COMP 527 – Logic and Computation
Assignment 4

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1 Induction principles for binary trees

1.

$$\frac{\Gamma \vdash t : \text{tree} \quad \Gamma \vdash A(\text{Empty}) \quad \Gamma, N : \text{nat}, T_1 : \text{tree}, T_2 : \text{tree}, ih_1 : A(T_1), ih_2 : A(T_2) \vdash A(\text{Node}(N, T_1, T_2))}{\Gamma \vdash A(t)} \text{ tree } E^{N,T_1,T_2,ih_1,ih_2}$$

2.

$$\frac{\Gamma \vdash t : \text{tree} \quad \Gamma \vdash M_E : A(\text{Empty}) \quad \Gamma, N : \text{nat}, T_1 : \text{tree}, T_2 : \text{tree}, \text{f}\, T_1 : A(T_1), \text{f}\, T_2 : A(T_2) \vdash M_{Node} : A(\text{Node}(N, T_1, T_2))}{\Gamma \vdash \text{rec}^{\forall x:\text{tree}\,.A(x)}\, t \,\text{with}\, \text{f Empty} \to M_E |\, \text{f Node}(N, T_1, T_2) \to M_{Node} : A(t)} \text{ tree } E^{N,T_1,T_2,\text{f}\,T_1,\text{f}\,T_2}$$

3.

$$\begin{aligned}
\text{rec}^A\, \text{Empty with } f\, \text{Empty} \to M_E \mid \text{f Node}(N, T_1, T_2) \to M_{Node} &\implies M_E \\
\text{rec}^A\, \text{Node}(n, t_1, t_2) \text{ with } f\, \text{Empty} \to M_E \mid \text{f Node}(N, T_1, T_2) \to M_{Node} &\implies [n/N][r/\text{f}\, N]M_{Node} \\
\text{where } r = \text{rec}^A\, n \text{ with f Empty} \to M_E \mid \text{f Node}(N, T_1, T_2) \to M_{Node}
\end{aligned}$$

4. Theorem: If $\overbrace{M \implies M'}^{\mathcal{D}}$ and $\Gamma \vdash M : C$ then $\Gamma \vdash M' : C$

   Proof by structural induction on $\mathcal{D} = M \implies_R M'$:

$$\begin{aligned}
\text{Case } \text{rec}^A\, \text{Node}(n, t_1, t_2) \text{ with } f\, \text{Empty} \to M_E \mid \text{f Node}(N, T_1, T_2) \to M_{Node} &\implies [n/N][r/\text{f}\, N]M_{Node} \\
\text{where } r = \text{rec}^A\, n \text{ with f Empty} \to M_E \mid \text{f Node}(N, T_1, T_2) \to M_{Node}
\end{aligned}$$

$\Gamma \vdash \text{rec}^A\, \text{Node}(n, t_1, t_2) \text{ with } f\, \text{Empty} \to M_E \mid \text{f Node}(N, T_1, T_2) \to M_{Node} : C$	by assumption
$\Gamma \vdash \text{Node}(n, t_1, t_2) : \text{tree}$	
$\Gamma \vdash M_E : A(\text{Empty})$	
$\Gamma T : \text{tree}, \text{f}\, T : A(T) \vdash M_{Node} : A() \text{ where } C = A(\text{Node}(n, t_1, t_2))$	by inversion on tree E
$\Gamma \vdash t : \text{tree}$	by tree I_{Node}
$\Gamma \vdash \text{rec}^A\, t \text{ with f Empty} \to M_E \mid \text{f}(sucn) \to M_{Node} : A(t)$	by tree E
$[t/n][r/fn]M_{Node} : A(suct)$	by substitution lemma (twice)

5.

$$\begin{aligned}
&\lambda t : \text{tree.}\, \text{rec}^{\forall x:\text{tree}\,.size(x)}\, t \text{ with} \\
&\qquad \mid \text{f Empty} \Rightarrow 0 \\
&\qquad \mid \text{f Node}(N, T_1, T_2) \Rightarrow N + size(T_1) + size(T_2)
\end{aligned}$$