Exercise 2

a) $\hat f(x) - f(x) = \hat f(x) - E[\hat f(x)] + E[\hat f(x)] - f(x)$

$$E[\hat f(x) - f(x)] = [\hat f(x) - E[\hat f(x)] + E[\hat f(x)] - f(x)]^2$$

MSE

$$= E\left([(\hat f(x) - E[\hat f(x)]]^2 - \overbrace{2[\hat f(x) - E[\hat f(x)]][E[\hat f(x)] - f(x)]}^{0} + [E[\hat f(x)] - f(x)]^2\right)$$

$$\text{MSE} = E[\hat f(x) - E[\hat f(x)]^2] + E[(E[\hat f(x)] - f(x))^2]$$

$$= Var(\hat f(x)) + [E[\hat f(x)] - f(x)]^2]$$

$$= Var(\hat f(x)) + (Bias)^2$$

b) $$\Pi_h \approx \int_{-h/2}^{h/2} f(0) + x f'(0) + \frac{x^2}{2} f''(0) = \left[f(0)x + \frac{x^2}{2} f'(0) + \frac{x^3}{6} f''(0)\right]_{-h/2}^{h/2} = f(0)h + \frac{h^3}{24} f''(0)$$

$$\text{MSE}\left(\hat f(0), f(0)\right) = E[\hat f(0)] - f(0)]^2 + Var(\hat f(0))$$

c) $$\text{MSE} = A\left(\left(\frac{B}{4An}\right)^{1/3}\right)^4 + B \qquad h = \left(\frac{B}{4an}\right)^{1/5}$$

$$= \frac{AB^{4/3}}{4^{4/3}A^{4/5}n^{4/5}} + \frac{f(0)}{nA\left(\frac{B}{4An}\right)^{1/5}} = O\frac{1}{n^{4/3}} + O\frac{1}{n^{1-1/5}}$$

Curve fitting by linear smoothing

A) $$\hat f(x^*) = \hat B x^* = ((X^TX^{-1})X^Ty)^T x^*$$

$$= y^T(X(X^TX)^{-1})x^*$$

$$= y^T X((X^TX)^{-1})^T x^*$$

$$= y^T X\left(\begin{pmatrix}x_{11}\\ \vdots \\ x_{n1}\end{pmatrix}(x_{11} \dots x_{n1})\right)^{-1} x^* = y^T X\left(\sum_{i=1}^{n} x_{n1}^2\right)^{-1} x^*$$

Scalar

$$= \left(\sum_{j=1}^{n} y_j x_j\right)\cdot\left(\sum_{i=1}^{n} x_{n1}^2\right)^{-1} x^*$$

c) JGS

$y_i = \hat{B} x_i + e_i$

$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$

$\hat{y}^* = \hat{\beta} x^*$

$= \sum \left[\frac{x_i x^*}{n s_x^2} \right] y_i$

weight function

more reasonable

$w_i(x_i, x^*) = \frac{x_i x^*}{c}$ vs $w_k(x_i, x^*) = \begin{cases} 1/k \\ 0 \end{cases}$

y variable is a local average

$y_i = f(x_i) + e_i$

$y = \hat{f}(x) + \hat{e}$

$y = X\beta + e$

$\hat{\beta} = (X^T X)^{-1} X^T y$

$Var(e) = \sigma^2 I$

$\underset{\mathbb{R}^n}{y} = \underset{\mathbb{R}^n}{\hat{y}} + \underset{\mathbb{R}^n}{e}$ geometric argument

$\hookrightarrow R(x) + [R(x)]^\perp$

$\hat{\sigma}^2 = \frac{\hat{e}^T \hat{e}}{n-p}$

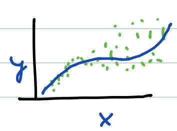
$\hat{\sigma}^2 = \frac{\sum (y_i - x^T \hat{\beta})^2}{n}$

$\hat{\sigma} = \frac{\|r\|^2}{n - \underbrace{[2\,tr(H) - tr(H^T H)]}_{\text{effective d.f.}}}$

$\hat{y}_i = h_i^T y$

$\hat{y}_n = H_{n \cdot n}\, y_n$

What happens when variance is not constant?



σ^2 is a function of $x \to \sigma^2(x)$

how to est. variance function?

heteroschedastic