L,U,L_1,U_1 have diagonal 1's so $D=D_1$. Then $L_1^{-1}L$ and U_1U^{-1} are both I.

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find ℓ and then one for the new pivot!). $T=$ bidiagonal L times bidiagonal U.

23 The 2 by 2 upper submatrix A_2 has the first two pivots $5,9$. Reason: Elimination on A starts in the upper left corner with elimination on A_2.

24 The upper left blocks all factor at the same time as A: A_k is L_kU_k.

25 The i,j entry of L^{-1} is j/i for $i\geq j$. And $L_{i\,i-1}$ is $(1-i)/i$ below the diagonal

26 $(K^{-1})_{ij}=j(n-i+1)/(n+1)$ for $i\geq j$ (and symmetric): $(n+1)K^{-1}$ looks good.

Problem Set 2.7, page 115

2 $(AB)^{\mathrm T}$ is not $A^{\mathrm T}B^{\mathrm T}$ *except when* $AB=BA$. Transpose that to find: $B^{\mathrm T}A^{\mathrm T}=A^{\mathrm T}B^{\mathrm T}$.

4 $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ has $A^2=0$. The diagonal of $A^{\mathrm T}A$ has dot products of columns of A with themselves. If $A^{\mathrm T}A=0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A=$ zero matrix.

6 $M^{\mathrm T}=\begin{bmatrix}A^{\mathrm T}&C^{\mathrm T}\\B^{\mathrm T}&D^{\mathrm T}\end{bmatrix}$; $M^{\mathrm T}=M$ needs $A^{\mathrm T}=A$ and $B^{\mathrm T}=C$ and $D^{\mathrm T}=D$.

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ exchange 2 pairs. $(1,2,3,4),(4,3,2,1)$ make 12 even P's.

14 The i,j entry of PAP is the $n-i+1,n-j+1$ entry of A. Diagonal will reverse order.

18 (a) $5+4+3+2+1=15$ independent entries if $A=A^{\mathrm T}$ (b) L has 10 and D has 5; total 15 in $LDL^{\mathrm T}$ (c) Zero diagonal if $A^{\mathrm T}=-A$, leaving $4+3+2+1=10$ choices.

20 $\begin{bmatrix}1&3\\3&2\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\0&-7\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}$; $\begin{bmatrix}1&b\\b&c\end{bmatrix}=\begin{bmatrix}1&0\\b&1\end{bmatrix}\begin{bmatrix}1&0\\0&c-b^2\end{bmatrix}\begin{bmatrix}1&b\\0&1\end{bmatrix}$

$$\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}=\begin{bmatrix}1&&\\-\frac12&1&\\0&-\frac23&1\end{bmatrix}\begin{bmatrix}2&&\\&\frac32&\\&&\frac43\end{bmatrix}\begin{bmatrix}1&-\frac12&0\\&1&-\frac23\\&&1\end{bmatrix}=\boldsymbol{LDL}^{\mathrm T}.$$

22 $\begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix}A=\begin{bmatrix}1&&\\0&1&\\2&3&1\end{bmatrix}\begin{bmatrix}1&0&1\\&1&1\\&&-1\end{bmatrix}$; $\begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix}A=\begin{bmatrix}1&&\\1&1&\\2&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\&-1&1\\&&1\end{bmatrix}$

24 $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and a_{12} is the pivot, $A = L_1 P_1 U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

26 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

31 $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x}$; $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix}\begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ 1 truck 1 plane

32 $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the *cost* of inputs while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the *value* of outputs.

33 $P^3 = I$ so three rotations for $360°$; P rotates around $(1, 1, 1)$ by $120°$.

36 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

37 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest $B = PL$ times southeast PU is $(PLP)U =$ upper triangular.

38 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \leq n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^6 = I$.

Problem Set 3.1, page 127

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$ (b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$ (b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$
(b) The plane and line probably intersect in the point $(0, 0, 0)$
(c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

20 (a) Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Solution only if $b_3 = -b_1$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.
$[\,A\ \ \boldsymbol{b}\,] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix}$ (larger column space) (no solution to $A\boldsymbol{x} = \boldsymbol{b}$) $\begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ ($\boldsymbol{b}$ is in column space) ($A\boldsymbol{x} = \boldsymbol{b}$ has a solution)

25 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S} + \boldsymbol{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\boldsymbol{S} + \boldsymbol{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: *a subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S} \cup \boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S} + \boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\,A\ \ B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A\ \ AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ when A is *invertible*.

Problem Set 3.2, page 140

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$
(b) Free variable x_3: solution $(1, -1, 1)$. Special solution for each free variable.

4 $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A.

6 (a) Special solutions $(3, 1, 0)$ and $(5, 0, 1)$ (b) $(3, 1, 0)$. Total of pivot and free is n.

8 $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 (a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I, R = I$.

14 If column 1 $=$ column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

16 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

20 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

24 This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.

26 30 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

32 Any zero rows come after these rows: $R = [\,1\ \ -2\ \ -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's !

35 The nullspace of $B = [\,A\ \ A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

36 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection*.

37 *Currents*: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops.

Problem Set 3.3, page 151

1 (a) and (c) are correct; (d) is false because R might have 1's in nonpivot columns.

3 $R_A = \begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$ $R_B = \begin{bmatrix}R_A & R_A\end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix}R_A & 0\\ 0 & R_A\end{bmatrix} \longrightarrow$ Zero rows go to the bottom

5 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

7 Special solutions in $N = [-2\ \ -4\ \ 1\ \ 0;\ -3\ \ -5\ \ 0\ \ 1]$ and $[1\ \ 0\ \ 0;\ 0\ \ -2\ \ 1]$.

13 P has $\operatorname{rank} r$ (the same as A) because elimination produces the same pivot columns.

14 The rank of R^{T} is also r. The example matrix A has $\operatorname{rank} 2$ with invertible S:

$$P = \begin{bmatrix}1&3\\2&6\\2&7\end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix}1&2&2\\3&6&7\end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix}1&2\\3&7\end{bmatrix} \qquad S = \begin{bmatrix}1&3\\2&7\end{bmatrix}.$$

16 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

18 If we know that $\operatorname{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \le \operatorname{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.

20 Certainly A and B have at most $\operatorname{rank} 2$. Then their product AB has at most $\operatorname{rank} 2$. Since BA is 3 by 3, it cannot be I even if $AB = I$.

21 (a) A and B will both have the same nullspace and row space as the R they share.
(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

22 A = (pivot columns)(nonzero rows of R) $= \begin{bmatrix}1&0\\1&4\\1&8\end{bmatrix}\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&0\\1&1&0\\1&1&0\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&4\\0&0&8\end{bmatrix}$. $B = \begin{bmatrix}2&2\\2&3\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} =$ columns times rows $= \begin{bmatrix}2&0\\2&0\end{bmatrix} + \begin{bmatrix}0&2\\0&3\end{bmatrix}$

26 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

27 $R = \begin{bmatrix}I&F\\0&0\end{bmatrix} = \begin{bmatrix}r \text{ by } r & r \text{ by } n-r\\ m-r \text{ by } r & m-r \text{ by } n-r\end{bmatrix}$; $\textbf{rref}(R^{\mathrm{T}}) = \begin{bmatrix}I&0\\0&0\end{bmatrix}$; $\textbf{rref}(R^{\mathrm{T}}R) =$ same R

28 The *row-column reduced echelon form* is always $\begin{bmatrix}I&0\\0&0\end{bmatrix}$; I is r by r.

Problem Set 3.4, page 163

2 $\begin{bmatrix}2&1&3&\mathbf{b}_1\\6&3&9&\mathbf{b}_2\\4&2&6&\mathbf{b}_3\end{bmatrix} \to \begin{bmatrix}2&1&3&\mathbf{b}_1\\0&0&0&\mathbf{b}_2-\mathbf{3b}_1\\0&0&0&\mathbf{b}_3-\mathbf{2b}_1\end{bmatrix}$ Then $[R\ \ \boldsymbol{d}] = \begin{bmatrix}1&1/2&3/2&\mathbf{5}\\0&0&0&\mathbf{0}\\0&0&0&\mathbf{0}\end{bmatrix}$
$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A)$ = line through $(2,6,4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5,0,0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3,1,0,0) + x_4(0,0,-2,1)$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

8 (a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.
(b) Need $b_3 = 2b_2$, because $(\text{row } 3) - 2(\text{row } 2) = \mathbf{0}$.

12 (a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$ (b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}, A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$
(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)
(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\, I \quad F \,]$ for any 3 by 2 matrix F.

18 Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

25 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

28 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

30 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$; $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

36 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b} =$ column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

Problem Set 3.5, page 178

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False*. The zero vector is always in the row space.

15 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \geq n)$.

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

20 One basis is $(2,1,0)$, $(-3,0,1)$. A basis for the intersection with the xy plane is $(2,1,0)$. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

25 Rank 2 if $c = 0$ and $d = 2$; rank 2 except when $c = d$ or $c = -d$.

28 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

32 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

34 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

37 The subspace of matrices that have $AS = SA$ has dimension *three*.

39 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

41 $\boldsymbol{I} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$. The six P's are dependent.

42 The dimension of $\mathbf{S}$ is (a) zero when $\boldsymbol{x} = \mathbf{0}$ (b) one when $\boldsymbol{x} = (1,1,1,1)$ (c) three when $\boldsymbol{x} = (1,1,-1,-1)$ because all rearrangements have $x_1 + \cdots + x_4 = 0$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** $\dim \mathbf{S} = 2$.

43 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients $=$ zero.

Key idea: The part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$, so the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now $\boldsymbol{x} - \boldsymbol{x} = \mathbf{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

44 The inputs to an m by n matrix fill $\mathbf{R}^n$. The outputs (column space!) have dimension r. The nullspace has $n - r$ special solutions. The formula becomes $r + (n - r) = n$.

Problem Set 3.6, page 190

1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(\boldsymbol{N}(A^{\mathrm{T}}))$ $= 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n - r)$ must be 3 (c) $[\,1 \;\; 1\,]$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

(e) *Impossible* Row space $=$ column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

6 A: $\dim$ $\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\boldsymbol{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: $\dim$ $\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\boldsymbol{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \mathbf{0}$ is in both $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

18 Row $3 - 2$ row $2 +$ row $1 =$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 1 independent solution $=$ last row of E^{-1}. ($E^{-1}A = R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

26 The rows of $C = AB$ are combinations of the rows of B. So rank $C \le$ rank B. Also rank $C \le$ rank A, because the columns of C are combinations of the columns of A.

29 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1$.

30 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \ne 0$.

31 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; $\dim(\text{nullspace}) = 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

32 The key is equal row spaces. First row of $A =$ combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

Problem Set 4.1, page 202

1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. The column space is perpendicular to the nullspace of A^{T} (two lines in $\mathbf{R}^2$ because rank $= 1$).

3 (a) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ (b) Impossible, $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular (d) Need $A^2 = 0$; take $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$
(e) $(1,1,1)$ in the nullspace (columns add to $\mathbf{0}$) and also row space; no such matrix.

6 Multiply the equations by $y_1, y_2, y_3 = 1, 1, -1$. Equations add to $0 = 1$ so no solution: $\boldsymbol{y} = (1, 1, -1)$ is in the left nullspace. $A\boldsymbol{x} = \boldsymbol{b}$ would need $0 = (\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

8 $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n = \boldsymbol{0}$ and $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n = A\boldsymbol{x}_r$. All $A\boldsymbol{x}$ are in $\boldsymbol{C}(A)$.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \boldsymbol{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$.

10 (a) With $A^{\mathrm{T}} = A$, the column and row spaces are the same (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

12 $\boldsymbol{x}$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n = (1, -1) + (1, 1) = (2, 0)$. Notice $\boldsymbol{N}(A^{\mathrm{T}})$ is a plane $(1, 0) = (1, 1)/2 + (1, -1)/2 = \boldsymbol{x}_r + \boldsymbol{x}_n$.

13 $V^{\mathrm{T}}W =$ zero makes each basis vector for $\boldsymbol{V}$ orthogonal to each basis vector for $\boldsymbol{W}$. Then every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$ (combinations of the basis vectors).

14 $A\boldsymbol{x} = B\widehat{\boldsymbol{x}}$ means that $[\,A \quad B\,]\begin{bmatrix}\boldsymbol{x}\\ -\widehat{\boldsymbol{x}}\end{bmatrix} = \boldsymbol{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $\boldsymbol{x} = (3, 1)$ and $\widehat{\boldsymbol{x}} = (1, 0)$ and $A\boldsymbol{x} = B\widehat{\boldsymbol{x}} = (5, 6, 5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must share a line.

16 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ leads to $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} = 0$. Then $\boldsymbol{y} \perp A\boldsymbol{x}$ and $\boldsymbol{N}(A^{\mathrm{T}}) \perp \boldsymbol{C}(A)$.

18 $\boldsymbol{S}^{\perp}$ is the nullspace of $A = \begin{bmatrix}1 & 5 & 1\\ 2 & 2 & 2\end{bmatrix}$. Therefore $\boldsymbol{S}^{\perp}$ is a *subspace* even if $\boldsymbol{S}$ is not.

21 For example $(-5, 0, 1, 1)$ and $(0, 1, -1, 0)$ span $\boldsymbol{S}^{\perp}$ =nullspace of $A = \begin{bmatrix}1 & 2 & 2 & 3\\ 1 & 3 & 3 & 2\end{bmatrix}$.

23 $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is perpendicular to any vector in $\boldsymbol{V}$. Since $\boldsymbol{V}$ contains all the vectors in $\boldsymbol{S}$, $\boldsymbol{x}$ is also perpendicular to any vector in $\boldsymbol{S}$. So every $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is also in $\boldsymbol{S}^{\perp}$.

28 (a) $(1, -1, 0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.

30 When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore the dimension of $\boldsymbol{C}(B) \leq$ dimension of $\boldsymbol{N}(A)$. This means $\operatorname{rank}(B) \leq 4 - \operatorname{rank}(A)$.

31 $\mathsf{null}(N')$ produces a basis for the *row space* of A (perpendicular to $\boldsymbol{N}(A)$).

32 We need $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{n} = 0$ and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\ell} = 0$. All possible examples have the form $a\boldsymbol{c}\boldsymbol{r}^{\mathrm{T}}$ with $a \neq 0$.

33 Both $\boldsymbol{r}$'s orthogonal to both $\boldsymbol{n}$'s, both $\boldsymbol{c}$'s orthogonal to both $\boldsymbol{\ell}$'s, each pair independent. All A's with these subspaces have the form $[\boldsymbol{c}_1\ \boldsymbol{c}_2]M[\boldsymbol{r}_1\ \boldsymbol{r}_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

Problem Set 4.2, page 214

1 (a) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 5/3$; $\boldsymbol{p} = 5\boldsymbol{a}/3$; $\boldsymbol{e} = (-2, 1, 1)/3$ (b) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = -1$; $\boldsymbol{p} = \boldsymbol{a}$; $\boldsymbol{e} = \boldsymbol{0}$.

3 $P_1 = \dfrac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$ and $P_1\boldsymbol{b} = \dfrac{1}{3}\begin{bmatrix}5\\ 5\\ 5\end{bmatrix}$. $P_2 = \dfrac{1}{11}\begin{bmatrix}1 & 3 & 1\\ 3 & 9 & 3\\ 1 & 3 & 1\end{bmatrix}$ and $P_2\boldsymbol{b} = \begin{bmatrix}1\\ 3\\ 1\end{bmatrix}$.

6 $\boldsymbol{p}_1=(\frac{1}{9},-\frac{2}{9},-\frac{2}{9})$ and $\boldsymbol{p}_2=(\frac{4}{9},\frac{4}{9},-\frac{2}{9})$ and $\boldsymbol{p}_3=(\frac{4}{9},-\frac{2}{9},\frac{4}{9})$. So $\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3=\boldsymbol{b}$.

9 Since A is invertible, $P=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}=AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}}=I$: project on all of $\mathbf{R}^2$.

11 (a) $\boldsymbol{p}=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}=(2,3,0)$, $\boldsymbol{e}=(0,0,4)$, $A^{\mathrm{T}}\boldsymbol{e}=\mathbf{0}$ (b) $\boldsymbol{p}=(4,4,6)$, $\boldsymbol{e}=\mathbf{0}$.

15 $2A$ has the same column space as A. $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

16 $\frac{1}{2}(1,2,-1)+\frac{3}{2}(1,0,1)=(2,1,1)$. So $\boldsymbol{b}$ is in the plane. Projection shows $P\boldsymbol{b}=\boldsymbol{b}$.

18 (a) $I-P$ is the projection matrix onto $(1,-1)$ in the perpendicular direction to $(1,1)$
(b) $I-P$ projects onto the plane $x+y+z=0$ perpendicular to $(1,1,1)$.

20 $\boldsymbol{e}=\begin{bmatrix}1\\-1\\-2\end{bmatrix}$, $Q=\dfrac{\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}}{\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}}=\begin{bmatrix}1/6&-1/6&-1/3\\-1/6&1/6&1/3\\-1/3&1/3&2/3\end{bmatrix}$, $I-Q=\begin{bmatrix}5/6&1/6&1/3\\1/6&5/6&-1/3\\1/3&-1/3&1/3\end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2=A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2=P$. $P\boldsymbol{b}$ is in the column space (where P projects). Then its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$.

24 The nullspace of A^{T} is *orthogonal* to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b}=\mathbf{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p}=\mathbf{0}$. Check $P\boldsymbol{b}=A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}=A(A^{\mathrm{T}}A)^{-1}\mathbf{0}$.

28 $P^2=P=P^{\mathrm{T}}$ give $P^{\mathrm{T}}P=P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of $P^{\mathrm{T}}P$ which is the length squared of column 2.

29 $A=B^{\mathrm{T}}$ has independent columns, so $A^{\mathrm{T}}A$ (which is BB^{T}) must be invertible.

30 (a) The column space is the line through $\boldsymbol{a}=\begin{bmatrix}3\\4\end{bmatrix}$ so $P_C=\dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}=\dfrac{1}{25}\begin{bmatrix}9&12\\12&25\end{bmatrix}$.
(b) The row space is the line through $\boldsymbol{v}=(1,2,2)$ and $P_R=\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_CA=A$ (columns of A project to themselves) and $AP_R=A$. Then $P_CAP_R=A$!

31 The error $\boldsymbol{e}=\boldsymbol{b}-\boldsymbol{p}$ must be perpendicular to all the $\boldsymbol{a}$'s.

32 Since $P_1\boldsymbol{b}$ is in $\boldsymbol{C}(A)$, $P_2(P_1\boldsymbol{b})$ equals $P_1\boldsymbol{b}$. So $P_2P_1=P_1=\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ where $\boldsymbol{a}=(1,2,0)$.

33 If $P_1P_2=P_2P_1$ then $\boldsymbol{S}$ is contained in $\boldsymbol{T}$ or $\boldsymbol{T}$ is contained in $\boldsymbol{S}$.

34 BB^{T} is invertible as in Problem 29. Then $(A^{\mathrm{T}}A)(BB^{\mathrm{T}})$ = product of r by r invertible matrices, so rank r. AB can't have rank $<r$, since A^{T} and B^{T} cannot increase the rank.
Conclusion: A (m by r of rank r) times B (r by n of rank r) produces AB of rank r.

Problem Set 4.3, page 226

1 $A=\begin{bmatrix}1&0\\1&1\\1&3\\1&4\end{bmatrix}$ and $\boldsymbol{b}=\begin{bmatrix}0\\8\\8\\20\end{bmatrix}$ give $A^{\mathrm{T}}A=\begin{bmatrix}4&8\\8&26\end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b}=\begin{bmatrix}36\\112\end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}}=\begin{bmatrix}1\\4\end{bmatrix}$ and $\boldsymbol{p}=A\widehat{\boldsymbol{x}}=\begin{bmatrix}1\\5\\13\\17\end{bmatrix}$ and $\boldsymbol{e}=\boldsymbol{b}-\boldsymbol{p}=\begin{bmatrix}-1\\3\\-5\\3\end{bmatrix}$
$E=\|\boldsymbol{e}\|^2=\mathbf{44}$

5 $E=(C-0)^2+(C-8)^2+(C-8)^2+(C-20)^2$. $A^{\mathrm{T}}=[1\ \ 1\ \ 1\ \ 1]$ and $A^{\mathrm{T}}A=[4]$. $A^{\mathrm{T}}\boldsymbol{b}=[36]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}=9=$ best height C. Errors $\boldsymbol{e}=(-9,-1,-1,11)$.

7 $A=[0\ \ 1\ \ 3\ \ 4]^{\mathrm{T}},\ A^{\mathrm{T}}A=[26]$ and $A^{\mathrm{T}}\boldsymbol{b}=[112]$. Best $D=112/26=56/13$.

8 $\widehat{\boldsymbol{x}}=56/13,\ \boldsymbol{p}=(56/13)(0,1,3,4)$. $(C,D)=(9,56/13)$ don't match $(C,D)=(1,4)$. Columns of A were not perpendicular so we can't project separately to find C and D.

9 Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix}1&0&0\\1&1&1\\1&3&9\\1&4&16\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix}=\begin{bmatrix}0\\8\\8\\20\end{bmatrix}$. $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=\begin{bmatrix}4&8&26\\8&26&92\\26&92&338\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix}=\begin{bmatrix}36\\112\\400\end{bmatrix}$.

11 (a) The best line $x=1+4t$ gives the center point $\widehat{b}=9$ when $\widehat{t}=2$.
(b) The first equation $Cm+D\sum t_i=\sum b_i$ divided by m gives $C+D\widehat{t}=\widehat{b}$.

13 $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b}-A\boldsymbol{x})=\widehat{\boldsymbol{x}}-\boldsymbol{x}$. When $\boldsymbol{e}=\boldsymbol{b}-A\boldsymbol{x}$ averages to $\boldsymbol{0}$, so does $\widehat{\boldsymbol{x}}-\boldsymbol{x}$.

14 The matrix $(\widehat{\boldsymbol{x}}-\boldsymbol{x})(\widehat{\boldsymbol{x}}-\boldsymbol{x})^{\mathrm{T}}$ is $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b}-A\boldsymbol{x})(\boldsymbol{b}-A\boldsymbol{x})^{\mathrm{T}}A(A^{\mathrm{T}}A)^{-1}$. When the average of $(\boldsymbol{b}-A\boldsymbol{x})(\boldsymbol{b}-A\boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2 I$, the average of $(\widehat{\boldsymbol{x}}-\boldsymbol{x})(\widehat{\boldsymbol{x}}-\boldsymbol{x})^{\mathrm{T}}$ will be the *output covariance matrix* $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\sigma^2 A(A^{\mathrm{T}}A)^{-1}$ which simplifies to $\sigma^2(A^{\mathrm{T}}A)^{-1}$.

16 $\dfrac{1}{10}b_{10}+\dfrac{\mathbf{9}}{\mathbf{10}}\widehat{x}_9=\dfrac{1}{10}(b_1+\cdots+b_{10})$. Knowing $\widehat{x}_9$ avoids adding all b's.

18 $\boldsymbol{p}=A\widehat{\boldsymbol{x}}=(5,13,17)$ gives the heights of the closest line. The error is $\boldsymbol{b}-\boldsymbol{p}=(2,-6,4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e}=P\boldsymbol{b}-P\boldsymbol{p}=\boldsymbol{p}-\boldsymbol{p}=\boldsymbol{0}$.

21 $\boldsymbol{e}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$; $\boldsymbol{p}$ is in $\boldsymbol{C}(A)$; $\widehat{\boldsymbol{x}}$ is in $\boldsymbol{C}(A^{\mathrm{T}})$; $\boldsymbol{N}(A)=\{\boldsymbol{0}\}=$ zero vector only.

23 The square of the distance between points on two lines is $E=(y-x)^2+(3y-x)^2+(1+x)^2$. Derivatives $\frac{1}{2}\partial E/\partial x=3x-4y+1=0$ and $\frac{1}{2}\partial E/\partial y=-4x+10y=0$. The solution is $x=-5/7, y=-2/7; E=2/7$, and the minimum distance is $\sqrt{2/7}$.

25 3 points on a line: *Equal slopes* $(b_2-b_1)/(t_2-t_1)=(b_3-b_2)/(t_3-t_2)$. Linear algebra: Orthogonal to $(1,1,1)$ and (t_1,t_2,t_3) is $\boldsymbol{y}=(t_2-t_3,t_3-t_1,t_1-t_2)$ in the left nullspace. $\boldsymbol{b}$ is in the column space. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}=0$ is the same equal slopes condition written as $(b_2-b_1)(t_3-t_2)=(b_3-b_2)(t_2-t_1)$.

27 The shortest link connecting two lines in space is *perpendicular to those lines*.

28 Only 1 plane contains $\boldsymbol{0},\boldsymbol{a}_1,\boldsymbol{a}_2$ unless $\boldsymbol{a}_1,\boldsymbol{a}_2$ are *dependent*. Same test for $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n$.

Problem Set 4.4, page 239

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries $1,4,9$.

6 Q_1Q_2 is orthogonal because $(Q_1Q_2)^{\mathrm{T}}Q_1Q_2=Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2=Q_2^{\mathrm{T}}Q_2=I$.

8 If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *orthonormal* vectors in $\mathbf{R}^5$ then $(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_1+(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_2$ is closest to $\boldsymbol{b}$.

11 (a) Two *orthonormal* vectors are $\boldsymbol{q}_1=\frac{1}{10}(1,3,4,5,7)$ and $\boldsymbol{q}_2=\frac{1}{10}(-7,3,4,-5,1)$
(b) Closest in the plane: *project* $QQ^{\mathrm{T}}(1,0,0,0,0)=(0.5,-0.18,-0.24,0.4,0)$.

13 The multiple to subtract is $\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$. Then $\boldsymbol{B}=\boldsymbol{b}-\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}=(4,0)-2\cdot(1,1)=(2,-2)$.

14 $\begin{bmatrix}1&4\\1&0\end{bmatrix}=\begin{bmatrix}\boldsymbol{q}_1&\boldsymbol{q}_2\end{bmatrix}\begin{bmatrix}\|\boldsymbol{a}\|&\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}\\0&\|\boldsymbol{B}\|\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2}&2\sqrt{2}\\0&2\sqrt{2}\end{bmatrix}=QR.$

15 (a) $\boldsymbol{q}_1 = \frac{1}{3}(1,2,-2),\ \boldsymbol{q}_2 = \frac{1}{3}(2,1,2),\ \boldsymbol{q}_3 = \frac{1}{3}(2,-2,-1)$ (b) The nullspace of A^{T} contains $\boldsymbol{q}_3$ (c) $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(1,2,7) = (1,2)$.

16 The projection $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is closest to $\boldsymbol{b}$; $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4,5,2,2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1,4,-4,-4)/7$ has $\|\boldsymbol{B}\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

18 $\boldsymbol{A} = \boldsymbol{a} = (1,-1,0,0)$; $\boldsymbol{B} = \boldsymbol{b}-\boldsymbol{p} = (\frac{1}{2},\frac{1}{2},-1,0)$; $\boldsymbol{C} = \boldsymbol{c}-\boldsymbol{p}_A-\boldsymbol{p}_B = (\frac{1}{3},\frac{1}{3},\frac{1}{3},-1)$. Notice the pattern in those orthogonal $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$. In $\mathbf{R}^5$, $\boldsymbol{D}$ would be $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-1)$.

20 (a) *True* (b) *True*. $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$.

21 The orthonormal vectors are $\boldsymbol{q}_1 = (1,1,1,1)/2$ and $\boldsymbol{q}_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4,-3,3,0)$ projects to $\boldsymbol{p} = (-7,-3,-1,3)/2$. And $\boldsymbol{b}-\boldsymbol{p} = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

22 $A = (1,1,2),\ B = (1,-1,0),\ C = (-1,-1,1)$. These are not yet unit vectors.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

28 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

30 The wavelet matrix W has orthonormal columns. Notice $W^{-1} = W^{\mathrm{T}}$ in Section 7.3.

32 $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ across plane $y + z = 0$.

33 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

Problem Set 5.1, page 251

1 $\det(2A) = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2 = \det(A^{\mathrm{T}})^{-1}$.

5 $|J_5| = 1,\ |J_6| = -1,\ |J_7| = -1$. Determinants $1, 1, -1, -1$ repeat so $|J_{101}| = 1$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so $\det$ can't blow up.

10 If the entries in every row add to zero, then $(1,1,\ldots,1)$ is in the nullspace: singular A has $\det = 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* $-\det DC$. If n is even we can have an invertible CD.

14 $\det(A) = 36$ and the 4 by 4 second difference matrix has $\det = 5$.

15 The first determinant is 0, the second is $1 - 2t^2 + t^4 = (1-t^2)^2$.

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3\det(K)$. This is $-\det(K)$. But always $\det(K^{\mathrm{T}}) = \det(K)$, so we must have $\det(K) = 0$ for 3 by 3.

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

23 $\det(A) = 10,\ A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix},\ \det(A^2) = 100,\ A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ has det $\frac{1}{10}$. $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = \mathbf{2}$ or $\lambda = \mathbf{5}$; those are eigenvalues.

27 $\det A = abc,\ \det B = -abcd,\ \det C = a(b-a)(c-b)$ by doing elimination.

32 Typical determinants of $\mathsf{rand}(n)$ are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. $\mathsf{randn}(n)$ with normal distribution gives $10^{31}, 10^{78}, 10^{186}$, Inf which means $\geq 2^{1024}$. MATLAB allows $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

Problem Set 5.2, page 263

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent.

4 $a_{11}a_{23}a_{32}a_{44}$ gives -1, because $2 \leftrightarrow 3$, $a_{14}a_{23}a_{32}a_{41}$ gives $+1$, $\det A = 1 - 1 = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms must be zero.

8 Some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is not zero! Move rows 1, 2, . . ., n into rows α, β, . . ., ω. Then these nonzero a's will be on the main diagonal.

9 To get $+1$ for the even permutations the matrix needs an *even* number of -1's. For the odd P's the matrix needs an *odd* number of -1's. So six 1's and $\det = 6$ are impossible five 1's and one -1 will give $AC = (ad - bc)I = (\det A)I$ $\max(\det) = 4$.

11 $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$. Puzzle: $\det D = 441 = (-21)^2$. *Why*?

12 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}} = C^{\mathrm{T}}/\det A$.

13 (a) $C_1 = 0$, $C_2 = -1$, $C_3 = 0$, $C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

15 The $1, 1$ cofactor of the n by n matrix is E_{n-1}. The $1, 2$ cofactor has a single 1 in its first column, with cofactor E_{n-2}: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then E_1 to E_6 is 1, 0, -1, -1, 0, 1 and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The $1, 1$ cofactor of the n by n matrix is F_{n-1}. The $1, 2$ cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the $1, 2$ entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

19 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x - a)(x - b)(x - c)$. Multiply by the cofactor V_3. The Vandermonde matrix $V_{ij} = (x_i)^{j-1}$ is for fitting a polynomial $p(\boldsymbol{x}) = \boldsymbol{b}$ at the points x_i. It has $\det V =$ product of all $x_k - x_m$ for $k > m$.

20 $G_2 = -1$, $G_3 = 2$, $G_4 = -3$, and $G_n = (-1)^{n-1}(n - 1) =$ (product of the λ's).

24 (a) All L's have $\det = 1$; $\det U_k = \det A_k = 2, 6, -6$ (b) Pivots $5, 6/5, 7/6$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} =$ cofactor C_{11}.

29 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+\,(1,1)(2,2)(3,3)(4,4)\; +\; (1,2)(2,1)(3,4)(4,3)\; -\; (1,2)(2,1)(3,3)(4,4)\; -\; (1,1)(2,2)(3,4)(4,3)\; -\; (1,1)(2,3)(3,2)(4,4)$. Total -1.

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule:
$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}$.

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor $= 1$: then $\det$ drops by 1.

34 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.

Problem Set 5.3, page 278

2 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad - bc)$ (b) $y = \det B_2 / \det A = (fg - id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0}/\mathbf{0}$: *undetermined*.

4 (a) $x_1 = \det\big([\,\boldsymbol{b} \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3\,]\big) / \det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\,\boldsymbol{a}_2\,\boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\,\boldsymbol{a}_2\,\boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{a}_3|$ which is $x_1 \det A$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\dfrac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $\det A = 3$. The $1,3$ cofactor of A is 0. Multiplying by 4 or 100: no change.

9 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

11 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

15 For $n = 5$, C contains *25* cofactors and each 4 by 4 cofactor has *24* terms. Each term needs *3* multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

17 Volume$=\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}=20$. Area of faces length of cross product $=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = $ $-2\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$ length$=6\sqrt{2}$

18 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

21 The maximum volume is $L_1L_2L_3L_4$ reached when the edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved.

23 $A^{\mathrm{T}}A = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} & 0 & 0 \\ 0 & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} \end{bmatrix}$ has $\begin{array}{lcl} \det A^{\mathrm{T}}A & = & (\|a\|\|b\|\|c\|)^2 \\ \det A & = & \pm\|a\|\|b\|\|c\| \end{array}$

25 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2+x)^n$ in Worked Example **2.4A**. Cube from $2I$ has volume 2^n.

26 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbf{R}^n$)

31 Base area 10, height 2, volume 20.

35 $S = (2, 1, -1)$, area $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$. The other four corners can be $(0, 0, 0)$, $(0, 0, 2)$, $(1, 2, 2)$, $(1, 1, 0)$. The volume of the tilted box is $|\det| = 1$.

39 $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1}$ is $1/\det A$, construct A^{-1} using the cofactors. *Invert to find A.*

Problem Set 6.1, page 293

1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1,1)$ and $\boldsymbol{x}_2 = (2,-1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to find $\boldsymbol{x}$.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $\boldsymbol{x}_1 = (1,2)$ and $\boldsymbol{x}_2 = (1,-1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Columns of $A - \lambda_1 I$ are in the nullspace of $A - \lambda_2 I$ because $M = (A - \lambda_2 I)(A - \lambda_1 I)$ $=$ zero matrix [this is the *Cayley-Hamilton Theorem* in Problem 6.2.32]. Notice that M has *zero eigenvalues* $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$ and $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$.

13 (a) $P\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}) = \boldsymbol{u}$ so $\lambda = 1$ (b) $P\boldsymbol{v} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}) = \boldsymbol{0}$ (c) $\boldsymbol{x}_1 = (-1,1,0,0)$, $\boldsymbol{x}_2 = (-3,0,1,0)$, $\boldsymbol{x}_3 = (-5,0,0,1)$ all have $P\boldsymbol{x} = 0\boldsymbol{x} = \boldsymbol{0}$.

15 The other two eigenvalues are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.

16 Set $\lambda = 0$ in $\det(A - \lambda I) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)$ to find $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{})$ add to $a + d$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$.

19 (a) rank $= 2$ (b) $\det(B^{\mathrm{T}}B) = 0$ (d) eigenvalues of $(B^2 + I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$.

20 Last rows are $-28, 11$ (check trace and $\det$) and $6, -11, 6$ (to match $\det(C - \lambda I)$).

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues $=$ trace $= \frac{1}{2}$).

23 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.32.

28 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have $\det = -3$.

32 (a) $\boldsymbol{u}$ is a basis for the nullspace, $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space
(b) $\boldsymbol{x} = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any $c\boldsymbol{u}$ from the nullspace
(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space: wrong dimension 3.

34 $\det(P - \lambda I) = 0$ gives the equation $\lambda^4 = 1$. This reflects the fact that $P^4 = I$. The solutions of $\lambda^4 = 1$ are $\lambda = 1, i, -1, -i$. The real eigenvector $\boldsymbol{x}_1 = (1,1,1,1)$ is not changed by the permutation P. Three more eigenvectors are (i, i^2, i^3, i^4) and $(1,-1,1,-1)$ and $(-i, (-i)^2, (-i)^3, (-i)^4)$.

36 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and $\det$. So does every $M^{-1}AM$!

Problem Set 6.2, page 307

1 $\begin{bmatrix}1&2\\0&3\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}$; $\begin{bmatrix}1&1\\3&3\end{bmatrix}=\begin{bmatrix}1&1\\-1&3\end{bmatrix}\begin{bmatrix}0&0\\0&4\end{bmatrix}\begin{bmatrix}\frac{3}{4}&-\frac{1}{4}\\\frac{1}{4}&\frac{1}{4}\end{bmatrix}$.

3 If $A=S\Lambda S^{-1}$ then the eigenvalue matrix for $A+2I$ is $\Lambda+2I$ and the eigenvector matrix is still S. $A+2I=S(\Lambda+2I)S^{-1}=S\Lambda S^{-1}+S(2I)S^{-1}=A+2I$.

4 (a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of S

6 The columns of S are nonzero multiples of (2,1) and (0,1): either order. Same for A^{-1}.

8 $A=S\Lambda S^{-1}=\begin{bmatrix}1&1\\1&0\end{bmatrix}=\dfrac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1&\lambda_2\\1&1\end{bmatrix}\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}\begin{bmatrix}1&-\lambda_2\\-1&\lambda_1\end{bmatrix}$. $S\Lambda^kS^{-1}=$ $\dfrac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1&\lambda_2\\1&1\end{bmatrix}\begin{bmatrix}\lambda_1^k&0\\0&\lambda_2^k\end{bmatrix}\begin{bmatrix}1&-\lambda_2\\-1&\lambda_1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\textit{2nd component}\text{ is }F_k\\(\lambda_1^k-\lambda_2^k)/(\lambda_1-\lambda_2)\end{bmatrix}$.

9 (a) $A=\begin{bmatrix}.5&.5\\1&0\end{bmatrix}$ has $\lambda_1=1,\ \lambda_2=-\frac{1}{2}$ with $\boldsymbol{x}_1=(1,1),\ \boldsymbol{x}_2=(1,-2)$

(b) $A^n=\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}1^n&0\\0&(-.5)^n\end{bmatrix}\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&-\frac{1}{3}\end{bmatrix}\to A^\infty=\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix}$

12 (a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 $A=\begin{bmatrix}8&3\\-3&2\end{bmatrix}$ (or other), $A=\begin{bmatrix}9&4\\-4&1\end{bmatrix}$, $A=\begin{bmatrix}10&5\\-5&0\end{bmatrix}$; only eigenvectors are $\boldsymbol{x}=(c,-c)$.

15 $A^k=S\Lambda^kS^{-1}$ approaches zero **if and only if every $|\boldsymbol{\lambda}|<1$**; $A_1^k\to A_1^\infty, A_2^k\to 0$.

17 $\Lambda=\begin{bmatrix}.9&0\\0&.3\end{bmatrix}$, $S=\begin{bmatrix}3&-3\\1&1\end{bmatrix}$; $A_2^{10}\begin{bmatrix}3\\1\end{bmatrix}=(.9)^{10}\begin{bmatrix}3\\1\end{bmatrix}$, $A_2^{10}\begin{bmatrix}3\\-1\end{bmatrix}=(.3)^{10}\begin{bmatrix}3\\-1\end{bmatrix}$, $A_2^{10}\begin{bmatrix}6\\0\end{bmatrix}=(.9)^{10}\begin{bmatrix}3\\1\end{bmatrix}+(.3)^{10}\begin{bmatrix}3\\-1\end{bmatrix}$ because $\begin{bmatrix}6\\0\end{bmatrix}$ is the sum of $\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}3\\-1\end{bmatrix}$.

19 $B^k=\begin{bmatrix}1&1\\0&-1\end{bmatrix}\begin{bmatrix}5&0\\0&4\end{bmatrix}^k\begin{bmatrix}1&1\\0&-1\end{bmatrix}=\begin{bmatrix}5^k&5^k-4^k\\0&4^k\end{bmatrix}$.

21 trace $ST=(aq+bs)+(cr+dt)$ is equal to $(qa+rc)+(sb+td)=$ trace TS. Diagonalizable case: the trace of $S\Lambda S^{-1}=$ trace of $(\Lambda S^{-1})S=\Lambda$: *sum of the* λ's.

24 The A's form a subspace since cA and A_1+A_2 all have the same S. When $S=I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

26 Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

27 $R=S\sqrt{\Lambda}S^{-1}=\begin{bmatrix}2&1\\1&2\end{bmatrix}$ has $R^2=A$. $\sqrt{B}$ needs $\lambda=\sqrt{9}$ and $\sqrt{-1}$, trace is not real. Note that $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ can have $\sqrt{-1}=i$ and $-i$, trace 0, real square root $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$.

28 $A^{\mathrm{T}}=A$ gives $\boldsymbol{x}^{\mathrm{T}}AB\boldsymbol{x}=(A\boldsymbol{x})^{\mathrm{T}}(B\boldsymbol{x})\le\|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$ by the Schwarz inequality. $B^{\mathrm{T}}=-B$ gives $-\boldsymbol{x}^{\mathrm{T}}BA\boldsymbol{x}=(B\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x})\le\|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$. Add to get Heisenberg's Uncertainty Principle when $AB-BA=I$. Position-momentum, also time-energy.

32 If $A = S\Lambda S^{-1}$ then $(A-\lambda_1 I)\cdots(A-\lambda_n I)$ equals $S(\Lambda-\lambda_1 I)\cdots(\Lambda-\lambda_n I)S^{-1}$. The factor $\Lambda-\lambda_j I$ is zero in row j. *The product is zero in all rows = **zero matrix***.

33 $\lambda = 2, -1, 0$ are in $\boldsymbol{\Lambda}$ and the eigenvectors are in S (below). $A^k = S\boldsymbol{\Lambda}^k S^{-1}$ is

$$\begin{bmatrix}2&1&0\\1&-1&1\\1&-1&-1\end{bmatrix}\boldsymbol{\Lambda^k}\frac{1}{6}\begin{bmatrix}2&1&1\\2&-2&-2\\0&3&-3\end{bmatrix} = \frac{2^k}{6}\begin{bmatrix}4&2&2\\2&1&1\\2&1&1\end{bmatrix} + \frac{(-1)^k}{3}\begin{bmatrix}1&-1&-1\\-1&1&1\\-1&1&1\end{bmatrix}$$

Check $k=4$. The $(2,2)$ entry of A^4 is $2^4/6+(-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Much harder to find the eleven 4-step paths that start and end at node 1.

35 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1\pm\sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

37 Columns of S times rows of ΛS^{-1} will give r rank-1 matrices ($r =$ rank of A).

Problem Set 6.3, page 325

1 $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{u}_2 = e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$. If $\boldsymbol{u}(0) = (5,-2)$, then $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$.

4 $d(v+w)/dt = (w-v)+(v-w) = \mathbf{0}$, so the total $v+w$ is constant. $A = \begin{bmatrix}-1&1\\1&-1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\\lambda_2 = -2\end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$; $\begin{matrix}v(1) = 20+10e^{-2} & v(\infty) = 20\\ w(1) = 20-10e^{-2} & w(\infty) = 20\end{matrix}$

8 $\begin{bmatrix}6&-2\\2&1\end{bmatrix}$ has $\lambda_1 = 5$, $\boldsymbol{x}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\lambda_2 = 2$, $\boldsymbol{x}_2 = \begin{bmatrix}1\\2\end{bmatrix}$; rabbits $r(t) = 20e^{5t}+10e^{2t}$, $w(t) = 10e^{5t}+20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; e^{5t} dominates.

12 $A = \begin{bmatrix}0&1\\-9&6\end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1,3)$.

14 When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ is $\|\boldsymbol{u}(0)\|$. So e^{At} is *orthogonal*.

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^t+4$; $\quad \boldsymbol{u}_p = \begin{bmatrix}4\\2\end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^t\begin{bmatrix}1\\t\end{bmatrix} + c_2e^t\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}4\\2\end{bmatrix}$.

16 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A-cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A-cI)^{-1}\boldsymbol{b}$ = particular solution. If c is an eigenvalue then $A-cI$ is not invertible.

20 The solution at time $t+T$ is also $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

21 $\begin{bmatrix}1&4\\0&0\end{bmatrix} = \begin{bmatrix}1&4\\0&-1\end{bmatrix}\begin{bmatrix}\mathbf{1}&0\\0&\mathbf{0}\end{bmatrix}\begin{bmatrix}1&4\\0&-1\end{bmatrix}$; $\begin{bmatrix}1&4\\0&-1\end{bmatrix}\begin{bmatrix}\boldsymbol{e^t}&0\\0&\mathbf{1}\end{bmatrix}\begin{bmatrix}1&4\\0&-1\end{bmatrix} = \begin{bmatrix}e^t&4e^t-4\\0&1\end{bmatrix}$.

22 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}\boldsymbol{At^2} + \cdots = I + (e^t-1)A = \begin{bmatrix}e^t&e^t-1\\0&1\end{bmatrix}$.

24 $A = \begin{bmatrix}1&1\\0&3\end{bmatrix} = \begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}\mathbf{1}&0\\0&\mathbf{3}\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}$. Then $e^{At} = \begin{bmatrix}\boldsymbol{e^t}&\frac{1}{2}(e^{3t}-e^t)\\0&\boldsymbol{e^{3t}}\end{bmatrix}$.

26 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$.

27 $(x, y) = (e^{4t}, e^{-4t})$ is a growing solution. The correct matrix for the exchanged $\boldsymbol{u} = (y, x)$ is $\begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

28 Centering produces $\boldsymbol{U}_{n+1} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1-(\Delta t)^2 \end{bmatrix} \boldsymbol{U}_n$. At $\Delta t = 1$, $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$ both have $\lambda^6 = 1$ so $A^6 = I$. $\boldsymbol{U}_6 = A^6\boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

29 First A has $\lambda = \pm i$ and $A^4 = I$
Second A has $\lambda = -1, -1$ and $A^n = (-1)^n \begin{bmatrix} 1-2n & -2n \\ 2n & 2n+1 \end{bmatrix}$ Linear growth.

30 With $a = \Delta t/2$ the trapezoidal step is $\boldsymbol{U}_{n+1} = \dfrac{1}{1+a^2} \begin{bmatrix} 1-a^2 & 2a \\ -2a & 1-a^2 \end{bmatrix} \boldsymbol{U}_n$.

Orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|\boldsymbol{U}_{n+1}\| = \|\boldsymbol{U}_n\|$

31 (a) $(\cos A)\boldsymbol{x} = (\cos\lambda)\boldsymbol{x}$ (b) $\lambda(A) = 2\pi$ and 0 so $\cos\lambda = 1, 1$ and $\cos A = I$ (c) $\boldsymbol{u}(t) = 3(\cos 2\pi t)(1,1) + 1(\cos 0t)(1,-1)$ [$\boldsymbol{u}' = A\boldsymbol{u}$ has **exp**, $\boldsymbol{u}'' = A\boldsymbol{u}$ has **cos**]

Problem Set 6.4, page 337

3 $\lambda = 0, 4, -2$; unit vectors $\pm(0,1,-1)/\sqrt{2}$ and $\pm(2,1,1)/\sqrt{6}$ and $\pm(1,-1,-1)/\sqrt{3}$.

5 $Q = \dfrac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of A
Each unit eigenvector could be multiplied by -1

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ gives $\Lambda = 0$. The only symmetric A is $Q\,0\,Q^{\mathrm{T}} =$ zero matrix.

10 If $\boldsymbol{x}$ is not real then $\lambda = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is *not* always real. Can't assume real eigenvectors!

11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{bmatrix} + 4\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0\begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

14 M is skew-symmetric and orthogonal; λ's must be $i, i, -i, -i$ to have trace zero.

16 (a) If $A\boldsymbol{z} = \lambda\boldsymbol{y}$ and $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{z}$ then $B[\,\boldsymbol{y};\ -\boldsymbol{z}\,] = [\,-A\boldsymbol{z};\ A^{\mathrm{T}}\boldsymbol{y}\,] = -\lambda[\,\boldsymbol{y};\ -\boldsymbol{z}\,]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}A\boldsymbol{z} = A^{\mathrm{T}}(\lambda\boldsymbol{y}) = \lambda^2\boldsymbol{z}$. (c) $\lambda = -1, -1, 1, 1$; $\boldsymbol{x}_1 = (1,0,-1,0)$, $\boldsymbol{x}_2 = (0,1,0,-1)$, $\boldsymbol{x}_3 = (1,0,1,0)$, $\boldsymbol{x}_4 = (0,1,0,1)$.

19 A has $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular for A
Not perpendicular for B
since $B^{\mathrm{T}} \neq B$

21 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$
(c) True from $A^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

22 A and A^{T} have the same λ's but the *order* of the $\boldsymbol{x}$'s can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\boldsymbol{x}_1 = (1, i)$ first for A but $\boldsymbol{x}_1 = (1, -i)$ first for A^{T}.

23 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows $QR, S\Lambda S^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

24 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no S if $b = -1$; singular if $b = 0$.

25 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $A = \pm I$ or $A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

27 The roots of $\lambda^2 + b\lambda + c = 0$ differ by $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.

29 (a) $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ times $\overline{A}^{\mathrm{T}} = Q\overline{\Lambda}^{\mathrm{T}}\overline{Q}^{\mathrm{T}}$ equals $\overline{A}^{\mathrm{T}}$ times A because $\Lambda\overline{\Lambda}^{\mathrm{T}} = \overline{\Lambda}^{\mathrm{T}}\Lambda$ (diagonal!) (b) step 2: The $1,1$ entries of $\overline{T}^{\mathrm{T}}T$ and $T\overline{T}^{\mathrm{T}}$ are $|a|^2$ and $|a|^2 + |b|^2$. This makes $b = 0$ and $T = \Lambda$.

30 a_{11} is $[q_{11} \ldots q_{1n}]\ [\lambda_1\overline{q}_{11} \ldots \lambda_n\overline{q}_{1n}]^{\mathrm{T}} \le \lambda_{\max}\left(|q_{11}|^2 + \cdots + |q_{1n}|^2\right) = \lambda_{\max}$.

31 (a) $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x} = -\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$. (b) $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z}$ is pure imaginary, its real part is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{y} = 0 + 0$ (c) $\det A = \lambda_1 \ldots \lambda_n \ge 0$: pairs of λ's $= ib, -ib$.

Problem Set 6.5, page 350

3 Positive definite for $-3 < b < 3$ $\quad \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9 - b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c - 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$.

4 $f(x,y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2;\ x^2 + 6xy + 9y^2 = (x + 3y)^2$.

8 $A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Pivots $3, 4$ outside squares, ℓ_{ij} inside. $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 3(x + 2y)^2 + 4y^2$

10 $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $\quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is singular; $B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

12 A is positive definite for $c > 1$; determinants $c, c^2 - 1, (c - 1)^2(c + 2) > 0$. B is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $\boldsymbol{x}^{\mathrm{T}}A^{-1}\boldsymbol{x} = (A^{-1}\boldsymbol{x})^{\mathrm{T}}A(A^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$.

17 If a_{jj} were smaller than all λ's, $A - a_{jj}I$ would have all eigenvalues > 0 (positive definite). But $A - a_{jj}I$ has a *zero* in the (j,j) position; impossible by Problem 16.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ by determinants.

22 $R = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $R = Q\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $1/\sqrt{\lambda} = \sqrt{2}$ and $\sqrt{2/3}$.

25 $A = C^{\mathrm{T}}C = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

29 $H_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$ is positive definite if $x \neq 0$; $F_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$; $H_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite, $(0, 1)$ is a saddle point of F_2.

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero on the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with $\det = 1$ are groups. Examples of subgroups are orthogonal matrices with $\det = 1$, exponentials e^{An} for integer n.

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2 - \sqrt{3},\ 2 - 1,\ 2,\ 2 + 1,\ 2 + \sqrt{3}$: product of eigenvalues $= 6 = \det K$.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

6 *Eight families* of similar matrices: six matrices have $\lambda = 0, 1$ (one family); three matrices have $\lambda = 1, 1$ and three have $\lambda = 0, 0$ (two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$; two have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$ (they are in one family).

7 (a) $(M^{-1}AM)(M^{-1}\boldsymbol{x}) = M^{-1}(A\boldsymbol{x}) = M^{-1}\mathbf{0} = \mathbf{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 Same Λ Same S But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$ and $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$ and $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

14 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1}J_iM_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those diagonal blocks M_i to get $M_0^{-1}JM_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}}J^{\mathrm{T}}M^{\mathrm{T}}$ equals $(M^{-1})^{\mathrm{T}}M_0^{-1}JM_0M^{\mathrm{T}} = (MM_0M^{\mathrm{T}})^{-1}A(MM_0M^{\mathrm{T}})$, and A^{T} *is similar to* A.

17 (a) *False*: Diagonalize a nonsymmetric $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar (b) *True*: A singular matrix has $\lambda = 0$. (c) *False*: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (they have $\lambda = \pm 1$) (d) *True*: Adding I increases all eigenvalues by 1

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *If* $AB\boldsymbol{x} = \lambda\boldsymbol{x}$ *then* $BA(B\boldsymbol{x}) = \lambda(B\boldsymbol{x})$.

19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeros.

22 $A = MJM^{-1}, A^n = MJ^nM^{-1} = 0$ (each J^k has 1's on the kth diagonal). $\det(A - \lambda I) = \lambda^n$ so $J^n = 0$ by the Cayley-Hamilton Theorem.

Problem Set 6.7, page 371

1 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3+\sqrt{5}}{2}$, $\sigma_2^2 = \dfrac{3-\sqrt{5}}{2}$. But A is indefinite
$\sigma_1 = (1+\sqrt{5})/2 = \lambda_1(A)$, $\sigma_2 = (\sqrt{5}-1)/2 = -\lambda_2(A)$; $\boldsymbol{u}_1 = \boldsymbol{v}_1$ but $\boldsymbol{u}_2 = -\boldsymbol{v}_2$.

5 A proof that ***eigshow*** finds the SVD. When $\boldsymbol{V}_1 = (1,0), \boldsymbol{V}_2 = (0,1)$ the demo finds $A\boldsymbol{V}_1$ and $A\boldsymbol{V}_2$ at some angle θ. A $90°$ turn by the mouse to $\boldsymbol{V}_2, -\boldsymbol{V}_1$ finds $A\boldsymbol{V}_2$ and $-A\boldsymbol{V}_1$ at the angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must produce $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ *at angle* $\pi/2$. Those orthogonal directions give $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

9 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

14 The smallest change in A is to set its smallest singular value σ_2 to zero.

15 The singular values of $A + I$ are *not* $\sigma_j + 1$. Need eigenvalues of $(A+I)^{\mathrm{T}}(A+I)$.

17 $A = U\Sigma V^{\mathrm{T}} =$ [cosines including $\boldsymbol{u}_4$] **diag**(**sqrt**$(2-\sqrt{2}, 2, 2+\sqrt{2})$) [sine matrix]$^{\mathrm{T}}$. $AV = U\Sigma$ says that differences of sines in V are cosines in U times σ's.

Problem Set 7.1, page 380

3 $T(\boldsymbol{v}) = (0,1)$ and $T(\boldsymbol{v}) = v_1 v_2$ are not linear.

4 (a) $S(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $S(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)) = S(T(\boldsymbol{v}_1)) + S(T(\boldsymbol{v}_2))$.

5 Choose $\boldsymbol{v} = (1,1)$ and $\boldsymbol{w} = (-1,0)$. $T(\boldsymbol{v}) + T(\boldsymbol{w}) = (0,1)$ but $T(\boldsymbol{v}+\boldsymbol{w}) = (0,0)$.

7 (a) $T(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $T(T(\boldsymbol{v})) = \boldsymbol{v} + (2,2)$ (c) $T(T(\boldsymbol{v})) = -\boldsymbol{v}$ (d) $T(T(\boldsymbol{v})) = T(\boldsymbol{v})$.

10 Not invertible: (a) $T(1,0) = \mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1) = \mathbf{0}$.

12 Write $\boldsymbol{v}$ as a combination $c(1,1) + d(2,0)$. Then $T(\boldsymbol{v}) = c(2,2) + d(0,0)$. $T(\boldsymbol{v}) = (4,4)$; $(2,2)$; $(2,2)$; if $\boldsymbol{v} = (a,b) = b(1,1) + \frac{a-b}{2}(2,0)$ then $T(\boldsymbol{v}) = b(2,2) + (0,0)$.

16 No matrix A gives $A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from $\mathbf{4}$ by $\mathbf{4}$ matrices. Those in Problems 13–15 were special.

17 (a) True (b) True (c) True (d) False.

19 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1,0) = (a_{11}, 0)$.

27 Also $\mathbf{30}$ emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0,0)$.)

Problem Set 7.2, page 395

3 (Matrix $A)^2 = B$ when (transformation $T)^2 = S$ and output basis = input basis.

5 $T(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) = 2\boldsymbol{w}_1 + \boldsymbol{w}_2 + 2\boldsymbol{w}_3$; A times $(1, 1, 1)$ gives $(2, 1, 2)$.

6 $\boldsymbol{v} = c(\boldsymbol{v}_2 - \boldsymbol{v}_3)$ gives $T(\boldsymbol{v}) = \mathbf{0}$; nullspace is $(0, c, -c)$; solutions $(1, 0, 0) + (0, c, -c)$.

8 For $T^2(\boldsymbol{v})$ we would need to know $T(\boldsymbol{w})$. If the $\boldsymbol{w}$'s equal the $\boldsymbol{v}$'s, the matrix is A^2.

12 (c) is wrong because $\boldsymbol{w}_1$ is not generally in the input space.

14 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

16 $MN = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

18 $(a, b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

20 $\boldsymbol{w}_2(x) = 1 - x^2$; $\boldsymbol{w}_3(x) = \frac{1}{2}(x^2 - x)$; $\boldsymbol{y} = 4\boldsymbol{w}_1 + 5\boldsymbol{w}_2 + 6\boldsymbol{w}_3$.

23 The matrix M with these nine entries must be invertible.

27 If T is not invertible, $T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)$ is not a basis. We couldn't choose $\boldsymbol{w}_i = T(\boldsymbol{v}_i)$.

30 S takes (x, y) to $(-x, y)$. $S(T(\boldsymbol{v})) = (\mathbf{-1}, \mathbf{2})$. $S(\boldsymbol{v}) = (-2, 1)$ and $T(S(\boldsymbol{v})) = (\mathbf{1}, \mathbf{-2})$.

34 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, -2, -2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

35 The wavelet basis is $(1, 1, 1, 1, 1, 1, 1, 1)$ and the long wavelet and two medium wavelets $(1, 1, -1, -1, 0, 0, 0, 0)$, $(0, 0, 0, 0, 1, 1, -1, -1)$ and 4 wavelets with a single pair $1, -1$.

36 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

37 Multiplication by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with this basis is represented by 4 by 4 $A = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}$

38 If $\boldsymbol{w}_1 = A\boldsymbol{v}_1$ and $\boldsymbol{w}_2 = A\boldsymbol{v}_2$ then $a_{11} = a_{22} = 1$. All other entries will be zero.

Problem Set 7.3, page 406

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

$A\boldsymbol{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 5 \\ 15 \end{bmatrix} = \sigma_1 \boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \mathbf{0}$. $\boldsymbol{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $AA^{\mathrm{T}}\boldsymbol{u}_1 = 50\, \boldsymbol{u}_1$.

3 $A = QH = \frac{1}{\sqrt{50}} \begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \frac{1}{\sqrt{50}} \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

4 $A^+ = V \begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}} = \frac{1}{50} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

7 $\begin{bmatrix} \sigma_1 \boldsymbol{u}_1 & \sigma_2 \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r \boldsymbol{u}_r \boldsymbol{v}_r^{\mathrm{T}}$.

9 A^+ is A^{-1} because A is invertible. Pseudoinverse equals inverse when A^{-1} exists!

11 $A = [\,1\,]\,[\,5 \;\; 0 \;\; 0\,]V^{\mathrm{T}}$ and $A^+ = V \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $AA^+ = [\,1\,]$

13 If $\det A = 0$ then rank$(A) < n$; thus rank$(A^+) < n$ and $\det A^+ = 0$.

16 $\boldsymbol{x}^+$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}} - \boldsymbol{x}^+$ in the nullspace of $A^{\mathrm{T}}A =$ nullspace of A. The right triangle has $c^2 = a^2 + b^2$.

17 $AA^+\boldsymbol{p} = \boldsymbol{p}, \;\; AA^+\boldsymbol{e} = \boldsymbol{0}, \;\; A^+A\boldsymbol{x}_r = \boldsymbol{x}_r, \;\; A^+A\boldsymbol{x}_n = \boldsymbol{0}.$

19 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1+3$ for LU, $1+2+1$ for LDU, $1+3$ for QR, $1+2+1$ for $U\Sigma V^{\mathrm{T}}$, $2+2+0$ for $S\Lambda S^{-1}$.

22 Keep only the r by r corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^{\mathrm{T}}$ has the required form $A = \widehat{U}M_1\Sigma_r M_2^{\mathrm{T}}\widehat{V}^{\mathrm{T}}$ with an invertible $M = M_1\Sigma_r M_2^{\mathrm{T}}$ in the middle.

23 $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{v} \\ A^{\mathrm{T}}\boldsymbol{u} \end{bmatrix} = \sigma \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}.$ The singular values of A are *eigenvalues* of this block matrix.

Problem Set 8.1, page 418

3 The rows of the free-free matrix in equation (9) add to $[\,0 \;\; 0 \;\; 0\,]$ so the right side needs $f_1 + f_2 + f_3 = 0$. $\boldsymbol{f} = (-1, 0, 1)$ gives $c_2u_1 - c_2u_2 = -1, c_3u_2 - c_3u_3 = -1, 0 = 0$. Then $\boldsymbol{u}_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}} = (1,1,1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx = -\left[c(x)\frac{du}{dx}\right]_0^1 = 0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx = 0$.

6 Multiply $A_1^{\mathrm{T}}C_1A_1$ as columns of A_1^{T} times c's times rows of A_1. The first 3 by 3 "*element matrix*" $c_1E_1 = [\,1 \;\; 0 \;\; 0\,]^{\mathrm{T}}c_1[\,1 \;\; 0 \;\; 0\,]$ has c_1 in the top left corner.

8 The solution to $-u'' = 1$ with $u(0) = u(1) = 0$ is $u(x) = \frac{1}{2}(x - x^2)$. At $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ this gives $\boldsymbol{u} = 2, 3, 3, 2$ (discrete solution in Problem 7) times $(\Delta x)^2 = 1/25$.

11 Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: $E = \mathsf{diag}(\mathsf{ones}(6,1),1)$; $K = 64 * (2 * \mathsf{eye}(7) - E - E')$; $D = 80 * (E - \mathsf{eye}(7))$; $(K + D)\backslash\mathsf{ones}(7,1)$; % forward; $(K - D')\backslash\mathsf{ones}(7,1)$; % backward; $(K + D/2 - D'/2)\backslash\mathsf{ones}(7,1)$; % centered is usually the best: more accurate

Problem Set 8.2, page 428

1 $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$; nullspace contains $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ for $\boldsymbol{y} = (1, -1, 1)$; current along edge 1, edge 3, back on edge 2 (full loop).

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is solvable for $\boldsymbol{f} = (1, -1, 0)$ and not solvable for $\boldsymbol{f} = (1, 0, 0)$; $\boldsymbol{f}$ must be orthogonal to $(1, 1, 1)$ in the nullspace: $f_1 + f_2 + f_3 = 0$.

6 $A^{\mathrm{T}}A\boldsymbol{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \boldsymbol{f}$ produces $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = 1, -1, 0$ and currents $-A\boldsymbol{x} = 2, 1, -1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 $A^{\mathrm{T}} \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$; $\boldsymbol{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ yields $\boldsymbol{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} +$ any $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = \frac{5}{4}, 1, \frac{7}{8}$ and currents $-CA\boldsymbol{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$.

9 Elimination on $A\boldsymbol{x} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

11 $A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$
diagonal entry = number of edges into the node
the trace is 2 times the number of nodes
off-diagonal entry $= -1$ if nodes are connected
$A^{\mathrm{T}}A$ is the **graph Laplacian**, $A^{\mathrm{T}}CA$ is **weighted** by C

13 $A^{\mathrm{T}}CA\boldsymbol{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$
gives four potentials $\boldsymbol{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$
I grounded $x_4 = 0$ and solved for $\boldsymbol{x}$
currents $\boldsymbol{y} = -CA\boldsymbol{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$

17 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

Problem Set 8.3, page 437

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^{\infty} = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; 1 and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1) =$ everyone is dead.

6 Add the components of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^{\infty}$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$

9 M^2 is still nonnegative; $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} M = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$ so multiply on the right by M to find $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} M^2 = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\boldsymbol{x}_1 = (b, 1 - a)$.

12 B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3, .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t}\boldsymbol{x}_1 = c_1\boldsymbol{x}_1$.

13 $\boldsymbol{x} = (1, 1, 1)$ is an eigenvector when the row sums are equal; $A\boldsymbol{x} = (.9, .9, .9)$.

15 The first two A's have $\lambda_{\max} < 1$; $\boldsymbol{p} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state $(.3, .3, .4)$ and $(30, 30, 40)$.

17 *No*, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

19 Λ times $S^{-1}\Delta S$ has the same diagonal as $S^{-1}\Delta S$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $A\boldsymbol{x} = \lambda_{\max}(A)\boldsymbol{x} > 0$ then $B\boldsymbol{x} > \lambda_{\max}(A)\boldsymbol{x}$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.

Problem Set 8.4, page 446

1 Feasible set = line segment $(6,0)$ to $(0,3)$; minimum cost at $(6,0)$, maximum at $(0,3)$.

2 Feasible set has corners $(0,0),(6,0),(2,2),(0,6)$. Minimum cost $2x-y$ at $(6,0)$.

3 Only two corners $(4,0,0)$ and $(0,2,0)$; let $x_i \to -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0,0,2)$ move to $\boldsymbol{x} = (0,1,1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\boldsymbol{x} = (1,0,1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

5 $\boldsymbol{c} = [\,3 \;\; 5 \;\; 7\,]$ has minimum cost 12 by the Ph.D. since $\boldsymbol{x} = (4,0,0)$ is minimizing. The dual problem maximizes $4y$ subject to $y \le 3$, $y \le 5$, $y \le 7$. Maximum = 12.

8 $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \le \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} = (A^{\mathrm{T}}\boldsymbol{y})^{\mathrm{T}}\boldsymbol{x} \le \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. The first inequality needed $\boldsymbol{y} \ge 0$ and $A\boldsymbol{x} - \boldsymbol{b} \ge 0$.

Problem Set 8.5, page 451

1 $\int_0^{2\pi} \cos((j+k)x)\,dx = \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x)\,dx = 0$
Notice $j - k \ne 0$ in the denominator. If $j = k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

4 $\int_{-1}^{1} (1)(x^3 - cx)\,dx = 0$ and $\int_{-1}^{1} (x^2 - \frac{1}{3})(x^3 - cx)\,dx = 0$ for all c (odd functions). Choose c so that $\int_{-1}^{1} x(x^3 - cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to the Fourier coefficients $a_1 = 0$, $b_1 = 4/\pi$, $b_2 = 0$.

6 From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ (*odd* k). The square wave has $\|f\|^2 = 2\pi$. Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$. That infinite series equals $\pi^2/8$.

8 $\|\boldsymbol{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1 + a^2 + a^4 + \cdots = 1/(1-a^2)$ so $\|\boldsymbol{v}\| = 1/\sqrt{1-a^2}$; $\int_0^{2\pi} (1 + 2\sin x + \sin^2 x)\,dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = (1 + \text{square wave})/2$ so the a's are $\frac{1}{2}$, 0, 0, ... and the b's are $2/\pi$, 0, $-2/3\pi, 0, 2/5\pi, \ldots$ (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, all other $a_k = 0$, $b_k = -2/k$.

11 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

13 $a_0 = \dfrac{1}{2\pi}\int F(x)\,dx = \dfrac{1}{2\pi}, a_k = \dfrac{\sin(kh/2)}{\pi kh/2} \to \dfrac{1}{\pi}$ for delta function; all $b_k = 0$.

Problem Set 8.6, page 458

3 If $\sigma_3 = 0$ the third equation is exact.

4 $0,1,2$ have probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ and $\sigma^2 = (0-1)^2\frac{1}{4} + (1-1)^2\frac{1}{2} + (2-1)^2\frac{1}{4} = \boldsymbol{\frac{1}{2}}$.

5 Mean $(\frac{1}{2}, \frac{1}{2})$. Independent flips lead to $\boldsymbol{\Sigma} = \mathbf{diag}(\frac{1}{4}, \frac{1}{4})$. Trace $= \sigma^2_{\text{total}} = \boldsymbol{\frac{1}{2}}$.

6 Mean $m = p_0$ and variance $\sigma^2 = (1-p_0)^2 p_0 + (0-p_0)^2(1-p_0) = p_0(1-p_0)$.

7 Minimize $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$ at $P' = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0$; $a = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$ recovers equation (2) for the statistically correct choice with minimum variance.

8 Multiply $L\boldsymbol{\Sigma}L^{\mathrm{T}} = (A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}A)^{-1}A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}A(A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}A)^{-1} = P = (A^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}A)^{-1}$.

9 Row 3 = $-$row 1 and row 4 = $-$row 2: A has rank 2.

Problem Set 8.7, page 464

1 (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \neq 0$.

4 $S = \text{diag}\,(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use $\boldsymbol{v}TS$!

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square, starting from an 8.5 by 11 page.

9 $\boldsymbol{n} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ has $P = I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$. Notice $\|\boldsymbol{n}\| = 1$.

10 We can choose $(0, 0, 3)$ on the plane and multiply $T_{-}PT_{+} = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3, 3, 3)$ projects to $\frac{1}{3}(-1, -1, 4)$ and $(3, 3, 3, 1)$ projects to $(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, 1)$. Row vectors!

13 That projection of a cube onto a plane produces a hexagon.

14 $(3, 3, 3)(I - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = \left(-\frac{11}{3}, -\frac{11}{3}, -\frac{1}{3}\right)$.

15 $(3, 3, 3, 1) \to (3, 3, 0, 1) \to (-\frac{7}{3}, -\frac{7}{3}, -\frac{8}{3}, 1) \to (-\frac{7}{3}, -\frac{7}{3}, \frac{1}{3}, 1)$.

17 Space is rescaled by $1/c$ because (x, y, z, c) is the same point as $(x/c, y/c, z/c, 1)$.

Problem Set 9.1, page 472

1 Without exchange, pivots .001 and 1000; with exchange, 1 and -1. When the pivot is larger than the entries below it, all $|\ell_{ij}| = |\text{entry/pivot}| \leq 1$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

4 The largest $\|\boldsymbol{x}\| = \|A^{-1}\boldsymbol{b}\|$ is $\|A^{-1}\| = 1/\lambda_{\min}$ since $A^{\mathrm{T}} = A$; largest error $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows $< wn$.

6 The triangular L^{-1}, U^{-1}, R^{-1} need $\frac{1}{2}n^2$ multiplications. Q needs n^2 to multiply the right side by $Q^{-1} = Q^{\mathrm{T}}$. So $QR\boldsymbol{x} = \boldsymbol{b}$ takes 1.5 times longer than $LU\boldsymbol{x} = \boldsymbol{b}$.

7 $UU^{-1} = I$: Back substitution needs $\frac{1}{2}j^2$ multiplications on column j, using the j by j upper left block. Then $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) =$ total to find U^{-1}.

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x} - \boldsymbol{x}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}$, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15} are of order 10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}.

11 (a) $\cos\theta = \frac{1}{\sqrt{10}}$, $\sin\theta = \frac{-3}{\sqrt{10}}$, $R = Q_{21}A = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$ (b) $\lambda = 4$; use $-\theta$; $\boldsymbol{x} = (1, -3)/\sqrt{10}$

13 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

Problem Set 9.2, page 478

1 $\|A\| = 2$, $\|A^{-1}\| = 2$, $c = 4$; $\|A\| = 3$, $\|A^{-1}\| = 1$, $c = 3$; $\|A\| = 2 + \sqrt{2} = \lambda_{\max}$ for positive definite A, $\|A^{-1}\| = 1/\lambda_{\min}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

3 For the first inequality replace $\boldsymbol{x}$ by $B\boldsymbol{x}$ in $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$; the second inequality is just $\|B\boldsymbol{x}\| \le \|B\|\|\boldsymbol{x}\|$. Then $\|AB\| = \max(\|AB\boldsymbol{x}\|/\|\boldsymbol{x}\|) \le \|A\|\|B\|$.

7 The triangle inequality gives $\|A\boldsymbol{x} + B\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$. Divide by $\|\boldsymbol{x}\|$ and take the maximum over all nonzero vectors to find $\|A + B\| \le \|A\| + \|B\|$.

8 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\| = |\lambda|$ for that particular vector $\boldsymbol{x}$. When we maximize the ratio over all vectors we get $\|A\| \ge |\lambda|$.

13 The residual $\boldsymbol{b} - A\boldsymbol{y} = (10^{-7}, 0)$ is much smaller than $\boldsymbol{b} - A\boldsymbol{z} = (.0013, .0016)$. But $\boldsymbol{z}$ is much closer to the solution than $\boldsymbol{y}$.

14 $\det A = 10^{-6}$ so $A^{-1} = 10^3 \begin{bmatrix} 659 & -563 \\ -913 & 780 \end{bmatrix}$: $\|A\| > 1$, $\|A^{-1}\| > 10^6$, then $c > 10^6$.

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $(|x_1| + \cdots + |x_n|)^2 = \|\boldsymbol{x}\|_1^2$.
$x_1^2 + \cdots + x_n^2 \le n \max(x_i^2)$ so $\|\boldsymbol{x}\| \le \sqrt{n}\|\boldsymbol{x}\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get $\|\boldsymbol{x}\|_1 = \boldsymbol{x} \cdot \boldsymbol{y} \le \|\boldsymbol{x}\|\|\boldsymbol{y}\| = \sqrt{n}\|\boldsymbol{x}\|$. $\boldsymbol{x} = (1, \ldots, 1)$ has $\|\boldsymbol{x}\|_1 = \sqrt{n}\,\|\boldsymbol{x}\|$.

Problem Set 9.3, page 489

2 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $(I - A)\boldsymbol{x} = (1 - \lambda)\boldsymbol{x}$. Real eigenvalues of $B = I - A$ have $|1 - \lambda| < 1$ provided λ is between 0 and 2.

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$. Small problem, fast convergence.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9}$ which is $(|\lambda|_{\max}$ for Jacobi$)^{\mathbf{2}}$.

9 Set the trace $2 - 2\omega + \frac{1}{4}\omega^2$ equal to $(\omega - 1) + (\omega - 1)$ to find $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07, a big improvement.

15 In the jth component of $A\boldsymbol{x}_1$, $\lambda_1 \sin\frac{j\pi}{n+1} = 2\sin\frac{j\pi}{n+1} - \sin\frac{(j-1)\pi}{n+1} - \sin\frac{(j+1)\pi}{n+1}$. The last two terms combine into $-2\sin\frac{j\pi}{n+1}\cos\frac{\pi}{n+1}$. Then $\lambda_1 = 2 - 2\cos\frac{\pi}{n+1}$.

17 $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $\boldsymbol{u}_1 = \dfrac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \dfrac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\boldsymbol{u}_3 = \dfrac{1}{27}\begin{bmatrix} 14 \\ 13 \end{bmatrix} \to \boldsymbol{u}_\infty = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix} 1 & \cos\theta\sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1 + \sin^2\theta) & -\sin^3\theta \\ \boldsymbol{-\sin^3\theta} & -\cos\theta\sin^2\theta \end{bmatrix}$.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues because A_1 is similar to A.

21 Multiply $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ by $\boldsymbol{q}_j^{\mathrm{T}}$ to find $\boldsymbol{q}_j^{\mathrm{T}}A\boldsymbol{q}_j = a_j$ (because the $\boldsymbol{q}$'s are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. The entries down the diagonals of T are the a's and b's.

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

26 If each center a_{ii} is larger than the circle radius r_i (this is diagonal dominance), then 0 is outside all circles: not an eigenvalue so A^{-1} exists.

Problem Set 10.1, page 498

2 In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.

4 $|z \times w| = 6,\ |z + w| \le 5,\ |z/w| = \frac{2}{3},\ |z - w| \le 5.$

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i;\ w^{12} = 1.$

9 $2+i;\ (2+i)(1+i) = 1+3i;\ e^{-i\pi/2} = -i;\ e^{-i\pi} = -1;\ \frac{1-i}{1+i} = -i;\ (-i)^{103} = i.$

10 $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad-bc)$; write $(a+d)^2 - 4(ad-bc)$ as $(a-d)^2 + 4bc$ which is positive when $bc > 0$.

14 $\det(P - \lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1, -1, i, -i$ with eigenvectors $(1,1,1,1)$ and $(1,-1,1,-1)$ and $(1,i,-1,-i)$ and $(1,-i,-1,i)$ = columns of Fourier matrix.

16 The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

18 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

21 $\cos 3\theta = \mathrm{Re}[(\cos\theta + i\sin\theta)^3] = \cos^3\theta - 3\cos\theta\sin^2\theta;\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$

23 e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e$; Infinitely many $i^e = e^{i(\pi/2 + 2\pi n)e}$.

24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 506

3 $\boldsymbol{z}$ = multiple of $(1+i, 1+i, -2)$; $A\boldsymbol{z} = \mathbf{0}$ gives $\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}} = \mathbf{0}^{\mathrm{H}}$ so $\boldsymbol{z}$ (not $\overline{\boldsymbol{z}}$!) is orthogonal to all columns of A^{H} (using complex inner product $\boldsymbol{z}^{\mathrm{H}}$ times columns of A^{H}).

4 The four fundamental subspaces are now $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{H}})$, $\boldsymbol{N}(A^{\mathrm{H}})$. A^{H} ***and not*** A^{T}.

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}A\boldsymbol{z} = \mathbf{0}$ then $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = 0$. This is $\|A\boldsymbol{z}\|^2 = 0$ so $A\boldsymbol{z} = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are always the ***same***.

6 (a) False (c) False $A = U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $A = A^{\mathrm{H}}$.

10 $(1,1,1)$, $(1, e^{2\pi i/3}, e^{4\pi i/3})$, $(1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2 + 5P + 4P^2$ has the Fourier eigenvector matrix F.
The eigenvalues are $2+5+4 = 11$, $2 + 5e^{2\pi i/3} + 4e^{4\pi i/3}$, $2 + 5e^{4\pi i/3} + 4e^{8\pi i/3}$.

13 Determinant = product of the eigenvalues *(all real)*. And $A = A^{\mathrm{H}}$ gives $\det A = \overline{\det A}$.

15 $A = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

18 $V = \frac{1}{L} \begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{L} \begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6 + 2\sqrt{3}$.
Unitary means $|\lambda| = 1$. $V = V^{\mathrm{H}}$ gives real λ. Then trace zero gives $\lambda = 1$ and -1.

19 The $\boldsymbol{v}$'s are columns of a unitary matrix U, so U^{H} is U^{-1}. Then $\boldsymbol{z} = UU^{\mathrm{H}}\boldsymbol{z} =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z}) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})$: a typical orthonormal expansion.

20 Don't multiply $(e^{-ix})(e^{ix})$. Conjugate the first, then $\int_0^{2\pi} e^{2ix}\,dx = [e^{2ix}/2i]_0^{2\pi} = 0$.

21 $R + iS = (R+iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

24 $[\,1\,]$ and $[\,-1\,]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$ and any angle ϕ

27 Unitary $U^{\mathrm{H}}U = I$ means $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A+iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$.
$A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

30 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \dfrac{1}{6} \begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$. Note real $\lambda = 1$ and 4.

Problem Set 10.3, page 514

8 $\boldsymbol{c} \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0) = F_8\boldsymbol{c}$.
$\boldsymbol{C} \to (0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to (4,0,0,0,-4,0,0,0) = F_8\boldsymbol{C}$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of $C = FEF^{-1}$ because F contains the eigenvectors.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Just transform column 0 of C. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n\log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times $\boldsymbol{x}$.

Conceptual Questions for Review

Chapter 1

1.1 Which vectors are linear combinations of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$?

1.2 Compare the dot product of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?

1.3 What is the cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in Question 1.2? What is the cosine of the angle between the x-axis and $\boldsymbol{v}$?

Chapter 2

2.1 Multiplying a matrix A times the column vector $\boldsymbol{x} = (2, -1)$ gives what combination of the columns of A? How many rows and columns in A?

2.2 If $A\boldsymbol{x} = \boldsymbol{b}$ then the vector $\boldsymbol{b}$ is a linear combination of what vectors from the matrix A? In vector space language, $\boldsymbol{b}$ lies in the _____ space of A.

2.3 If A is the 2 by 2 matrix $\begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}$ what are its pivots?

2.4 If A is the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ how does elimination proceed? What permutation matrix P is involved?

2.5 If A is the matrix $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ find $\boldsymbol{b}$ and $\boldsymbol{c}$ so that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution and $A\boldsymbol{x} = \boldsymbol{c}$ has a solution.

2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?

2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?

2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB? How many *column times row* products go into AB? How many separate small multiplications are involved (the same for both)?

2.9 Suppose $A = \left[\begin{smallmatrix} \mathbf{I} & \mathbf{U} \\ \mathbf{0} & \mathbf{I} \end{smallmatrix}\right]$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

2.10 How can you find the inverse of A by working with $[\,A \quad I\,]$? If you solve the n equations $A\boldsymbol{x} =$ columns of I then the solutions $\boldsymbol{x}$ are columns of _____.

2.11 How does elimination decide whether a square matrix A is invertible?

2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U?

2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?

2.14 What is the transpose of the inverse of AB?

2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

Chapter 3

3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?

3.2 If every column of A is a multiple of the first column, what is the column space of A?

3.3 What are the two requirements for a set of vectors in $\mathbf{R}^n$ to be a subspace?

3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?

3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $A\boldsymbol{x} = \boldsymbol{b}$?

3.6 From the row reduced form R, how would you decide the rank of A?

3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.

3.8 Describe in words the complete solution to a linear system $A\boldsymbol{x} = \boldsymbol{b}$.

3.9 If $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{b}$, what can you say about A?

3.10 Give an example of vectors that span $\mathbf{R}^2$ but are not a basis for $\mathbf{R}^2$.

3.11 What is the dimension of the space of 4 by 4 symmetric matrices?

3.12 Describe the meaning of *basis* and *dimension* of a vector space.

3.13 Why is every row of A perpendicular to every vector in the nullspace?

3.14 How do you know that a column $\boldsymbol{u}$ times a row $\boldsymbol{v}^{\mathrm{T}}$ (both nonzero) has rank 1?

3.15 What are the dimensions of the four fundamental subspaces, if A is 6 by 3 with rank 2?

3.16 What is the row reduced form R of a 3 by 4 matrix of all 2's?

3.17 Describe a *pivot column* of A.

3.18 True? The vectors in the left nullspace of A have the form $A^{\mathrm{T}}\boldsymbol{y}$.

3.19 Why do the columns of every invertible matrix yield a basis?

Chapter 4

4.1 What does the word *complement* mean about orthogonal subspaces?

4.2 If $\boldsymbol{V}$ is a subspace of the 7-dimensional space $\mathbf{R}^7$, the dimensions of $\boldsymbol{V}$ and its orthogonal complement add to _____ .

4.3 The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the vector _____ .

4.4 The projection matrix onto the line through $\boldsymbol{a}$ is $P =$ _____ .

4.5 The key equation to project $\boldsymbol{b}$ onto the column space of A is the *normal equation* _____ .

4.6 The matrix $A^{\mathrm{T}}A$ is invertible when the columns of A are _____ .

4.7 The least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ minimizes what error function?

4.8 What is the connection between the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ and the idea of projection onto the column space?

4.9 If you graph the best straight line to a set of 10 data points, what shape is the matrix A and where does the projection $\boldsymbol{p}$ appear in the graph?

4.10 If the columns of Q are orthonormal, why is $Q^{\mathrm{T}}Q = I$?

4.11 What is the projection matrix P onto the columns of Q?

4.12 If Gram-Schmidt starts with the vectors $\boldsymbol{a} = (2, 0)$ and $\boldsymbol{b} = (1, 1)$, which two orthonormal vectors does it produce? If we keep $\boldsymbol{a} = (2, 0)$ does Gram-Schmidt always produce the same two orthonormal vectors?

4.13 True? Every permutation matrix is an orthogonal matrix.

4.14 The inverse of the orthogonal matrix Q is _____ .

Chapter 5

5.1 What is the determinant of the matrix $-I$?

5.2 Explain how the determinant is a linear function of the first row.

5.3 How do you know that $\det A^{-1} = 1/\det A$?

5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?

5.5 Suppose the first row of A is $0, 0, 0, 3$. What does the "big formula" for the determinant of A reduce to in this case?

5.6 Is the ordering $(2, 5, 3, 4, 1)$ even or odd? What permutation matrix has what determinant, from your answer?

5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?

5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.

5.9 How does Cramer's Rule give the first component in the solution to $I\boldsymbol{x} = \boldsymbol{b}$?

5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?

5.11 What is the connection between determinants and volumes?

5.12 Find the cross product of $\boldsymbol{u} = (0, 0, 1)$ and $\boldsymbol{v} = (0, 1, 0)$ and its direction.

5.13 If A is n by n, why is $\det(A - \lambda I)$ a polynomial in λ of degree n?

Chapter 6

6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?

6.2 If A is singular what does this say about its eigenvalues?

6.3 If A times A equals $4A$, what numbers can be eigenvalues of A?

6.4 Find a real matrix that has no real eigenvalues or eigenvectors.

6.5 How can you find the sum and product of the eigenvalues directly from A?

6.6 What are the eigenvalues of the rank one matrix $[\,1\ \ 2\ \ 1\,]^{\mathrm{T}}[\,1\ \ 1\ \ 1\,]$?

6.7 Explain the diagonalization formula $A = S\Lambda S^{-1}$. Why is it true and when is it true?

6.8 What is the difference between the algebraic and geometric multiplicities of an eigenvalue of A? Which might be larger?

6.9 Explain why the trace of AB equals the trace of BA.

6.10 How do the eigenvectors of A help to solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$?

6.11 How do the eigenvectors of A help to solve $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$?

6.12 Define the matrix exponential e^A and its inverse and its square.

6.13 If A is symmetric, what is special about its eigenvectors? Do any other matrices have eigenvectors with this property?

6.14 What is the diagonalization formula when A is symmetric?

6.15 What does it mean to say that A is *positive definite*?

6.16 When is $B = A^{\mathrm{T}}A$ a positive definite matrix (A is real)?

6.17 If A is positive definite describe the surface $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ in $\mathbf{R}^n$.

6.18 What does it mean for A and B to be *similar*? What is sure to be the same for A and B?

6.19 The 3 by 3 matrix with ones for $i \geq j$ has what Jordan form?

6.20 The SVD expresses A as a product of what three types of matrices?

6.21 How is the SVD for A linked to $A^{\mathrm{T}}A$?

Chapter 7

7.1 Define a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ and give one example.

7.2 If the upper middle house on the cover of the book is the original, find something nonlinear in the transformations of the other eight houses.

7.3 If a linear transformation takes every vector in the input basis into the next basis vector (and the last into zero), what is its matrix?

7.4 Suppose we change from the standard basis (the columns of I) to the basis given by the columns of A (invertible matrix). What is the change of basis matrix M?

7.5 Suppose our new basis is formed from the eigenvectors of a matrix A. What matrix represents A in this new basis?

7.6 If A and B are the matrices representing linear transformations S and T on $\mathbf{R}^n$, what matrix represents the transformation from $\boldsymbol{v}$ to $S(T(\boldsymbol{v}))$?

7.7 Describe five important factorizations of a matrix A and explain when each of them succeeds (what conditions on A?).