PH-112 (2023 Spring): Tutorial Sheet 2

Notes:

1. * marked problems will be solved in the Wednesday tutorial class.
2. Please make sure that you do the assignment by yourself. You can consult your classmates and seniors and ensure you understand the concept. However, do not copy assignments from others.

Wave packets: Group and Phase Velocity

1. Consider two wave functions $\psi_1(y,t) = 5y\cos 7t$ and $\psi_2(y,t) = -5y\cos 9t$, where y and t are in meters and seconds, respectively. Show that their superposition generates a wave packet. Plot it and identify the modulated and modulating functions.

2. *Two harmonic waves which travel simultaneously along a wire are represented by

$$y_1 = 0.002\cos(8.0x - 400t) \quad \& \quad y_2 = 0.002\cos(7.6x - 380t)$$

where x, y are in meters and t is in sec.

   (a) Find the resultant wave and its phase and group velocities

   (b) Calculate the range Δx between the zeros of the group wave. Find the product of Δx and Δk ?

3. The angular frequency of the surface waves in a liquid is given in terms of the wave number k by $\omega = \sqrt{gk + Tk^3/\rho}$, where g is the acceleration due to gravity, ρ is the density of the liquid, and T is the surface tension (which gives an upward force on an element of the surface liquid). Find the phase and group velocities for the limiting cases when the surface waves have:
(a) very large wavelengths and
(b) very small wavelengths.

4. *Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

5. Consider an electromagnetic (EM) wave of the form $A\exp(i[kx - \omega t])$. Its speed in free space is given by $c = \frac{\omega}{k} = 1/\sqrt{\epsilon_0\mu_0}$, where ϵ_0, μ_0 is the electric permittivity, magnetic permeability of free space, respectively.

   (a) Find an expression for the speed (v) of EM waves in a medium, in terms of its permittivity ε and permeability μ.

(b) Suppose the permittivity of the medium depends on the frequency, given by $\epsilon = \epsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right)$ where ω_p is a constant called the plasma frequency, find the dispersion relation for the EM waves in a medium. wp is a constant and is called the plasma frequency of the medium (assume $\mu = \mu_0$).

(c) Consider waves with $\omega = 3\omega_p$. Find the phase and group velocity of the waves. What is the product of group and phase velocities?

6. A wave packet describes a particle having momentum p. Starting with the relativistic relationship $E^2 = p^2c^2 + E_0^2$, show that the group velocity is βc and the phase velocity is c/β (where $\beta = v/c$). How can the phase velocity physically be greater than c?

7. *Consider a squre 2-D system with small balls (each of mass m) connected by springs. The spring constants along the x- and y-directions are β_x and β_y, respectively. The dispersion relation for this system is given by

$$-\omega^2 m + 2\beta_x\left(1-\cos k_x a_x\right) + 2\beta_y\left(1-\cos k_y a_y\right) = 0$$

where $\vec{k} = k_x\hat{i} + k_y\hat{j}$ is the wave vector and a_x,a_y are the natural distances between the two successive masses along the x-, y-directions, respectively. Find the group velocity and the angle that it makes with the x-axis

Fourier Transform

1. * If $\phi(k) = A(a-|k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
(a) Find the Fourier transform for $\phi(k)$
(b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

2. A wave packet is of the form $f(x) = \cos^2\left(\frac{x}{2}\right)$ (for $-\pi \leq x \leq \pi$) and $f(x) = 0$ elsewhere
(a) Plot $f(x)$ versus x.
(b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
(c) At what value of k, $|g(k)|$ attains its maximum value?
(d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
(e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

3. Find the Fourier transform of the following functions:

a) $f(x) = \begin{cases} a, & -\ell < x < 0, a > 0 \\ 0, & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} a, & -\ell < x < 0 \\ b, & 0 < x < \ell \quad a > 0, b > 0 \\ 0, & \text{otherwise} \end{cases}$

4. A wave packet is of the form $f(x) = \exp(-\alpha|x|) \cdot \exp(ik_0 x)$ (for $-\infty \leq x \leq \infty$) where α, k_0 are positive constants.

   (a) Plot $|f(x)|$ versus x.

   (b) At what values of x does $|f(x)|$ attain half of its maximum value? Consider the full width at half maxima (FWHM) as a measure of the spread (uncertainty) in x, find Δx

   (c) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x) e^{ikx} dx$

   (d) Plot $g(k)$ versus k.

   (e) Find the values of k at which $g(k)$ attains half its maximum value? Using the same concept of FWHM as in part (b), calculate Δk ? Hence calculate the product $\Delta x . \Delta k$

   [Given : $\int_0^{\infty} e^{-(\alpha - ik)x} dx = \frac{1}{\alpha - ik}$]
