Homework 8

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Problem 1

(a)

Since g is a natural cubic spline interpolant to the pairs (x_n, z_n) $(n = 1, ..., N)$, where $a < x_1 < ... < x_N < b$, so $g''(x) = 0$ and $g'''(x) = 0$ for x in the two end intervals $[a, x_1] \cup [x_N, b]$, and that $g'''(x) = g'''(x_n^+)$ for $\forall x \in [x_n, x_{n+1}]$

Also, since $\tilde{g}$ is any other twice differentiable function on $[a, b]$ that interpolates the pairs (x_n, z_n) $(n = 1, ..., N)$, so at the knots x_n's we have: $\tilde{g}(x_n) = g(x_n) = z_n$ $(n = 1, ..., N)$, thus $h(x_n) = g(x_n) - \tilde{g}(x_n) = 0$ $(n = 1, ..., N)$.

Therefore:

$$\int_a^b g''(x)h''(x)dx = \int_a^b g''(x)dh'(x) = g''(x)h'(x)|_a^b - \int_a^b h'(x)dg''(x) = g''(b)h'(b) - g''(a)h'(a) - \int_a^b g'''(x)h'(x)dx$$

$$= 0 - \int_{x_1}^{x_N} g'''(x)h'(x)dx = -\sum_{n=1}^{N-1} \int_{x_n}^{x_{n+1}} g'''(x)h'(x)dx = -\sum_{n=1}^{N-1} g'''(x_n^+) \int_{x_n}^{x_{n+1}} h'(x)dx = -\sum_{n=1}^{N-1} g'''(x_n^+)\, h(x)|_{x_n}^{x_{n+1}}$$

$$= -\sum_{n=1}^{N-1} g'''(x_n^+)\ [h(x_{n+1}) - h(x_n)] = -\sum_{n=1}^{N-1} g'''(x_n^+)\ [0 - 0] = 0$$

(b)

In question (a), we have derived that:

$$\int_a^b g''(x)h''(x)dx = 0 \quad \Rightarrow \quad \int_a^b g''(x)\ [g''(x) - \tilde{g}''(x)]dx = 0 \quad \Rightarrow \quad \int_a^b g''(x)^2 dx = \int_a^b g''(x)\tilde{g}''(x)dx$$

Therefore, by Cauchy-Schwarz Inequality, we can obtain that:

$$\left(\int_a^b g''(x)^2 dx\right)^2 = \left(\int_a^b g''(x)\tilde{g}''(x)dx\right)^2 \le \left(\int_a^b g''(x)^2 dx\right)\left(\int_a^b \tilde{g}''(x)^2 dx\right)$$

$$\Rightarrow \qquad \int_a^b g''(x)^2 dx \le \int_a^b \tilde{g}''(x)^2 dx$$

where the equality holds when there exists a nonzero constant C such that $\tilde{g}''(x) = Cg''(x)$ for $\forall x \in [a, b]$.

Since g is a natural cubic spline interpolant to the pairs (x_n, z_n) $(n = 1, ..., N)$, so when $\tilde{g}''(x) = Cg''(x)$, we have: $\tilde{g}(x) = Cg(x) + A + Bx$ for $\forall x \in [a, b]$, where A and B are also constants. Because $\tilde{g}(x_n) = g(x_n) = z_n$ at all the knots x_n's $(n = 1, ..., N)$, hence at the knots we have: $(1 - C)g(x_n) = A + Bx_n$. If $C \neq 1, A \neq 0, B \neq 0$, then g becomes a linear or constant function that can pass through all the (x_n, z_n) points, which contradicts the fact that g is a natural cubic spline. As a result, it must be that $C = 1,\ A = B = 0$. So we obtain that the equality holds only when $\tilde{g}''(x) = g''(x)$ and $\tilde{g}(x) = g(x)$ for $\forall x \in [a, b]$.

Therefore, the equality holds only when $h(x) = g(x) - \tilde{g}(x) = 0$ for $\forall x \in [a, b]$.

(c)

Suppose $\hat{f}$ is the minimizer of the penalized least squares problem, i.e.:

$$\min_{f\in F}\left[\sum_{n=1}^{N}(Y_n - f(X_n))^2 + \lambda\int_a^b f''(x)^2dx\right] = \sum_{n=1}^{N}\left(Y_n - \hat{f}(X_n)\right)^2 + \lambda\int_a^b \hat{f}''(x)^2dx$$

Set $\hat{f}(X_n) = Z_n$ $(n = 1, ..., N)$, since $\hat{f}$ is a function with continuous second derivatives on $[a, b]$, so $\hat{f}$ can be seen as a twice differentiable function on $[a, b]$ that interpolates the pairs (X_n, Z_n) $(n = 1, ..., N)$, which has all the properties that the funciton $\tilde{g}$ has in question (a) and (b).

Next, we can generate a function g, which is a natural cubic spline interpolant to the pairs (X_n, Z_n) $(n = 1, ..., N)$, so at the knots X_n's we have: $g(X_n) = \hat{f}(X_n) = Z_n$ $(n = 1, ..., N)$, and that: $\int_a^b \hat{f}''(x)^2dx \geq \int_a^b g''(x)^2dx$, where the equality holds only when $\hat{f}(x) = g(x)$ for $\forall x \in [a, b]$.

Since $\lambda > 0$, therefore:

$$\min_{f\in F}\left[\sum_{n=1}^{N}(Y_n - f(X_n))^2 + \lambda\int_a^b f''(x)^2dx\right] = \sum_{n=1}^{N}\left(Y_n - \hat{f}(X_n)\right)^2 + \lambda\int_a^b \hat{f}''(x)^2dx \ \geq \ \sum_{n=1}^{N}(Y_n - g(X_n))^2 + \lambda\int_a^b g''(x)^2dx$$

Since $\hat{f}$ is the minimizer, thus the equality must holds here, otherwise function g will further minimize the penalized least squares problem which contradicts the fact that $\hat{f}$ is the minimizer. And the equality holds only when $\hat{f}(x) = g(x)$ for $\forall x \in [a, b]$.

Therefore, the minimizer of the penalized least squares problem must be a natural cubic spline with knots at the points X_n $(n = 1, ..., N)$.