Constrained Optimization
Simplex Method

WQD7011 Numerical Optimization

Recall

- Solving linear optimization problem using graphical solution…
  - Feasible in real-life problem?

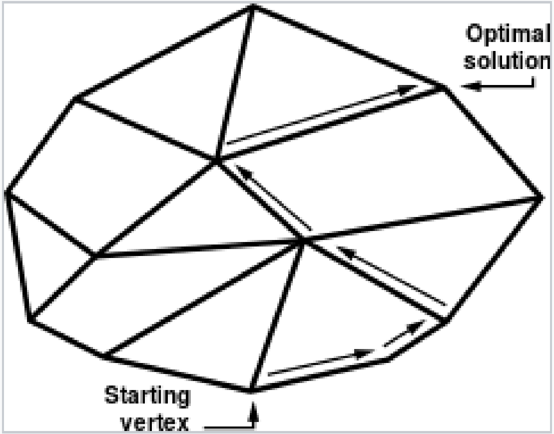
Simplex Method

- Creator: **George Dantzig** (1946)
- 1940s marks the start of modern era in optimization
- Enable economists to formulate large models and analyze them in a systematic and efficient way
- Still continually improved and now it has reached high level of sophistication

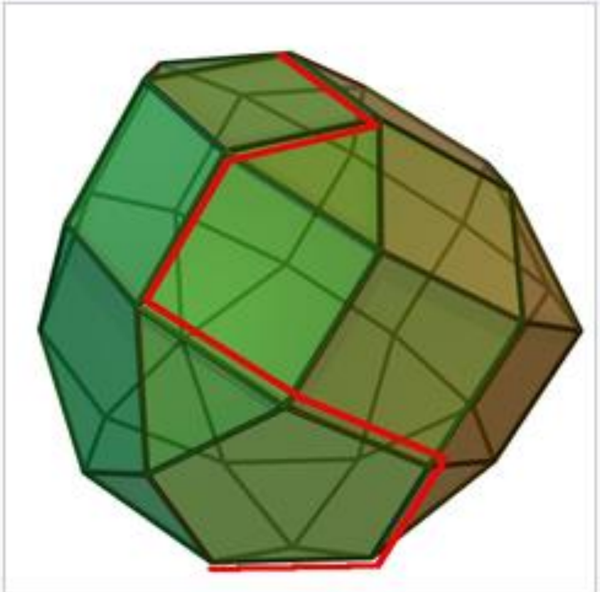
Simplex Method

- Still the most widely used of all optimization tools. **WHY?**
  - ✓ Advanced state of software
  - ✓ Guaranteed convergence to a global minimum
  - ✓ Uncertainty in the model makes linear model more appropriate than an overly complex non-linear model.

Simplex Method



A system of linear inequalities defines a polytope as a feasible region. The simplex algorithm begins at a starting vertex and moves along the edges of the polytope until it reaches the vertex of the optimal solution.



Polyhedron of simplex algorithm in 3D

Simplex Method

- Linear programs are usually stated and analyzed in the following standard form:

$$\min c^T x, \text{subject to } Ax = b, x \geq 0$$

- where c and x are vectors $\mathbb{R}^n$, b is a vector in $\mathbb{R}^n$, and A is an $m \times n$ matrix.
- Given

$$\min c^T x, \text{subject to } Ax \leq b$$

  - How do you **convert the inequality constraint to equality**?

Simplex Method

- Given

$$\min c^T x, \text{ subject to } Ax \le b$$

  - How do you convert the inequality constraint to equality?
- **Adding slack or surplus variable.**
  - If $\boldsymbol{Ax} \le \boldsymbol{b}$, we use slack variable:
    - Becomes $Ax + S_1 = b$
  - If $\boldsymbol{Ax} \ge \boldsymbol{b}$, we use surplus variable:
    - Becomes $Ax - S_1 = b$

Simplex Method *Example*

- Solve the following LP problem by using simplex method:

Maximize $Z = 12x_1 + 16x_2$, subject to

$$10x_1 + 20x_2 \leq 120$$
$$8x_1 + 8x_2 \leq 80$$
$$x_1 \text{ and } x_2 \geq 0$$

Simplex Method *Example*

- Step 1: Convert the objective function and constraints to standard form.

Maximize $Z = 12x_1 + 16x_2 + 0S_1 + 0S_2$,

subject to,

$$10x_1 + 20x_2 + S_1 = 120$$
$$8x_1 + 8x_2 + S_2 = 80$$
$$x_1, x_2, S_1, S_2 \geq 0$$

Simplex Method Example

- Step 2: Transform the standard form to initial simplex table.

Maximize $Z = 12x_1 + 16x_2 + 0S_1 + 0S_2$, subject to

$$10x_1 + 20x_2 + S_1 = 120$$
$$8x_1 + 8x_2 + S_2 = 80$$
$$x_1, x_2, S_1, S_2 \geq 0$$

CB_i	C_j	**12**	**16**	**0**	**0**	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	
0	S_2	8	8	0	1	80	
	Z_j						
	$Z_j - C_j$						

Simplex Method Example

- Step 3: Calculate $Z_j = \sum_{i=1}^{2} CB_i \times a_{ij}$ and $C_j - Z_j$.

$\boldsymbol{CB_i}$	$\boldsymbol{C_j}$	**12**	**16**	**0**	**0**	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	
0	S_2	8	8	0	1	80	
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0	0	

Optimality conditions

For maximization:

$(C_i - Z_j) \leq 0$

For maximization:

$(C_i - Z_j) \geq 0$

Simplex Method Example

- Step 4: Check the optimality condition for $C_j - Z_j$.

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	
0	S_2	8	8	0	1	80	
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	$\boxed{12}$	$\boxed{16}$	0	0	0	

Have not found optimal solution yet! Continue to the next step…

Simplex Method *Example*

- Step 5: Prepare Table for Iteration 1.
  - a) Find **key columns**, key rows, key elements first.

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	
0	S_2	8	8	0	1	80	
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0	0	

Key column because largest value of $(C_j - Z_j)$

Simplex Method *Example*

- Step 5: Prepare Table for Iteration 1.
  - b) Find key columns, **key rows**, key elements first.

CB_i	C_j	**12**	**16**	**0**	**0**	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	120/20=6
0	S_2	8	8	0	1	80	80/8=10
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0	0	

Key row because smallest value of $ratio$

Simplex Method *Example*

- Step 5: Prepare Table for Iteration 1.
  - c) Find key columns, key rows, **key elements** first.

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	120/20=6
0	S_2	8	8	0	1	80	80/8=10
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0	0	

Intersection between key row and key column = key element

Simplex Method *Example*

- Step 5: Prepare Table for Iteration 1.
  - d) identify entering and departing variables

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	120/20=6
0	S_2	8	8	0	1	80	80/8=10
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0	0	

x_2 is entering variable while S_1 is departing variable.

Simplex Method *Example*

- Step 5: Prepare Table for Iteration 1.
  - e) filling in the elements

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
16	x_2	1/2	1	1/20	0	6	
0	S_2						
	Z_j						
	$C_j - Z_j$						

Divide all initial value with key element.

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	120/20=6
0	S_2	8	8	0	1	80	80/8=10
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0	0	

Simplex Method Example

1st. 8 – (8*10)/20 = 4	2nd . 8 – (8*20)/20 = 0
3rd. 0 – (8*1)/20 = -2/5	4th. 1 – (8*0)/20 = 1
5th. 80 – (8*120)/20 = 32	

- Step 5: Prepare Table for Iteration 1.
  - e) filling in the elements

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
16	x_2	1/2	1	1/20	0	6	
0	S_2	4	0	-2/5	1	32	

New Value = Old Value – (Cor Key Col * Cor Key Row)/Key Element

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
0	S_1	10	20	1	0	120	120/20=6
0	S_2	8	8	0	1	80	80/8=10
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	12	16	0	0	0	

Simplex Method *Example*

- Step 6: Calculate Z_j and $C_j - Z_j$.

CB_i	C_j	**12**	**16**	**0**	**0**	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
16	x_2	1/2	1	1/20	0	6	
0	S_2	4	0	-2/5	1	32	
	Z_j	8	16	4/5	0		
	$C_j - Z_j$	4	0	-4/5	0		

Simplex Method *Example*

- Step 7: Is the optimality condition met?

CB_i	C_j	12	16	0	0	Solution	Ratio
	Basic Variable	x_1	x_2	S_1	S_2		
16	x_2	1/2	1	1/20	0	6	
0	S_2	4	0	-2/5	1	32	
	Z_j	8	16	2/5	0		
	$C_j - Z_j$	4	0	-2/5	0		

Continue…

Simple Method
Exercise

- Solve the following linear programming problem using Table solution:

$$\text{Maximize } 3x_1 + 2x_2$$

$$\text{With subject to, } 2x_1 + x_2 \leq 18$$
$$2x_1 + 3x_2 \leq 42$$
$$3x_1 + x_2 \leq 24$$
$$x_1, x_2 \geq 0$$

Exercise this week…

- Code the simplex method using Table Solution in Python, then test the codes using the two examples in this slides.

Discuss on Group Assignment