Question: STRAIN-DISPLACEMENT RELATION FROM CYLINDRICAL TO SPHERICAL COORDINATE SYSTEM

Answer:-

The relation between cylindrical and spherical coordinates is:

$$r = \rho \sin\phi \quad , \quad z = \rho \cos\phi \quad , \quad \theta = \theta$$

where

$$\rho = \sqrt{r^2+z^2} \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad , \quad \phi = \arccos\left(\frac{z}{\rho}\right)$$

The partial derivatives for the above equations are:

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r}\cdot\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial r}\cdot\frac{\partial}{\partial \phi}$$

$$= \sin\phi \frac{\partial}{\partial \rho} + \frac{r^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\cdot\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial \rho}{\partial z}\cdot\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial}{\partial \phi}$$

$$= \cos\phi \frac{\partial}{\partial \rho} + \frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\cdot\frac{\partial}{\partial \phi}$$

Now

$$u_r = u_\rho \sin\phi + u_\phi \frac{r^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}} \quad ; \quad u_z = u_\rho\cos\phi + u_\phi\frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{3/2}} \quad ; \quad u_\theta = u_\theta$$

calculating $e_\rho = \frac{\partial u_r}{\partial r}$

$$\hat{e}_\rho = \sin\phi\left[\frac{\partial}{\partial\rho}\left(u_\rho \sin\phi + u_\phi\frac{r^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\right)\right] + \frac{r^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\cdot\frac{\partial}{\partial\phi}\left[u_\rho\sin\phi + u_\phi\frac{r^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\right]$$

$$= \left[\frac{\partial u_\rho}{\partial\rho}\sin^2\phi + \frac{\partial u_\phi}{\partial\rho}\cdot\frac{r^2\sin\phi}{\rho^{3/2}\sqrt{r^2-z^2}} + \frac{u_\phi r^2}{\sqrt{r^2-z^2}}\cdot\frac{\sin\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial\phi}\cdot\frac{\sin\phi\, r^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}} + \frac{r^2 u_\rho\cos\phi}{\sqrt{r^2-z^2}} + \frac{\partial u_\phi}{\partial\phi}\cdot\frac{r^4}{\rho^3(r^2-z^2)}\right]$$

$$\hat{e}_\rho = \frac{\partial u_\rho}{\partial\rho}\sin^2\phi + \left(\frac{\partial u_\phi}{\partial\rho}\cdot\frac{1}{\rho^{3/2}} + \frac{u_\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial\phi}\cdot\frac{1}{\rho^{3/2}}\right)\frac{r^2\sin\phi}{\sqrt{r^2-z^2}} + \left(u_\rho\cos\phi + \frac{\partial u_\phi}{\partial\phi}\cdot\frac{1}{\rho^3}\cdot\frac{r^2}{\sqrt{r^2-z^2}}\right)\frac{r^2}{\sqrt{r^2-z^2}}$$

$$\hat{e}_\phi = \frac{\partial u_z}{\partial z}$$

$$\hat{e}_\phi = \cos\phi\frac{\partial}{\partial\rho}\left[u_\rho\cos\phi + u_\phi\cdot\frac{rz}{\rho^{3/2}\cdot\sqrt{r^2-z^2}}\right] + \frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\frac{\partial}{\partial\phi}\left[u_\rho\cos\phi + u_\phi\frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\right]$$

$$= \frac{\partial u_\rho}{\partial \rho}\cos^2\phi + \frac{\partial u_\phi}{\partial \rho}\cdot\frac{rz\cos\phi}{\rho^{\frac{3}{2}}\cdot\sqrt{r^2-z^2}} + \frac{u_\phi\, rz}{\sqrt{r^2-z^2}}\cdot\frac{\cos\phi}{\rho^{-\frac{5}{2}}} + \frac{\partial u_\rho}{\partial \phi}\,\frac{\cos\phi\, rz}{\rho^{\frac{3}{2}}\cdot\sqrt{r^2-z^2}} - \frac{\sin\phi\cdot rz\, u_\rho}{\sqrt{r^2-z^2}\cdot\rho^{\frac{3}{2}}}$$

$$+ \frac{\partial u_\phi}{\partial \phi}\cdot\frac{r^2z^2}{(r^2-z^2)\rho^3}.$$

$$\hat{e}_\phi = \frac{\partial u_\rho}{\partial \rho}\cos^2\phi + \left(\frac{\partial u_\phi}{\partial \rho}\cdot\frac{1}{\rho^{\frac{3}{2}}} + \frac{u_\phi}{\rho^{-\frac{5}{2}}} + \frac{\partial u_\rho}{\partial \phi}\cdot\frac{1}{\rho^{\frac{3}{2}}}\right)\frac{\cos\phi\cdot rz}{\sqrt{r^2-z^2}} + \left(\frac{\partial u_\phi}{\partial \phi}\cdot\frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{\frac{3}{2}}} - \frac{u_\rho\sin\phi}{\rho^{\frac{3}{2}}}\right)\frac{rz}{\sqrt{r^2-z^2}}.$$

Therefore the strain-displacement relation becomes;

$$e_\rho = \frac{\partial u_r}{\partial r}, \quad e_\phi = \frac{1}{\rho}\left(u_r + \frac{\partial u_z}{\partial \phi}\right)$$

$$e_\theta = \frac{1}{\rho\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + \sin\phi\, u_r + \cos\phi\, u_z\right)$$

$$e_{\rho\phi} = \frac{1}{2}\left(\frac{1}{\rho}\,\frac{\partial u_r}{\partial \phi} + \frac{\partial u_z}{\partial \rho} - \frac{u_z}{\rho}\right)$$

$$e_{\phi\theta} = \frac{1}{2\rho}\left(\frac{1}{\sin\phi}\cdot\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - \cot\phi\, u_\theta\right)$$

$$e_{\theta\rho} = \frac{1}{2}\left(\frac{1}{\rho\sin\phi}\cdot\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \rho} - \frac{u_\theta}{\rho}\right).$$

End