Conditionals + Marginals

$x \sim N(\mu, \Sigma)$

If $z = Ax + b$

$z \sim N(A\mu + b, A\Sigma A^T)$

A)

Proof:

Let's define $B \equiv [I_{k \times k} \; O_{k \times p}]$

$$p(x_1 = Bx) = N\left([I\,O]\begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}, [I\,O]\begin{bmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\begin{bmatrix}I \\ 0\end{bmatrix}\right)$$

B) $\qquad = N(\mu_1, \Sigma_{11})$

want: $p(x_1 | x_2)$

$z = Ax_1 + Bx_2$; $cov(z, x_2) = 0$

$Ax_1 = z - Bx_2 \quad cov(z, x_2) = cov(Ax_1 + Bx_2, x_2)$

$$= A\,cov(x_1, x_2) + B\,cov(x_2)$$
$$= A\Sigma_{12} + B\Sigma_{22} = 0$$

$$E(Ax_1 | x_2) = E(z - Bx_2 | x_2)$$
$$= E(z|x_2) - E[Bx_2 | x_2]$$
$$= E(z) - Bx_2$$
$$= A\mu_1 + B(\mu_2 - x_2)$$

$$cov(Ax_1 | x_2)$$
$$= cov(z - Bx_2 | x_2)$$
$$= cov(z|x_2) + B(x_2|x_2) - B\,cov(x_2, z|x_2) - cov(z, x_2|x_2)B^T$$
$$= cov(z|x_2)$$
$$= cov(Ax_1 + Bx_2)$$
$$= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$x_1 | x_2 \sim MVN(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

B) another way

$f(x_1|x_2) \propto \dfrac{f(x_1, x_2)}{f(x_2)}$

$f(x_1) \sim N(M_1, \Sigma_{11})$

$f(x_2) \sim N(M_2, \Sigma_{22})$

$$\log f(x_1|x_2) \propto \log f(x_1, x_2) - \log f(x_2) - \tfrac{1}{2} X^T \Sigma^{-1} X^T - (-\tfrac{1}{2}) X_2^T \Sigma_{22}^{-1} x_2$$
$$\propto X^T \Sigma^{-1} X^T - X_2^T \Sigma_{22}^{-1} X_2$$

$$[X_1^T X_2^T]\begin{bmatrix}\Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22}\end{bmatrix}\begin{bmatrix}X_1 \\ X_2\end{bmatrix}$$
$$= X_1^T \Omega_{11} X_1 + X_2^T \Omega_{21} X_1 + X_1^T \Omega_{12} X_2 + X_2^T \Omega_{22} X_2$$
$$= (X_1 - \mu)^T A (X_1 - \mu) + C$$
$$= x_1^T A x_1 - \mu^T A x_1 - X_1^T A\mu + \mu^T A\mu + \mu^T A\mu + C$$

$X_1^T \Omega_{12} X_2 = X_1^T A\mu = -X_1^T \Omega_{11}\mu$

$\mu = -\Omega_{11}^{-1}\Omega_{12}X_2$

$$f(x_1|X_2) \sim N(-\Omega_{11}^{-1}\Omega_{12}X_{22}, \Omega_{11}^{-1})$$