The Cut-Elimination Theorem for a Sequent-Calculus for the Predicate Calculus

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1 Background

1.1 Formulas, terms and languages

1.1.1 Definition: Language

A *language*, L, is a tuple which contains a triple of sequences

$$L = (\underline{f_1}, ..., \underline{f_k}, \underline{p_1}, ..., \underline{p_l}, \underline{c_1}, ..., \underline{c_m})$$

together with a function $a : \{\underline{f_1}, ..., \underline{f_k}, \underline{p_1}, ..., \underline{p_l}\} \longrightarrow \mathbb{N}$ where:

- $\underline{f_i}$ are function symbols and where $a(\underline{f_i})$ is the arity of $\underline{f_i}$ for $1 \leq i \leq k$
- $\underline{p_i}$ are relation symbols and where $a(\underline{p_i})$ is the arity of $\underline{p_i}$ for $1 \leq i \leq l$
- $\underline{c_i}$ are constant symbols for $1 \leq i \leq m$

Remark: When considering languages and related concepts below, the focus will be on *predicate logic*, and not the closely related concept of *first order languages*. While both notions allow for variables being quantified over predicates, predicate logic does not include the notion of *equality*.

For first order languages, the equality symbol '=' would be included in **1.1.2** in the symbols of a language and in **1.1.4** two terms connected by an equality symbol would be considered a formula.

To any first order language, five axioms of equality are considered and used to make inferences based on formulas containing the equality symbol. Since this project focuses on predicate logic, these concepts will not be discussed further, but the discussed principles could be applied to first order languages analogously.

1.1.2 Symbols of a Language

To talk inside a language L, certain symbols are required. These are:

- the symbols of L, which are $\underline{f_1}, ..., \underline{f_k}, \underline{p_1}, ..., \underline{p_l}$
- logical connectives: '$\rightarrow$' and '$\perp$'
- quantifier(s): '$\exists$'
- variables: $a_1, a_2, ..., b_1, ..., c_1, ..., x_1, x_2, ..., y_1, ..., z_1, ...$
- auxiliary symbols: parentheses '(', ')' and comma ','

1.1.3 Definition: Terms

Terms are defined inductively:

- any variable or a constant is a term
- if $\tau_1, ..., \tau_n$ are terms, then $\underline{f_i}(\tau_1, ..., \tau_n)$ is a term

If a term t does not contain any variables, it is a *constant term*.

1.1.4 Definition: Formulas

Formulas are defined inductively:

- if $\tau_1, ..., \tau_n$ are terms and $\underline{p}$ is an n-ary relation symbol, then $\underline{p}(\tau_1, ..., \tau_n)$ is an *atomic formula*
- '$\bot$' is always a formula
- if ϕ and ψ are formulas, then $(\phi \to \psi)$ is a formula
- if x is a variable, then $\exists x\phi$ is a formula

For formulas, it is conventional to omit the outermost set of parentheses.

Remark: Formulas, as defined above, only contain '$\to$' and '$\bot$' as logical connectives, and '$\exists$' as their quantifiers. In other treatments of predicate logic, '$\neg\phi$' is often used as a shorthand for '$\phi \to \bot$' and up to five logical connectives and two quantifiers are used. These are '$\wedge$', '$\vee$', '$\to$', '$\leftrightarrow$', '$\neg$' and '$\exists$', '$\forall$' respectively.

These treatments are however equally as powerful in what they can express as the predicate logic currently considered, as the additional logical connectives can be simulated with only '$\to$' and '$\neg$'. Likewise, '$\forall x\phi$' is logically equivalent to '$\neg\exists x\neg\phi$'.

Going forwards, only this system of predicate logic with fewer logical connectives and quantifiers will be considered.

1.1.5 Definition: Substitution and Sentences

Suppose t is a term and ϕ is a logical formula. *The substitution of x to t*, denoted '$\phi(t/x)$', is ϕ but with all free occurrences of x changed to t.

A *substitution is correct* if no variable of t becomes bound after the substitution.

A formula is called a *sentence* if it contains no free variables.

1.2 Proof Systems in General

A *proof system*, P, is defined in relation to a language L and comprises of *axioms* and *inference rules*. Inference rules are used for deriving *theorems* of P, which are formulas of L, from the proof system's axioms. Inference rules generally build upon a set, most often 1 or 2, of previously derived formulas, or axioms, to produced a new formula. Axioms are formulas of L which are taken to be true in P. Different proof systems have different conventions of how a proof is presented, with two examples being tree-like derivations and indented sequences like those of Fitch systems.

Proofs within a proof system P are ordered collections of axioms of P and inference rules of P, which together end with a formula of L which is the theorem to be proved.

Within a system, a formula - or a sequent as will be introduced soon - can be said to be a *premise*, meaning that the formula or sequent can be treated as an axiom. If a formula or sequent is provable from a set of formulas or sequents Σ as premises, it is said that the formula at hand is *provable* from Σ. If $\Sigma = \emptyset$, it is simply said that the formula is *provable*.

1.3 The Modified Gentzen System G^S

For the sake of this paper, a modified *Gentzen system* will be used. The proof system is denoted G^S, G for Gentzen and super-scripted with S to denote the set-theoretic foundation which will be introduced. The basic unit of Gentzen systems is the *sequent*. In Gentzen's original system, a sequent is defined as two sequences of formulas (possible empty), $\phi_1, ..., \phi_k$ and $\psi_1, .., \psi_l$, connected by $\vdash$ as in $\phi_1, ..., \phi_k \vdash \psi_1, .., \psi_l$. The intuitive logical meaning of a sequent $\phi_1, ..., \phi_k \vdash \psi_1, .., \psi_l$ is: $\bigwedge_{i=1}^{k} \phi_i \rightarrow \bigvee_{i=1}^{l} \psi_i$, although a sequent is not part of the language L itself since it contains the meta-logical symbol '$\vdash$'. The first sequence is called the *antecedent* of the sequent and the second sequence is called the *consequent* of the sequent.

In both the classical Gentzen proof System and in G^S, axioms and inference rules are based upon sequents instead of the usual focus on formulas of L in other proof systems.

In the modified system G^S, sequents are built upon two (possibly empty) sets of formulas connected by '$\vdash$'. A sequent in G^S therefore looks like $A \vdash B$ where A and B are sets of formulas of L. For ease of notation, a set with a super-scripted formula A^ϕ is used to denote the underlying set A but with ϕ removed from the set. Analogously to the normal Gentzen system, the first set is called the *antecedent* of the sequent and the second set is called the *consequent* of the sequent.

Inference rules in Gentzen systems take either one or two previously derived sequents, called upper sequents of the inference, and produce a singular new sequent, called the end-sequent or lower sequent of the inference. Based on this structure, a Gentzen system proof naturally forms a rooted tree where each inference rule is a node, the leaves are axioms and the end-sequent of the proof is the root.

In this treatment, the modified system G^S will only be discussed when based on a restricted first order language L containing only '$\rightarrow$' and '$\perp$' as logical connectives and '$\exists$' as a quantifier. Treating $\neg\phi$ as a shorthand for $\phi \rightarrow \neg$, it is apparent that restricted languages only using '$\rightarrow$','$\perp$' and '$\exists$' are equally as expressive as languages using four or five connectives and two quantifiers, since the additional connectives and quantifiers can be expressed as a combination of '$\rightarrow$','$\perp$' and '$\exists$'. The advantage of working with fewer connectives and quantifiers is that there are fewer inference rules in the system, since there usually are two inference rules associated with every quantifier and connective (except for '$\perp$' since it is a nullary connective).

Explicitly using '$\perp$' instead of $\neg$ does however introduce the need for an additional axiom, compared to languages using '$\neg$'.

1.3.1 Axiom Scheme of G^S

The axiom scheme of G^S only contains two axiom inferences called $A1$ and $A2$. The lower sequent of such an inference is referred to as an *initial sequent.* In more traditional treatments of the sequent calculus, the axiom scheme is often only $A1$, but in the current treatment is is required to have an introduction of '$\bot$' since it is used explicitly.

$$\frac{}{\{\phi\} \vdash \{\phi\}}\ \text{A1} \qquad\qquad \frac{}{\{\bot\} \vdash \emptyset}\ \text{A2}$$

1.3.2 Inference Rules of G^S

The inference rules of G^S for the specified restricted languages is shown below, where ϕ and ψ are any formulas of L, t any term of L and A_i and B_i are sets of formulas of L. The weakening rule is referred to as a *structural rule*, which remains from the original Gentzen system in which there are additional structural rules.

Weakening:

$$\frac{A \vdash B}{A \cup \{\phi\} \vdash B}\ \text{WL} \qquad\qquad \frac{A \vdash B}{A \vdash B \cup \{\phi\}}\ \text{WR}$$

$\rightarrow$ inferences:

$$\frac{A_1 \vdash B_1 \cup \{\phi\} \qquad A_2 \cup \{\psi\} \vdash B_2}{A_1 \cup A_2 \cup \{\phi \rightarrow \psi\} \vdash B_1 \cup B_2}\ {\rightarrow}\text{L} \qquad\qquad \frac{A \cup \{\phi\} \vdash B \cup \{\psi\}}{A \vdash B \cup \{\phi \rightarrow \psi\}}\ {\rightarrow}\text{R}$$

$\exists$ inferences:

$$\frac{A \cup \{\phi(a)\} \vdash B}{A \cup \{\exists x \phi(x)\} \vdash B}\ \exists\text{L}\ \textbf{Restr.} \qquad\qquad \frac{A \vdash B \cup \{\phi(t)\}}{A \vdash B \cup \{\exists x \phi(x)\}}\ \exists\text{L}$$

Restr.: In the $\exists$L inference, the proper variable of the inference, a, cannot appear free in the lower sequent, meaning it cannot occur in any formulas in A, B and $\phi(x)$.

Cut:

$$\frac{A_1 \vdash B_1 \cup \{\phi\} \qquad A_2 \cup \{\phi\} \vdash B_2}{A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}\ (\phi)$$

For the cut-rule, it is conventional only to include the formula which is being cut in parenthesis as the inference description on the right of the inference line.

1.3.3 Admissible Rules in G^S

For clarity, the derivations for introducing $\phi \rightarrow \neg$, equivalent to introducing $\neg\phi$ in both the antecedent and succedent of a sequent are shown below.

Derivations equivalent to ¬ inferences:

$$\dfrac{A \vdash B \cup \{\phi\} \qquad \dfrac{}{\{\bot\} \vdash \emptyset}\text{A2}}{A \cup \{\phi \to \bot\} \vdash B}\to\text{L} \qquad\qquad \dfrac{\dfrac{A \cup \{\phi\} \vdash B}{A \cup \{\phi\} \vdash B \cup \{\bot\}}\text{WR}}{A \vdash B \cup \{\phi \to \bot\}}\to\text{R}$$

Lemma (Law of Excluded Middle): The *Law of Excluded Middle* can be derived in G^S as shown below.

$$\dfrac{\dfrac{\dfrac{}{\{\phi\} \vdash \{\phi\}}\text{A1}}{\{\phi\} \vdash \{\phi, \bot\}}\text{WR}}{\emptyset \vdash \{\phi, \phi \to \bot\}}\to\text{R}$$

It is also often desirable to be able to reason through *reductio ad absurdum.* Intuitionistic style systems do not allow for this type of inference, whereas it is an admissible rule in G^S. Below is first the desired rule and secondly a derivation of said rule using only the inference rules of G^S. This derivation crucially uses the law of excluded middle.

RAA:

$$\dfrac{A \cup \{\phi \to \bot\} \vdash B}{A \vdash B \cup \{\phi\}}\text{RAA}$$

Equivalent Derivation to RAA in G^S:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\phi\} \vdash \{\phi\}}\text{A1}}{\{\phi\} \vdash \{\phi, \bot\}}\text{WR}}{\emptyset \vdash \{\phi, \phi \to \bot\}}\to\text{R} \qquad A \cup \{\phi \to \bot\} \vdash B}{A^{\phi} \vdash B \cup \{\phi\}}(\phi \to \bot)$$

1.4 Grade, Branches and Rank

To further simplify the treatment of proof trees in G^S a concepts will be introduced below. These will be used mainly later when proving the *cut-elimination theorem.*

1.4.1 Definition: Grade

The *grade* of a formula of a language L is the number of logical symbols in the formula. In the restricted languages currently considered, the only logical symbols contributing to the grade of a formula are '$\to$' and '$\exists$'. Any *atomic formula* is said to have a grade of 0 since they contain no logical symbols. If ϕ is a formula of grader n, $g(\phi) = n$ is the notation which will be used going forward, where $g(x)$ is a function from the set of L-formulas to the natural numbers defined by the grade of the input formula as above.

1.4.2 Definition: Branch

A *branch* in a G^S proof tree is a sequence of sequents whose where the first sequent is the lower sequent of an axiom introduction and where the last sequent in the sequence is the end-sequent of the proof tree. Every sequent in the branch, except the end-sequent, is an upper sequent of an inference of which the next sequent in the sequence is the lower sequent.

1.4.3 Definition: Rank

The rank of a G^S-proof tree, call it Π, is defined in relation to the cut-rule, where Π must have a cut as its final inference. Given such a proof tree Π, the *left rank* of Π, denoted $\text{rank}_l(\Pi)$, is the number of consecutive sequents, counting upward from the left upper sequent of the mix, that contains the mix formula in its succedent. The right rank, denoted $\text{rank}_r(\Pi)$, is defined similarly as the number of consecutive sequents, counting upward from the right upper sequent of the mix, that contains the mix formula in its antecedent.

The rank of a proof tree Π with a cut as its last inference is the sum of the right and left rank, meaning that the minimum rank of any such proof tree is 2.

2 Models, Satisfaction and Completeness

2.1 Models and Interpretations

2.1.1 Definition: Model

A *model* is a tuple

$$M = (A, f_1, ..., f_{k'}, p_1, ..., p_{l'}, c_1, ..., c_{m'})$$

where:

- A is a non-empty set and is called the universe of M, denoted $\|M\|$
- each f_i is a function $f_i : A^{n_i} \longrightarrow A$ where n_i is the arity of f_i
- each p_i is a relation $P_i \subseteq A^{k_i}$ where k_i is the arity of P_i
- each $c_i \in A$ is a *distinguished element*

If $l' = 0$, then M is called an *algebraic structure*, and if $k' = 0$, then M is called a *relational structure*. Examples of algebraic structures are groups, rings and fields and examples of relational structures are partially ordered sets.

2.1.2 Interpreting Symbols

Given a language L, a model M, and:

$$\begin{aligned} k &= k' \\ l &= l' \\ m &= m' \\ a(\underline{f_i}) &= \text{arity of } f_i \text{ (for } 1 \leq i \leq k) \\ a(\underline{p_i}) &= \text{arity of } p_i \text{ (for } 1 \leq i \leq l) \end{aligned}$$

it is said that M is a model for L and that the interpretations of the symbols of L, denoted by a super-scripted M, are as shown below.

$$\begin{aligned} \underline{f_i}^M &= f_i \text{ (for } 1 \leq i \leq k) \\ \underline{p_i}^M &= p_i \text{ (for } 1 \leq i \leq l) \\ \underline{c_i}^M &= c_i \text{ (for } 1 \leq i \leq m) \end{aligned}$$

Constant terms in a model M are interpreted inductively as shown below. Let t be the term to be interpreted and $t_1, ..., t_j$ be arbitrary terms.

$$\text{If } t = \underline{c_i} \text{ then } t^M = \underline{c_i}^M$$
$$\text{If } t = \underline{f_j}(t_1, ..., t_j) \text{ then } t^M = \underline{f_j}^M(t_1, ..., t_j)$$

2.2 Satisfaction

2.2.1 Extending a Language

Suppose $M = (A, f_1, ..., f_{k'}, p_1, ..., p_{l'}, c_1, ..., c_{m'})$ is a model for the language $L = (\underline{f_1}, ..., \underline{f_k}, \underline{p_1}, ..., \underline{p_l}, \underline{c_1}, ..., \underline{c_m})$. Consider the set $\{\underline{a} : a \in A\}$ as a set of new constant symbols corresponding to the objects in the universe of the model. From this set it is possible to create a new extended language, denoted $L(M)$, and defined as below.

$$L(M) = L \cup \{\underline{a} : a \in A\}$$

The interpretation of terms of $L(M)$ is analogous to that of the previously defined interpretation and is shown below. The interpretation of function symbols and relational symbols are identical since they are not affected by the extension and is omitted.

$$\text{If } t = \underline{c_i} \text{ then } t^M = \underline{c_i}^M$$
$$\text{If } t = \underline{a} \text{ then } t^M = a$$
$$\text{If } t = \underline{f_j}(t_1, ..., t_j) \text{ then } t^M = \underline{f_j}^M(t_1, ..., t_j)$$

2.2.2 Satisfaction of Sentences of an Extended Language

Given a model M, which is a model for a language L, and a sentence of $L(M)$, say ϕ, it is said that M satisfies ϕ, denoted by $M \models \phi$, if:

2.2.2.1 Atomic Sentences:

If ϕ is a atomic sentence, meaning $\phi = p_i(t_1, ..., t_j)$ for some terms $t_1, ..., t_j$, then $M \models \phi$ if $p_i^M(t_1^M, ..., t_j^M)$ is true.

By definition of falsehood, it is always true that $M \not\models \bot$.

2.2.2.2 Logical Connective as Outermost Logical Symbol:

In the restricted system currently being considered, the only non-nullary logical connective is implication. Thus, ϕ will be of the form $\phi \equiv \psi \to \chi$ for some sentences ψ and χ. It is true that $M \models \phi$ if it is NOT true that $M \models \psi$ and $M \not\models \chi$.

2.2.2.3 Quantifier as Outermost Logical Symbol:

In the restricted system currently being considered, '$\exists$' is the only available quantifier, meaning that $\phi \equiv \exists x \psi(x)$ for some formula $\psi(x)$. It is true that $M \models \psi$ if there exists an $a \in A$ such that $M \models \psi(a/x)$.

2.3 Satisfaction of a Sequent

Given a model M, which is a model for a language L, and a sequent of sentences from $L(M)$, say $S \equiv \{a_1, ..., a_i\} \vdash \{b_1, ..., b_j\}$ for some $i, j \in \mathbf{N}$, it is said that $M \models S$ if it is true that:

- If $M \models a_1$ and $M \models a_2$, ... , and $M \models a_i$
- then: $M \models b_1$ or $M \models b_2$, ... , $M \models b_j$

2.4 Consistent Sets of Sequents

A set of sequents, Σ, is said to be *inconsistent* if the sequent $\emptyset \vdash \bot$ is provable from Σ. Σ is said to be *consistent* if Σ is not inconsistent.

Theorem: The above definition of an inconsistent set of sequents is equivalent to defining Σ as *inconsistent* if Σ proves every sequent.

Proof: The reverse direction is trivial since if Σ proves every sequent, then Σ must prove $\emptyset \vdash \bot$.

To prove the other direction ,WLOG, let '$A \vdash B$' be the sequent which is to be proven and suppose that Σ proves $\emptyset \vdash \bot$.

For clarity, let a_i be any formulas in A and b_i be any formula in B.

Through a cut and weakenings, it is possible to prove $A \vdash B$ as is shown below, which completes the proof.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\emptyset \vdash \bot$}
\AxiomC{}
\RightLabel{A2}
\UnaryInfC{$\bot \vdash \emptyset$}
\RightLabel{$(\bot)$}
\BinaryInfC{$\emptyset \vdash \emptyset$}
\RightLabel{WL}
\UnaryInfC{$\{a_1\} \vdash \emptyset$}
\RightLabel{Several WL}
\UnaryInfC{$\vdots$}
\UnaryInfC{$A \vdash \emptyset$}
\RightLabel{Several WR}
\UnaryInfC{$\vdots$}
\UnaryInfC{$A \vdash B$}
\end{prooftree}
$$

2.5 Gödel's Theorem on the Existence of a Model

Theorem (Gödel's Theorem on the Existence of a Model): Σ is a consistent set of sequents if and only if Σ has a model.

Definition: A set of sequents Σ is said to *have a model* if there exists a model M with a non-empty universe such that $M \models S$ for all sequents $S \in \Sigma$.

Proof: The proof of this theorem is omitted for the sake of brevity. Henkin's proof is often preferred over Gödel's proof and is reliant on constructing a model for Σ using Zorn's Lemma.

2.6 Gödel's Completeness Theorem

Theorem (Gödel's Completeness Theorem): The sequent $A \vdash B$ is provable if and only if $A \models B$.

Definition: It is said that $A \models B$ is true if for every model M, $M \models A \vdash B$

In order to prove Gödel's Completeness Theorem, the following Lemmma will be used. The proof of this lemma can be constructed using induction on the length of the proof and is omitted.

Provability Lemma: $A \vdash B$ is provable if and only if $\emptyset \vdash B$ is provable from $\emptyset \vdash a_1$, $\emptyset \vdash a_2$, ... ,$\emptyset \vdash a_n$ where a_i $(1 \leq i \leq n)$ are all formulas of A.

Proof Sketch: The forwards direction of this proof is proving *soundness* of G^S which can be done through induction on the length of the proof. The details of this part of the proof is omitted.

The backwards direction of this proof is proving *completeness* of G^S.

For ease of notation, let a_i $(1 \leq i \leq n)$ be all formulas of A and σ_i $(1 \leq i \leq m)$ be all formulas in B and suppose that $A \models B$. Now consider the set of all of these formulas transformed into their trivial sequents, $\Sigma = \{\emptyset \vdash \{a_1\}, ..., \emptyset \vdash \{a_n\}, \emptyset \vdash \{\sigma_1\}, ..., \emptyset \vdash \{\sigma_m\}\}$.

Now since $A \models B$, meaning that $\{a_1, ..., a_n\} \models \{\sigma_1, ..., \sigma_m\}$, it is known that $A \cup \{\neg\sigma_1, ..., \neg\sigma_m\}$ does not have a model since it would be inconsistent with $A \models B$, meaning that $A \cup \{\neg\sigma_1, ..., \neg\sigma_m\}$ is inconsistent. By the equivalent definitions of inconsistency, it is therefore know that $\{a_1, ..., a_n, \neg\sigma_1, ..., \neg\sigma_m\} \vdash \bot$. From this, the following derivation can be made.

$$\dfrac{\dfrac{\dfrac{\vdots}{\{a_1, ..., a_n, \neg\sigma_1, ..., \neg\sigma_m\} \vdash \{\bot\}} \qquad \dfrac{}{\{\bot\} \vdash \emptyset}\ \text{A2}}{\{a_1, ..., a_n, \neg\sigma_1, ..., \neg\sigma_m\} \vdash \emptyset}\ (\bot)}{\dfrac{\vdots}{\{a_1, ..., a_n\} \vdash \{\sigma_1, ..., \sigma_m\}}}\ m \text{ number of RAA inferences}$$

But this means that for any sequent for which $A \models B$ holds, $A \vdash B$ must also hold, which completes the proof.

3 Main Lemma

Main Lemma: If Π is a proof of a sequent S in G^m which contains only one cut, occurring as the last inference, then S is provable without a cut.

Going forwards, the upper two sequents of the cut-inference will be called S_1 and S_2 respectively. The upper sequent of S_1 will be referred to as U_1 and the upper sequent of S_2 as U_2. If S_2 has two upper sequents, the right of these two will also be referred to as U_3. Assuming that Π is as stated in the Main Lemma, the sub-proofs of S_1 and S_2 in Π are cut-free and Π must look like below.

$$\dfrac{A_1 \vdash B_1 \qquad A_2 \vdash B_2}{A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}\ (\phi)$$

4 Proof of Main Lemma

Proof of Main Lemma: Let Π be as stated in the Main Lemma. The proof of the main lemma is divided into two cases - **Case 1** and **Case 2**. **Case 1** corresponds to when $\text{rank}(\Pi) = 2$ and **Case 2** corresponds to when $\text{rank}(\Pi) > 2$.

Induction on both the grade and rank of the cut formula will be used, with the "outer induction" being on the rank. Below is stated the base case and induction hypothesis.

- **Base case:** The base case here is to prove that it is possible to eliminate the cut of Π when $\text{rank}(\Pi) = 2$, which is what case 1 covers. This will be proven in case 1.
- **Induction hypothesis:** As the induction hypothesis, suppose that for any proof Π' for which $\text{rank}(\Pi') < n$, where $n = \text{rank}(\Pi)$, and where Π' only contains one cut as its last inference with the cut-formula ϕ', where $g(\phi') = g(\phi)$, it is possible to construct a cut-free proof of the same end-sequent as in Π'.

4.1 Case 1: Rank(Π)= 2

For case 1, suppose that rank(Π)= 2, meaning that $\text{rank}_r = \text{rank}_l = 1$.

4.1.1 S_1 is an initial sequent

Suppose S_1 is an initial sequent, meaning that Π must look like below. Notice that S_1 cannot be an initial sequent from A2, since A2 will never be able to produce anything but an empty set in the succedent of the initial sequent.

$$\dfrac{\dfrac{}{\{\phi\} \vdash \{\phi\}}\text{A1} \qquad A_2 \vdash B_2}{\{\phi\} \cup A_2^{\phi} \vdash B_2}\,(\phi)$$

Notice however that $\{\phi\} \cup A_2^{\phi} \vdash B_2$ is equal to $A_2 \vdash B_2$, meaning that the subproof for S_2 proves the desired sequent.

4.1.2 S_2 is an initial sequent

Suppose S_2 is an initial sequent. This can be proved the same way as **4.1.1**.

4.1.3 S_1 and S_2 are not initial sequents and S_1 is the lower sequent of weakening

Suppose S_1 and S_2 are not initial sequents and S_1 is the lower sequent of a weakening. Since it is assumed that rank(Π) = 2 this means that the only possible weakening which could have been applied to produce S_1 is WR with ϕ as the weakening formula. This means that the final three levels of Π must look like below, and that B_1 cannot contain ϕ.

$$\dfrac{\dfrac{A_1 \vdash B_1}{A_1 \vdash B_1 \cup \{\phi\}}\text{WR} \qquad A_2 \vdash B_2}{A_1 \cup A_2^{\phi} \vdash B_1 \cup B_2}\,(\phi)$$

The cut can be eliminated by starting from the sequent above S_1 in Π, as shown below.

$$\dfrac{\dfrac{A_1 \vdash B_1}{\text{Possibly several WL and WR}}}{A_1 \cup A_2^{\phi} \vdash B_1 \cup B_2}$$

4.1.4 S_1 and S_2 are not initial sequents and S_2 is the lower sequent of a weakening

Suppose S_1 and S_2 are not initial sequents and S_2 is the lower sequent of a weakening. Since it is assumed that rank(Π) = 2 this means that the only possible structural rule which could have been applied to produce S_2 is WL with ϕ as the weakening formula. This case is analogous to **3.1.3**.

4.1.5 S_1 and S_2 are the lower sequents of logical inferences

Suppose that both S_1 and S_2 are the lower sequents of logical inferences. In this case, since $\text{rank}_l = 1$ and $\text{rank}_r = 1$, the cut formula on each side must be the principal formula of the logical inference. Here, strong mathematical induction will be used on the grade of the formula, with two different cases corresponding to the outermost logical symbol of the cut-formula ϕ.

The base case and the induction hypothesis are the same for all cases.

- **Base Case:** The base case is when $g(\phi) = 0$, meaning that ϕ is atomic. The only way to introduce an atomic formula is through weakening or axiom introduction, meaning that this case has already been covered in **1.1)** to **1.4)**.

- **Induction Hypothesis:** As the induction hypothesis, suppose that for any formula ψ satisfying $g(\psi) < n$, it is possible to obtain a cut-free proof of the same end-sequent as the proof with its final inference being a cut with ψ as its cut-formula as in:

$$\frac{A_1 \vdash B_1 \cup \{\psi\} \qquad \{\psi\} \cup A_2 \vdash B_2}{A_1 \cup A_2^{\psi} \vdash B_1^{\psi} \cup B_2}\ (\psi)$$

where $g(\phi) = n$ and ϕ is the cut-formula of the cut which is to be eliminated in Π.

4.1.5.1 Implication as outermost symbol

Suppose that '$\rightarrow$' is the outermost symbol, meaning that $\phi \equiv \psi \rightarrow \chi$ for some logical formulas ψ and χ. So since $\text{rank}_l = 1$ and $\text{rank}_r = 1$, the three final levels of Π must look like below.

$$\frac{\dfrac{A_1 \cup \{\psi\} \vdash B_1 \cup \{\chi\}}{A_1 \vdash B_1 \cup \{\psi \rightarrow \chi\}}\ {\rightarrow}\text{R} \qquad \dfrac{A_2 \vdash B_2 \cup \{\psi\} \qquad A_3 \cup \{\chi\} \vdash B_3}{A_2 \cup A_3 \cup \{\psi \rightarrow \chi\} \vdash B_2 \cup B_3}\ {\rightarrow}\text{L}}{A_1 \cup A_2 \cup A_3 \vdash B_1 \cup B_2 \cup B_3}\ (\psi \rightarrow \chi)$$

By the assumption that there is only one cut as the last inference in Π, the proofs of U_1, U_2 and U_3 do not contain any cuts. Notice also that $\phi = \psi \rightarrow \chi$ does not appear in B_1, A_2, A_3 due to the rank-restriction.

It is now possible to use U_1, U_2, U_3 to get the same end-sequent by constructing the proof tree below, and keeping the cut-free proofs of U_1, U_2, U_3.

$$\frac{\dfrac{A_2 \vdash B_2 \cup \psi \qquad A_1 \cup \{\psi\} \vdash B_1 \cup \{\chi\}}{A_1 \cup A_2^{\psi} \vdash B_1^{\psi} \cup B_2 \cup \{\chi\}}\ (\psi) \qquad A_3 \cup \{\chi\} \vdash B_3}{A_1 \cup A_2^{\psi} \cup A_3^{\chi} \vdash B_1^{\psi,\chi} \cup B_2^{\chi} \cup B_3}\ (\chi)$$

Since:

$$g(\psi) < g(\phi) = g(\psi \rightarrow \chi)$$
$$g(\chi) < g(\phi) = g(\psi \rightarrow \chi)$$

the induction hypothesis states that there is a cut-free proof for the end-sequent of the cut in the tree above, meaning it is possible produce a cut-free proof of the sequent $A_1 \cup A_2^{\psi} \cup A_3^{\chi} \vdash B_1^{\psi,\chi} \cup B_2^{\chi} \cup B_3$, on which weakenings can be applied to produce a proof with end-sequent $A_1 \cup A_2 \cup A_3 \vdash B_1 \cup B_2 \cup B_3$, meaning that there is a cut-free proof of $A_1 \cup A_2 \cup A_3 \vdash B_1 \cup B_2 \cup B_3$.

4.1.5.2 Existential quantification as outermost symbol

Suppose that existential quantification is the outermost logical symbol, meaning that $\phi \equiv \exists x\psi(x)$ for some logical formula ψ. Since $\text{rank}_l = 1$ and $\text{rank}_r = 1$, the final three levels of Π must look like below.

$$\dfrac{\dfrac{A_1 \vdash B_1 \cup \{\psi(t)\}}{A_1 \vdash B_1 \cup \{\exists x\psi(x)\}}\ \exists\text{R} \qquad \dfrac{A_2 \cup \{\psi(a)\} \vdash B_2}{A_2 \cup \{\exists x\psi(x)\} \vdash B_2}\ \exists\text{L}}{A_1 \cup A_2 \vdash B_1 \cup B_2}\ (\psi)$$

Where a is not free in A_2 and B_2, in accordance with the restriction for applying $\exists$ Left, and where it is assumed that the proof of U_1 and U_2 contain no cuts. By replacing a by t in the proof tree with end-sequent U_2 it is possible to obtain a cut-free proof of the sequent $A_2 \cup \psi(t) \vdash B_2$ which will be called U_2'. This is possible due to the variable restriction of $\exists$L, which states that a cannot be in a formula in A_2 and B_2 nor in $\psi(x)$.

Now from U_1 and U_2' it is possible to obtain the below inference.

$$\dfrac{A_1 \vdash B_1 \cup \{\psi(t)\} \qquad A_2 \cup \{\psi(t)\} \vdash B_2}{A_1 \cup A_2^{\psi(t)} \vdash B_1^{\psi(t)} \cup B_2}\ (\psi)$$

Since:

$$g(\psi) < g(\exists x\psi(x))$$

the induction hypothesis states that there is a cut-free proof for the cut in the tree above, meaning it is possible produce a cut-free proof of the sequent $A_1 \cup A_2^{\psi(t)} \vdash B_1^{\psi(t)} \cup B_2$, on which weakenings can be applied to produce a proof with end-sequent $A_1 \cup A_2 \vdash B_1 \cup B_2$, meaning that there is a cut-free proof of $A_1 \cup A_2 \vdash B_1 \cup B_2$.

4.2 Case 2: Rank(Π) > 2

For case 2, suppose that rank(Π) > 2, meaning that at either the left or right rank is greater than 1 (or both). Case 2 will be split into two analogous cases which together cover the three possibilities there are for the values of the left and right rank. Only one of these two cases will be covered fully, since they are analogous.

The general idea for this case is to reduce the rank of Π through strong mathematical induction on the rank. The intuitive strategy is to "move the cut up" the inference tree while still getting the same end-sequent and thus reducing the rank of the tree with the help of the induction hypothesis. The base case and induction hypothesis which will be used from here on out were introduced in the beginning of the proof.

4.2.1 Rank$_r$(Π) > 1

Suppose that $\text{rank}_r(\Pi) > 1$, meaning that $\text{rank}_l(\Pi) = 1$ or $\text{rank}_l(\Pi) > 1$. The left rank does not matter in this case.

In all cases but **4.2.1.1**, where the cut can be eliminated directly, the elimination of the cut will be reliant on the induction hypothesis.

4.2.1.1 ϕ in antecedent of S_1

Suppose that ϕ is in the antecedent of S_1, which implies that the end of Π looks like below.

$$\frac{A_1 \cup \{\phi\} \vdash B_1 \qquad A_2 \vdash B_2}{A_1 \cup \{\phi\} \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}\ (\phi)$$

It is possible to construct a cut-free proof starting from S_2, and ending with the same end-sequent as above. S_2 is rewritten into its equivalent form: $\{\phi\} \cup A_2^{\phi} \vdash B_2$ for clarity. This is shown below.

$$\frac{\dfrac{\{\phi\} \cup A_2^{\phi} \vdash B_2}{\text{Possibly several WL and WR}}}{A_1 \cup \{\phi\} \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}$$

4.2.1.2 ϕ not in antecedent of S_1

Suppose that ϕ not in antecedent of S_1. This case will be split into several subcases depending on what inference rule S_2 is the end-sequent of. For the purpose of simplicity, the inference rule will be referred to as I.

4.2.1.2.1 I is weakening in antecedent of S_2

Suppose that I is a weakening in the antecedent of S_2, meaning that the end of Π will look like below.

$$\frac{A_1 \vdash B_1 \qquad \dfrac{A_2 \vdash B_2}{A_2 \cup \{\psi\} \vdash B_2}\ \text{WL}}{A_1 \cup A_2^{\phi} \cup \{\psi\} \vdash B_1^{\phi} \cup B_2}\ (\phi)$$

The case where $\psi \equiv \phi$ is trivial since that means that $A_2 \vdash B_2$ and $A_2 \cup \{\psi\} \vdash B_2$ are equivalent. The weakening can therefore be removed, reducing the right rank of Π by 1, meaning that the cut can be eliminated through the induction hypothesis.

In this case where $\psi \not\equiv \psi$ the cut can be moved up the tree before the weakening and the weakening applied after to get the same end-sequent, as is shown below.

$$\frac{\dfrac{A_1 \vdash B_1 \qquad A_2 \vdash B_2}{A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}\ (\phi)}{A_1 \cup A_2^{\phi} \cup \{\psi\} \vdash B_1^{\phi} \cup B_2}\ \text{WL}$$

As can be seen, the rank of the proof for the end-sequent $A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2$ is less than rank(Π), meaning that through the induction hypothesis, a cut-free proof can be obtained for the same end-sequent, which in turn means that a cut-free proof can be obtained for the original end-sequent $A_1 \cup A_2^{\phi} \cup \{\psi\} \vdash B_1^{\phi} \cup B_2$.

4.2.1.2.2 I is an inference rule with one upper sequent

Suppose I is an inference rule with one upper sequent but not WL. This means that the end of Π looks like below.

$$\dfrac{A_1 \vdash B_1 \qquad \dfrac{C \cup A_2 \vdash B_2}{D \cup A_2 \vdash B_2'}\,I}{A_1 \cup D^{\phi} \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2'}\,(\phi)$$

Depending on what rule I is, C can either be empty or contain a singular side-formula to I and likewise, D can either be empty or contain only the principle formula of I.

To start off, the end of Π from S_1 and U_1 is transformed into the proof below.

$$\dfrac{\dfrac{\dfrac{A_1 \vdash B_1 \qquad C \cup A_2 \vdash B_2}{A_1 \cup C^{\phi} \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}\,(\phi)}{C \cup A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}\,\text{Possible WL}}{D \cup A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2'}\,I$$

The rank of the subproof ending with the sequent $A_1 \cup C^{\phi} \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2$ is less than rank(Π), meaning that the cut in the new proof can be eliminated through the induction hypothesis. Call this new cut-free proof Π'.

In the case that D does not contain the original cut-formula ϕ, and is thus a set only containing the principle formula of I, the end-sequent of Π' is $D \cup A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2'$. But since $D^{\phi} = D$, the end-sequent of Π is equivalent to $D \cup A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2'$, meaning that Π' is a cut-free proof of the desired end-sequent.

In the case that D contains the original cut-formula ϕ, ϕ is the principal formula of I and $D = \{\phi\}$. From the final two levels of Π' we continue the proof as below (reusing the cut-free proof of S_1 which is a sub-proof of Π).

$$\dfrac{A_1 \vdash B_1 \qquad \dfrac{C \cup A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2}{\{\phi\} \cup A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2'}\,I}{A_1 \cup A_1^{\phi} \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_1^{\phi} \cup B_2'}\,(\phi)$$

The above end-sequent is equivalent to $A_1 \cup A_2^{\phi} \vdash B_1^{\phi} \cup B_2'$, which is the desired end-sequent. Now, consider the rank of the cut of the proof above. The left rank is the same as in Π since the original subproof of S_1 is being reused. The right rank is 1, since none of the sets in the antecedent of the upper sequent of I can contain ϕ. C can only contain a side-formula to I, which means it cannot contain ϕ since ϕ is the principal formula. A_1 cannot contain ϕ by the assumption of **4.2.1.2** and A_2^{ϕ} obviously does not contain ϕ. Thus, the rank of the cut in the above proof is lower than the cut in Π, meaning the cut can be eliminated through the induction hypothesis and a cut-free proof can be constructed of the desired end-sequent.

4.2.1.2.3 I is an inference rule with two upper sequents

Suppose I is an inference rule with two upper sequents. By the reduced system currently considered, the only such possible inference rule is $\to$L. This means that the end of Π must look like below for some formulas ψ and χ.

$$\dfrac{A_1 \vdash B_1 \qquad \dfrac{A_2 \vdash B_2 \cup \{\psi\} \qquad \{\chi\} \cup A_3 \vdash B_3}{A_2 \cup A_3 \cup \{\psi \to \chi\} \vdash B_2 \cup B_3}\ {\to}\mathrm{L}}{A_1 \cup A_2^\phi \cup A_3^\phi \cup \{\psi \to \chi\}^\phi \vdash B_1^\phi \cup B_2 \cup B_3}\ (\phi)$$

Since it is assumed rank(Π) ≥ 2, either A_2 or A_3 must contain ϕ, or both. In the case that $\phi \in A_2$ and $\phi \in A_3$, a cut-free proof of the desired end-sequent can be created from S_1, U_1 and U_2 as shown below.

$$\dfrac{\dfrac{A_1 \vdash B_1 \qquad A_2 \vdash B_2 \cup \{\psi\}}{A_1 \cup A_2^\phi \vdash B_1^\phi \cup B_2 \cup \{\psi\}}\ (\phi) \qquad \dfrac{\dfrac{A_1 \vdash B_1 \qquad \{\chi\} \cup A_3 \vdash B_3}{A_1 \cup \{\chi\}^\phi \cup A_3^\phi \vdash B_1^\phi \cup B_3}\ (\phi)}{\{\chi\} \cup A_1 \cup A_3^\phi \vdash B_1^\phi \cup B_3}\ \text{Possible WL}}{\{\psi \to \chi\} \cup A_1 \cup A_1 \cup A_2^\phi \cup A_3^\phi \vdash B_1^\phi \cup B_1^\phi \cup B_2 \cup B_3}\ {\to}\mathrm{L}$$

The end-sequent is obviously equivalent to $\{\psi \to \chi\} \cup A_1 \cup A_2^\phi \cup A_3^\phi \vdash B_1^\phi \cup B_2 \cup B_3$. For both cuts in the newly constructed proof above, the rank of the cut is lower since the left rank is the same as in Π while the right rank is less than in Π. This means that by the induction hypothesis, the cuts can be eliminated and a cut-free proof of $\{\psi \to \chi\} \cup A_1 \cup A_2^\phi \cup A_3^\phi \vdash B_1^\phi \cup B_2 \cup B_3$ can be obtained.

In the case where $\phi \not\equiv \psi \to \chi$, it is true that $\{\psi \to \chi\}^\phi = \{\psi \to \chi\}$, meaning that the new end-sequent is equivalent the end-sequent of Π.

In the case where $\phi \equiv \psi \to \chi$, the same procedure as in **4.2.1.2.2** can be applied to produce a cut-free proof. The corresponding proof tree is shown below. The details are the same as in **4.2.1.2.2** and is left to the reader. Notice that it is possible to remove the possible WL since ϕ is known.

$$\dfrac{A_1 \vdash B_1 \qquad \dfrac{\dfrac{A_1 \vdash B_1 \qquad A_2 \vdash B_2 \cup \{\psi\}}{A_1 \cup A_2^\phi \vdash B_1^\phi \cup B_2 \cup \{\psi\}}\ (\phi) \qquad \dfrac{A_1 \vdash B_1 \qquad \{\chi\} \cup A_3 \vdash B_3}{A_1 \cup \{\chi\} \cup A_3^\phi \vdash B_1^\phi \cup B_3}\ (\phi)}{\{\phi\} \cup A_1 \cup A_1 \cup A_2^\phi \cup A_3^\phi \vdash B_1^\phi \cup B_1^\phi \cup B_2 \cup B_3}\ {\to}\mathrm{L}}{A_1 \cup A_1^\phi \cup A_1^\phi \cup A_2^\phi \cup A_3^\phi \vdash B_1^\phi \cup B_1^\phi \cup B_1^\phi \cup B_2 \cup B_3}\ (\phi)$$

In the case that $\phi \notin A_2$ and $\phi \in A_3$, a cut-free proof can be constructed as below. The case where $\phi \in A_2$ and $\phi \notin A_3$ is entirely analogous and is left to the reader.

$$\dfrac{A_2 \vdash B_2 \cup \{\psi\} \qquad \dfrac{\dfrac{A_1 \vdash B_1 \qquad \{\chi\} \cup A_3 \vdash B_3}{A_1 \cup A_3^\phi \cup \{\chi\}^\phi \vdash B_1^\phi \cup B_3}\ (\phi)}{\{\chi\} \cup A_1 \cup A_3^\phi \vdash B_1^\phi \cup B_3}\ \text{Possible WL}}{\{\psi \to \chi\} \cup A_1 \cup A_2 \cup A_3^\phi \vdash B_1^\phi \cup B_2 \cup B_3}\ {\to}\mathrm{L}$$

Just like in the case where $\phi \in A_2$ and $\phi \in A_3$, the cut can be eliminated through the induction hypothesis. The subcases of when $\phi \equiv \psi \to \chi$ and $\phi \not\equiv \psi \to \chi$ can be treated exactly the same as in the above mentioned case, and the details are left to the reader.

4.2.2 Rank$_r$(Π) = 1

Suppose that rank$_r(\Pi) = 1$, meaning that that rank$_l(\Pi) > 1$ since rank$(\Pi) > 2$. This case focuses on moving the cut up the left side of the inference tree almost analogously to **4.2.1**, with the only difference being in the asymmetrical inference figures.

For the sake of brevity, let the naming conventions used in **4.2.1** translate to this case.

4.2.2.1 I is an inference rule with one upper sequent

In this case there is a need to rewrite the sequents which were S_2 and U_2 in **4.2.1**. Instead of:

$$\frac{C \cup A_2 \vdash B_2}{D \cup A_2 \vdash B_2'}\ I$$

the sequents will be written as below in order to deal with the cases which arise from C and D.

$$\frac{A_2' \vdash B_2 \cup C}{A_2' \vdash B_2 \cup D}\ I$$

4.2.2.2 I is an inference rule with two upper sequents

As in **4.2.1.2.3**, this case can only mean that S_1 is the lower sequent of →L. Call the upper two sequent of S_1 U_1 and U_2 respectively. The end of Π therefore looks like below.

$$\frac{\dfrac{A_1 \vdash B_1 \cup \{\psi\} \qquad \{\chi\} \cup A_2 \vdash B_2}{A_1 \cup A_2 \cup \{\psi \to \chi\} \vdash B_1 \cup B_2} \to \text{Left} \qquad A_3 \vdash B_3}{\{\psi \to \chi\} \cup A_1 \cup A_2 \cup A_3^{\phi} \vdash B_1^{\phi} \cup B_2^{\phi} \cup B_3}\ (\phi)$$

Just like in **4.2.1.2.3**, there are different cases depending on if ϕ is in B_1 and/or B_2.

Suppose that $\phi \in B_1$ and $\phi \in B_2$. To produce a cut-free proof, a proof can be constructed from U_1, U_2 and S_2 as below. The cuts can be eliminated through the induction hypothesis and through rewriting and simplification of the set-notation it can be seen that the end-sequent is the desired sequent.

$$\frac{\dfrac{\dfrac{A_1 \vdash B_1 \cup \{\psi\} \qquad A_3 \vdash B_3}{A_1 \cup A_3^{\phi} \vdash B_1^{\phi} \cup \{\psi\}^{\phi} \cup B_2}\ (\phi)}{A_1 \cup A_3^{\phi} \vdash B_1^{\phi} \cup \{\psi\} \cup B_2}\ \text{Possible WR} \qquad \dfrac{\{\chi\} \cup A_2 \vdash B_2 \qquad A_3 \vdash B_3}{\{\chi\} \cup A_2 \cup A_3^{\phi} \vdash B_2^{\phi} \cup B_3}\ (\phi)}{\{\psi \to \chi\} \cup A_1 \cup A_2 \cup A_3^{\phi} \cup A_3^{\phi} \vdash B_1^{\phi} \cup B_2^{\phi} \cup B_2 \cup B_3}\ \to\text{L}$$

Now suppose that $\phi \in B_1$ and $\phi \notin B_2$. The reversed case is treated analogously and is left to the reader. The end of Π is then transformed into the proof below. Through the newly structured tree, the rank of the cut is lower and the cut can be eliminated through the induction hypothesis, and through simplification of the set-notation in the end-sequent, a cut-free proof of the desired end-sequent can be achieved.

$$\frac{A_1 \vdash B_1 \cup \{\psi\} \qquad \dfrac{\{\chi\} \cup A_2 \vdash B_2 \qquad A_3 \vdash B_3}{\{\chi\} \cup A_2 \cup A_3^{\phi} \vdash B_2^{\phi} \cup B_3}\ (\phi)}{\{\psi \to \chi\} \cup A_1 \cup A_2 \cup A_3^{\phi} \vdash B_1 \cup B_2^{\phi} \cup B_3}\ \to\text{L}$$

4.3 Conclusion

Through careful analysis of the possible structure of a G^S proof tree Π as stated in the Main Lemma, is has been shown that the cut of Π can be eliminated by first reducing the rank of Π to 2 and then by induction on the grade of the cut-formula, eliminate the cut.

5 The Cut-Elimination Theorem and its Proof

The *Cut-Elimination Theorem*, also known as *Gentzen's Hauptsatz*, states that any sequent which is provable in G^S is also provable without a cut in G^S.

5.1 Proof of the Cut-Elimination Theorem

Suppose that Π is a proof tree in G^S which contains n cuts. The proof of the Cut-Elimination Theorem will be constructed using induction on the number of cuts of Π.

- **Base Case:** Suppose $n = 1$. In this case, consider the sub-proof of Π which has the cut as its last inference. Applying the Main Lemma, a cut-free proof of the sub-proof can be constructed and substituted, making Π cut-free.
- **Induction Step:** As the **induction hypothesis**, suppose that for any proof Π' containing less than n cuts, it is possible to construct a cut-free proof of the end-sequent of Π'.

  Given Π, find a cut above which no cuts occur. Using the Main Lemma, substitute the end-sequent of the cut under consideration with a cut-free proof, meaning that Π now contains $n - 1$ cuts. Through the induction hypothesis then, it is possible to construct a cut-free proof of the end-sequent of Π.
- **Conclusion:** Through Strong Mathematical induction, it has been shown that from any proof-tree in G^S, a proof-tree without any cuts but with the same end-sequent as the original tree can be constructed, thus proving the Cut-Elimination Theorem.

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