1) a independent observations $X_1, \dots, X_N$ $\quad X_i \sim \text{Bernoulli}(\omega)$

$$P(x_i|\omega) = \omega^{x_i}(1-\omega)^{(1-x_i)}$$

$x \in \{0,1\}$ sample space
$\omega \in [0,1]$

$$p(\omega) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \omega^{a-1}(1-\omega)^{b-1} \qquad \omega \sim \text{Beta}(a,b)$$

Bayes Rule $$p(\omega|x_1 \dots x_N) = \frac{p(x_1 \dots x_N|\omega)p(\omega|a,b)}{p(x_1 \dots x_N)} = \prod_{i=1}^{N}\left[\omega^{x_i}(1-\omega)^{(1-x_i)}\right] \cdot \omega^{a-1}(1-\omega)^{b-1} \cdot \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{1}{p(x_1 \dots x_N)}}_{k}$$

$$\propto \omega^{\sum_{i=1}^{N} x_i + a - 1} \cdot (1-\omega)^{\sum_{i=1}^{N}(1-x_i)+b-1}$$

$$\propto \omega|x_1 \dots x_N, a, b \sim \text{Beta}\left(\sum_{i=1}^{N} x_i + a, \sum_{i=1}^{N}(1-x_i)+b\right)$$

1) b $$f(y_1,y_2) = f_{x_1x_2}\left(g_1^{-1}(y_1,y_2)\, g_2^{-1}(y_1,y_2)\right)|J(y_1,y_2)|$$

$$|J(y_1,y_2)| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = |y_2|$$

$$y_1 = x_1/(x_1+x_2) \qquad y_2 = x_1 + x_2 \longrightarrow \begin{cases} x_1 = y_1 y_2 = g_1^{-1}(y_1,y_2) \\ x_2 = y_2(1-y_1) = g_2^{-1}(y_1,y_2) \end{cases}$$

Assume that $x_1 + x_2$ are independent

$$f_{x_1x_2} = f_{x_1} f_{x_2}$$

$$= \frac{1}{\Gamma(a)} x_1^{a-1} \exp(-bx_1) \cdot \frac{1}{\Gamma(a_2)} x_2^{a_2-1} \exp(-bx_2)$$

$$f_{x_1x_2}(y_1,y_2) = \frac{1}{\Gamma(a_1)\Gamma(a_2)} (y_1y_2)^{a_1-1} \exp(-y_1y_2)(y_2(1-y_1))^{a_2-1} \exp(-y_2(1-y_1)) \cdot y_2$$

$$= \underbrace{\frac{1}{\Gamma(a_1)\Gamma(a_2)}}\ y_1^{a_1-1}(1-y_1)^{a_2-1}\ \underbrace{y_2^{(a_1+a_2-1)-1+1} \exp(-y_2)}$$

$$\propto \underbrace{y_1^{a_1-1}(1-y_1)^{a_2-1}}_{\text{Beta}(a_1,a_2)} \quad \underbrace{y_2^{a_1+a_2-1} \exp(-y_2)}_{\text{Gamma}(a_1+a_2,1)}$$

1) c $\quad x_i \sim N(\theta, \sigma^2) \quad \sigma^2$ is known

$\theta \sim N(m, v)$

Find $p(\omega | x_1, \dots x_N)$

$p(\theta | x_1 \dots x_N) = p(x_1 \dots x_N | \theta) p(\theta)$

$= p(x_1|\theta) \dots p(x_N|\theta) p(\theta) \propto \exp -\left(\frac{x_1-\theta}{2\sigma^2}\right) \dots \exp -\frac{(x_N-\theta)^2}{2\sigma^2} \exp -\frac{(\theta-m)^2}{2v}$

$= \exp\left(-\sum_{i=1}^{N} \frac{(x_i-\theta)^2}{2\sigma^2} - \frac{(\theta-m)^2}{2v}\right) \propto \exp\left(\theta^2\left(\frac{-N}{2\sigma^2} + \frac{1}{2v}\right) + \theta\left(\frac{\sum_{i=1}^{N} x_i}{\sigma^2} + \frac{m}{v}\right)\right)$

$$\propto \exp\left(-\frac{1}{2} \frac{\left(\theta - \left(\frac{\sum_{i=1}^{N} \frac{x_i}{\sigma^2} + \frac{m}{v}}{\frac{N}{\sigma^2} + \frac{1}{v}}\right)\right)^2}{\frac{1}{\left(\frac{N}{\sigma^2} + \frac{1}{v}\right)}}\right)$$

1) d $\quad x_1 \dots x_N$ iid $\sim N(\bar{\theta}, \sigma^2 = 1/\omega)$

$\omega \sim \text{Gamma}(a,b) \propto \omega^{a-1} e^{-b\omega}$

$\omega \in (0, \infty)$

$p(\omega | x) \propto \left[\omega^{n/2} \exp\left(-\frac{1}{2}\omega \sum (x_i-\theta)^2\right)\right] \left[\omega^{a-1} e^{-b\omega}\right]$

Likelihood $\qquad$ Prior

$\propto w^{\frac{n}{2}+a-1} e^{w\left(-\left(b + \frac{1}{2}\sum (x_i-\theta)^2\right)\right)}$

$\sim \text{Gamma}\left(\frac{n}{2} + a, b + \frac{1}{2}\sum (x_i-\theta)^2\right)$

$\omega = \frac{1}{\sigma^2} \rightarrow \frac{d\omega}{d\sigma^2} = \frac{1}{-(\sigma^2)^2}$

$p_{\sigma^2}(\sigma^2) = p_w(w) \det\left(\frac{-1}{(\sigma^2)^2}\right) = \sigma^{2\left(-a-\frac{n}{2}-1\right)} \exp\left(\frac{-1}{\sigma^2}\left[b + \frac{1}{2}\sum (x_i-\theta)^2\right]\right)$

$IG \sim x^{-a-1} e^{-\frac{b}{x}}$

$\sigma^2 \sim IG\left(a + \frac{n}{2}, b + \frac{1}{2}\sum (x_i-\theta)^2\right)$

1)e $x_1, \dots, x_N \overset{iid}{\sim} N(\bar{\theta}, \sigma^2 = 1/\omega)$

$\omega \sim \text{Gamma}(a, b) \sim \omega^{a-1} e^{-b\omega}$

$x_i \sim N(\theta, \sigma_i^2)$

$$p(\theta \mid x_1 \dots x_N) \propto p(\theta)\, p(x_1 \dots x_N \mid \theta$$

$$\propto p(\theta / \prod_{i=1}^{N} p(x_i \mid \theta)$$

$$\propto \exp\left(-\frac{1}{2v}(\theta - m)^2 \prod_{i=1}^{N} \exp\left(\frac{-1}{2\sigma_i^2}(x_i - \theta)^2\right)\right.$$

$$\propto \exp\left(\frac{-1}{2v}(\theta^2 - 2m\theta)\right) \cdot \exp\left(\sum_{i=1}^{N} \frac{-1}{2\sigma_i^2}(\theta^2 - 2x_i\theta)\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\left(\frac{1}{v} + \sum_{i=1}^{N} \frac{1}{\sigma_i^2}\right)\theta^2 - 2\left(\frac{m}{v} + \sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}\right)\theta\right]\right.$$

$$\propto \exp\left(-\frac{1}{2}\left(\frac{1}{v} + \sum_{i=1}^{N} \frac{1}{\sigma_i^2}\right)\left(\theta - \frac{\frac{1}{v}m + \sum_{i=1}^{N} \frac{1}{\sigma_i^2} x_i}{\frac{1}{v} + \sum_{i=1}^{N} \frac{1}{\sigma_i^2}}\right)^2\right.$$

$$\sim N\left(\frac{\frac{1}{v}m + \sum \frac{1}{\sigma_i^2} x_i}{\frac{1}{v} + \sum \frac{1}{\sigma_i^2}}, \sqrt{\frac{1}{\left(\frac{1}{v} + \sum \frac{1}{\sigma_i^2}\right)}}^{\,2}\right.$$

$x_i \sim N(\theta, \sigma^2)$
$\theta \sim N(m, v)$

$$\hat{\theta} = \frac{\frac{1}{v}m + \frac{n}{\sigma^2}\bar{x}}{\frac{1}{v} + \frac{n}{\sigma^2}}$$

$$= cm + (1-c)\bar{x}$$

$$c = \frac{\frac{1}{v}}{\frac{1}{v} + \frac{n}{\sigma^2}}$$

1)f $(x \mid \sigma^2) \sim N(0, \sigma^2)$

$\frac{1}{\sigma^2} \sim Ga(a,b)$ prior

$$f(x \mid \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$f(1/\sigma^2) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{\sigma^2}\right)^{a-1} \exp\left(-b/\sigma^2\right)$$