1D Ring Network

1. Network Topology

The one dimensional ring network is generated as a *Watts-Strogatz* small world graph with random connection to n-1 nearest neighbours where n is the number of neurons in the network :

```
networkx.watts_strogatz_graph(n, k, p)
```

where n (int) is The number of nodes, each node is joined with its $k = n - 1$ nearest neighbors in a ring topology and $p = 0$ (float) is The probability of rewiring each edge

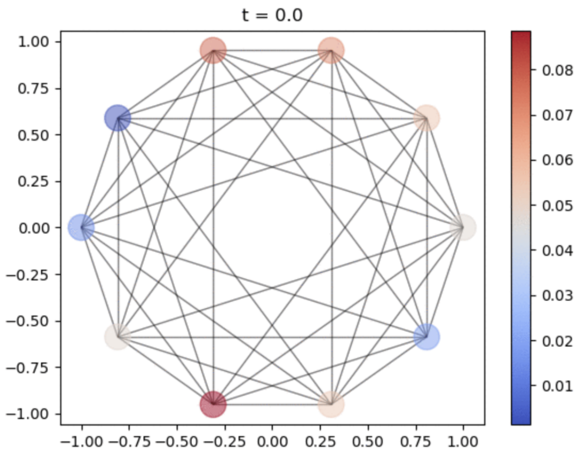

Figure 1: A ring network with $n = 10$.

2. The dynamics equation

The time evolution of the system has "standard" simplified continuous dynamics [1]:

$$\tau \frac{du}{dt} = -u + w * f \tag{1}$$

which is solved by using the forward Euler method as:

```
for i in range(1, tf):
    u[:, i] = deltat * (- u[:, i-1] + w.dot(sig(u[:, i-1])))
```

where the network is initialized by a random noise.
The function f is the sigmoidal function:

$$f = \sigma(u)$$

where

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

.

The weight distribution w is a a circulant matrix, i.e. a square matrix in which each row vector is rotated one element to the right relative to the preceding row vector. The vector generating the circulant matrix is an array of n values (floats) uniformly distributed over one period of the cosine function.

3. Network animation

The network animation plots the time evolution of the average net input u of each neuron - from initial random noise to the steady state.

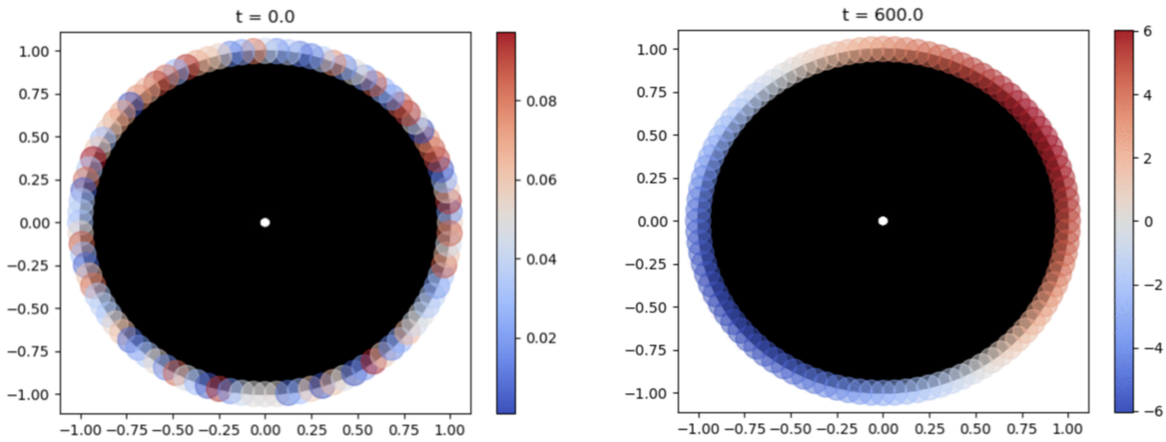

Figure 2: Time evolution from random noise to steady state.

In the case of steady state, the firing rate f and average net input u correspond.

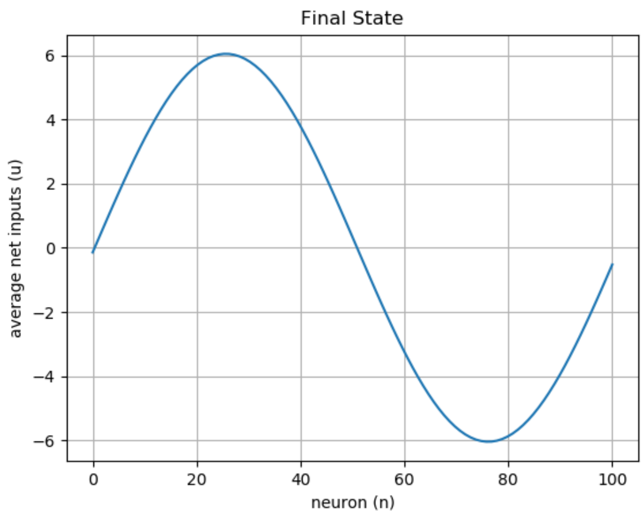

Figure 3: Steady state configuration of the network.

From initial random noise, the neurons abruptly reach their steady state.

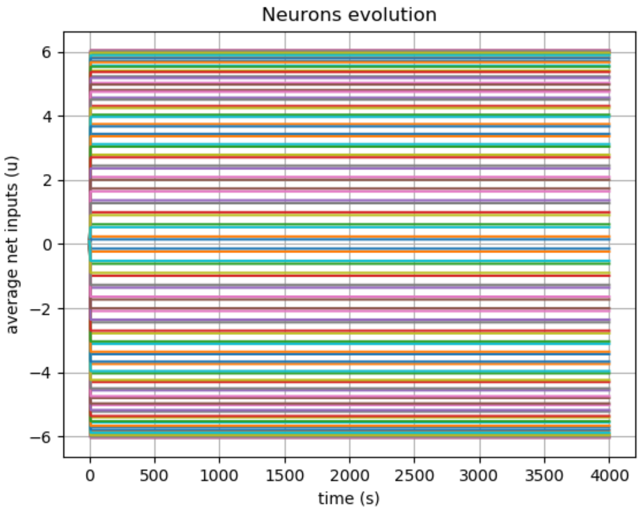

Figure 4: Time evolution of each single neuron.

4. Lyapunov Stability

From the firing-rate model given by

$$\tau \frac{d\mathbf{I}}{dt} = -\mathbf{I} + \mathbf{h} + \mathbf{M} * \mathbf{F(I)} \tag{2}$$

where $\mathbf{I} = I_s$ is the total synaptic current, $\mathbf{h} = \mathbf{W} \cdot \mathbf{u}$ is the total feedforward input to each neuron in the network, $\mathbf{u} = u_b$ represents the firing rates of neurons b, $\mathbf{W}$ is the matrix of synaptic weights, with the matrix component W_{ab} representing the strenght of the synapse from input unit b to output unit a and $\mathbf{M}$ describes the synaptic weights between output neurons.

Per single component this corresponds to

$$\tau \frac{dI_a}{dt} = -I_a + h_a + \sum_{k=1}^{N_\nu} M_{ak} F(I_k) \tag{3}$$

With a symmetric recurrent weight matrix, Cohen and Grossberg [2] demonstrated that the function

$$\begin{aligned} L(\mathbf{I}) &= \sum_{a=1}^{N_\nu} \left(\int_o^{I_a} dz_a z_a F'(z_a) - h_a F(I_a) - \frac{1}{2} \sum_{a'=1}^{N_\nu} F(I_a) M_{aa'} F(I_{a'}) \right) \\ &= \sum_{a=1}^{N_\nu} \int_o^{I_a} dz_a z_a F'(z_a) - \sum_{a=1}^{N_\nu} h_a F(I_a) - \frac{1}{2} \sum_{a=1}^{N_\nu} \sum_{a'=1}^{N_\nu} F(I_a) M_{aa'} F(I_{a'}) \end{aligned} \tag{4}$$

where, using $F(z_a) = \sigma(z_a)$ and $F'(z_a) = \sigma(z_a)(1 - \sigma(z_a))$

$$\begin{aligned} \int_o^{I_a} dz_a z_a F'(z_a) = I_a \sigma(I_a) + \log(2 - 2\sigma(I_a)) \\ + \frac{1}{2} \left(\pi^2 - (\log(1 - \sigma(I_a)) - \log(\sigma(I_a) - 1))(\log(1 - \sigma(I_a)) \right. \\ \left. + \log(\sigma(I_a) - 1) - 2\log(\sigma(I_a)) + 2I_a \right) \end{aligned} \tag{5}$$

has $dL/dt < 0$ whenever $d\mathbf{I}/dt \neq 0$.

To see this take the time derivative of (4)

$$\frac{dL(\mathbf{I})}{dt} = \sum_{a=1}^{N_\nu} \frac{dL(\mathbf{I})}{dI_a} \frac{dI_a}{dt} \tag{6}$$

where

$$\frac{dL(\mathbf{I})}{dI_a} = \frac{d}{dI_a} \int_o^{I_a} dz_a z_a F'(z_a) - \frac{d}{dI_a} h_a F(I_a) - \frac{1}{2} \frac{d}{dI_a} \sum_{a'=1}^{N_\nu} F(I_a) M_{aa'} F(I_{a'}) \tag{7}$$

The first term is:

$$\frac{d}{dI_a} \int_0^{I_a} dz_a z_a F'(z_a) = F'(I_a) I_a \tag{8}$$

The second term is:

$$\frac{d}{dI_a} h_a F(I_a) = h_a F'(I_a) \tag{9}$$

The third term is:

$$
\begin{aligned}
\frac{d}{dI_k}\left(\frac{1}{2}\sum_{a'=1}^{N_\nu} F(I_a)M_{aa'}F(I_{a'})\right) &= \frac{1}{2}\sum_{a'=1}^{N_\nu}\left(F'(I_a)M_{aa'}F(I_{a'}) + F(I_a)M_{aa'}F'(I_{a'})\right) \\
&= \sum_{a'=1}^{N_\nu}\left(F'(I_a)M_{aa'}F(I_{a'})\right) \\
&= F'(I_a)\left[\mathbf{M}\ \mathbf{F(I)}\right]_a
\end{aligned}
\tag{10}
$$

Leading to

$$
\begin{aligned}
\frac{dL(\mathbf{I})}{dI_a} &= \frac{d}{dI_a}\int_o^{I_a} dz_a z_a F'(z_a) - \frac{d}{dI_a}h_a F(I_a) - \frac{d}{dI_a}\sum_{a'=1}^{N_\nu} F(I_a)M_{aa'}F(I_{a'}) \\
&= I_a F'(I_a) - h_a F'(I_a) - F'(I_a)\left[\mathbf{M}\ \mathbf{F(I)}\right]_a \\
&= F'(I_a)\left(I_a - h_a - \left[\mathbf{M}\ \mathbf{F(I)}\right]_a\right) \\
&= -\frac{1}{\tau}F'(I_a)\left(\frac{dI_a}{dt}\right)
\end{aligned}
\tag{11}
$$

Therefore

$$
\begin{aligned}
\frac{dL(\mathbf{I})}{dt} &= \sum_{a=1}^{N_\nu}\frac{dL(\mathbf{I})}{dI_a}\frac{dI_a}{dt} \\
&= -\frac{1}{\tau}\sum_{a=1}^{N_\nu}F'(I_a)\left(\frac{dI_a}{dt}\right)^2
\end{aligned}
\tag{12}
$$

5. Conversion between firing rate and synaptic current

The synaptic output/current (up to the weights) is given by

$$
\frac{1}{\tau}\dot{s} = -s_j + r_j \Leftrightarrow s_j(t) = f * r_j(t) \tag{13}
$$

where $f * r_j(t)$ is a convolution between the exponential kernel f and the time-dependent firing rate $r_j(t)$ given by:

$$
r_i = \phi\underbrace{\left[\sum_j w_{ij}s_j + b_i\right]}_{I_i} \tag{14}
$$

where ϕ represents a non-linearity (e.g. threshold linear) acting on I_i which is the input current.
Expressing $\dot{s}_i = \dot{s}_i(I_i)$ in terms of I_i:

$$
\begin{aligned}
\frac{1}{\tau}\dot{s} &= -s_j + r_j \\
&= -s_j + \phi\left[\sum_j w_{ij}s_j + b_i\right] \\
&= -s_i + \phi(I_i)
\end{aligned}
\tag{15}
$$

The time derivative of I_i is given by

$$\begin{aligned}
\dot{I}_i &= \frac{d}{dt}\left(\sum_j w_{ij} s_j + b_i\right) \\
&= \sum_j w_{ij} \dot{s}_j \\
&= \sum_j w_{ij} \tau \left[-s_j + \phi(I_i)\right]
\end{aligned} \tag{16}$$

Finally, the input current model is given by

$$\begin{aligned}
\frac{1}{\tau}\dot{I}_i &= \sum_j w_{ij} \left[-s_j + \phi(I_i)\right] \\
&= \underbrace{-\sum_j w_{ij} s_j}_{I_i - b_i} + \sum_j w_{ij} \phi(I_i) \\
&= -I_i + b_i + \sum_j w_{ij} \phi(I_i) \\
&= -I_i + b_i + \sum_j w_{ij} r_i
\end{aligned} \tag{17}$$

References

[1] *K. Zhang, Representation of spatial orientation by the intrinsic dynamics of the head-direction cell ensemble: a theory, Journal of Neuroscience 16 (6) (1996) 2112–2126. arXiv:https://www.jneurosci.org/content/16/6/2112.full.pdf, doi:10.1523/JNEUROSCI.16-06-02112.1996.*
URL `https://www.jneurosci.org/content/16/6/2112`

[2] *M. A. Cohen, S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, IEEE Transactions on Systems, Man, and Cybernetics SMC-13 (5) (1983) 815–826.*

Appendix A - Ring network code