STAT33600 Homework 3

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Section 3.1 Autoregressive Moving Average Models

3.1

For model MA(1): $x_t = w_t + \theta w_{t-1}$, the autocovariance function is:

$$\gamma_x(h) = Cov(x_{t+h}, x_t) = Cov(w_{t+h} + \theta w_{t+h-1}, w_t + \theta w_{t-1})$$

$$= (\theta^2+1)\gamma_w(h) + \theta\left[\gamma_w(h+1) + \gamma_w(h-1)\right] = \begin{cases} (\theta^2+1)\sigma_w^2 & if\ h=0 \\ \theta\sigma_w^2 & if\ |h|=1 \\ 0 & if\ |h|\geq 2 \end{cases}$$

So the ACF function is:

$$\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \begin{cases} 1 & if\ h=0 \\ \frac{\theta}{\theta^2+1} & if\ |h|=1 \\ 0 & if\ |h|\geq 2 \end{cases}$$

Because $\theta^2 + 1 \geq |2\theta| \quad \forall\theta$, and the equation holds when $|\theta| = 1$

$$\Rightarrow |\rho_x(1)| = \left|\frac{\theta}{\theta^2+1}\right| = \frac{|\theta|}{\theta^2+1} \leq \frac{|\theta|}{|2\theta|} = \frac{1}{2} \qquad \forall\theta$$

And $\max\rho_x(1) = 1/2$, when $\theta = 1$; $\min\rho_x(1) = -1/2$, when $\theta = -1$

3.2

(a)

Since $x_t = \phi x_{t-1} + w_t$, and $x_0 = w_0$, so:

$$x_t = \phi x_{t-1} + w_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t = \phi^2 x_{t-2} + \phi w_{t-1} + w_t = \cdots = \sum_{j=0}^{t}\phi^j w_{t-j} \qquad (t = 0, 1, 2, ...)$$

(b)

$$E(x_t) = E\left(\sum_{j=0}^{t}\phi^j w_{t-j}\right) = \sum_{j=0}^{t}\phi^j E(w_{t-j}) = 0 \qquad (t = 0, 1, 2, ...)$$

(c)

$$Var(x_t) = Var(\sum_{j=0}^{t}\phi^j w_{t-j}) = \sum_{i=0}^{t}\sum_{j=0}^{t}Cov(\phi^i w_{t-i}, \phi^j w_{t-j}) = \sum_{j=0}^{t}Var(\phi^j w_{t-j})$$

$$= \sum_{j=0}^{t}\phi^{2j}\sigma_w^2 = \frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)}) \qquad (t = 0, 1, 2, ...)$$

(d)

For $h \geq 0$, we have:

$$Cov(x_{t+h}, x_t) = Cov\left(\sum_{i=0}^{t+h} \phi^i w_{t+h-i}, \sum_{j=0}^{t} \phi^j w_{t-j}\right) = \sum_{i=0}^{t+h}\sum_{j=0}^{t} Cov(\phi^i w_{t+h-i}, \phi^j w_{t-j})$$

$$= \sum_{j=0}^{t} Cov(\phi^{j+h} w_{t-j}, \phi^j w_{t-j}) = \sum_{j=0}^{t} \phi^{2j+h}\sigma_w^2 \;\; = \phi^h Var(x_t)$$

(e)

The mean function is constant 0, but the autocovariance function is:

$$\gamma_x(h) = Cov(x_{t+h}, x_t) = \phi^h Var(x_t) = \frac{\phi^h \sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)})$$

which depends on time t.

Therefore, series x_t is not stationary.

(f)

Since $|\phi| < 1$, so we have:

$$\gamma_x(h) = \frac{\phi^h \sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)}) \;\longrightarrow\; \frac{\phi^h \sigma_w^2}{1-\phi^2} \quad (\text{ as } t \to \infty)$$

So when $t \to \infty$, the autocovariance function is converged to a constant which does not depend on time t but only on the lag h.

Therefore, the series x_t is asymptotically stationary.

(g)

From (f) we get that as $t \to \infty$, the AR(1) model is asymptotically stationary. Therefore, it will converge to a stationary process while the time t is large enough.

Based on this conclusion, we can first simulate number m (m is much larger than n) samples of iid N(0,1) values, assign a constant $|\phi| < 1$, then compute the accumulated sum of the samples using the formula: $x_t = \sum_{j=0}^{t} \phi^j w_{t-j}$. This will generate number m observations of the AR(1) model. In the end, we only take the last number n of the m observations, so this last number n samples will be a stationary series.

(h)

Suppose $x_0 = w_0/\sqrt{1-\phi^2}$, then we have:

$$x_t = \sum_{j=0}^{t-1} \phi^j w_{t-j} + \phi^t x_0 = \sum_{j=0}^{t-1} \phi^j w_{t-j} + \frac{\phi^t w_0}{\sqrt{1-\phi^2}} \qquad (t = 1, 2, ...)$$

(1) Mean function

When $t = 0$, the mean function is $E(x_t) = E(x_0) = 0$; when $t \geq 0$, the mean function is:

$$E(x_t) = E\left(\sum_{j=0}^{t-1} \phi^j w_{t-j} + \frac{\phi^t w_0}{\sqrt{1-\phi^2}}\right) = \sum_{j=0}^{t-1} \phi^j E(w_{t-j}) + \frac{\phi^t E(w_0)}{\sqrt{1-\phi^2}} = 0 \qquad (t = 1, 2, ...)$$

(2) Autocovariance function:

$$Var(x_t) = Var\left(\sum_{j=0}^{t-1} \phi^j w_{t-j} + \frac{\phi^t w_0}{\sqrt{1-\phi^2}}\right) = \sum_{j=0}^{t-1} Var(\phi^j w_{t-j}) + \frac{\phi^{2t}\sigma_w^2}{1-\phi^2}$$

$$= \sum_{j=0}^{t-1} \phi^{2j}\sigma_w^2 + \frac{\phi^{2t}\sigma_w^2}{1-\phi^2} = \frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2t}) + \frac{\phi^{2t}\sigma_w^2}{1-\phi^2} = \frac{\sigma_w^2}{1-\phi^2} \qquad (t = 0, 1, 2, ...)$$

which is a constant.

Thus, for $h \geq 0$, the autocovariance function is:

$$\gamma_x(h) = Cov(x_{t+h}, x_t) = Cov\left(\sum_{i=0}^{t+h-1} \phi^i w_{t+h-i} + \frac{\phi^{t+h} w_0}{\sqrt{1-\phi^2}}, \sum_{j=0}^{t-1} \phi^j w_{t-j} + \frac{\phi^t w_0}{\sqrt{1-\phi^2}}\right)$$

$$= \sum_{i=0}^{t+h-1} \sum_{j=0}^{t-1} Cov(\phi^i w_{t+h-i}, \phi^j w_{t-j}) + \frac{\phi^{2t+h}\sigma_w^2}{1-\phi^2} = \sum_{j=0}^{t-1} Cov(\phi^{j+h} w_{t-j}, \phi^j w_{t-j}) + \frac{\phi^{2t+h}\sigma_w^2}{1-\phi^2}$$

$$= \sum_{j=0}^{t-1} \phi^{2j+h}\sigma_w^2 + \frac{\phi^{2t+h}\sigma_w^2}{1-\phi^2} = \phi^h Var(x_t) = \frac{\phi^h \sigma_w^2}{1-\phi^2}$$

which only depends on lag h but not depends on time t.

As a result, based on (1) and (2), the series x_t is stationary.

3.4 ARMA(p,q) models

(a)

$$\begin{aligned} x_t &= 0.80x_{t-1} - 0.15x_{t-2} + w_t - 0.30w_{t-1} \\ x_t - 0.80x_{t-1} + 0.15x_{t-2} &= w_t - 0.30w_{t-1} \\ (1 - 0.80B + 0.15B^2)\ x_t &= (1 - 0.30B)\ w_t \end{aligned}$$

So:

$$\phi(B) = 1 - 0.80B + 0.15B^2 = (1 - 0.30B)(1 - 0.50B)$$

and:

$$\theta(B) = 1 - 0.30B$$

Hence the common factor $1 - 0.30B$ can be canceled and get that:

$$(1 - 0.50B)\ x_t = \ w_t$$

$$x_t = 0.50x_{t-1} + w_t$$

which is a AR(1) model.

This AR(1) model is causal because the root of $\phi(z) = 1 - 0.50z = 0$ is $z = 2$, which is outside the unit circle.

This AR(1) model is also intertible, because it can be written as $w_t = x_t - 0.50x_{t-1} = (1 - 0.50B)\ x_t$.

(b)

$$\begin{aligned}
x_t &= x_{t-1} - 0.50x_{t-2} + w_t - w_{t-1} \\
x_t - x_{t-1} + 0.50x_{t-2} &= w_t - w_{t-1} \\
(1 - B + 0.50B^2)\ x_t &= (1 - B)\ w_t
\end{aligned}$$

So:

$$\phi(B) = 1 - B + 0.50B^2$$

and:

$$\theta(B) = 1 - B$$

There is no common factor, so this is a ARMA(2,1) model.

This ARMA(2,1) model is causal because the root of $\phi(z) = 1 - z + 0.50z^2 = 0$ are complex roots: $z = 1 \pm \sqrt{-1} = 1 \pm i$. Since $|1 \pm i|^2 = 2$, which are both outside the unit circle.

This ARMA(2,1) model is not intertible, because the root of $\theta(z) = 1 - z$ is $z = 1$, which is not outside the unit circle.

3.5

For the AR(2) model:

$$\begin{aligned}
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t \\
x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} &= w_t \\
(1 - \phi_1 B - \phi_2 B^2)\ x_t &= w_t
\end{aligned}$$

So:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2$$

And autoregressive polynomial is:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

And the roots of it are:

$$z_1, z_2 = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

where z_1, z_2 can be real and distinct, real and equal, or a complex conjugate pair.

And we can get that:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = (1 - \frac{z}{z_1})(1 - \frac{z}{z_2}) = 1 - (\frac{1}{z_1} + \frac{1}{z_2})z + \frac{1}{z_1 z_2} z^2$$

$$\Rightarrow \qquad \phi_1 = \frac{1}{z_1} + \frac{1}{z_2}, \qquad \phi_2 = -\frac{1}{z_1 z_2}$$

Therefore, this AR(2) process is causal when the roots z_1, z_2 both lie outside the unit circle, i.e. $|z_1| > 1, |z_2| > 1$.

Now we prove that the causal conditions $|z_1| > 1$ and $|z_2| > 1$ are equivalent to the following three conditions:

$$\phi_1 + \phi_2 - 1 = \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{z_1 z_2} - 1 = -(1 - \frac{1}{z_1})(1 - \frac{1}{z_2}) < 0 \tag{1}$$

$$\phi_2 - \phi_1 - 1 = -\frac{1}{z_1 z_2} - \frac{1}{z_1} - \frac{1}{z_2} - 1 = -(1+\frac{1}{z_1})(1+\frac{1}{z_2}) < 0 \qquad (2)$$

$$|\phi_2| = \left|\frac{1}{z_1 z_2}\right| < 1 \iff |z_1 z_2| > 1 \qquad (3)$$

Note that in these three inequations, the positions of z_1, z_2 are symmetric.

(1) Sufficiency

When $\phi_1 + \phi_2 < 1,\ \phi_2 - \phi_2 < 1,\ |\phi_2| < 1$ holds:

If z_1, z_2 are real, then $|z_1 z_2| = |z_1||z_2| > 1$ shows that at least one of z_1, z_2 is larger than 1. Since their positions in the three equations are symmetric, we can suppose that $|z_1| > 1$. Let's multiply (1) by (2) and get that: $(1-\frac{1}{z_1^2})(1-\frac{1}{z_2^2}) > 0$, since $|z_1| > 1$, so $(1-\frac{1}{z_1^2}) > 0$, thus $(1-\frac{1}{z_2^2}) > 0$, i.e. we also have $|z_2| > 1$.

If z_1, z_2 are a complex conjugate pair $a \pm bi$, then we have $|z_1 z_2| = a^2 + b^2 > 1$, so we can get that both $|z_1| = |z_2| = |a \pm bi| = \sqrt{a^2+b^2} > 1$, i.e. $|z_1| > 1, |z_2| > 1$.

Therefore, in both cases, we can get that $|z_1| > 1, |z_2| > 1$.

(2) Necessity

When $|z_1| > 1, |z_2| > 1$ holds:

If z_1, z_2 are real, then we have $|z_1 z_2| > 1$, so (3) holds. And $(1 \pm \frac{1}{z_1}) > 0$, $(1 \pm \frac{1}{z_1}) > 0$, so that both (1) and (2) hold.

If z_1, z_2 are a complex conjugate pair $a \pm bi$, then $|z_1| = |z_2| = |a \pm bi| = \sqrt{a^2+b^2} > 1$, thus we have $|z_1 z_2| = a^2 + b^2 > 1$, so (3) holds. Further, since the reciprocals of a complex conjuate pair $\frac{1}{z_1}, \frac{1}{z_2} = \frac{a \mp bi}{a^2+b^2}$ are also a complex conjugate pair, and that $|\frac{1}{z_1}| = |\frac{1}{z_2}| = \frac{\sqrt{a^2+b^2}}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}} < 1$. As a result, for (1), $(1-\frac{1}{z_1})(1-\frac{1}{z_2}) = |1-\frac{1}{z_1}|^2 > 0$, so (1) holds. And for (2), $(1+\frac{1}{z_1})(1+\frac{1}{z_2}) = |1+\frac{1}{z_1}|^2 > 0$, so (2) holds.

Therefore, in both cases, we can get that $\phi_1 + \phi_2 < 1,\ \phi_2 - \phi_2 < 1,\ |\phi_2| < 1$.

Conclusion:

Based on the above sufficiency and necessity, we have proved that:

$$\text{AR(2) is causal} \iff |z_1| > 1, |z_2| > 1 \iff \phi_1 + \phi_2 < 1,\ \phi_2 - \phi_2 < 1,\ |\phi_2| < 1$$

Section 3.2 Difference Equations

3.6

$$x_t = -0.9 x_{t-2} + w_t$$
$$x_t + 0.9 x_{t-2} = w_t$$
$$(1 + 0.9B^2)\ x_t = w_t$$

So the autoregressive polynomial is:

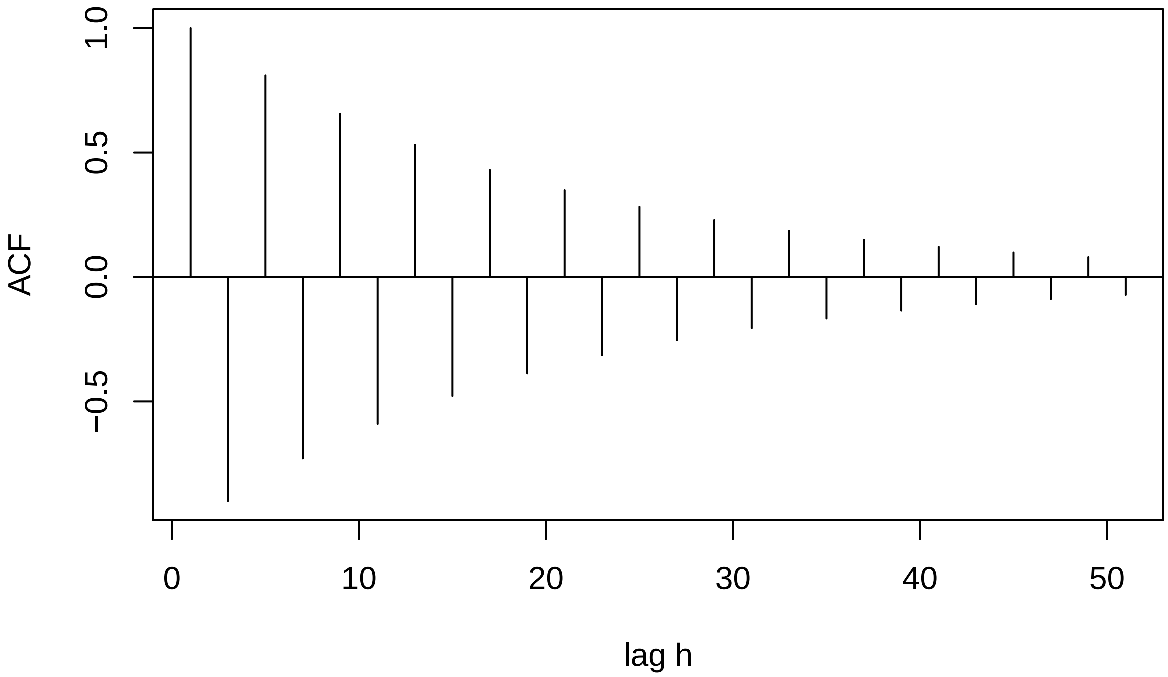
$$\phi(z) = 1 + 0.9z^2$$

And the roots of it are:

$$z = \pm \frac{i}{\sqrt{0.9}}$$

The plot of the ACF $\rho(h)$ is shown as below:

```
ACF = ARMAacf(ar=c(0,-0.9), ma=0, lag.max=50)
plot(ACF, type="h", xlab="lag h")
abline(h=0)
```



ACF
1.0
0.5
0.0
−0.5
0
10
20
30
40
50
lag h