Stochastic Resonance in a simple climate model

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Abstract

Contrary to conventional wisdom, background noise can play a beneficial role in many natural phenomena. In this brief report, we study the mechanism of Stochastic resonance (SR) in which adding noise to a nonlinear system's input signal can enhance the coherence between the output response and the signal (by coherence we mean the degree to which the response *mimcs* the driving signal). SR was discovered in trying to explain the periodicity of the Earth's glacial-interglacial cycles which seem to have no powerful enough driving mechanism. We simulate a toy model of the climate and show that white noise can drive state transitions in climate between glacial and interglacial states.

1 Introduction

Earth's climate is a *complex system* which has posed major modelling challenges to researchers for decades. Past features of the climate can be extracted from measurements of deep sea ice cores by isotope analysis. One such feature is the so called 100,000 year cycles observed in paleoclimatic records. These cycles are characterized by phases of sudden warming followed by a slow decrease in temperature. A major challenge in climatology has been to explain this regularity in such glaciation cycles. One important observation is that the periodicity of ice ages happens to be *in phase* with the so called *Milankovitch cycles* which describe how variations in the earth's astronomical parameters such as orbital eccentricity, axial tilt and precession have a *cyclical effect* on the *incoming solar radiation.* However, mathematical modelling reveals that such cyclical variations themselves are not sufficient to drive glacial-interglacial state transitions. To remove this paradox, scientists proposed the role played by *noise* which corresponds to random fluctuations in *insolation* from short term weather instabilities. They reasoned that noise *and* the regular astronomical variations together might be the reason for the periodicity of the ice ages. This *cooperative effect* of noise with a periodic signal was termed *stochastic resonance.* Though numerical simulations have been succesful in partially validating this hypothesis, the problem still stands to be solved more rigorously. Nevertheless, stochastic resonance has given rise to numerous applications in other fields including electronics, laser physics, biophysics, neuroscience, etc. Here, we simulate stochastic resonance for a toy system that *corresponds* to a simple *energy balance model* for the climate.

1.1 Literature survey

Stochastic resonance was first proposed by an Italian group of scientists, Roberto Benzi, Alfonso Sutera and Angelo Vulpiani in 1981.[1,2] The initial work was based on numerical simulations of a mathematical *energy balance model* with an analysis of the underyling *stochastic differential equations*. Rigorous theoretical results were developed later by Bruce Mcnamara and Kurt Wiesenfeld in 1989.[3] The first experimental demonstration of stochastic resonance was shown by Fauve and Heslot

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in 1983 using an electronic *Schmitt trigger*.[4] Further experimental studies consisted of a variety of systems including a ring laser, electrosensory apparatus of crayfish, etc but we do not list them here. Subsequent climatology simulations have demonstrated SR in *millenial-scale* climate variability during glacial times as opposed to the larger timescales originally used (100ky).[5] For our simulations, we have chosen a review paper by Benzi.[6] For an introductory and popular science reading, we also refer to a *Scientific American* article published by Moss and Weisenfeld.[7]

1.2 Objectives

In this brief report, we first introduce a simple energy balance model (EBM) used in standard climatology simulations. Then, we state (without derivation) how this model can be mapped onto a *quasipotential* which determines the time dynamics of the Earth' temperature. We then choose appropriate simulation parameters and describe the computational details of the algorithms used. Finally, we show the results of the climatic (temperature) trajectory and how white noise can amplify a weak signal and cause what corresponds to glacial-interglacial transitions.

2 Methods and numerical details

For this work, we use a simple energy balance model requiring a few parameters to model the time dynamics of the *averaged Earth temperature* T. Therefore, glacial and interglacial periods are characterized by a specific value of T. The rate of incoming solar radiation is given by the variable R_{in} and the rate of outgoing radiation by R_{out}. Reflection from the earth surface is accounted for by a parameter for *albedo* which we will call α. In addition to the albedo, the outgoing radiation also consists of *infrared emission* from the earth surface which is given by the parameter E_I (rate of infrared emission). The *difference* between incoming and outgoing rate of radiation must be equal to the rate at which energy is pumped into the earth surface which can be calculated from the *average thermal capacity/inertia* C_E of the earth. Putting all of the above paramters together, we can write a differential equation for the average temperature as a function of time:

$$C_E \frac{dT}{dt} = R_{in} - R_{out} \qquad (1)$$

This is simply an equation for the conservation of energy. R_{out} consists of two terms, one for reflection due to albedo effects and another for the infrared emission. Thus, $R_{out} = \alpha R_{in} + E_I$. Substituting this in equation (1), we obtain

$$C_E \frac{dT}{dt} = R_{in}(1-\alpha) - E_I \qquad (2)$$

Both α and E_I depend on the temperature T. The simplest approximation for the functional form of $\alpha(T)$ and $E_I(T)$ are shown in the diagram below:

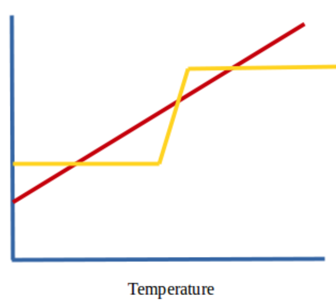

Figure 1: The yellow line corresponds to $R_{in}(1-\alpha)$ and the red line corresponds to E_I.

The 'climate' here corresponds to a *stationary solution* of equation (1) which means that $\frac{dT}{dt} = 0$. Notice from figure 1 that stationary solutions correspond to the *intersections of the two curves.* This is because at the points where the curves intersect, $R_{in}(1-\alpha) = E_I$ and therefore $\frac{dT}{dt} = 0$. The leftmost intersection point corresponds to the lowest temperature and hence is called an *ice-cover earth.* The warmest possible temperature (rightmost intersection point) is observed to be close to the *current day* temperature. The intermediate temperature is *unstable (a small perturbation from it grows exponentially)* whereas the right and left temperatures are *stable (a small perturbation from it returns back to the original point).* The small *orbital forcing* corresponding to the Milankovitch cycle (discussed in the introduction) can be modelled as a small-amplitude sinusoidal variation in time of the incoming radiation R_{in}. That is, $R_{in}(t) = R_{in}^s + A\cos(\omega t)$. This new form of $R_{in}(t)$ can be substituted in equation (1) and the resulting effect analysed. We state without proof here that introducing the above form of $R_{in}(t)$ results in a deviation of about $0.5K$ from the stationary state. This is *much smaller* than the $10K$ temperature change observed in paleoclimatic records. In other words, the glacial-interglacial transition requires a temperature change of $10K$ but introducing a Milankovitch/orbital forcing only gives us a change of $0.5K$. Thus, it is clear that *only orbital forcing cannot cause glacial-interglacial transitions.* To account for the required temperature change, some modifications need to be made to the model. A *new form for the albedo* is taken which gives two more stationary solutions as shown in the following diagram:

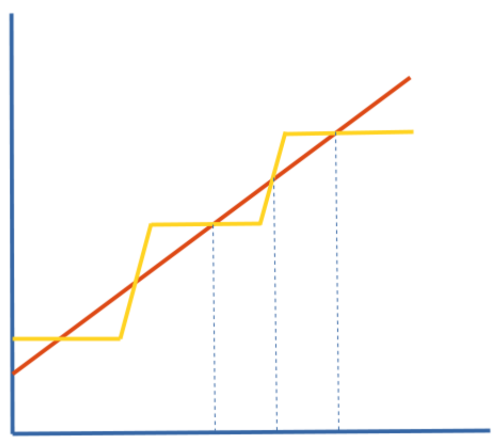

Figure 2: Modified energy balance model with a new albedo function. Red curve is for E_I and the yellow curve is for α.

As before, the intermediate temperature T_i is unstable. Furthermore, from empirical observations, it is assumed that $T_0 - T_1 = 10K$ and $T_0 - T_i = T_i - T_1$. Notice that we have not considered noise so far. Hence, a *noise term* is added in the form of a *white noise* $\eta(t)$ *with unit variance* (in other words, $\eta(t)$ is a *gaussian random variable).* For simulation purposes, a *rescaling* of the variables makes the equations simple and a variable X is introduced as follows:

$T = T_i + \Delta T X$ with $\Delta T = 5K$ (3)

Equation (1) can therefore be recast in terms of the variable X as follows (derivation not shown):

$\frac{dX}{dt} = X - X^3 + A\sin(2\pi\nu_0 t) + \sqrt{\sigma}\eta(t)$ (4)

The parameter σ is the *standard deviation* of the gaussian (white) noise. Geometrically, equation (4) corresponds to *movement of a particle in a double-well potential.* We call such a potential a *quasipotential or pseudo-potential.* Intuitively, we can think of the earth as a particle sitting in one of the wells (T_0 or T_1) of the double-well potential. For the variable X however, the wells lie at the points $X = +1$ and $X = -1$ respectively. Adding a periodic drive (sinusoidal signal) to this well corresponds to the *Milankovitch cycles* (oscillating R_{in} due to cyclical changes in earth's astronomical parameters). The sinusoidal term alternatively *rasies and lowers the two wells in time.* Below, we show how the potential well is modulated by the sinusoidal signal:

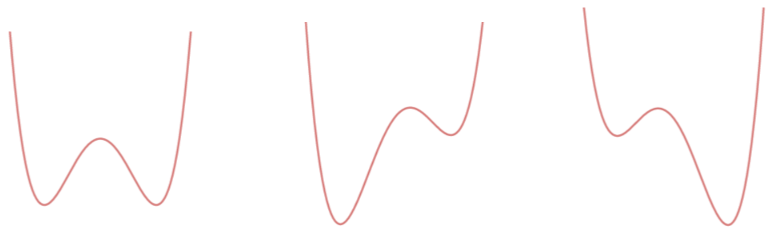

Figure 3: The double well potential(left) Modulation of this potential by the sinusoidal term (middle and right)

Now if *only* the Milankovitch cycles are considered, it is like the particle oscillating due to the sinusoidal drive but *unable to cross over to the other well.* Adding a noise component to the drive however can induce *inter-well transitions.* We show this result from our simulations in the next section.

3 Results and Discussion

As was explained in the previous section, we can cast the energy balance model into equation (4) which we then simulate. Notice that the stochastic term $\eta(t)$ makes a straightforward analytical integration impossible. Equation (4) is a *stochastic differential equation.* This means, analytically only the *probability distribution of X as a function of time* can be obtained. We do not apply any analytical methods in this report. We numerically integrate equation (4) using *the fourth order Runge-Kutta method.* Below, we show the trajectory i.e. $X(t)$ for a *subthreshold* value of A i.e. a value which is not sufficient to cause inter-well transitions. No noise is added.

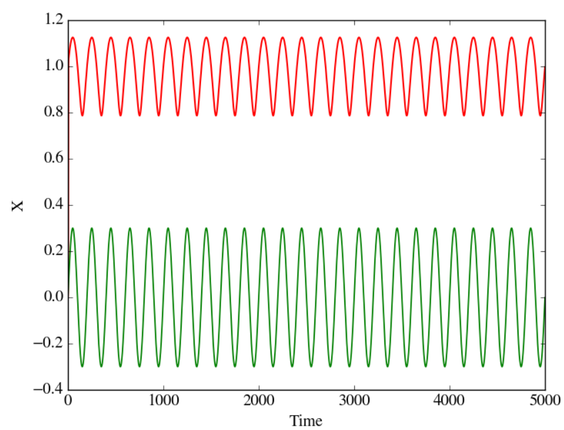

Figure 4: $X(t)$ trajectory for $A = 0.3$, $\eta(t) = 0$, $\nu_0 = 0.005$ Hz. The green curve corresonds to the signal $A\sin(2\pi\nu_0 t)$ and the red curve corresponds the particle trajectory.

Notice that the particle *stays in the right well* ($X = +1$). Adding *only noise* on the other hand also cannot cause inter-well transitions for low noise intensities. For high noise intensities, there are inter-well transitions but they are totally random. Below, we show the two cases:

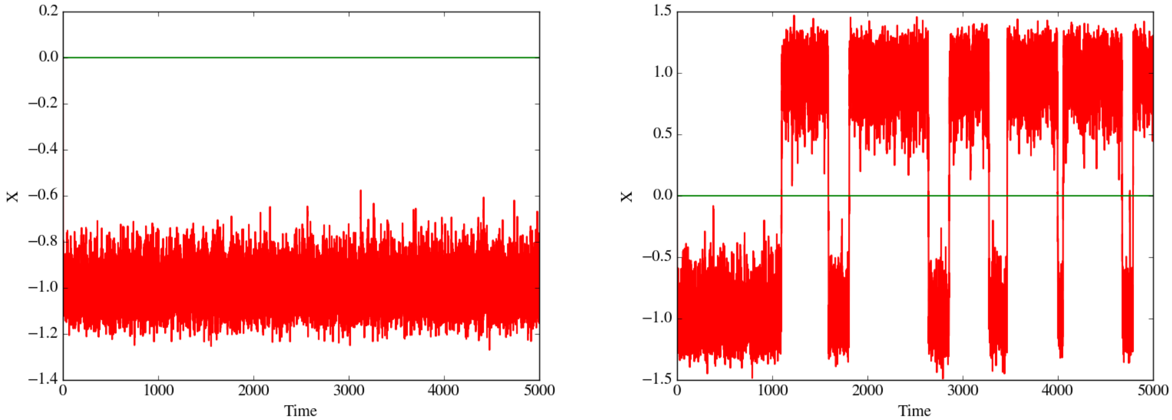

Figure 5: Absence of inter-well transitions for low noise intensity $\sigma = 0.5$ (left) and random transitions for high noise intensity $\sigma = 1.0$ (right) The green line is the signal which here is taken to be a constant ($A = 0$).

The remarkable effect of stochastic resonance arises when *both* a subthreshold (insufficient for inter-well transitions) periodic signal and noise are superposed. Below, we use a subthreshold signal (corresponding to orbital forcing) $A = 0.3$ and add white noise to it ($\sigma = 0.5$). The noise *introduces a coherence between the response and the driving signal (by coherence we mean the response mimics the driving signal).* We also plot the *power spectrum* of the response which shows a *peak* at the signal frequency ν_0

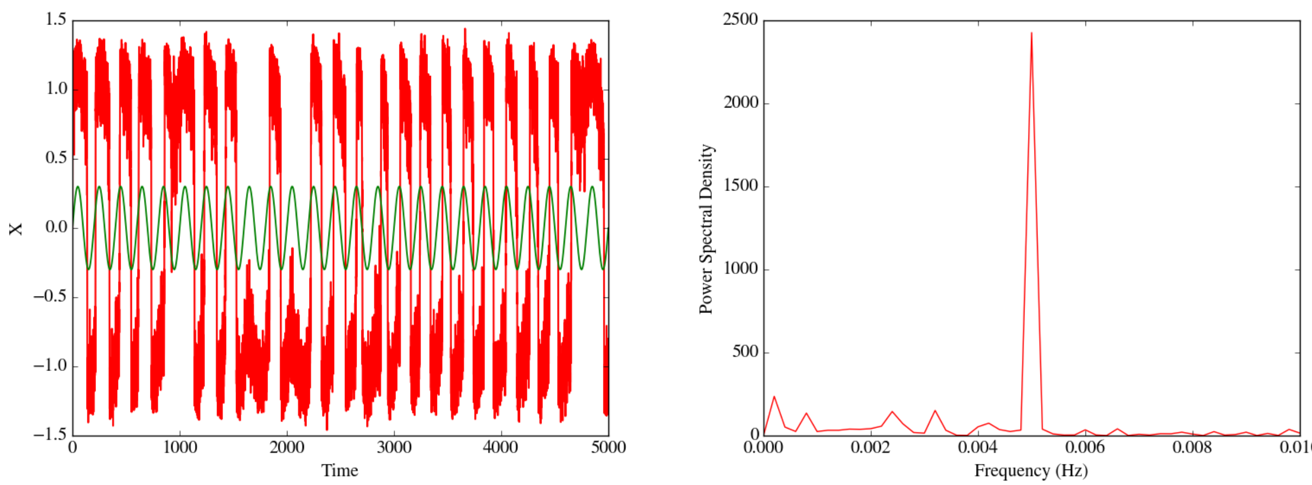

Figure 6: (Left) Noise induces inter-well transitions for a subthreshold signal with $A = 0.3$, $\nu_0 = 0.005$ Hz and $\sigma = 0.8$. Notice the coherence between the response (red) and the signal (green). (Right) The power spectral density of the response as a function of frequency. Notice the peak at $\nu_0 = 0.005$ Hz corresponding to the signal frequency.

Stochastic resonance not only implies that adding noise increases the coherence between the response and the driving signal but also that *there is an optimum noise intensity at which maximum coherence occurs.* To show this, we run the simulation for *a range of noise intensities* σ and observe the corresponding peak in the power spectrum (at ν_0). The height of the peak which is the signal power is a good measure for the degree to which the response carries the information in the signal. Below, we show stochastic resonance for a subthreshold signal.

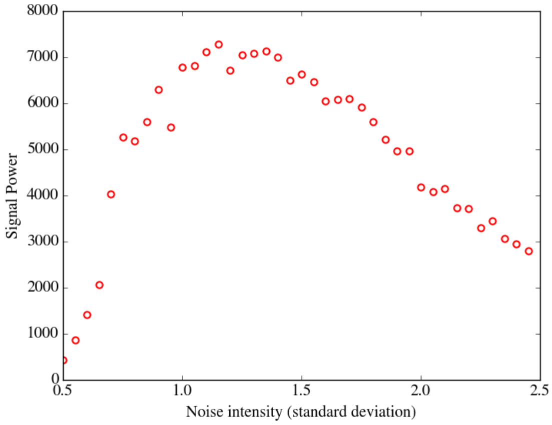

Figure 7: Stochastic resonance in a subthreshold signal with $A = 0.3$, $\nu_0 = 0.005$ Hz, for a range of noise intensities. Notice the peak at the optimum noise intensity $\sigma \sim 1.1$

An often used measure to quantify the degree of coherence between the response and driving signal is the *Signal to Noise ratio (SNR)*. The SNR takes into account the background noise power as well and reflects the degree of coherence *relative* to the noise present in the system. Multiple definitions of the SNR are used. We define it as *the ratio of the signal and noise power integrated on a small region around the driving frequency* ν_0. Specifically, if the signal power is given by $P_s(\nu)$ and the noise power is given by $P_n(\nu)$, then the SNR is defined as follows:

$$SNR = 10\log\left(\frac{\int_{\nu_0-\Delta\nu}^{\nu_0+\Delta\nu} P_s(\nu)d\nu}{\int_{\nu_0-\Delta\nu}^{\nu_0+\Delta\nu} P_n(\nu)d\nu}\right)$$

Here, $\Delta\nu$ is the small region around the driving frequency ν_0 in which the signal and noise powers are integrated. The *SNR* is measured in *decibels (dB)*. Notice that the above definition of *SNR* is an approximation and the exact value is reached in the limit of $\Delta\nu$ going to 0 . Below, we plot the *SNR* as a function of noise intensity σ . Again, we notice a peak like behavior with the peak at the optimum noise intensity.

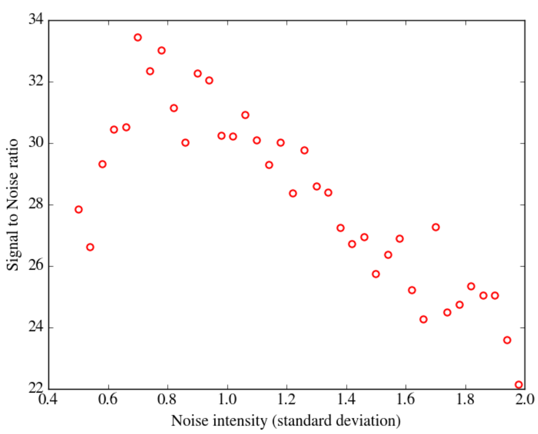

Figure 8: Stochastic resonance in a subthreshold signal with $A = 0.3$, $\nu_0 = 0.005$ Hz, for a range of noise intensities. We have used $\Delta\nu = 0.0015$ Hz. Notice the peak at the optimum noise intensity.

An interesting observation we make is that *the optimum noise intensity can differ depending on the measure of coherence used.* Our simulations do not give a smooth peak when the SNR is plotted. We suspect this to be happening because we have used only a *single run* for the simulation. Instead, if we decide to use an *ensemble of simulation runs* and then plot the *average SNR*, we may obtain the required smooth behaviour. To summarize, we have shown stochastic resonance for the energy balance model in equation (1) when it is transformed into the equivalent problem of a particle in a double-well potential. We note that we have *not* used parameter values that *directly* correspond to the climate system (used by Benzi[6]) but have chosen arbitrary but simple values for convenience of simulation. Thus, noise plays a major role in increasing the *sensitivity* of climate models to a weak periodic signal. We emphasize that rigorously establishing the role of stochastic resonance in glaciation-interglaciation cycles will require accurate measurements from several sources and additional simulations based on detailed and more complex models.

4 Acknowledgements

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5 References

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