COMP 527 – Logic and Computation
Homework 2

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Exercise 1 If and only if

Task 1 Annotated introduction and elimination rules for $\equiv$.

$$\frac{\begin{array}{cc}\overline{x : A}^{\,u} & \overline{y : B}^{\,v} \\ \vdots & \vdots \\ M : B & N : A\end{array}}{\langle \lambda x \in A.M, \lambda y \in B.N \rangle A \equiv B} \equiv I^{u,v}$$

$$\frac{M : A \equiv B \quad x : A}{(\text{fst}\, M)x : B} \equiv E_L \qquad \frac{M : A \equiv B \quad x : B}{(\text{snd}\, M)x : A} \equiv E_R$$

Task 2 Local reduction:

$$\frac{\dfrac{\begin{array}{cc}\overline{x : A}^{\,u} & \overline{y : B}^{\,v} \\ \mathcal{D} & \mathcal{E} \\ M : B & N : A\end{array}}{\langle \lambda x \in A.M, \lambda y \in B.N \rangle A \equiv B} \equiv I^{u,v} \quad \begin{array}{c}\mathcal{F} \\ w : A\end{array}}{(\text{fst}\langle \lambda x \in A.M, \lambda y \in B.N \rangle)w : B} \equiv E_L \Rightarrow_R \begin{array}{c}[\mathcal{F}/x]\mathcal{D} \\ M : B\end{array}$$

$$\frac{\dfrac{\begin{array}{cc}\overline{x : A}^{\,u} & \overline{y : B}^{\,v} \\ \mathcal{D} & \mathcal{E} \\ M : B & N : A\end{array}}{\langle \lambda x \in A.M, \lambda y \in B.N \rangle A \equiv B} \equiv I^{u,v} \quad \begin{array}{c}\mathcal{F} \\ w : B\end{array}}{(\text{snd}\langle \lambda x \in A.M, \lambda y \in B.N \rangle)w : A} \equiv E_L \Rightarrow_R \begin{array}{c}[\mathcal{F}/y]\mathcal{E} \\ N : A\end{array}$$

Local expansion:

$$\begin{array}{c}\mathcal{D} \\ M : A \equiv B\end{array} \Rightarrow_E \frac{\dfrac{\begin{array}{c}\mathcal{D} \\ M : A \equiv B\end{array} \quad \overline{x : A}^{\,u}}{(\text{fst}\, M)x : B} \equiv E_L \quad \dfrac{\begin{array}{c}\mathcal{D} \\ M : A \equiv B\end{array} \quad \overline{y : B}^{\,u}}{(\text{snd}\, M)y : A} \equiv E_L}{\langle \text{fst}\, M, \text{snd}\, M \rangle : A \equiv B} \equiv I^{u,v}$$

Task 3 Here are three cases:

$$\mathcal{D} = \frac{\Gamma, x : A, \Gamma', u : B_2 \vdash B_1 \quad \Gamma, x : A, \Gamma', v : B_1 \vdash B_2}{\Gamma, x : A, \Gamma' \vdash B_1 \equiv B_2} \equiv I^{u,v}$$

$$\mathcal{D} = \frac{\Gamma, x : A, \Gamma' \vdash B_1 \equiv B_2 \quad \Gamma, x : A, \Gamma' \vdash B_1}{\Gamma, x : A, \Gamma' \vdash B_2} \equiv E_l$$

$$\mathcal{D} = \frac{\Gamma, x : A, \Gamma' \vdash B_1 \equiv B_2 \quad \Gamma, x : A, \Gamma' \vdash B_2}{\Gamma, x : A, \Gamma' \vdash B_1} \equiv E_r$$

Task 4

$$\frac{\Gamma^{\downarrow}, u : A{\downarrow} \vdash M : B \uparrow \quad \Gamma^{\downarrow}, v : B{\downarrow} \vdash N : A \uparrow}{\Gamma^{\downarrow} \vdash \langle \lambda u \in A.M, \lambda v \in B.N \rangle A \equiv B \uparrow} \equiv I^{u,v}$$

$$\frac{\Gamma^{\downarrow} \vdash M : A \equiv B \downarrow \quad \Gamma^{\downarrow} \vdash x : A \uparrow}{\Gamma^{\downarrow} \vdash (\mathrm{fst}\, M)x : B \downarrow} \equiv E_L \qquad \frac{\Gamma^{\downarrow} \vdash M : A \equiv B \downarrow \quad \Gamma^{\downarrow} \vdash x : B \uparrow}{\Gamma^{\downarrow} \vdash (\mathrm{snd}\, M)x : A \downarrow} \equiv E_R$$

Exercise 2 We prove this by structural induction on the proof tree $\mathcal{D}$.

- First we show the base cases: $\mathcal{D} = \dfrac{}{\Gamma \vdash () : \top} \top I \quad \mathcal{D}' = \dfrac{}{\Gamma \vdash () : \top} \top I$

  $\top = \top$ by definition

- case $\mathcal{D} = \dfrac{\overset{\mathcal{E}_1}{\Gamma \vdash M : A_1} \quad \overset{\mathcal{E}_2}{\Gamma \vdash N : B_1}}{\Gamma \vdash \langle M, N \rangle : A_1 \wedge B_1} \wedge I \quad \mathcal{D}' = \dfrac{\overset{\mathcal{F}_1}{\Gamma \vdash M : A_2} \quad \overset{\mathcal{F}_2}{\Gamma \vdash N : B_2}}{\Gamma \vdash \langle M, N \rangle : A_2 \wedge B_2} \wedge I$

  $A_1 = A_2$ by IH on $\mathcal{E}_1$ and $\mathcal{F}_1$
  $B_1 = B_2$ by IH on $\mathcal{E}_2$ and $\mathcal{F}_2$
  $\Gamma \vdash \langle M, N \rangle : A_1 \wedge B_1$ by $\wedge I$ on $\mathcal{F}_1$ and $\mathcal{F}_2$
  $A_1 \wedge B_1 = A_2 \wedge B_2$ because we were able to derive $\Gamma \vdash \langle M, N \rangle : A_1 \wedge B_1$ from $\mathcal{F}_1, \mathcal{F}_2$

- case $\mathcal{D} = \dfrac{\overset{\mathcal{E}}{\Gamma \vdash M : A_1 \wedge B}}{\Gamma \vdash \mathrm{fst}\, M : A_1} \wedge E_l \quad \mathcal{D}' = \dfrac{\overset{\mathcal{E}'}{\Gamma \vdash M : A_2 \wedge B}}{\Gamma \vdash \mathrm{fst}\, M : A_2} \wedge E_l$

  $A_1 \wedge B = A_2 \wedge B$ by IH on $\mathcal{E}$ and $\mathcal{E}'$
  $\Gamma \vdash \mathrm{fst}\, M : A_1$ by $\wedge E_l$ on $\mathcal{E}'$
  $A_1 = A_2$ because we derived $\Gamma \vdash \mathrm{fst}\, M : A_1$ from $\mathcal{E}'$

- case $\mathcal{D} = \dfrac{\overset{\mathcal{E}}{\Gamma \vdash M : A \wedge B_1}}{\Gamma \vdash \mathrm{snd}\, M : B_1} \wedge E_r \quad \mathcal{D}' = \dfrac{\overset{\mathcal{E}'}{\Gamma \vdash M : A \wedge B_2}}{\Gamma \vdash \mathrm{snd}\, M : B_2} \wedge E_r$

  $A \wedge B_1 = A \wedge B_2$ by IH on $\mathcal{E}$ and $\mathcal{E}'$
  $\Gamma \vdash \mathrm{snd}\, M : B_1$ by $\wedge E_r$ on $\mathcal{E}'$
  $B_1 = B_2$ because we derived $\Gamma \vdash \mathrm{snd}\, M : B_1$ from $\mathcal{E}'$

- case $\mathcal{D} = \dfrac{\overset{\mathcal{E}}{\Gamma \vdash M : A_1}}{\Gamma \vdash \mathrm{inl}^B\, M : A_1 \vee B} \vee I_l \quad \mathcal{D}' = \dfrac{\overset{\mathcal{E}'}{\Gamma \vdash M : A_2}}{\Gamma \vdash \mathrm{inl}^B\, M : A_2 \vee B} \vee I_l$

  $A_1 = A_2$ by IH on $\mathcal{E}$ and $\mathcal{E}'$
  $\Gamma \vdash \mathrm{inl}^B\, M : A_1 \vee B$ by $\vee I_l$ on $\mathcal{E}'$
  $A_1 \vee B = A_2 \vee B$ because we derived $\Gamma \vdash \mathrm{inl}^B\, M : A_1 \vee B$ from $\mathcal{E}'$

- case $\mathcal{D} = \dfrac{\overset{\mathcal{E}}{\Gamma \vdash M : B_1}}{\Gamma \vdash \mathrm{inr}^A\, M : A \vee B_1} \vee I_r \quad \mathcal{D}' = \dfrac{\overset{\mathcal{E}'}{\Gamma \vdash M : B_2}}{\Gamma \vdash \mathrm{inr}^A\, M : A \vee B_2} \vee I_r$

  $B_1 = B_2$ by IH on $\mathcal{E}$ and $\mathcal{E}'$
  $\Gamma \vdash \mathrm{inl}^A\, M : A \vee B_1$ by $\vee I_r$ on $\mathcal{E}'$
  $A \vee B_1 = A \vee B_2$ because we derived $\Gamma \vdash \mathrm{inr}^A\, M : A \vee B_1$ from $\mathcal{E}'$

- case $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{E}\\ \Gamma, u : A_1 \vdash M : B_1\end{array}}{\Gamma \vdash \lambda u \in A_1.M : A_1 \supset B_1} \supset I^u \quad \mathcal{D}' = \dfrac{\begin{array}{c}\mathcal{E}'\\ \Gamma, u : A_2 \vdash M : B_2\end{array}}{\Gamma \vdash \lambda u \in A_2.M : A_2 \supset B_2} \supset I^u$

$$
\begin{array}{ll}
A_1 = A_2 & \text{by IH on } \mathcal{E} \text{ and } \mathcal{E}' \\
B_1 = B_2 & \text{by IH on } \mathcal{E} \text{ and } \mathcal{E}' \\
\Gamma, u : A_1 \vdash M : B_1 & \text{by } \supset I \text{ on } \mathcal{E}' \\
A_1 \supset B_1 = A_2 \supset B_2 & \text{because we derived } A_1 \supset B_1 \text{ from } \mathcal{E}'
\end{array}
$$

The other cases follow a very similar argument.

Exercise 3
1. Proof and term given as `conj1` in `ex3.tut`.
2. The proof breaks down because doing an $\exists E^{x,p}$ gives us $x : \tau$ and $A \supset P(x)$ yet in order to prove $\forall y \in \tau.P(y)$ we need an arbitrary $y \in \tau$.
3. Same reason as the previous conjecture.
4. Proof and term given as `conj4` in `ex3.tut`.
5. As shown in the course notes, we need a witness for $x \in \tau$ such that $\neg B(x)$ yet we know nothing about the domain τ. Therefore we cannot produce one just fdrom the assumption $\neg(\forall x \in \tau.B(x))$.
6. Proof and term given as `conj6` in `ex3.tut`.

Exercise 4 Provable

1. Proof and term given as `conj1` in `ex4.tut`.
2. Proof and term given as `conj2` in `ex4.tut`.
3. Proof and term given as `conj3` in `ex4.tut`.