COMP 531 – Advanced Theory of Computation
Assignment #4

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3. Disjunction of sets of naturals

First, we define a *fooling set* for a function f as a subset $F \subseteq X \times Y$ such that for any distinct pair $(x_1, y_1), (x_2, y_2) \in F$:

1. $f(x_1, y_1) = f(x_2, y_2)$
2. $f(x_1, y_2) \neq f(x_1, y_1)$ or $f(x_2, y_1) \neq f(x_1, y_1)$.

Note that every f-monochromatic rectangle contains at most one element in F.
Take a set $F = \big\{(S, S^c) : S \subseteq \{1, \ldots, n\}\big\}$. F is a fooling set for $DISJ$:

1. $\forall S \in F, DISJ(S, S^c) = 1$
2. for $S \neq T$, either $S \cap T^c \neq \emptyset$ or $T \cap S^c \neq \emptyset$ or both.

Since all inputs in F return 1, there must be at least one rectangle that returns 0. Since $|F| = 2^n$, the minimum # of monochromatic rectangles in a disjoint cover of $X \times Y$ is $2^n + 1$.
Using the lemma seen in class, we conclude that $D(DISJ) \geq \lceil \log(2^n + 1) \rceil \geq n + 1$.
The naive protocol sends x (of length n) to the other party, who then computes $f(x, y)$ and returns the value (of length 1). The naive protocol has cost $n+1$ and therefore $D(DISJ) \leq n + 1$.
Therefore $D(DISJ) = n + 1$.

4. Intersection of a clique and an independent set

Here is a protocol with a total communication cost of $O(\log^2 n)$. Note that we use the word *return* to indicate that the protocol ends and whoever declares the f-value sends it to the other player so they get it too (cost of 1 bit).
Follow these steps procedurally:

1. If $\exists v \in C : deg(v) < \frac{n}{2}$:
   (a) Player 1 sends v to Player 2 ($\log n$ bits)
   (b) Player 2 can then check whether $v \in I$. If it is, return 1 and communicates it to Player 1 too (1 bit).
   (c) Otherwise, repeat this entire protocol on $H = \{u : u = v \text{ or } (u, v) \in E(G)\} \subseteq G$ (i.e. the closed neighbourhood of v). Note that $|V(H)| \leq \frac{1}{2}|V(G)|$ and $C \cap I \subseteq C \subseteq H$. Therefore recursing on H will find the vertex in $C \cap I$ if it exists.
2. Else if $\exists v \in I : deg(v) \geq \frac{n}{2}$:
   (a) Player 2 sends v to Player 1 ($\log n$ bits)
   (b) Player 1 can then check whether $v \in C$. If it is, return 1 and communicates it to Player 2 (1 bit).
   (c) Otherwise, repeat this entire protocol on $H = \{u : u = v \text{ or } (u, v) \notin E(G)\} \subseteq G$ (i.e. v and its non-neighbours). Note that $|V(H)| \leq \frac{1}{2}|V(G)|$ and $C \cap I \subseteq I \subseteq H$. Therefore recursing on H will find the vertex in $C \cap I$ if it exists.
3. Else, Player 2 returns 1 and communicates it to Player 1 (1 bit). This is because this step is only reached if all nodes in C have degree $< \frac{n}{2}$ and all nodes in I have degree $\geq \frac{n}{2}$ meaning that C and I are disjoint.

Since every recursion step cuts the input size in half, there will be at most $\log n$ recursion steps, who each send $O(\log n)$ bits. Therefore the communication complexity of this function is $O(\log^2 n)$.