Ans(1) (a) Any $x \in S_K$ can be written as

$$\cancel{S_K}\ x = \alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_K V_K$$

Now $Ax = \alpha_1 AV_1 + \alpha_2 AV_2 + \cdots + \alpha_K AV_K$

since $AV_i = \sigma_i u_i$

$\Rightarrow Ax = \sigma_1 \alpha_1 u_1 + \sigma_2 \alpha_2 u_2 + \cdots + \sigma_K \alpha_K u_K$

In matrix form x & Ax can be shown as

$$x = \left[v_1 \mid v_2 \mid \cdots \mid v_K \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_K \end{bmatrix} = V \cdot P$$

$$\& \quad Ax = \left[u_1 \mid u_2 \mid \cdots \mid u_K \right] \begin{bmatrix} \sigma_1 \alpha_1 \\ \sigma_2 \alpha_2 \\ \vdots \\ \sigma_K \alpha_K \end{bmatrix} = U \cdot Q$$

Now $\|x\|_2 = \|VP\|_2 = \|P\|_2$ because V is unitary

Similarly $\|Ax\|_2 = \|UQ\|_2 = \|Q\|_2$ as Q is unitary

$$\Rightarrow \frac{\|Ax\|_2}{\|x\|_2} = \frac{\sqrt{\sigma_1^2 \alpha_1^2 + \sigma_2^2 \alpha_2^2 + \cdots + \sigma_K^2 \alpha_K^2}}{\sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_K^2}}$$

since $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_K$

this ratio is minimum when

$\alpha_1 = \alpha_2 = \cdots = \alpha_{K-1} = 0$

so $\inf\limits_{x \in S_K} \dfrac{\|Ax\|_2}{\|x\|_2} = \sigma_K$

Ans(2) (a) Since S & T are two complementary subspaces in $\mathbb{R}^m$ so,
any vector $x \in \mathbb{R}^m$ can be expressed uniquely as

$$x = s + t \quad \text{where } s \in S \ \& \ t \in T$$

& Basis for $\mathbb{R}^m$ = (Basis of S) $\cup$ (Basis of T)

because $x = (\alpha_1 a_1 + \alpha_2 a_2 + \cdots) + (\beta_1 b_1 + \beta_2 b_2 + \cdots)$
where $\{a_1, a_2 \cdots\}$ & $\{b_1, b_2 \cdots\}$ are bases for S&T
x is 0 only when $s = t = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = b_1 = \cdots = 0$
Hence $\{a_1, a_2 \cdots\} \cup \{b_1, b_2 \cdots\}$ is a linear independent set that spans $\mathbb{R}^m$.

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Ans 2 (b) Take

$$Q = \left[a_1 \,|\, a_2 \,|\, \cdots \,|\, b_1 \,|\, b_2 \cdots \right]$$

Suppose P is projector that projects onto S along T

So $Pa_1 = a_1$, $Pa_2 = a_2$ ...

& $Pb_1 = 0$; $Pb_2 = 0$ ...

because basis of T lies in the subspace of T so its projection onto S along T is zero

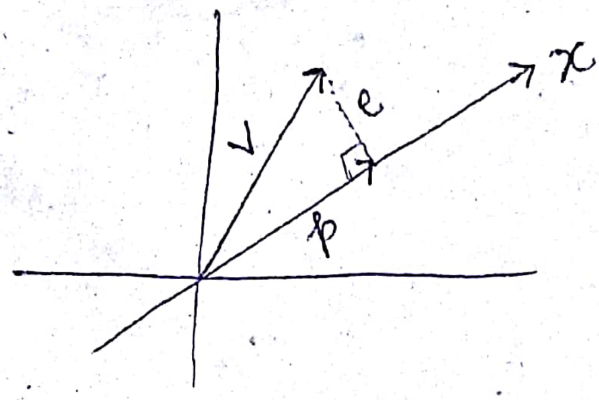
Hence $PQ = \left[a_1 \,|\, a_2 \,|\, \cdots \,|\, 0 \,|\, 0 \,|\, 0 \cdots \right]$

$$P = Q^{-1} \cdot T$$

where $Q = \left[a_1 \,|\, a_2 \,|\, \cdots \,|\, b_1 \,|\, b_2 \cdots \right]$

& $T = \left[a_1 \,|\, a_2 \,|\, \cdots \,|\, 0 \,|\, 0 \cdots \right]$

Ans(3) (a)



projection   scalar

$p = \alpha x$

since $p + e = v$

$e = v - p = v - \alpha x$

as e is $\perp$ to x

$f(x,e) = f(e,x) = 0$ (using modified inner product

$\Rightarrow f(x,e) = x^T C e = 0$

$\Rightarrow x^T C (v - \alpha x) = 0$ gives $\boxed{\alpha = \dfrac{x^T C v}{x^T C x}}$

we can also write $p = x\alpha$

so $p = x \dfrac{x^T C v}{x^T C x} = \underbrace{\dfrac{x x^T C}{x^T C x}}_{P} v$

$p = Pv$

so $\boxed{P = \dfrac{x x^T C}{x^T C x}}$ Ans

Ans 3 (b) From 3(a) we have

$$\alpha = \frac{x^T C v}{x^T C x}$$

& projection (p) of v onto x is

$$p = \alpha x = \frac{x^T C v}{x^T C x} x$$

From classical Gram-schmidt algorithm for

$$\begin{array}{ccc} A & = & X & R \end{array}$$

$$\begin{bmatrix} a_1 | a_2 | \cdots a_n \end{bmatrix} = \begin{bmatrix} q_1 | q_2 | \cdots | q_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix}$$

we had :

for $j = 1$ to n
  $v_j = a_j$
  for $i = 1$ to $j-1$
    $r_{ij} = q_i^T a_j$
    $v_j = v_j - r_{ij} q_i$
  $r_{jj} = \|v_j\|_2$
  $q_j = v_j / r_{jj}$

Here we subtracted projection of a_j ~~along~~ onto q_i
& that subtracted value was $r_{ij} q_i = \dfrac{(q_i^T a_j) q_i}{q_i^T q}$

From above projection formula
new projection value to be subtracted

$$= \left(\frac{q_i^T C a_j}{q_i^T C q_i} \right) q_i$$

So new $r_{ij} = \dfrac{q_i^T C a_j}{q_i^T C q_i}$

Ans(4) In householder transformation

$$P = I - 2\omega\omega^T$$

$$\omega^T\omega = 1$$

Now given $F = I - 2\frac{vv^T}{v^Tv}$ where $v = x - x'$

$$F = I - 2\left(\frac{(x-x')}{\|x-x'\|}\right)\left(\frac{(x-x')^T}{\|x-x'\|}\right)$$

take $\omega = \frac{x-x'}{\|x-x'\|}$, clearly $\omega^T\omega = 1$

$$F = I - 2\omega\omega^T \quad \text{where } \omega = \frac{x-x'}{\|x-x'\|}$$

$$\begin{aligned}
F^TF &= (I - 2\omega\omega^T)^T(I - 2\omega\omega^T) \\
&= (I^T - (2\omega\omega^T)^T)(I - 2\omega\omega^T) \\
&= (I - 2\omega\omega^T)(I - 2\omega\omega^T) \\
&= I - 4\omega\omega^T + 4\omega(\omega^T\omega)\omega^T \\
&= I - 4\omega\omega^T + 4\omega\omega^T \quad (\text{as } \omega^T\omega = 1)
\end{aligned}$$

$$\boxed{F^TF = I}$$

Now to show $Fx = x'$

$$Fx = \left(I - \frac{2(x-x')(x-x')^T}{\|x-x'\|^2} \right) x$$

$$= x - \frac{2(x-x')(x-x')^T x}{\|x-x'\|^2}$$

$$= \left((x-x') - \frac{2(x-x')(x-x')^T x}{\|x-x'\|^2} \right) + x'$$

$$= \frac{(x-x')}{\|x-x'\|^2} \left(\|x-x'\|^2 - 2(x-x')^T x \right) + x'.$$

$$= \frac{x-x'}{\|x-x'\|^2} \left((x-x')^T (x-x') - 2(x-x')^T x \right) + x'$$

$$= \frac{x-x'}{\|x-x'\|^2} \left(x^T x + (x')^T x' - x^T x' - (x')^T x - 2x^T x + 2(x')^T x \right) + x'$$

$$Fx = 0 + x'$$

0 comes from the fact as $\|x\| = \|x'\|$
$\Rightarrow x^T x = (x')^T x'$

and $x^T (x') = (x')^T x$ → comes from dot product property

So $Fx = x'$

(P.S. norms in above equations are 2 norm)

Ans(5) We can use unitary matrix for this case.

Since for Unitary matrix(Q) we have

$$\|Qx\| = \|x\|$$

to derive this take $\|Qx\|^2 = (Qx)^T(Qx)$

$$\|Qx\|^2 = x^T Q^T Q x$$

Since $Q^T Q = I \Rightarrow \|Qx\|^2 = x^T x = \|x\|^2$

$$\Rightarrow \|Qx\| = \|x\|$$

So for any $a_i \in R^m$, $b_i = Q a_i$ where Q is $n \times m$ unitary matrix

Now take any two vectors $a_i, a_j \in R^m$

$$b_i = Q a_i, \quad b_j = Q a_j$$

$$b_i^T b_j = (Qa_i)^T (Qa_j)$$

$$= a_i^T Q^T Q a_j$$

$$b_i^T b_j = a_i^T a_j \qquad (\text{as } Q^T Q = I)$$

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