ISTANBUL TECHNICAL UNIVERSITY

UCK 419E

Computational Aerodynamics

CRN : 12167

HomeWork 6

NAME : NESLİHAN GÜLSOY

STUDENT ID : 110160111

DEPARTMENT : AEROSPACE ENGINEERING

INSTRUCTORS : Prof.Dr. Melike Nikbay

QUESTION 1 : Using the following periodic function, evaluate both first and second derivatives using the following formulas:

$$u = -sin(x) + (sin(2x))cos(x) \tag{1}$$

1. First Derivative

(a) 1st order backward

(b) 2nd order central

(c) 2nd order backward

(d) 4th order central

2. Second Derivative

(a) 1st order backward

(b) 2nd order central

(c) 4th order central

Start with a Δx that yields five subdivisions for one period of the function, and then halve Δx until you reach "grid independence". Analytically determine the derivatives from the given function.

• Compare the analytic results with the numerical results obtained using up to twelve levels of halving.

• Plot total error versus Δx on a log-log scale. Write an explanation of what you observed. • Perform a hand calculation of each of the derivatives for the first Ax at x = 3.0 and compare your answers with the results from the spreadsheet.

SOLUTION 1 :

Analytical first and second order derivatives are :

$$\frac{\partial u}{\partial x} = -cos(x) + 2sin(4x)cos(x) - sin(2x)^2 sin(x) \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} = sin(x) + 8cos(4x)cos(x) - 2sin(x)sin(4x) - 2sin(4x)sin(x) - cos(x)sin(2x)^2 \tag{3}$$

Period of u is equal to 2π. So first $\Delta x_1 = 2\pi/5$. After that, Δx is halved 12 times. Formulas for finite differences are taken from Ref. [1] and they are implemented on MatLab. To check grid in-dependency, numeric solutions are compared and it is seen that after 9 halving, grid in-dependency is achieved.

Numeric solutions are conducted on $x = 3$. Total error versus Δx on a log-log scale are given in Fig. 1 and Fig. 2. By looking the Figs. 1, 2 it is seen that, total error decreases as Δx decreases. Also 4th order central finite difference method gives more accurate results that others. As expected 1st order backward method gives less accurate results.

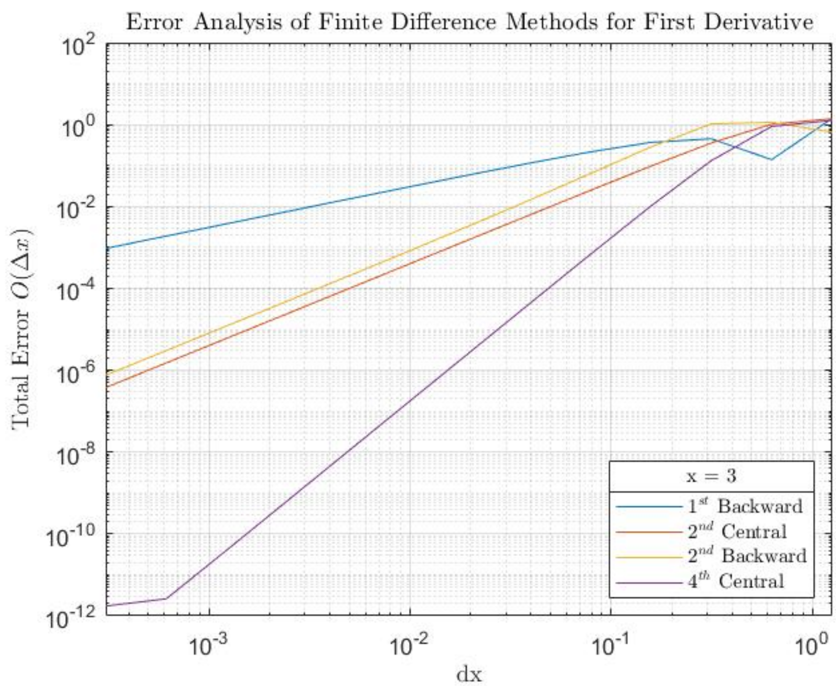

Fig. 1 Total error versus Δx, First Derivative

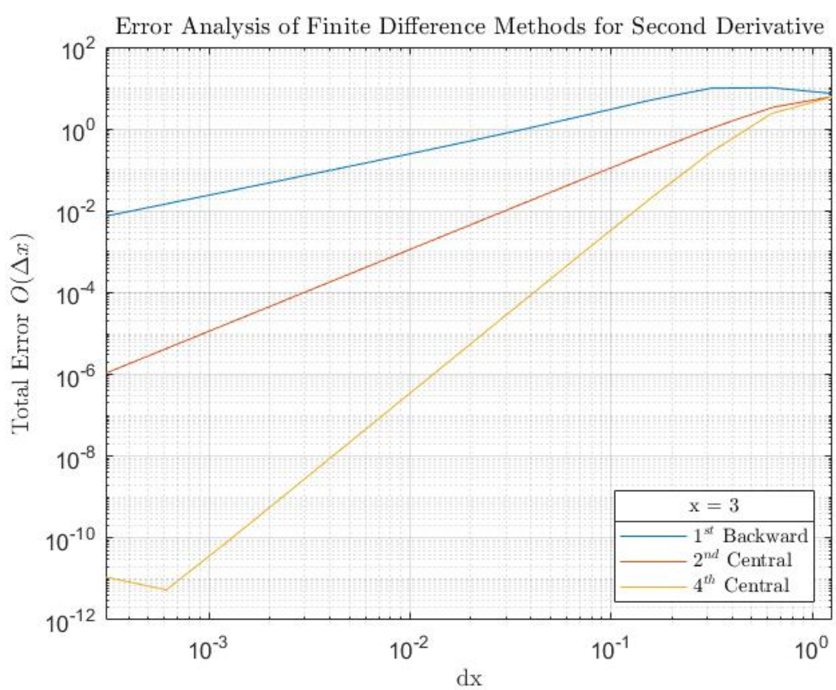

Fig. 2 Total error versus Δx, Second Derivative

QUESTION 2 : Solve the Heat Equation

$$u_t = 0.2u_{xx} \tag{4}$$

on a computer using the simple explicit method for the initial conditions

$$u(x,0) = 100sin\left(\frac{\pi x}{L}\right) \tag{5}$$

use $\pi = 4tan^{-1}(1) =$, $L = 1$ and the boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{6}$$

Compute to t=1.5 for the following cases (if possible)

Case	Number of Grid Points	CFL
1	19	0.25
2	9	0.50
3	19	0.50
4	19	Find the CFL for instability
5	19	2.00

Where the CFL number is $\alpha\frac{\Delta t}{\Delta x^2}$ and $\alpha = 0.2$ (the coefficient of u,, in the governing equation). Repeat the problem using Laasonen's simple implicit method. Compare all of your results with the exact solution.

$$u(x,t) = 100e^{-\alpha\pi^2 t}sin(\pi x) \tag{7}$$

Comment on the results relative to stability, accuracy, type of error, number of grid points, method used, etc.

SOLUTION 2 :

Formulation for simple explicit and Laasonen's simple implicit methods are taken from Refs. [1], [2]. CFL number is taken as 0.72 for Case4. Fig. 3 represents comparison of analytical results with explicit method at $t = 0$ and $t = 1.5$ s for first 3 cases. At Fig. 4, simple implicit method solutions are represented for first 3 cases. Solution for Case 4and Case 5 are shown at Fig. 5.

	max. ε%	
	Explicit	Implicit
Case1	0.3765	1.8458
Case2	8.1726	13.2610
Case3	1.5239	2.9221

Table 1 Error Analysis of Methods

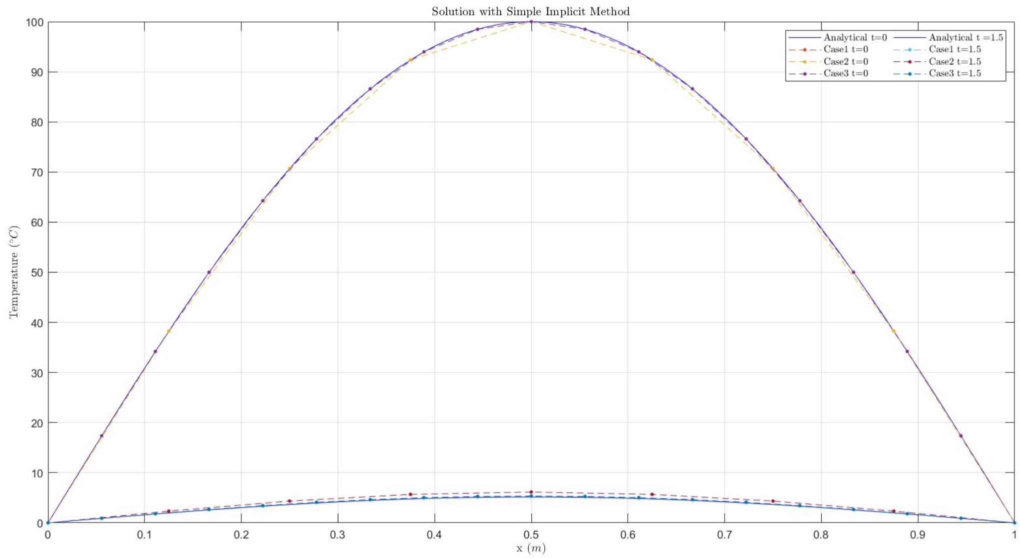

Fig. 3 Simple Explicit Method

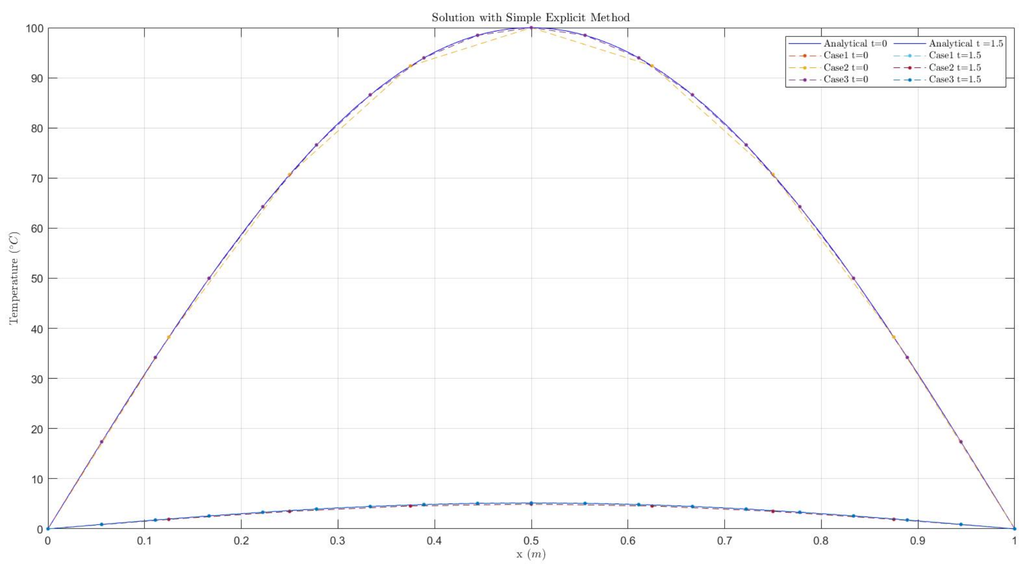

Fig. 4 Simple Implicit Method

As seen above figure and Tab. 1 there is a strong relationship between accuracy and grid number. Also, CFL number affects accuracy. Accuracy increases as the number of grids increases and CFL number decreases. Maximum error occurs at $t = 1.5\ s$ at both methods because of cumulative error. Cumulative error occurs mainly due to truncation error of numeric approaches to derivatives.

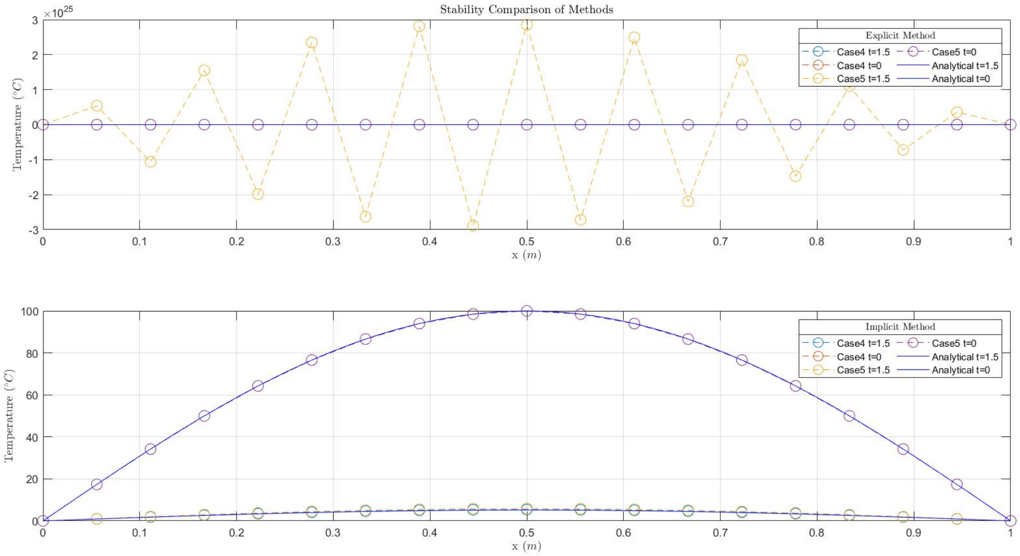

Fig. 5 Solutions for Case 4 and Case 5

For first 3 cases simple implicit method gives less accurate results than simple explicit method. But as seen at Fig 5, explicit method becomes unstable for CFL numbers greater than a certain number. Stability condition is given with $CFL \leq 0.5$ [1]. Unlike the first 3 cases, random errors are also seen when stability conditions are not satisfied in explicit method. However, the implicit method remains stable under higher CFL numbers.

References

[1] Cummings, R. M., Mason, W. H., Morton, S. A., and McDaniel, D. R., *Applied Computational Aerodynamics: A Modern Engineering Approach*, Cambridge University Press, 2015. https://doi.org/10.1017/CBO9781107284166.

[2] Chapra, S. C., and Canale, R. P., *Numerical Methods for Engineers*, 7^{th} ed., McGraw-Hill Education, 2015.