Q-01

For the given matrix/vector pairs, compute the following quantities a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_ib_j, b_ib_i. For each case, point out whether the result is a scalar, vector or matrix. Note that $a_{ij}b_j$ is actually a matrix product $[a][b]$, while $a_{ij}a_{jk}$ is the product $[a][a]$.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $b_i \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

↳ $a_{ii} = a_{11} + a_{22} + a_{23} = 1+4+1 = 6$ (scalar)

↳ $a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+1+1+0+16+4+0+1+1 = 25$ (scalar)

↳ $a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix}$ (matrix)

↳ $a_{ij}b_j = a_{11}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (vector

↳ $a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$.

$= 1+0+2+0+0+0+0+0+4 = 7$ (scalar)

↳ $b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ (matrix)

↳ $b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1+0+4 = 5$ (scalar)

(b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$, $b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

↳ $a_{ii} = a_{11} + a_{22} + a_{33} = 1+2+2 = 5$ (scalar)

↳ $a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21} + \cancel{a_{22}} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31}$

$+ a_{33}a_{33}$.

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$= 1+4+0+0+4+1+0+16+4 = 30$ (scalar)

$\Rightarrow a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix}$ (matrix)

$\Rightarrow a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ (vector)

$\Rightarrow a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3$
$+ a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 4+4+0+0+2+1+0+4+2 = 17$ (scalar)

$\Rightarrow b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (matrix)

$\Rightarrow b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4+1+1 = 6$ (scalar)

(c) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow a_{ii} = a_{11} + a_{22}\, a_{33} = 1+0+4 = 5$ (scalar)

$\Rightarrow a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32}$
$+ a_{33}a_{33}$.

$= 1+1+1+1+0+4+0+1+16 = 25$ (scalar)

$\Rightarrow a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix}$ (matrix)

$\Rightarrow a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (vector)

$\Rightarrow a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3$
$+ a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$.

$= 1+1+0+1+0+0+0+0+0 = 3$ (scalar).

$\hookrightarrow b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (matrix)

$\hookrightarrow b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3 + 1 + 1 + 0 = 2$ (scalar)

Q-02

Use the decomposition result (1.2-10) to express a_{ij} from Ex-1 in terms of the sum of the symmetric and anti-symmetric matrices. verify that $a_{(ij)}$ and $a_{[ij]}$ satisfy the conditions given in last paragraph of section 1.2.

Answer:

(a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 01 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satify the conditions below:

- a_{ij} is symenetric, it has only six independent components.
- $a_{[ij]}$ is anti-symmetric, its diagonal terms a_{ii} must be zero. and has only 3 independent components.

(b) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ which is the symmetric part $\frac{1}{2}(a_{ij} + a_{ji})$, and $a_{[ij]}$ which is antisymmetric satisfy the appropriate conditions.

(c) $\frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 6 \end{bmatrix}$

$a_{(ij)}$ and $a_{[ij]}$ satisfy appropriate conditions.

Q-03

If a_{ij} is symmetric and b_{ij} is anti-symmetric, prove in general that $a_{ij}b_{ij}$ is zero. verify ~~that~~ the specific case by using symmetric and anti-symmetric terms from Ex-2.

Answer:

$$a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij} \Rightarrow 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0.$$

From Ex-02 (a):

$$a_{(ij)}a_{[ij]} = \frac{1}{4}\operatorname{tr}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right)$$

$$= \frac{1}{4}\operatorname{tr}\left[\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}\right]$$

$$= \frac{1}{4}\operatorname{tr}\begin{bmatrix} 2 & -1 & -3 \\ 11 & 2 & -9 \\ 5 & 1 & -4 \end{bmatrix}$$

$$= \frac{1}{4}(2+2+(-4))$$

$$= \frac{1}{4}(4-4)$$

$$= 0$$

From Ex-02 (b):

$$a_{(ij)}a_{[ij]} = \frac{1}{4}\operatorname{tr}\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right)$$

$$= 0$$

From Ex-02 (c):

$$a_{(ij)}a_{[ij]} = \frac{1}{4}\operatorname{tr}\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right)$$

$$= 0.$$

Q-04 Explicitly verify the following properties of kronecker delta:

$$\delta_{ij} a_j = a_i$$

$$\delta_{ij} a_{jk} = a_{ik}$$

Answer:

$$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 = \begin{bmatrix} \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij} a_{jk} = \begin{bmatrix} \delta_{11} a_{11} + \delta_{12} a_{21} + \delta_{13} a_{31} & \delta_{11} a_{12} + \delta_{12} a_{22} + \delta_{13} a_{32} & \delta_{11} a_{13} + \delta_{12} a_{23} + \delta_{13} a_{33} \\ \delta_{21} a_{11} + \delta_{22} a_{21} + \delta_{23} a_{31} & \delta_{21} a_{12} + \delta_{22} a_{22} + \delta_{23} a_{32} & \delta_{21} a_{13} + \delta_{22} a_{23} + \delta_{23} a_{33} \\ \delta_{31} a_{11} + \delta_{32} a_{21} + \delta_{33} a_{31} & \delta_{31} a_{12} + \delta_{32} a_{22} + \delta_{33} a_{32} & \delta_{31} a_{13} + \delta_{32} a_{23} + \delta_{33} a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & 1a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

Q-05 Formally expand the expression (1.3.4) for the determinant and justify that either index notation form yields a result that matches the traditional form for det $[a_{ij}]$.

Answer:

$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

$$= \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{231} a_{12} a_{23} a_{31} + \varepsilon_{312} a_{13} a_{21} a_{32} + \varepsilon_{321} a_{13} a_{22} a_{31}$$

$$+ \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{213} a_{12} a_{21} a_{33}.$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

$$= a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} a_{31}) + a_{13}(a_{21} a_{32} - a_{22} a_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Q-06

Determine the components of vector b_i and matrix a_{ij} given in Ex-01 in a new coordinate system found through a rotation of 45° ($\pi/4$ rad) about the x_1-axis. The rotation direction follows the positive sense presented in Example 1-2.

Answer:

45° rotation about x_1-axis $\Rightarrow$ Q_{ij}

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) & \cos(x_1', x_3) \\ \cos(x_2', x_1) & \cos(x_2', x_2) & \cos(x_2', x_3) \\ \cos(x_3', x_1) & \cos(x_3', x_2) & \cos(x_3', x_3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

From Ex-1 part (a)

$$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

From Ex-1 (b)

$$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

From Ex-1 (c):

$$b_i' = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & 3.5 & 2.5 \\ -\frac{1}{\sqrt{2}} & 1.5 & 0.5 \end{bmatrix}$$

Q-07

Consider the two dimensional coordinate transformation shown in Fig 1-7. Through the counterclockwise rotation θ, a new polar coordinate system is created. Show that the transformation matrix for this case is given by

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

If $b_i = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $a_{ij} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ are the components of a first and second order tensor in the x_1, x_2 system; calculate their components in the rotated polar coordinate system.

Answer:

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90^\circ-\theta) \\ \cos(90^\circ+\theta) & \cos\theta \end{bmatrix}$$

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2 \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

Q-08:

Show that the second order tensor $a\delta_{ij}$, where a is an arbitra constant, retain its form under any transformation Q_{ij}. This form is then an isotropic second order tensor.

Answer.

$$a'\delta_{ij}' = Q_{ip} Q_{jq} a \delta_{pq}$$
$$= a\, Q_{ip} Q_{jp}$$
$$= a\,\delta_{ij}$$

Q-09

The most general form of a forth order isotropic tensor ca be expressed by $\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$. where α, β and γ are arbitrary constants. Verify that this form remains the sam under the general transformation given by (1.5.1).

Answer:

$$\alpha'\delta_{ij}'\delta_{kl}' + \beta'\delta_{ik}'\delta_{jl}' + \gamma'\delta_{il}'\delta_{jk}$$
$$= Q_{im} Q_{jn} Q_{kp} (\alpha\delta_{mn}\delta_{pq} + \beta\delta_{mn}\delta_{nq} + \gamma\delta_{mq}\delta_{np})$$
$$= \alpha Q_{im} Q_{jm} Q_{kp} Q_{lp} + \beta Q_{im} Q_{jn} Q_{km} Q_{ln} + \gamma Q_{im} Q_{jn} Q_{kn} Q_{lm}$$
$$= \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}.$$

Q 1.10/.

For 4th order tensor given in Ex. Show that if $\beta = \gamma$, ...

Sol./.

From E.g 1-9,

$$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}$$

Using $\beta = \gamma$

$$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$= \alpha \delta_{kl}\delta_{ij} + \beta(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li})$$

$$= C_{klij}$$

Q 1.11/.

Show that fundamental invariants can be expressed ....

Sol./.

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$

$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix}$

$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$

$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$

Q 1.12/.

Determine invariants and principal values, directions of foll. matrices...

a) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution/.

$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_a = -1$

$II_a = -2$ and $III_a = 0$

Characteristic Eq. is $-\lambda^3 - \lambda^2 + 2\lambda = 0$

$\Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda(\lambda+2)(\lambda-1) = 0$

$\lambda_1 = -2, \lambda_2 = 0, \lambda_3 = 1$

For case 1.

when $\lambda_1 = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0$$

$n_1^{(1)} = -n_2^{(1)}$

$= \pm\sqrt{2}/2$, $n^{(1)} = \pm(\sqrt{2}/2)(-1,1,0)$

For $\lambda_2 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$

$n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1$

For $\lambda_3 = 1$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$-2n_1^{(1)} + n_2^{(3)} = 0$

$n^{(3)} = \pm(0,0,1)$

$n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1$

Rotation matrix is given by:

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \cancel{\tfrac{1}{2}} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution/.

$$a_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -4, \; II_a = 3$$

$III_a = 0$

Ch. eq is $-\lambda^3 - 4\lambda^2 - 3\lambda = 0$

$\lambda(\lambda^2 + 4\lambda + 3) = 0$

$\lambda_1 = -3, \; \lambda_2 = -1 \;\&\; \lambda_3 = 0$

For $\lambda_1 = -3$,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0$$

$$n^{(1)} = \pm \left(\sqrt{2}/2 \right) (-1, 1, 0)$$
$$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$$

For $\lambda_2 = -1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$n^{(2)} = \pm \sqrt{2}/2 \ (1, 1, 0)$$
$$n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1$$

For $\lambda_3 = 0$,

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm (0, 0, 1)$$
$$n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1$$

Rot. matrix is,

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $\lambda_1 = -3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \qquad n_1^{(1)} + n_2^{(1)} = 0$$

$$n^{(1)} = \pm \left(\sqrt{2}/2 \right) (-1, 1, 0)$$
$$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$$

For $\lambda_2 = -1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow -n_1^{(2)} + n_2^{(2)} = 0$$

$$n_1 = n_2 = \pm \sqrt{2}/2 \Rightarrow n^{(2)} = \pm \sqrt{2}/2 \ (1, 1, 0)$$
$$n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1$$

For $\lambda_3 = 0$,

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -2n_1^{(3)} + n_2^{(1)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \end{matrix}$$

$$n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm (0, 0, 1)$$

$$n_1^{(1)2} + n_2^{(3)2} + n_3^{(3)2} = 1$$

Rot. matrix is:

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$a'_{ij} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution/.

$I_a = -2$, $II_a = 0$, $III_a = 0$

Ch. eq. is $-\lambda^3 - 2\lambda^2 = 0$

$\lambda^2(\lambda+2) = 0$

$\lambda_1 = -2$, $\lambda_2 = \lambda_3 = 0$

For $\lambda_1 = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0$$

$n_1^{(1)} + n_2^{(1)} = 0$

$n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2$

$n^{(1)} = \pm\sqrt{2}/2 \; (-1, 1, 0)$

For $\lambda_2 = \lambda_3 = 0$,

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -n_1 + n_2 = 0 \\ n = \pm(k, k, \sqrt{1-2k^2}) \end{array}$$

For arbitrary k & thus derivatives aren't uniquely determined.

let $\cancel{k = \sqrt{2}/\sqrt{2}}$ $k = \sqrt{2}/2$ & 0

$n^{(2)} = \pm\sqrt{2}/2 \; (1,1,0)$

$n^{(3)} = \pm(0,0,1)$

Rot. matrix is:

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$a'_{ij} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q1.14/.

Calculate quantities $\nabla . u$, $\nabla \times u$, $\nabla^2 u$, ...

a) $u = x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3$

$\nabla . u = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1 x_2$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1 x_2 & 2x_1x_2x_3 \end{vmatrix} = 2x_1 x_3 e_1 - 2x_2 x_3 e_2 + x_2 e_3$$

$\nabla^2 u = 0e_1 + 0e_2 + 0e_3 = 0$

$$\nabla u = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & x_2 & 0 \\ 2x_2 x_3 & 2x_1 x_3 & 2x_1 x_2 \end{bmatrix}$$

$tr(\nabla u) = 1 + x_1 + 2x_1 x_2$

b) $u = x_1^2 e_1 + 2x_1 x_2 e_2 + x_3^3 e_3$

$\nabla . u = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1 x_2 & x_3^3 \end{vmatrix}$$

$= 0e_1 - 0e_2 + 2x_2 e_3$

$\nabla^2 u = 0$

$$\nabla u = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}$$

$tr(\nabla u) = 4x_1 + 3x_3^2$

c) $u = x_2^2 e_1 + 2x_2 x_3 e_2 + 4x_1^2 e_3$

$\nabla . u = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2 x_3 & 4x_1^3 \end{vmatrix} = -2x_2 e_1 - 8x_1 e_2 - 2x_2 e_3$$

$\nabla^2 u = 2e_1 + 0e_2 + 8e_3 = 0$

$$\nabla u = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}$$

$tr(\nabla u) = 3x_3$

Q 1.15/.

The dual vector a_i of antisymmetric second order tensor a_{ij} is defined by $a_i = -\frac{1}{2}\varepsilon_{ijk} a_{jk}$ · Show that ...

Sol/. $a_i = -\frac{1}{2} \varepsilon_{ijk} a_{jk}$

$$\varepsilon_{imn} a_i = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{imn} a_{jk} = -\frac{1}{2} \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

$$= -\frac{1}{2} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) a_{jk}$$

$$= -\frac{1}{2} (a_{mn} - a_{nm}) = -\frac{1}{2} (a_{mn} + a_{mn})$$

$$= -a_{mn}$$

$$\therefore \; a_{jk} = -\varepsilon_{ijk} a_i$$

Q 1.16/.

Using index notation, explicitily verify vector identities.

(a) $(1.8.5)_{1,2,3}$

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k}$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi \cdot \nabla\psi$$

$$\nabla \cdot (\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k} u_k = \nabla\phi u + \phi(\nabla \cdot u)$$

b). $$\nabla \times (\phi u) = \varepsilon_{ijk} (\phi u_k)_{,j}$$

$$= \varepsilon_{ijk} (\phi u_{k,j} + \phi_{,j} u_k)$$

$$= \nabla\phi \times u + \phi(\nabla \times u)$$

$$\nabla \cdot (u \times v) = (\varepsilon_{ijk} u_j v_k)_{,i}$$

$$= v_k \varepsilon_{ijk} u_{j,i} + u_j \varepsilon_{ijk} v_{k,i}$$

$$= v \cdot (\nabla \times u) - u \cdot (\nabla \times v)$$

$$\nabla \times \nabla\phi : \varepsilon_{ijk} (\phi_{,k})_{,j} = \varepsilon_{ijk} \phi_{,kj} = 0$$ (symmetry & anti-symmetry in jk)

$$\nabla \cdot \nabla\phi = \nabla^2\phi$$

c) $$\nabla \cdot (\nabla \times u) = (\varepsilon_{ijk} u_{k,j})_{,i} = \varepsilon_{ijk} u_{k,ji} = 0$$

$$\nabla \times (\nabla \times u) = (\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}) u_{k,jn}$$

$$= \nabla(\nabla \cdot u) - \nabla^2 u$$

$$u \times (\nabla \times u) = \varepsilon_{ijk} u_j (\varepsilon_{kmn} u_{n,m}) = \varepsilon_{kji} \varepsilon_{kmn} u_j u_{n,m}$$

$$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) u_j u_{n,m}$$

$$= u_n u_{n,i} - u_m u_{i,m}$$

Q1.17/

Extend results found in example 1-5, & determine ...

Sol/

Cylindrical coord. are:

$$\xi^1 = r,\ \xi^2 = \theta,\ \xi^3 = z$$

$$(ds)^2 = (dr)^2 + (r d\theta)^2 + (dz)^2$$

$$h_1 = 1,\ h_2 = r,\ h_3 = 1$$

$$\frac{\partial \hat{e_r}}{\partial \theta} = \hat{e}_\theta,\quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r,\quad \frac{\partial \hat{e_r}}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e_z}}{\partial z} = 0$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e_z}\frac{\partial}{\partial z}$$

$$\nabla f = \hat{e_r}\frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e_z}\frac{\partial f}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e_r} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\hat{e_z}$$

Q1.18/

For spherical coord. system, (R, ϕ, θ) in fig., show that ....

Sol./ Spherical coord.

$$\xi^1 = R,\ \xi^2 = \phi,\ \xi^3 = \theta$$

$$x^1 = \xi^1 \sin\xi^2 \xi^3,\ x^2 = \xi^1 \sin\xi^2 \sin\xi^3,\ x^3 = \xi^1 \cos\xi^2$$

Scale factors:

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1$$

$$(h_2)^2 = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit vectors:

$$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3$$

$$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3$$

$$\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2$$

$$\frac{\partial \hat{e}_R}{\partial R} = 0,\quad \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e_\phi},\quad \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0,\quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_r = \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta$$

Using (1.9.12) – (1.9.16),

$$\nabla = \hat{e}_R\frac{\partial}{\partial R} + \hat{e}_\phi\frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta\frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}$$

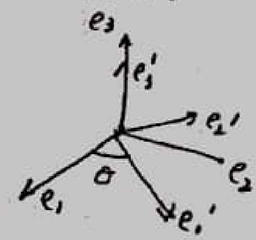
$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla \cdot u = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}(u_\theta)$$

$$\nabla^2 f = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

Example/.

Suppose basis $\{e_1', e_2', e_3'\}$ is obtained by rotating basis $\{e_1, e_2, e_3\}$ through θ about unit e_3. Write out rule for 2-tensors.



$$e_1' = \cos\theta\, e_1 + \sin\theta\, e_2$$
$$e_2' = -\sin\theta\, e_1 + \cos\theta\, e_2$$
$$e_3' = e_3$$

$$[Q] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$[A'] = [Q][A][Q^T]$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ A_{21}' & A_{22}' & A_{23}' \\ A_{31}' & A_{32}' & A_{33}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_{11}\cos\theta + A_{12}\sin\theta & -A_{11}\sin\theta + A_{12}\cos\theta & A_{13} \\ A_{21}\cos\theta + A_{22}\sin\theta & -A_{21}\sin\theta + A_{22}\cos\theta & A_{23} \\ A_{31}\cos\theta + A_{32}\sin\theta & -A_{31}\sin\theta + A_{32}\cos\theta & A_{33} \end{bmatrix}$$

R.H.S =

$$\begin{bmatrix} A_{11}\cos^2\theta + A_{22}\sin^2\theta + (A_{12}+A_{21})\sin\theta\cos\theta & A_{12}\cos^2\theta - A_{21}\sin^2\theta + (A_{22}-A_{11})\cos\theta\sin\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ A_{21}\cos^2\theta - A_{12}\sin^2\theta + (A_{22}-A_{11})\sin\theta\cos\theta & A_{22}\cos^2\theta + A_{11}\sin^2\theta - (A_{12}+A_{21})\cos\theta\sin\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

Using half-angle identities, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$

$$= \begin{bmatrix} \left(\frac{A_{11}+A_{22}}{2}\right) + \left(\frac{A_{11}-A_{22}}{2}\right)\cos 2\theta + \frac{A_{12}+A_{21}}{2}\sin 2\theta & \left(\frac{A_{12}-A_{21}}{2}\right) + \left(\frac{A_{12}+A_{21}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ \left(\frac{A_{21}-A_{12}}{2}\right) + \left(\frac{A_{21}+A_{12}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta & \left(\frac{A_{22}+A_{11}}{2}\right) + \left(\frac{A_{22}-A_{11}}{2}\right)\cos 2\theta - \left(\frac{A_{12}+A_{21}}{2}\right)\sin 2\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

Comparing b/s,

$$A_{11}' = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + \frac{A_{12}+A_{21}}{2}\sin 2\theta$$
$$A_{12}' = \frac{A_{12}-A_{21}}{2} + \frac{A_{12}+A_{21}}{2}\cos 2\theta + \frac{A_{22}-A_{11}}{2}\sin 2\theta$$
$$A_{13}' = A_{13}\cos\theta + A_{23}\sin\theta$$
$$A_{21}' = \frac{A_{21}-A_{12}}{2} + \frac{A_{21}+A_{12}}{2}\cos 2\theta + \frac{A_{22}-A_{11}}{2}\sin 2\theta$$
$$A_{22}' = \frac{A_{22}+A_{11}}{2} + \frac{A_{22}-A_{11}}{2}\cos 2\theta - \frac{A_{12}+A_{21}}{2}\sin 2\theta$$
$$A_{23}' = A_{23}\cos\theta - A_{13}\sin\theta$$
$$A_{31}' = A_{31}\cos\theta + A_{32}\sin\theta$$
$$A_{32}' = A_{32}\cos\theta - A_{31}\sin\theta$$
$$A_{33}' = A_{33}$$

When [A] is symmetric,

$A_{13} = A_{23} = 0$, so, 9 eq's simplify to:

$$A_{11}' = \frac{A_{11} + A_{22}}{2} + \frac{A_{11} - A_{22}}{2} \cos 2\theta + A_{12} \sin 2\theta$$

$$A_{22}' = \frac{A_{11} + A_{22}}{2} - \frac{A_{11} - A_{22}}{2} \cos 2\theta - A_{12} \sin 2\theta$$

$$A_{12}' = -\frac{A_{11} - A_{22}}{2} \sin 2\theta$$

together $A_{13}' = A_{23}' = 0$ & $A_{33}' = A_{33}$.

They are well known eq's underlying Mohr's circle for transforming 2-tensors in 2-D.

(11) Transform the strain-displacement relations given below

$$e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad e_z = \frac{\partial w}{\partial z}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

to cylindrical and spherical coordinates.

I Cylindrical Coordinates

The relation between cartesian and cylindrical coordinates is;

$$x = r\cos\theta \quad ; \quad y = r\sin\theta \quad ; \quad z = z$$

where $r = \sqrt{x^2+y^2+z^2}$; $\theta = \arctan\left(\frac{y}{x}\right)$

The partial derivatives of above relations are:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

Now, we can determine the expression;

$e_x = \frac{\partial u_x}{\partial x}$ where $u_x = u_r\cos\theta - u_\theta\sin\theta$

$$e_x = \cos\theta\left(\frac{\partial}{\partial r}\left(u_r\cos\theta - u_\theta\sin\theta\right)\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(u_r\cos\theta - u_\theta\sin\theta\right)$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta - \frac{\partial u_\theta}{\partial r}\sin\theta\cos\theta - \frac{\partial u_r}{\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{u_r}{r}\sin^2\theta + \frac{\partial u_\theta}{\partial\theta}\frac{\sin^2\theta}{r} + \frac{u_\theta}{r}\sin\theta\cos\theta$$

$$= \frac{\partial u_r}{r}\cos^2\theta + \left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\sin\theta\cos\theta + \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\sin^2\theta$$

Similarly,

$e_y = \frac{\partial u_y}{\partial y}$ where $u_y = u_r\sin\theta + u_\theta\cos\theta$

$$e_y = \sin\theta\frac{\partial}{\partial r}\left(u_r\sin\theta + u_\theta\cos\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(u_r\sin\theta + u_\theta\cos\theta\right)$$

$$= \frac{\partial u_r}{\partial r}\sin^2\theta + \sin\theta\cos\theta\frac{\partial u_\theta}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(u_r\sin\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(u_\theta\cos\theta\right)$$

$$= \frac{\partial u_r}{\partial r}\sin^2\theta + \frac{\partial u_\theta}{\partial r}\sin\theta\cos\theta + \frac{\partial u_r}{\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{u_r}{r}\cos^2\theta + \frac{\partial u_\theta}{\partial\theta}\frac{\cos^2\theta}{r} + \frac{\cos\theta(-\sin\theta)\,u_\theta}{r}$$

$$e_y = \frac{\partial u_r}{\partial r}\sin^2\theta + \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\sin\theta\cos\theta + \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)\cos^2\theta$$

$$\because e_{xy} = 2\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Hence we obtain,

$$e_r = \frac{\partial u_r}{\partial r} \; ; \; e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) \; ; \; e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \; ; \; e_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \; ; \; e_z = \frac{\partial u_z}{\partial z}$$

II Spherical coordinates

The relation between cartesian and cylindrical coordinates is:

$$x = R\cos\theta\sin\phi \; ; \; y = R\sin\theta\sin\phi \; ; \; z = R\cos\phi$$

where $R = \sqrt{x_r^2 + y^2 + z^2}$; $\phi = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$; $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$u = u_R e_R + u_\phi e_\phi + u_\theta e_\theta$$

$$\nabla \cdot u = \frac{\hat{e}_R}{h_1}\frac{\partial u}{\partial R} + \hat{e}_\phi \frac{1}{h_2}\frac{\partial u}{\partial \phi} + \hat{e}_\theta \frac{1}{h_3}\frac{\partial u}{\partial \theta}$$

$$(h_1)^2 = \frac{\partial x}{\partial \xi^1}\frac{\partial x}{\partial \xi^1}$$

$$= (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

$$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3$$
$$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3$$
$$\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2.$$

$$\frac{\partial \hat{e}_R}{\partial R} = 0 \; ; \; \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e}_\phi \; , \; \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0 \; , \; \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_r \; , \; \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0 \; , \; \frac{\partial \hat{e}_\theta}{\partial \phi} = 0 \; , \; \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi\, \hat{e}_\phi$$

$$\nabla u = \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial R}(R^2 \sin\phi\, u_R) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi\, u_\phi) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \theta}(R u_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}(u_\theta)$$

Substituting the above expression in

$$e = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

the strain-displacement relation for the spherical coordinates become:

$$e_R = \frac{\partial u_R}{\partial R}, \quad e_\phi = \frac{1}{R}\left(u_R + \frac{\partial u_\phi}{\partial \phi}\right)$$

$$e_\theta = \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + \sin\phi\, u_R + \cos\phi\, u_\phi\right)$$

$$e_{R\phi} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right)$$

$$e_{\phi\theta} = \frac{1}{2R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - \cot\phi\, u_\theta\right)$$

$$e_{\theta R} = \frac{1}{2}\left(\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right)$$