MA108 ODE: Existence and Uniqueness
Lecture 4 (D4)

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Existence & Uniqueness Theorem

Let $R = \{(x, y) : |x - x_0| < a, |y - y_0| < b\}$. Let

- $f(x, y)$ be defined and continuous on R, and
- $|f(x, y)| \leq K$ for all (x, y) in R.

Then the IVP $y' = f(x, y)$, $y(x_0) = y_0$ has at least one solution on $(x_0 - \alpha, x_0 + \alpha)$ where

$$\alpha = \min\{a, b/K\}.$$

Moreover, if f satisfies the Lipschitz condition with respect to y on R, i.e. there is $M > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in R$ then the IVP admits a unique solution on the interval $(x_0 - \alpha, x_0 + \alpha)$.

Remarks

1. The theorem guarantees existence and uniqueness only in an interval.
2. The theorem DOES NOT give the largest interval where the solution exists.

Example

Example: Solve the IVP:

$$\frac{dy}{dt} = y^2; \; y(0) = 1$$

and find the interval in which the solution exists.

The IVP has a unique solution on an interval containing $t = 0$ by the above theorem. The given DE is separable and we have

$$\frac{dy}{y^2} = dt.$$

Thus,

$$-\frac{1}{y} = t + c;$$

i.e.,

$$y = -\frac{1}{t + c}.$$

Example continued

The initial condition $y(0) = 1$ gives $c = -1$. Hence

$$y(t) = \frac{1}{1-t}$$

is the solution to the given DE. This function is unbounded as $t \to 1$. Hence,

$$y : (-\infty, 1) \to \mathbb{R}$$

defined by

$$y(t) = \frac{1}{1-t}$$

is the solution to the given IVP.

Example

Example: Consider the IVP $y' = y^2 + \cos(x^2), y(0) = 0$ on $R = \{(x, y) \in \mathbb{R}^2 : \ |x| < 1, |y| < 1\}$. Does the IVP have a solution on the interval $(-\frac{1}{2}, \frac{1}{2})$? If yes, is the solution unique?

Ans. Here, $f(x, y) = y^2 + \cos(x^2)$ is continuous on R, $|f(x, y)| \leq 2$ for all $(x, y) \in R$. The function f satisfies the Lipschitz condition wrt y on R.

$$|f(x, y_1) - f(x, y_2)| = |y_1^2 - y_2^2| \leq 2|y_1 - y_2|,$$

for all $(x, y_1), (x, y_2) \in R$.

It follows from the theorem that the IVP admits a unique solution on $(-\alpha, \alpha)$, where $\alpha = \min\{a, b/K\}$, $a = 1, b = 1$ and $K = 2$. Hence, the IVP has a solution on $(-\frac{1}{2}, \frac{1}{2})$ and the solution is unique on this interval.

Existence and Uniqueness

Example: Consider the IVP

$$y' = \frac{10}{3}xy^{2/5}, y(x_0) = y_0. \tag{1}$$

(i) For what points (x_0, y_0), does the Theorem imply that (1) has a solution?

(ii) For what points (x_0, y_0), does the Theorem imply that (1) has a unique solution on some open interval that contains x_0?

Ans.(i) The IVP admits a solution for every (x_0, y_0) on an interval around the point x_0.

(ii) f is not Lipschitz with respect to y on every rectangle R containing the points $(x, 0), x \in \mathbb{R}^2$. Also, f satisfies the Lipschitz condition on any rectangle R not containing $(x, 0), x \in \mathbb{R}^2$. Therefore, if $y_0 \neq 0$, there is an open rectangle on which f satisfies the Lipschitz condition with respect to y, and hence the IVP has a unique solution on an interval around x_0. If $y_0 = 0$, then f does not satisfy the Lipschitz condition wrt y on any rectangle containing $(x_0, 0)$, and thus the uniqueness theorem is not applicable to this IVP for $y_0 = 0$.

Linear first order ODEs

Consider the linear equation

$$y' + p(t)y = q(t), \tag{2}$$

where $p(\cdot)$ and $q(\cdot)$ are continuous on an interval $I \subseteq \mathbb{R}$.

(i) The general solution of the ODE (2), containing an arbitrary constant is given by

$$y(t) = e^{-\int p(t)dt} \left(\int e^{\int p(t)dt} \cdot q(t)dt + c \right), t \in I.$$

A particular solution that satisfies the given initial condition can be picked out by choosing the appropriate value of the arbitrary constant.

(ii) The solution of the IVP $y' + p(t)y = q(t), y(t_0) = y_0,\ t_0 \in I$ exists throughout the interval I. Moreover, the solution is unique. Why?

Linear ODE contd...

Let ϕ_1 and ϕ_2 be two solutions of the IVP

$$y' + p(t)y = q(t), y(t_0) = y_0$$

. Consider

$$w(t) = \phi_1(t) - \phi_2(t), \ t \in I.$$

Then $w(\cdot)$ satisfies $w(t_0) = 0$ and

$$w'(t) + p(t)w(t) = 0, \tag{3}$$

on I. Use the Integrating factor $\mu(t) = e^{\int p(t)\,dt}$ for (3) to deduce that

$$\frac{d}{dt}\Big(\mu(t)w(t)\Big) = 0,$$

on I. Hence, $\mu(t)w(t) = \mu(t_0)w(t_0) = 0$ for all $t \in I$. I.e. $w(t) = 0$ for all $t \in I$, and $\phi_1(t) = \phi_2(t)$ throughout I.

Picard's Iteration Method

Picard's iteration method gives us a rough idea on how to construct solutions to IVP's. Consider the IVP

$$y' = f(t, y); \; y(t_0) = y_0.$$

Assume that all the hypotheses as in the Existence & Uniqueness Theorem are satisfied. Suppose $y = \phi(t)$ is a solution to the IVP. That is,

$$\frac{d\phi}{dt} = f(t, \phi(t)), \; \phi(t_0) = y_0.$$

Then,

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} f(s, \phi(s))ds,$$

so that

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds,$$

The above equation is called an integral equation in the unknown function ϕ.

Picard's Iteration Method

Conversely, if there is a continuous function ϕ that satisfies the integral equation, i.e.,

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds,$$

then note that $\phi(t_0) = y_0$. Moreover, by the Fundamental Theorem of Calculus,

$$\frac{d\phi}{dt} = f(t, \phi(t)),$$

so that $y = \phi(t)$ is a solution to the IVP $y' = f(t, y);\ y(t_0) = y_0$. Thus, solving the integral equation is equivalent to solving the IVP.

Picard's Iteration Method

Picard's iteration describes a way to look for solutions of the integral equation

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds. \tag{4}$$

We define iteratively a sequence of functions $\phi_n(t)$ for every integer $n \geq 0$ as follows: Let

$$\begin{aligned}
\phi_0(t) &\equiv y_0 \\
\phi_1(t) &= y_0 + \int_{t_0}^{t} f(s, \phi_0(s))ds \\
\phi_2(t) &= y_0 + \int_{t_0}^{t} f(s, \phi_1(s))ds \\
&. \\
&. \\
\phi_{n+1}(t) &= y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds.
\end{aligned}$$

Picard's Iteration Method

Note: Each ϕ_n satisfies the initial condition $\phi_n(t_0) = y_0$. None of the ϕ_n may satisfy $y' = f(t, y)$. However, it is possible to show that under the hypotheses of the Existence & Uniqueness theorem, the sequence $\{\phi_n\}$ converges to a function

$$\phi(t) = \lim_{n \to \infty} \phi_n(t)$$

which is the solution to the given IVP.

Example

Example: Solve the IVP:

$$y' = 2t(1+y);\ \ y(0) = 0$$

by the method of successive approximation.
If $y = \phi(t)$ is a solution of the IVP, the corresponding integral equation is

$$\phi(t) = \int_0^t 2s(1+\phi(s))ds.$$

Let $\phi_0(t) \equiv 0$. Then,

$$\begin{aligned}
\phi_1(t) &= \int_0^t 2sds = t^2, \\
\phi_2(t) &= \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}, \\
\phi_3(t) &= \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}.
\end{aligned}$$

Example continued

We claim:

$$\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!}.$$

Use induction to prove this:

$$\begin{aligned}
\phi_{n+1}(t) &= \int_0^t 2s(1+\phi_n(s))ds \\
&= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2} + \ldots + \frac{s^{2n}}{n!}\right) ds \\
&= t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \ldots + \frac{t^{2n}}{n!} + \frac{t^{2n+2}}{(n+1)!}.
\end{aligned}$$

Hence $\phi_n(t)$ is the n-th partial sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Example continued

Recall that $\phi_n(t)$ is the n-th partial sum of the series $\displaystyle\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$.

Applying the ratio test, we get:

$$\left| \frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \to 0$$

for all t as $k \to \infty$. Thus,

$$\lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1.$$

Hence, $y(t) = e^{t^2} - 1$ is the solution of the IVP.

Outline of the Proof of the Uniqueness statement

Let $R = \{(x, y) : |x - x_0| < a,\ |y - y_0| < b\}$.

- $f(x, y)$ is continuous on R, and
- $|f(x, y)| \leq K$ for all $(x, y) \in R$.
- f satisfies the Lipschitz condition (wrt to y) on R, i.e., there is $M > 0$ such that

  $$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

  for all $(x, y_1), (x, y_2)$ in R.

Suppose $\phi(\cdot)$ and $\psi(\cdot)$ are solutions of $y' = f(x, y), y(x_0) = y_0$ on an interval $(x_0 - h, x_0 + h)$. Thus, both these satisfy the integral equation as well. Then, for $x_0 < x < x_0 + h$,

$$\phi(x) - \psi(x) = \int_{x_0}^{x} (f(s, \phi(s)) - f(s, \psi(s)))ds.$$

Thus,

$$|\phi(x) - \psi(x)| \leq \int_{x_0}^{x} |f(s, \phi(s) - f(s, \psi(s)|ds.$$

Using the Lipschitz condition, we have

$$|\phi(x) - \psi(x)| \leq \int_{x_0}^{x} M\,|\phi(s) - \psi(s)|ds = M\int_{x_0}^{x} |\phi(s) - \psi(s)|ds.$$

Let $U(x) = \int_{x_0}^{x} |\phi(s) - \psi(s)|ds$. Clearly, $U(x_0) = 0, U(x) \geq 0$.
Also, $U'(x) = |\phi(x) - \psi(x)|$ and from above we get

$$U'(x) - MU(x) \leq 0.$$

It yields $\frac{d}{dx}\left(e^{-Mx}U(x)\right) \leq 0$.
Integrating both sides from (x_0, x), we get

$$\int_{x_0}^{x} \frac{d}{ds}\left(e^{-Ms}U(s)\right)\,ds \leq 0,$$

and thus $e^{-Mx}U(x) - e^{-Mx_0}U(x_0) \leq 0$, for all $x_0 < x < x_0 + h$.
Hence, $U(x) \leq 0$ for all $x_0 < x < x_0 + h$.
Similarly, derive $U(x) \leq 0$, for all $x_0 - h < x < x_0$.
We conclude that $U(x) \leq 0$, and hence,

$$U(x) \equiv 0,$$

on $(x_0 - h, x_0 + h)$. Thus, $\phi(x) \equiv \psi(x)$.