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Assignment 0

2) Linear Algebra

Q.1) Prove or disprove : Empty set is a vector space

⇒ To determine whether empty set is a vector space, we need to check if it satisfies defn of vector space.

A vector space V over a field F is a set with 2 operations

1) Vector addition : for any $u, v \in V$, $u + v \in V$
2) Scalar multn : for any $v \in V$ & any scalar $c \in F$, $c \cdot v \in V$

Vector space should satisfy following properties

1) Associativity of addition
2) Commutativity of addition
3) Identity element of addition
4) Inverse element of addition
5) Distributivity
6) Identity element of scalar multiplication
7) closed

For a set to be a vector space, it must include the zero vector. The empty set does not contain any elements, so it cannot contain a zero vector.

The operations of vector addition & scalar multiplication cannot be performed on elements of the empty set because there are no elements to operate on.

Empty space does not satisfy necessary cond's of vector space. Therefore empty set is not a vector space.

Que. 2) Show that inverse of $M = I + (uv^T)$ is of the type $I + \alpha (uv^T)$ where $u, v \in R^n$, $v^T u \neq 0$. continuing from previous question find α. For what u & v is M singular ? Find the null space of M, if it is singular.

$\Rightarrow$ We want to find inverse of M, denoted as M^{-1}, & show that it is of the form $M^{-1} = I + \alpha uv^T$ for some scalar α.

1) Assume that inverse M^{-1} is of form $I + \alpha uv^T$, α is scalar to be determined.

$$M^{-1} = I + \alpha uv^T$$

$$MM^{-1} = I$$

$$(I + uv^T)(I + \alpha uv^T) = I$$

$$LHS = I + \alpha uv^T + uv^T + \alpha (uv^T)(uv^T)$$

$$= I + \alpha uv^T + uv^T + \alpha u \underbrace{(v^T u)}_{\text{scalar}} v^T$$

$$= I + (1+\alpha) uv^T + \alpha (v^T u) uv^T$$

$$= I + [1 + \alpha + \alpha (v^T u)] uv^T$$

2) $MM^{-1} = I$, the 2nd term should be zero

$$1 + \alpha + \alpha (v^T u) = 0$$

$$\alpha (1 + v^T u) = -1$$

$$\boxed{\alpha = \frac{-1}{1 + v^T u}}$$

The inverse of $M = I + uv^T$ is indeed of the form $M^{-1} = I + \alpha uv^T$, where $\alpha = \frac{-1}{1 + v^T u}$

So, $M^{-1} = I + \frac{(-1)}{1+v^Tu} uv^T$

3) If M is singular, M^{-1} must not exist. This happens if denominator of α is zero

$$1 + v^Tu = 0$$

$$\boxed{v^Tu = -1}$$

4) M is singular then

$$v^Tu = -1$$

We need find x s.t.

$$Mx = 0$$

$$(1 + uv^T)x = 0$$

$$x + \cancel{x}\, uv^T x = 0$$

$$x = -uv^T x$$

Since $v^Tu = -1$, $v^Tx = 0$

$\therefore$ $x = \lambda u$

where λ is any scalar

The null space of M when M is singular is span of vector u

$$\text{Null}(M) = \{\lambda u \mid \lambda \in R\}$$

Sol^n 3) 1) $A = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -2-\lambda & 2 \\ -6 & 5-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 2 \\ -6 & 5-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(5-\lambda) + 12$$

$$= (\lambda+2)(\lambda-5) + 12$$

$= (\lambda+2)(\lambda-5)+12$

$= \lambda^2 - 3\lambda - 10 + 12$

$|A-\lambda I| \quad = \lambda^2 - 3\lambda + 2 = 0$

$(\lambda-1)(\lambda-2) = 0$

$\therefore \boxed{\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= 2\end{aligned}}$

For finding eigen vectors

$(A-\lambda I)x = 0$

For $\lambda = 1$, $\quad (A-I)x = 0$

$$\left(\begin{bmatrix}-2 & 2\\ -6 & 5\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)x = 0$$

$$\begin{bmatrix}-3 & 2\\ -6 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$-3x_1 + 2x_2 = 0, \quad -6x_1 + 4x_2 = 0$

$x_2 = \frac{3x_1}{2}$

Eigenvector corresponding to $\lambda_1 = 1$ is any scalar multiple of

$$\begin{bmatrix}x_1\\ \frac{3}{2}x_1\end{bmatrix} = x_1\begin{bmatrix}1\\ 3/2\end{bmatrix} \qquad \boxed{x_1 = \begin{bmatrix}2\\ 3\end{bmatrix}}$$

For $\lambda = 2$, $\quad (A-\lambda I)x = 0$

$$A\left(\begin{bmatrix}-2 & 2\\ -6 & 5\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-4 & 2\\ -6 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$-4x_1 + 2x_2 = 0 \qquad -6x_1 + 3x_2 = 0$

$x_2 = 2x_1$

for $\lambda = 2$, scalar multiple of, $\begin{bmatrix}x_1\\ 2x_1\end{bmatrix} = x\begin{bmatrix}1\\ 2\end{bmatrix}$

$\boxed{x_2 = \begin{pmatrix}1\\ 2\end{pmatrix}}$

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2) Since A is ~~ei~~ diagonal matrix, eigen values will be dia. ele.

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

To satisfy $AU = U\Lambda$

The matrix U is formed placing eigenvectors as columns,

$$U = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

3) ~~$A = UAU$~~ ~~$A = U\Lambda A U^{-1}$~~

~~$A = UAA$~~

$$A = U\Lambda U^{-1}$$

$$U = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$|U| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$U^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\boxed{U^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}}$$

Verify : $A = U\Lambda U^{-1}$

$$U\Lambda U^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$U\Lambda U^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -6 & 4 \end{bmatrix}$$

$$U\Lambda U^{-1} = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$$

$$U\Lambda U^{-1} = A$$

$\therefore$ $A = U\Lambda U^{-1}$ is verified

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Q. 3) Show that for any square matrix, A the eigenvectors of A are also eigenvectors of A^2. What are eigenvalues of A^2.

⇒

i) Lets prove that for any square matrix A, the eigenvectors of A are ~~to~~ also eigenvectors of A^2.

Let x be an eigenvector of A with eigen values λ.

by def^n, $Ax = \lambda x$

Now consider action of A^2 on x

$$A^2 x = A(Ax)$$

substitute $Ax = \lambda x$

$$A^2 x = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$$

$$A^2 x = \lambda^2 x$$

This shows that x is also eigenvector of A^2 with corresponding eigenvalue λ^2.

2) From the derivation above, if λ is eigenvalue of A, then λ^2 is an eigenvalue of A^2.

Thus, eigenvalues of A^2 are squares of the eigenvalues of A.

3) Probability

Q.1) If 2 binary random variables X & Y are independent, are $\bar{X}$ & Y are also independent? Prove your claim.

⇒ If X & Y are independant

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

x	$\bar{x}$
x	$1-x$

To show $\bar{X}$ & Y are independant, we need to show

$$P(\bar{X}=\bar{x}, Y=y) = P(\bar{X}=\bar{x}) \cdot P(Y=y)$$

For binary variable, $\bar{X} = 1-X$,

$$\therefore P(\bar{X}=1-x) = P(X=x)$$

Hence,

$$P(\bar{X}=\bar{x}, Y=y) = P(X=1-\bar{x}, Y=y)$$
$$= P(X=1-\bar{x}) \cdot P(Y=y)$$
$$= P(\bar{X}=\bar{x}) \cdot P(Y=y)$$

$\therefore$ This shows $\bar{X}$ & Y are independant.

Q.2) Show that if two variables x & y are indepedent, then their covarience is zero.

⇒ $$\text{cov}(X,Y) = E\left[(X-E[X])(Y-E[Y])\right]$$

X & Y are independent.

$$\therefore E[XY] = E[X] \cdot E[Y]$$

$$\therefore \text{cov}(X,Y) = E[XY] - E[X]E[Y]$$
$$= E[X]E[Y] - E[X]E[Y]$$
$$= 0$$

This shows that covarience of 2 independant variables is zero.

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Sol^n 3)
$$N(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

We want to verify that,

$$E(X) = \int_{-\infty}^{\infty} N(x|\mu, \sigma^2)\, x\, dx = \mu$$

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) . x\, dx$$

put $\boxed{\frac{x-\mu}{\sigma} = z}$

$x - \mu = z.\sigma$ $\quad \therefore \boxed{dx = \sigma\, dz}$

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\cancel{\sigma^2}} z^2 \cancel{\sigma^2}\right) x . \cancel{dx}^{\sigma} dz$$

$$= \cancel{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \left(\sigma . z\right) \exp\left(-\frac{z^2}{2}\right) dx}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} (\sigma z + \mu) \exp\left(-\frac{z^2}{2}\right) \sigma\, dz$$

$$= \int_{-\infty}^{\infty} \frac{(\sigma z + \mu) . \exp\left(-\frac{z^2}{2}\right) dz . \cancel{\sigma}}{\sqrt{2\pi} . \cancel{\sigma}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z \exp\left(-\frac{z^2}{2}\right) dz + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz$$

$= \quad 0 \quad + \quad \mu . 1$

(odd function) (Normalization condition)

$\therefore \boxed{E(X) = \mu}$

$\to \int_{-\infty}^{\infty} N(x|\mu, \sigma^2)\, dx = 1$

diff. w.r.t to σ^2

$$\frac{d}{d\sigma^2}\left(\int_{-\infty}^{\infty} N(x|\mu, \sigma^2)\, dx\right) = \frac{d}{d\sigma^2}(1)$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial \sigma^2} N(x|\mu, \sigma^2)\, dx = 0$$

Now compute the derivative,

$$\frac{\partial}{\partial \sigma^2} N(x|\mu, \sigma^2) = \frac{\partial}{\partial \sigma^2} \cancel{N(x|\mu, \sigma^2)} \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)\right)$$

$$= N(x|\mu, \sigma^2)\left(\frac{-1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4}\right)$$

Integrating,

$$\int_{-\infty}^{\infty} N(x|\mu, \sigma^2) \cdot \cancel{N(x} \left(\frac{(x-\mu)^2 - \sigma^2}{2\sigma^4}\right) dx = 0$$

$$E[(x-\mu)^2] = \sigma^2$$

$$\therefore E[x^2] = \mu^2 + \sigma^2$$

Sol^n (4)

$P(C_1 = H) = 0.5$
$P(C_1 = T) = 0.5$

$P(C_2 = H \mid C_1 = H) = 0.7$
$P(C_2 = H / C_1 = T) = 0.5$

$S = C_1 + C_2 = 1$

by bayes th^m,

$P(C_1 = T \text{ and } C_2 = H \mid S = 1)$

$$= \frac{P(S=1 \mid C_1 = T \text{ and } C_2 = H) \quad P(C_1 = T \text{ and } C_2 = H)}{P(S=1)}$$

S = 1 can happen in 2 scenarios.

1) $C_1 = H$ & $C_2 = T$ (probability $0.5 \times 0.3 = 0.15$)
2) $C_1 = T$ & $C_2 = H$ (probability $0.5 \times 0.5) = 0.25$

$\therefore \; P(S=1) = 0.15 + 0.25 = 0.4$

$\therefore \; P(C_1 = T \text{ and } C_2 = H \mid S = 1) = \frac{0.25}{0.4} = \frac{5}{8} = 0.625$

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sol^n (5)

$p(r) = 0.2$

$p(b) = 0.2$

$p(g) = 0.6$

Box r : 3 apples, 4 oranges, 3 limes, Total = 10

Box b : 1 apple, 1 orange, 0 limes, Total = 2

box g : 3 apple, 3 orange, 4 limes, total = 10

Probability of selecting an apple :

$$P(\text{apple}) = p(r) \times \frac{3}{10} + p(b) \times \frac{1}{2} + p(g) \times \frac{3}{10}$$

$$= 0.2 \times 0.3 + 0.2 \times 0.5 + 0.6 \times 0.3$$

$$= \cancel{0.06 + 0.} = 10^{-2} [6 + 10 + 18]$$

$$P(\text{apple}) = 0.34$$

Probability that orange came from green box

$$P(g/\text{orange}) = \frac{P(\text{orange} | g) \times P(g)}{P(\text{orange})}$$

$$P(\text{orange}/g) = \frac{3}{10}$$

$$P(\text{orange}) = 0.2 \times \frac{4}{10} + 0.2 + \frac{1}{2}\left(\frac{5}{5}\right) + 0.6 \times \frac{3}{10}$$

$$P(\text{orange}) = \frac{0.8 + 1 + 1.8}{10} = 0.36$$

$$P(g/\text{orange}) = \frac{\frac{3}{10} \times 0.6}{0.36} = \frac{3 \times 0.6}{0.36 \times 10} = \frac{\cancel{60} \times \cancel{8}}{\cancel{36} \times \cancel{10}} = \frac{1}{2}$$

$$P(g/\text{orange}) = 0.5$$