STAT30040 Homework 3

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Problem 1

The general form of a rotated ellipse that is centered at the origin (rotate counterclockwise about the origin through an angle of α) is:

$$\frac{(x\cos\alpha + y\sin\alpha)^2}{a^2} + \frac{(x\sin\alpha - y\cos\alpha)^2}{b^2} = m \qquad (1)$$

where a, b, m are constants, and $\sqrt{m}a$ is the length of original horizontal x-axis while $\sqrt{m}b$ is the length of original vertical y-axis, so $a > 0,\ b > 0,\ m > 0$

Expanding the binomial squares and collecting like terms gives:

$$\Rightarrow \quad (\frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2})x^2 + 2\cos\alpha\sin\alpha(\frac{1}{a^2} - \frac{1}{b^2})xy + (\frac{\sin^2\alpha}{a^2} + \frac{\cos^2\alpha}{b^2})y^2 = m \qquad (2)$$

which is in the form $Ax^2 + Bxy + Cy^2 = m$, where $m > 0,\ A > 0,\ C > 0,\ B \neq 0$, and $B^2 - 4AC < 0$.

Set the joint distribution of (U, V) equals to constant c_1, hence we have:

$$f(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\{-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\} = constant\ c_1$$

where $\rho \in (-1, 1)$ and $\rho \neq 0$ (otherwise they will be independent and have circular contours)

$$\Rightarrow \quad u^2 - 2\rho uv + v^2 = -2(1-\rho^2)log(2\pi c_1\sqrt{1-\rho^2}) = constant\ c_2 \qquad (3)$$

Comparing equation (3) with the ellipse formula form $Ax^2 + Bxy + Cy^2 = m$, here we have $A = C = 1 > 0,\ B = -2\rho \neq 0,\ B^2 - 4AC = 4(\rho^2 - 1) < 0$, and $c_2 = u^2 - 2\rho uv + v^2 = (u - \rho v)^2 + (1-\rho^2)v^2 > 0$. Therefore, the contours of the density $\{(u,v) : f(u,v) = constant\}$ are ellipses.

Further, comparing equation (3) with equation (2), we can set the following relationships (note that in equation (3) u and v are symmetric):

$$m = c_2 \times c_3, \quad \frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2} = \frac{\sin^2\alpha}{a^2} + \frac{\cos^2\alpha}{b^2} = 1, \quad 2\cos\alpha\sin\alpha(\frac{1}{a^2} - \frac{1}{b^2}) = -2\rho$$

Thus we get:

$$\alpha = \frac{\pi}{4} + k\pi \ \ or \ \ \frac{3\pi}{4} + k\pi, \quad \frac{1}{a^2} + \frac{1}{b^2} = 2, \quad \frac{1}{a^2} - \frac{1}{b^2} = \frac{-\rho}{\sin\alpha\cos\alpha}$$

When $\alpha = \frac{\pi}{4} + k\pi$, we get $\frac{1}{a^2} + \frac{1}{b^2} = 2,\ \ \frac{1}{a^2} - \frac{1}{b^2} = -2\rho$, so $a^2 = \frac{1}{1-\rho}$ and $b^2 = \frac{1}{1+\rho}$, thus $\sqrt{m}a$, the length of original horizontal axis, is also the length of major axis. Therefore, we are rotating the horizontal major axis counterclockwise about the origin through an angle of $\alpha = \frac{\pi}{4} + k\pi$, so the direction of the major axis falls in the line $u = v$ after the rotation.

When $\alpha = \frac{3\pi}{4} + k\pi$, we get $\frac{1}{a^2} + \frac{1}{b^2} = 2,\ \ \frac{1}{a^2} - \frac{1}{b^2} = 2\rho$, so $a^2 = \frac{1}{1+\rho}$ and $b^2 = \frac{1}{1-\rho}$, thus $\sqrt{m}b$, the length of original vertical axis, is now the length of major axis. Therefore, we are rotating the vertical major axis counterclockwise about the origin through an angle of $\alpha = \frac{3\pi}{4} + k\pi$, so the direction of the major axis still falls in the line $u = v$ after the rotation.

So we have proved that the line $u = v$ is the direction of the major axis of the elliptical contours.

(b)

In question (a), we have proved that whether $\alpha = \frac{\pi}{4} + k\pi$ or $\alpha = \frac{3\pi}{4} + k\pi$, the line $u = v$ is the direction of the major axis of the elliptical contours, and the ratio of the lengths of major axis and minor axis of the elliptical contours is always $(\frac{\sqrt{m}}{\sqrt{1-\rho}})/(\frac{\sqrt{m}}{\sqrt{1+\rho}}) = \sqrt{\frac{1+\rho}{1-\rho}}$

Codes for other problems

Problem 3

(d)

```
x1 = 7810
x2 = 6688
n1 = 14736
n2 = 12673

t = (x1+x2)/(n1+n2)
a = x1/n1
b = x2/n2

LR = (t/a)^x1 * (t/b)^x2 * ((1-t)/(1-a))^(n1-x1) * ((1-t)/(1-b))^(n2-x2); LR
```

```
## [1] 0.9326387
```

```
-2*log(0.9326)
```

```
## [1] 0.1395578
```

```
pchisq(0.1396,1,lower.tail=FALSE)
```

```
## [1] 0.708679
```

Problem 5

(c)

```
pchisq(-2*log(0.9331),1,lower.tail=FALSE)
```

```
## [1] 0.709791
```

(d)

```
2 * (1-pbinom(33,65,1/2))
```

```
## [1] 0.804317
```