STAT33600 Homework 7

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Section 4.4 Nonparametric Spectral Estimation

4.17

(1) Prove (4.71)

In general, for series x_t, the periodogram can be written as:

$$I(w_j) = |d(w_j)|^2 = \left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t e^{-2\pi i w_j t}\right|^2 = \frac{1}{n}\left(\sum_{t=1}^{n} x_t e^{-2\pi i w_j t}\right)\left(\sum_{t=1}^{n} x_t e^{2\pi i w_j t}\right)$$

$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} x_t x_s e^{-2\pi i w_j t} e^{2\pi i w_j s} = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} x_t x_s e^{-2\pi i w_j (t-s)}$$

Then use Property 4.2:

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i w h} f(w) dw$$

If $y_t = h_t x_t$, where x_t is a stationary process with zero mean, h_t is a taper, then we can have that:

$$E[I_y(w_j)] = E\left[\frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} y_t y_s e^{-2\pi i w_j (t-s)}\right] = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} E[y_t y_s] e^{-2\pi i w_j (t-s)}$$

$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} h_t h_s E[x_t x_s] e^{-2\pi i w_j (t-s)} = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} h_t h_s \gamma_x(t-s) e^{-2\pi i w_j (t-s)}$$

$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} h_t h_s \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i w (t-s)} f_x(w) dw\right) e^{-2\pi i w_j (t-s)}$$

$$= \frac{1}{n}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{t=1}^{n}\sum_{s=1}^{n} h_t h_s e^{2\pi i w (t-s)} e^{-2\pi i w_j (t-s)}\right) f_x(w) dw$$

$$= \frac{1}{n}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{t=1}^{n}\sum_{s=1}^{n} h_t h_s e^{-2\pi i (w_j - w) t} e^{2\pi i (w_j - w) s}\right) f_x(w) dw$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h_t e^{-2\pi i (w_j - w) t}\right)\left(\frac{1}{\sqrt{n}}\sum_{s=1}^{n} h_s e^{2\pi i (w_j - w) s}\right) f_x(w) dw$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h_t e^{-2\pi i (w_j - w) t}\right|^2 f_x(w) dw$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} W_n(w_j - w)\ f_x(w)\ dw$$

where

$$W_n(w) = |H_n(w)|^2 \qquad , \qquad H_n(w) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t e^{-2\pi i w t}$$

(2) Prove (4.74)

If $h_t = 1$ for all t, i.e. $y_t = x_t$, we have $I_y(w_j) = I_x(w_j)$ is simply the periodogram of the data, and:

$$E[I_x(w_j)] = E[I_y(w_j)] = \int_{-\frac{1}{2}}^{\frac{1}{2}} W_n(w_j - w)\ f_x(w)\ dw$$

where

$$W_n(w) = |H_n(w)|^2 = \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-2\pi i w t} \right|^2 = \frac{1}{n} \left(\sum_{t=1}^{n} e^{-2\pi i w t} \right) \left(\sum_{t=1}^{n} e^{2\pi i w t} \right)$$

Then when $w \neq 0$, the window $W_n(w)$ is:

$$W_n(w) = \frac{1}{n} \times \frac{e^{-2\pi i w}(1 - e^{-2\pi i w n})}{1 - e^{-2\pi i w}} \times \frac{e^{2\pi i w}(1 - e^{2\pi i w n})}{1 - e^{2\pi i w}} = \frac{1}{n} \times \frac{(1 - e^{-2\pi i w n})(1 - e^{2\pi i w n})}{(1 - e^{-2\pi i w})(1 - e^{2\pi i w})}$$

$$= \frac{1}{n} \times \frac{(1 - \cos(2\pi w n))^2 + \sin^2(2\pi w n)}{(1 - \cos(2\pi w))^2 + \sin^2(2\pi w)} = \frac{1}{n} \times \frac{2 \times (1 - \cos(2\pi w n))}{2 \times (1 - \cos(2\pi w))}$$

$$= \frac{1}{n} \times \frac{4 \sin^2(\pi w n)}{4 \sin^2(\pi w)} \quad = \quad \frac{\sin^2(n\pi w)}{n \sin^2(\pi w)} \qquad (w \neq 0)$$

And when $w = 0$:

$$W_n(0) \ = \ \frac{1}{n} \times n \times n \ = \ n$$

(3) Prove (4.75)

From (2), we have proved that if $h_t = 1$ for all t, i.e. $y_t = x_t$, then $I_y(w_j) = I_x(w_j)$ is simply the periodogram of the data, and:

$$E[I_x(w_j)] = E[I_y(w_j)] = \int_{-\frac{1}{2}}^{\frac{1}{2}} W_n(w_j - w)\ f_x(w)\ dw$$

where

$$W_n(w) = |H_n(w)|^2 = \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-2\pi i w t} \right|^2 = \begin{cases} \frac{\sin^2(n\pi w)}{n \sin^2(\pi w)} & w \neq 0 \\ n & w = 0 \end{cases}$$

If use the averaged periodogram as the estimator of $f_x(w)$:

$$\hat{f}_x(w_j) = \frac{1}{L} \sum_{k=-m}^{m} I_x(w_j + \frac{k}{n}) \qquad\qquad (L = 2m + 1)$$

Then we have that:

$$E[\hat{f}_x(w_j)] = E\left[\frac{1}{L}\sum_{k=-m}^{m} I_x(w_j+\frac{k}{n})\right] \;=\; \frac{1}{L}\sum_{k=-m}^{m} E[I_x(w_j+\frac{k}{n})]$$

$$= \frac{1}{L}\sum_{k=-m}^{m}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} W_n(w_j+\frac{k}{n}-w)\; f_x(w)\; dw\right)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{L}\sum_{k=-m}^{m} W_n(w_j+\frac{k}{n}-w)\right) f_x(w)\; dw$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{W}_n(w_j-w)\; f_x(w)\; dw$$

So the new form of the window $W_n(w)$ here is:

$$\widetilde{W}_n(w) = \frac{1}{L}\sum_{k=-m}^{m} W_n(w+\frac{k}{n}) = \frac{1}{L}\sum_{k=-m}^{m}\frac{\sin^2[n\pi(w+\frac{k}{n})]}{n\sin^2[\pi(w+\frac{k}{n})]} = \frac{1}{nL}\sum_{k=-m}^{m}\frac{\sin^2[n\pi(w+\frac{k}{n})]}{\sin^2[\pi(w+\frac{k}{n})]}$$

Section 4.7 Linear Filters

4.26

(a)

$$x_t = w_t, \qquad y_t = \phi x_{t-D} + v_t$$

where w_t, v_t are independent white noise processes with common vairance σ^2.

So we have that $f_x(w) = f_w(w) = f_v(w) = \sigma^2$, and the cross-covariance is:

$$\gamma_{xy} = Cov(x_{t+h}, y_t) = Cov(x_{t+h},\; \phi x_{t-D} + v_t) = \phi\gamma_x(h+D) = \begin{cases} \phi\sigma^2 & h=-D \\ 0 & \text{otherwise} \end{cases}$$

$$f_{xy}(w) = \sum_{h=-\infty}^{\infty} \gamma_{xy}(h)e^{-2\pi iwh} = \phi\sigma^2 e^{2\pi iwD}$$

$$A_{xy}(w) = \frac{f_{xy}(w)}{f_{xx}(w)} = \frac{\phi\sigma^2 e^{2\pi iwD}}{\sigma^2} = \phi e^{2\pi iwD}$$

Thus the amplitude is:

$$|A_{xy}(w)| = |\phi|$$

And the phase between x_t and y_t is:

$$\phi_{xy}(w) = 2\pi wD \qquad \text{(i.e.:} \quad \phi_{yx}(w) = -2\pi wD)$$

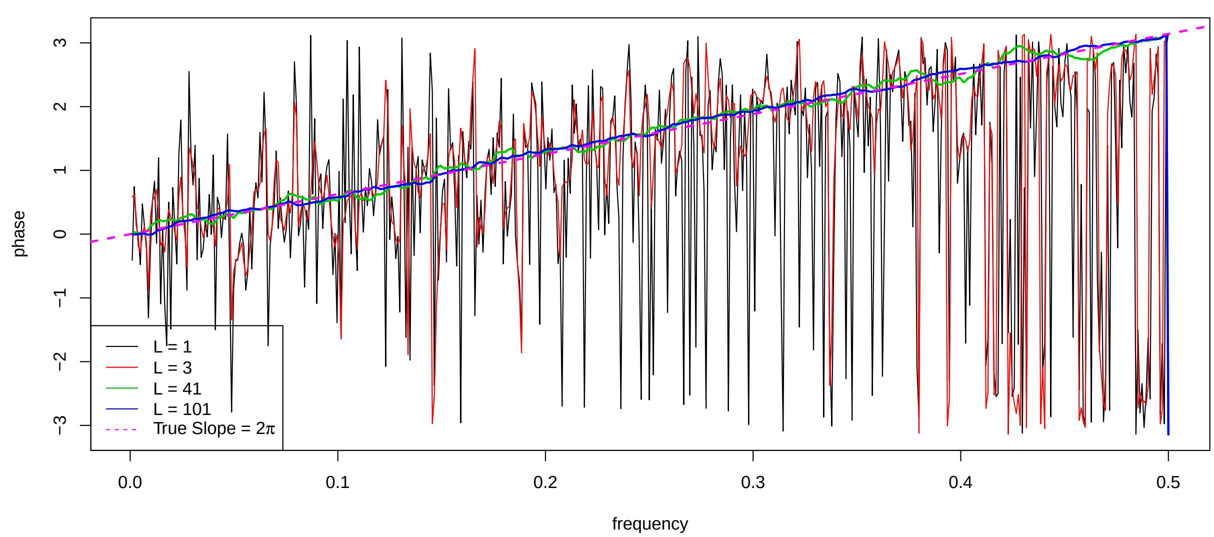
(b)

```r
library(astsa)

# simulate series data
set.seed(123)
n = 1024     # already have: n = 2^p
D = 1
phi = 0.9
x = ts(rnorm(n+1,0,1))
y = phi*lag(x,-D) + rnorm(n+1,0,1)
xy = ts.intersect(x,y)

# estimate phase using different L values
xy_phase1 = mvspec(xy, plot=FALSE)$phase   # no spans means L=1
xy_phase3 = mvspec(xy, plot=FALSE, spans=3)$phase
xy_phase41 = mvspec(xy, plot=FALSE, spans=41)$phase
xy_phase101 = mvspec(xy, plot=FALSE, spans=101)$phase

# plot the estimated phases for different L values
freq = 1:(n/2) / n
plot(freq, xy_phase1, type="l", col=1, lty=1,
     ylim=c(-pi,pi), xlab="frequency", ylab="phase")
lines(freq, xy_phase3, col=2, lty=1)
lines(freq, xy_phase41, col=3, lty=1, lwd=2)
lines(freq, xy_phase101, col=4, lty=1, lwd=2)
abline(a=0, b=2*pi, col=6, lty=2, lwd=2) # true slope
legend("bottomleft", legend=c("L = 1","L = 3","L = 41","L = 101", expression(paste("True Slope = ", 2*p
```



From question (a), we have that the true slope should be:

$$\frac{\phi_{xy}(w)}{w} = 2\pi D = 2\pi \qquad (D = 1)$$

which is also plotted in the above figure.

From the above plots we can see that as the value of L increases, the estimates of the phases, i.e. $\hat{\phi}_{xy}(w) =$

$2\pi w\hat{D}$, is closer to the true values of the phase, i.e. $\phi_{xy}(w) = 2\pi w$. Therefore, larger value of L gives better estimates of the delay D as well as the phase between x_t and y_t.