Assignment #4
Numerical Computing (COMP 350)

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1. (a) After many hours of staring at this problem, clicking every Google result on the topic and not having a single clue, I give up. No idea what is going on or even where to start. I don't see why Newton's method doesn't converge quadratically with multiple roots. I don't even understand why adding the m in this "modified" method is better than Newton's method or why it changes the rate of convergence.

   (b) We start with the Taylor series expansion of $f(x)$ about x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$
$$f(x) = f(x_0) + \Big(f'(x_0) + \frac{f''(x_0)}{2}(x - x_0)\Big)(x - x_0)$$

   We use the facts that $f(x) = 0$ and $(x - x_0) = -\frac{f(x_0)}{f'(x_0)}$ (Newton's method):

$$0 = f(x_0) + \Big(f'(x_0) - \frac{f''(x_0)}{2} \cdot \frac{f(x_0)}{f'(x_0)}\Big)(x - x_0)$$
$$x - x_0 = \frac{-f(x_0)}{f'(x_0) - \frac{f''(x_0)f(x_0)}{2f'(x_0)}}$$
$$x = x_0 - \frac{2f(x_0)f'(x_0)}{2\big[f'(x_0)\big]^2 - f(x_0)f''(x_0)}$$

   This is also known as Halley's method. See attached file `halley.m` for its implementation.

2. The computed roots for each methods are in Table 1 and the results of each of their iteration steps are in Tables 2, 3, 4 and 5. The graph of $y = f(x)$ (in green) and $y = f'(x)$ (in red) are in Figure 1.

   See attached function files `halley.m` and `secant.m` for the implementation of the new root-finding methods.

   See attached function files `f.m`, `fd.m` and `fdd.m` for the implementation of $f(x)$, $f'(x)$ and $f''(x)$.

   See attached script file `q2.m` for the commands generating the results and the plot. In this case, we used $n_{max} = 10^5$, $x_0 = 1$, $x_1 = 2$.

   As we can see, Halley's method is the fastest with 3 iterations since it has cubic convergence but it requires $f'(x)$ and $f''(x)$. Newton's method comes as a close second with 4 iterations since it converges quadratically but requires $f'(x)$. Not far after, the secant method converging super-linearly takes 8 iterations to reach the same degree of precision without requiring a derivative of $f(x)$. Then far behind the others, the bisection method took 40 iterations. That is 5 times more than the next fastest method and more than 13 times more than the fastest, because it has linear convergence.

Method	Computed root	Number of iterations
Bisection	1.834243184314801	40
Newton's	1.834243184314032	4
Halley's	1.834243184313922	3
Secant	1.834243184313922	8

Table 1: Comparison of all methods

a	b	c	$f(c)$	error_bound
$1.000000000000000e+00$	$2.000000000000000e+00$	$1.500000000000000e+00$	$-1.125000000000000e+00$	$5.000000000000000e-01$
$1.500000000000000e+00$	$2.000000000000000e+00$	$1.750000000000000e+00$	$-3.906250000000000e-01$	$2.500000000000000e-01$
$1.750000000000000e+00$	$2.000000000000000e+00$	$1.875000000000000e+00$	$2.167968750000000e-01$	$1.250000000000000e-01$
$1.750000000000000e+00$	$1.875000000000000e+00$	$1.812500000000000e+00$	$-1.081542968750000e-01$	$6.250000000000000e-02$
$1.812500000000000e+00$	$1.875000000000000e+00$	$1.843750000000000e+00$	$4.891967773437500e-02$	$3.125000000000000e-02$
$1.812500000000000e+00$	$1.843750000000000e+00$	$1.828125000000000e+00$	$-3.095626831054688e-02$	$1.562500000000000e-02$
$1.828125000000000e+00$	$1.843750000000000e+00$	$1.835937500000000e+00$	$8.645534515380859e-03$	$7.812500000000000e-03$
$1.828125000000000e+00$	$1.835937500000000e+00$	$1.832031250000000e+00$	$-1.123923063278198e-02$	$3.906250000000000e-03$
$1.832031250000000e+00$	$1.835937500000000e+00$	$1.833984375000000e+00$	$-1.317836344242096e-03$	$1.953125000000000e-03$
$1.833984375000000e+00$	$1.835937500000000e+00$	$1.834960937500000e+00$	$3.658599220216274e-03$	$9.765625000000000e-04$
$1.833984375000000e+00$	$1.834960937500000e+00$	$1.834472656250000e+00$	$1.169069320894778e-03$	$4.882812500000000e-04$
$1.833984375000000e+00$	$1.834472656250000e+00$	$1.834228515625000e+00$	$-7.471149729099125e-05$	$2.441406250000000e-04$
$1.834228515625000e+00$	$1.834472656250000e+00$	$1.834350585937500e+00$	$5.470969099405920e-04$	$1.220703125000000e-04$
$1.834228515625000e+00$	$1.834350585937500e+00$	$1.834289550781250e+00$	$2.361722065415961e-04$	$6.103515625000000e-05$
$1.834228515625000e+00$	$1.834289550781250e+00$	$1.834259033203125e+00$	$8.072522976476648e-05$	$3.051757812500000e-05$
$1.834228515625000e+00$	$1.834259033203125e+00$	$1.834243774414063e+00$	$3.005585032411773e-06$	$1.525878906250000e-05$
$1.834228515625000e+00$	$1.834243774414063e+00$	$1.834236145019531e+00$	$-3.585327642952052e-05$	$7.629394531250000e-06$
$1.834236145019531e+00$	$1.834243774414063e+00$	$1.834239959716797e+00$	$-1.642392577316798e-05$	$3.814697265625000e-06$
$1.834239959716797e+00$	$1.834243774414063e+00$	$1.834241867065430e+00$	$-6.709190389031505e-06$	$1.907348632812500e-06$
$1.834241867065430e+00$	$1.834243774414063e+00$	$1.834242820739746e+00$	$-1.851807683195261e-06$	$9.536743164062500e-07$
$1.834242820739746e+00$	$1.834243774414063e+00$	$1.834243297576904e+00$	$5.768874231648624e-07$	$4.768371582031250e-07$
$1.834242820739746e+00$	$1.834243297576904e+00$	$1.834243059158325e+00$	$-6.374604426540032e-07$	$2.384185791015625e-07$
$1.834243059158325e+00$	$1.834243297576904e+00$	$1.834243178367615e+00$	$-3.028658746018209e-08$	$1.192092895507813e-07$
$1.834243178367615e+00$	$1.834243297576904e+00$	$1.834243237972260e+00$	$2.733003983124149e-07$	$5.960464477539063e-08$
$1.834243178367615e+00$	$1.834243237972260e+00$	$1.834243208169937e+00$	$1.215069005411351e-07$	$2.980232238769531e-08$
$1.834243178367615e+00$	$1.834243208169937e+00$	$1.834243193268776e+00$	$4.561015476411967e-08$	$1.490116119384766e-08$
$1.834243178367615e+00$	$1.834243193268776e+00$	$1.834243185818195e+00$	$7.661783207879580e-09$	$7.450580596923828e-09$
$1.834243178367615e+00$	$1.834243185818195e+00$	$1.834243182092905e+00$	$-1.131240257024047e-08$	$3.725290298461914e-09$
$1.834243182092905e+00$	$1.834243185818195e+00$	$1.834243183955550e+00$	$-1.825309681180443e-09$	$1.862645149230957e-09$
$1.834243183955550e+00$	$1.834243185818195e+00$	$1.834243184886873e+00$	$2.918237207438779e-09$	$9.313225746154785e-10$
$1.834243183955550e+00$	$1.834243184886873e+00$	$1.834243184421212e+00$	$5.464642072183779e-10$	$4.656612873077393e-10$
$1.834243183955550e+00$	$1.834243184421212e+00$	$1.834243184188381e+00$	$-6.394227369810324e-10$	$2.328306436538696e-10$
$1.834243184188381e+00$	$1.834243184421212e+00$	$1.834243184304796e+00$	$-4.648015305974695e-11$	$1.164153218269348e-10$
$1.834243184304796e+00$	$1.834243184421212e+00$	$1.834243184363004e+00$	$2.499920270793155e-10$	$5.820766091346741e-11$
$1.834243184304796e+00$	$1.834243184363004e+00$	$1.834243184333900e+00$	$1.017559370097843e-10$	$2.910383045673370e-11$
$1.834243184304796e+00$	$1.834243184333900e+00$	$1.834243184319348e+00$	$2.763833606422850e-11$	$1.455191522836685e-11$
$1.834243184304796e+00$	$1.834243184319348e+00$	$1.834243184312072e+00$	$-9.420908497759228e-12$	$7.275957614183426e-12$
$1.834243184312072e+00$	$1.834243184319348e+00$	$1.834243184315710e+00$	$9.109157872444484e-12$	$3.637978807091713e-12$
$1.834243184312072e+00$	$1.834243184315710e+00$	$1.834243184313891e+00$	$-1.563194018672220e-13$	$1.818989403545857e-12$
$1.834243184313891e+00$	$1.834243184315710e+00$	$1.834243184314801e+00$	$4.476419235288631e-12$	$9.094947017729282e-13$

Table 2: Results of the bisection method

n	x	$f(x)$
0	$2.000000000000000e+00$	$1.000000000000000e+00$
1	$1.857142857142857e+00$	$1.195335276967926e-01$
2	$1.834787350054526e+00$	$2.773253015902810e-03$
3	$1.834243503918507e+00$	$1.627856713426468e-06$
4	$1.834243184314032e+00$	$5.622169396701793e-13$

Table 3: Results of the Newton's method

n	x	$f(x)$	$f'(x)$	$f''(x)$
0	$2.000000000000000e+00$	$1.000000000000000e+00$	$7.000000000000000e+00$	$1.200000000000000e+01$
1	$1.837209302325581e+00$	$1.515589822279662e-02$	$5.126014061654949e+00$	$1.102325581395349e+01$
2	$1.834243209456139e+00$	$1.280579668971882e-07$	$5.093344454307868e+00$	$1.100545925673683e+01$
3	$1.834243184313922e+00$	$8.881784197001252e-16$	$5.093344177606227e+00$	$1.100545910588353e+01$

Table 4: Results of Halley's method

n	x_1	$f(x_1)$
0	$2.000000000000000e+00$	$1.000000000000000e+00$
1	$1.500000000000000e+00$	$-1.125000000000000e+00$
2	$1.764705882352941e+00$	$-3.279055566863436e-01$
3	$1.873599540361965e+00$	$2.090397299390610e-01$
4	$1.831205833909406e+00$	$-1.541953360163717e-02$
5	$1.834118127083818e+00$	$-6.368734578741098e-04$
6	$1.834243595856441e+00$	$2.096128623563232e-06$
7	$1.834243184258313e+00$	$-2.832365453286911e-10$
8	$1.834243184313922e+00$	$-1.776356839400251e-15$

Table 5: Results of the secant method

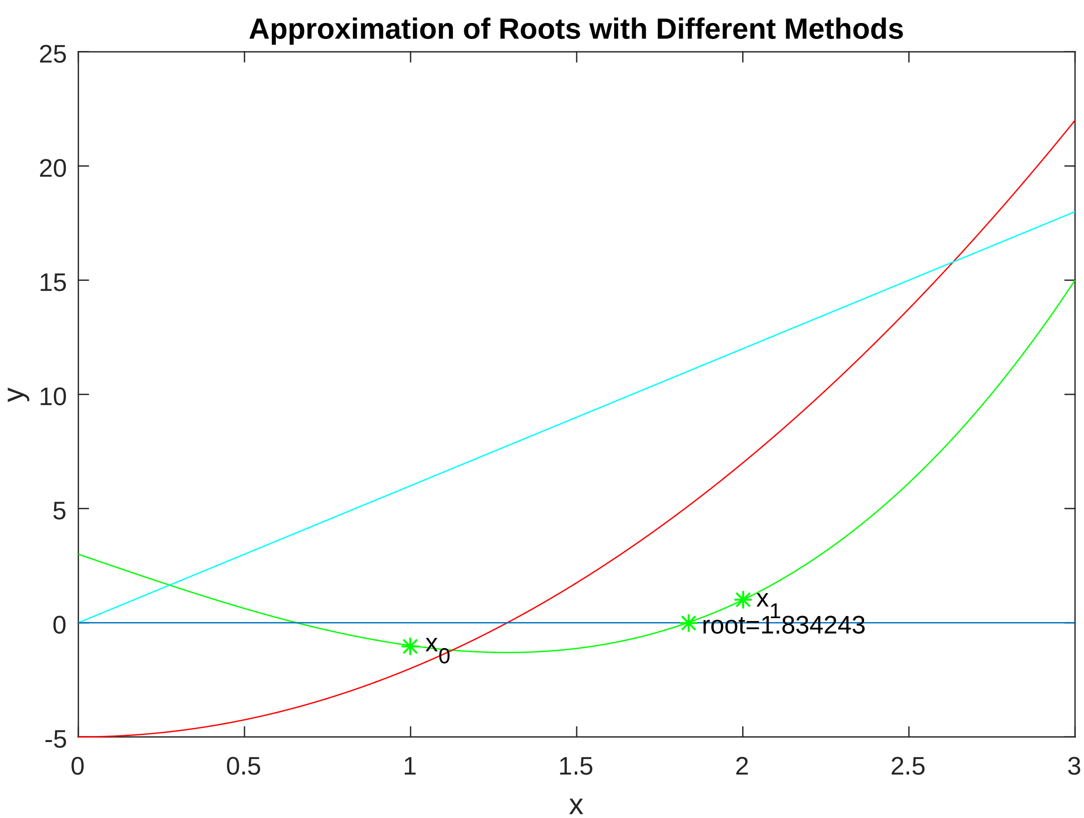

Figure 1: $f(x)$ is shown in green, with the points $\big(x_0, f(x_0)\big)$ and $\big(x_1, f(x_1)\big)$. $f'(x)$ and $f''(x)$ are shown in red and cyan.