Theoretical Cosmology

Part I: Introduction to Big-Bang cosmology

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1. *Introduction to Big-Bang cosmology*	*15 November*
2. Hot thermal Universe	19 November
3. Inflation	22 November
4. Dark energy	26 November

Plan for today

0. Very brief review of General Relativity
1. Geometry of the Universe (FLRW)
2. Contents of the Universe
3. Cosmological parameters, distance-redshift relations
4. Present Universe: ΛCDM

References

- *Big-Bang cosmology*, Keith Olive and John Peacock, in *Review of Particle Physics* (http://pdg.lbl.gov/)
- Past lecture notes by Kazuya Koyama, Vincent Vennin and Hans Winther

- Book suggestions *(University library has all of these except Gorbunov/Rubakov)*
  For cosmology, any reputable book should be fine. Some books that I use:
  - "Modern Cosmology," Dodelson 2020 *(very detailed "Analysis" section, useful for interpreting observations)*
  - "Physical Foundations of Cosmology," Mukhanov 2005 *(one of the best inflation/perturbation treatments)*
  - "Primordial Cosmology," Peter and Uzan 2009 *(very detailed, covers a lot of topics, e.g. alternative models)*
  - "Introduction to the Theory of the Early Universe," Gorbunov and Rubakov 2011 *(separate volumes for inflation and big bang)*

  For general relativity, the levels of books vary greatly. Some books emphasise the mathematical aspects and may be very tricky to get into. A few recommendations:
  - "Spacetime and Geometry," Carroll 2004 *(very accessible introduction to GR written by a cosmologist)*
  - "General Relativity," Wald 1984 *(graduate level text. Appendix on Hamiltonian formulation of GR is excellent)*
  - "Gravity," Poisson and Will 2014 *(extra focus on the weak field limit)*

Natural Units

$$\hbar = c = k_B = 1$$

A convenient choice which measures all angular momenta relative to the reduced Planck's constant, all velocities relative to the speed of light and energy per absolute temperature relative to the Boltzmann's constant.

- Natural units allow us to convert different units into each other

$$1\,\mathrm{s} = 3\times 10^8 \mathrm{m} = \frac{1}{6.6\times 10^{-16}\mathrm{eV}}\,, \qquad 1\,\mathrm{K} = 8.6\ \times 10^{-5}\mathrm{eV}$$

- All scales in a problem can be converted to ***mass units***. I will use square brackets to denote the mass unit of a quantity

$$[\mathrm{time}] = [\mathrm{length}] = [\mathrm{temperature}]^{-1} = [\mathrm{energy}]^{-1} = [\mathrm{mass}]^{-1} = -1$$

- Since we want to keep track of scales, we will **not** set the Newton's constant to unity (although sometimes people do, especially in the context of GR). In our units,

$$8\,\pi\, G_N = \frac{1}{M_{Pl}^2} \Rightarrow \quad [G_N] = -2$$

M_{Pl}: Reduced Planck mass

General Theory of Relativity

Special vs General Relativity

- **Metric**: Tensor that defines the rules for calculating distances between two points.
- In **special relativity**, metric fixed, flat space-time: ***Minkowski*** *metric*. Distance invariant when we change inertial frames: ***global*** invariance under Lorentz boost + rotatation + translation.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

- **In general relativity** (i.e. with gravity), invariance is local: any system with gravity is *locally* equivalent to a system without gravity (Einstein's equivalence principle)
- The ***line element*** $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is invariant under ***local coordinate transformations*** $x^\mu \to y^\mu = y^\mu(x)$
- Objects classified according to their transformation properties:

Scalar: $\phi(x) \to \phi'(y) = \phi(x(y))$

Vector: $A_\mu(x) \to A'_\mu(y) = \frac{\partial x^\nu}{\partial y^\mu} A_\nu(x(y))$

Tensor: $T^{\mu_1 \dots \mu_n}{}_{\alpha_1 \dots \alpha_m} \to \frac{\partial y^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial y^{\mu_n}}{\partial x^{\nu_n}} \frac{\partial x^{\beta_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{\beta_m}}{\partial y^{\alpha_m}} T^{\nu_1 \dots \nu_n}{}_{\beta_1 \dots \beta_m}$

Conventions:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$$

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu$$

$$\hbar = c = k_B = 1$$

unless noted otherwise.

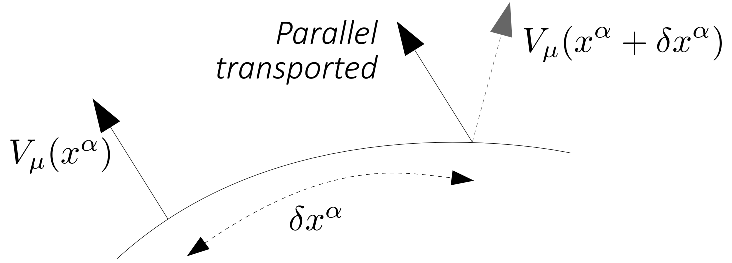
General Theory of Relativity

Parallel transport, connection

- *WARNING!* Not all objects with indices are tensors. Let V_μ be a vector.

$$\partial_\alpha V_\beta \equiv \frac{\partial V_\beta}{\partial x^\alpha} \rightarrow \frac{\partial V'_\beta}{\partial y^\alpha} = \frac{\partial}{\partial y^\alpha}\left(\frac{\partial x^\nu}{\partial y^\mu} V_\nu\right) = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\alpha}\frac{\partial V_\nu}{\partial x^\beta} + \overbrace{\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\mu}}^{\text{not a tensor!}} V_\nu$$

- Why? Vectors (and tensor) are defined on the tangent space of each point. Partial derivatives, by definition, take the difference between two different points.
- Instead *parallel transport* vector at x to $x + \delta x$, then take the difference



Parallel transported:

$$V_\mu \rightarrow V_\mu + \Gamma^\alpha_{\mu\beta} V_\alpha \delta x^\alpha$$

connection

- *Covariant derivative*

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} V_\alpha \longrightarrow \text{Now } \textit{this} \text{ is a tensor.}$$

General Theory of Relativity

Geometric quantities: Christoffel symbol, curvature

- Connection not unique. Simple choice: *Levi-civita connection*, for which $\nabla_\alpha g_{\mu\nu} = 0$. In coordinate basis, the connection is given by *Christoffel symbols*

$$\Gamma^\alpha_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$$

- *Inverse metric* $g^{\alpha\beta}$ defined by $g^{\alpha\beta} g_{\beta\mu} = \delta^\alpha_\mu$, used to raise indices, e.g. $V^\alpha = g^{\alpha\beta} V_\beta$
- *Riemann curvature tensor* $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\alpha = -R^\beta{}_{\alpha\mu\nu} V_\beta$

  in terms of connections $R^\beta{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\beta_{\nu\alpha} - \partial_\nu \Gamma^\beta_{\mu\alpha} + \Gamma^\beta_{\mu\rho}\Gamma^\rho_{\nu\alpha} - \Gamma^\beta_{\nu\rho}\Gamma^\rho_{\mu\alpha}$

- *Ricci tensor* $R_{\alpha\nu} = R^\mu{}_{\alpha\mu\nu}$
- *Ricci scalar* $R = g^{\alpha\nu} R_{\alpha\nu}$
- *Einstein tensor* $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}\dfrac{R}{2}$

General Theory of Relativity

Action, Einstein's field equations

- The action for gravity and matter:

$$S = \frac{1}{16\,\pi\,G_N} \overbrace{\int \sqrt{-g}\,d^4x\,(R \underbrace{-\,2\,\Lambda)}_{\text{vacuum energy}}}^{\substack{\text{gravity piece} \\ \text{(Einstein-Hilbert action)}}} + \overbrace{\int \sqrt{-g}\,d^4x\,\mathcal{L}_{\text{matter}}}^{\text{matter piece}}$$

Volume element: $\sqrt{-g}d^4x$

where $g \equiv \det g$

- Field equations (equations of motion) for the metric follow from the least action principle i.e. $\frac{\delta S}{\delta g^{\mu\nu}} = 0$

$$\boxed{G_{\mu\nu} = 8\,\pi\,G_N\,T_{\mu\nu} - \Lambda g_{\mu\nu}}$$

Einstein field equations

where the ***energy-momentum tensor*** is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\text{matter}})}{\partial g^{\mu\nu}}$$

- Next step is to build the cosmological model.

Foundations of Physical Cosmology

1. Cosmological Principle

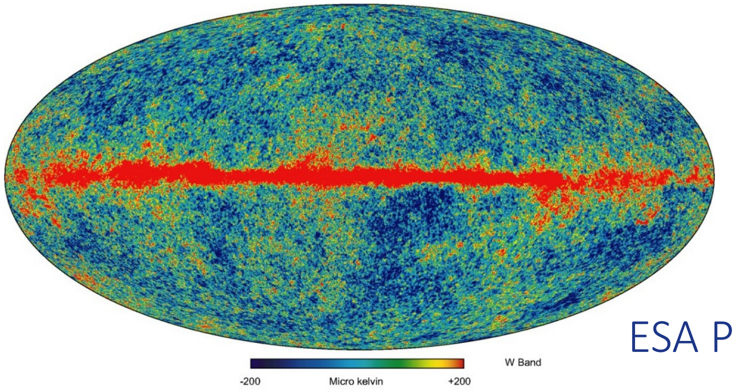
The Universe is homogeneous and isotropic when viewed on a large enough scale

- ***Isotropy***: Universe looks the same in all directions

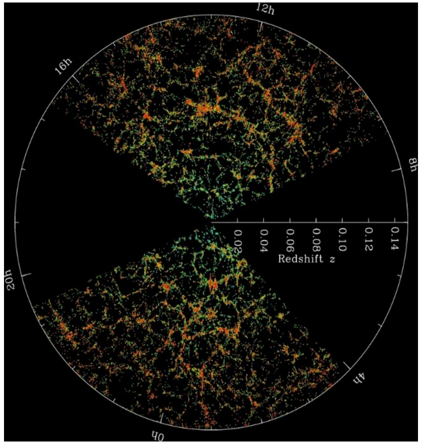
  Justification: CMB photons with temperature fluctuations $\frac{\Delta T}{T} \sim 10^{-5}$

- ***Homogeneity***: Universe looks the same everywhere

  Justification: Galaxy distribution above $\mathcal{O}(10^2)\mathrm{Mpc/h}$ is homogeneous



ESA Planck



SDSSIII

Foundations of Physical Cosmology

2. Universal physical laws

- Gravitation described by ***General Relativity***

- Matter described by the ***Standard Model*** (SM) of Particle physics

- Very likely, we might need new matter beyond the SM: *dark matter, dark energy, inflaton*

Isotropic & homogeneous spaces

Constant spatial curvature K, the sign of which dictates the topology.

finite	*Spherical:*	$\sum_{\triangle} \text{angles} > \pi$	$K > 0$
infinite	*Hyperbolic:*	$\sum_{\triangle} \text{angles} < \pi$	$K < 0$
	Flat:	$\sum_{\triangle} \text{angles} = \pi$	$K = 0$

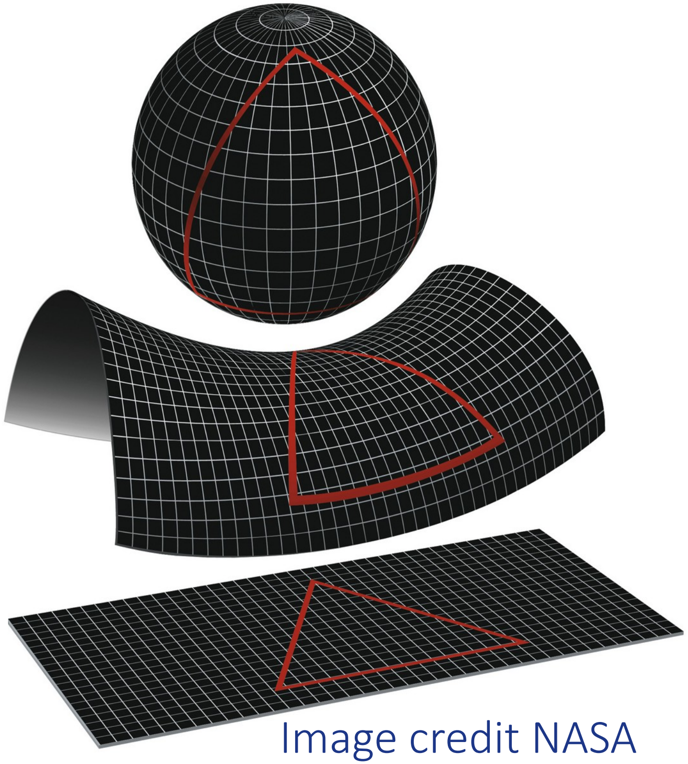


Image credit NASA
map.gsfc.nasa.gov

Mathematical model of the Universe

Friedmann-Lemaître-Robertson-Walker (FLRW) metric

- Extending Minkowski to allow time evolution, ***flat FLRW metric*** is

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right)$$

Scale factor

- For generic spatial curvature, ***FLRW metric*** is given by

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - K\,r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$= -dt^2 + a(t)^2 \left[d\chi^2 + f_K(\chi)^2 (d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$f_K(\chi) = \begin{cases} \dfrac{\sin(\sqrt{K}\chi)}{\sqrt{K}} & K > 0 \quad \text{Spherical} \quad (closed) \\ \chi & K = 0 \quad \text{Euclidean} \quad (flat) \\ \dfrac{\sinh(\sqrt{-K}\chi)}{\sqrt{-K}} & K < 0 \quad \text{Hyperbolic} \quad (open) \end{cases}$$

Is it Friedmann, FL, RW, FRW or FLRW?

Alexander Friedmann
Russian mathematician.
Solved Einstein equations for homogeneous/isotropic matter in 1922.

Georges Lemaître
Belgian physicist.
Reached Friedmann's conclusion independently in 1927.

Howard P. Robertson
American mathematician and physicist.

Arthur G. Walker
British mathematician.

in 1935, Robertson and Walker showed that the metric for a homogeneous and isotropic space is uniquely given by FLRW.

Depending on which country you are in, you might hear different abbreviations for the cosmological metric. ***They are all the same!*** *To be safe, I will refer to it as FLRW.*

Remarks on FLRW spacetime

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - K\, r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]$$

- The spatial coordinates in FLRW are ***comoving***, i.e. they move with the expansion. The time dependence due to expansion is taken out as the ***scale factor*** $a(t)$.
- The metric can be used to calculate space-time distances between two events:
  - *Light-like (null):* trajectory of light moving a distance dr in time dt

$$ds^2 = 0 = -dt^2 + a(t)^2 \frac{dr^2}{1 - K\, r^2}$$

  - *Time-like:* e.g. Proper time $|ds|$ in the rest frame of an observer with $ds^2 < 0$
  - ***Space-like:*** e.g. Two simultaneous events separated by proper distance ds with $ds^2 > 0$ (not causally connected!)
- We can also define a new time coordinate, e.g. *conformal time*: $d\tau \equiv dt/a(t)$

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \left(\frac{dr^2}{1 - K\, r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right) \right]$$

Kinematics of a free falling particle

- Test particles follow the shortest path on the 4-geometry.
- Shortest path in curved space: ***geodesics***.
- Action for a test particle (assuming time-like trajectory)

$$S = \int ds = \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$$

- Variation $\frac{\delta S}{\delta x^\mu} = 0$ gives the **geodesic equation**

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0$$

s : proper time

$ds = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$

Kinematics in FLRW

- Geodesic equation:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0$$

- For a particle with proper 4-velocity $u^\mu = \frac{dx^\mu}{ds}$, the time component of the geodesic equation becomes

$$\frac{du^0}{ds} + \frac{\dot{a}}{a}|\vec{u}|^2 = 0$$

- Since the proper velocity has unit norm $-(u^0)^2 + |\vec{u}|^2 = -1$ and cosmological time is $dt = u^0 ds$, we have

$$\frac{d|\vec{u}|}{dt} + \frac{\dot{a}}{a}|\vec{u}| = 0$$

- Solution: $|\vec{u}| = \frac{\text{constant}}{a}$ $\Rightarrow$ momentum $m|\vec{u}|$ decreases ("redshifts") with expansion, de Broglie wavelength $\lambda = \frac{1}{p} \propto a$ increases!

- In FLRW:

  $\Gamma^0_{ij} = \frac{1}{2}\partial_0 g_{ij}$

  $= \frac{\dot{a}}{a} g_{ij}$

  where $(i,j) = 1,2,3$

- $|\vec{u}| \equiv \sqrt{u^i u^j g_{ij}}$

- $u^\mu u_\mu = -1$

Kinematics of a photon

- Same conclusion applies to massless particles moving on a null geodesic $ds = 0$, as we will now see.
- Consider light emitted at $t = t_e$ and $r = r_e$, observed today $t = t_0$ at $r = 0$

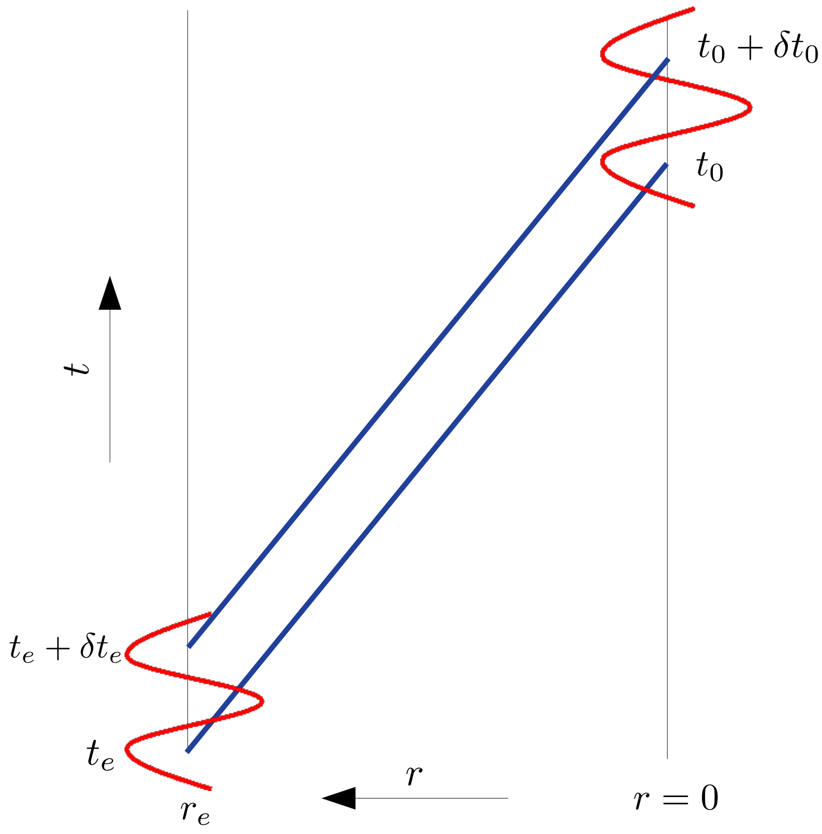
$$\int_{t_e}^{t_0} \frac{dt}{a} = \pm \int_{r_e}^{0} \frac{dr}{\sqrt{1 - K\, r^2}}$$

- The next peak will be emitted at $t = t_e + \delta t_e$ and observed at $t = t_0 + \delta t_0$ although the comoving coordinates will stay the same

$$\cancel{\int_{t_e}^{t_0} \frac{dt}{a}} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a} = \cancel{\int_{t_e}^{t_0} \frac{dt}{a}} - \int_{t_e}^{t_e+\delta t_e} \frac{dt}{a} + \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a}$$

- For small δt we have $\frac{\delta t_e}{\delta t_0} = \frac{a(t_e)}{a(t_0)}$, or since the proper time interval between two peaks is the wavelength,

$$\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = 1 + z$$ ⇨ z : ***redshift***. The change in wavelength due to expansion.



$$\int_a^{a+\epsilon} f(x)dx = f(a)\epsilon + \mathcal{O}(\epsilon^2)$$

Describing the matter sector

- The matter sector consists of bosonic (gauge fields, Higgs) and fermionic (leptons, quarks/hadrons) particles.
- At cosmological scales, matter distribution is consistent with an idealised ***perfect fluid*** description

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$$

Definitions:

ρ	Energy density
P	Pressure
u_μ	4-velocity of the fluid with $u_\mu u^\mu = -1$

- Consistency with FLRW requires isotropic (and uniform) pressure and uniform density, i.e. $\rho = \rho(t)\,, \quad P = P(t)$.
- Isotropy also implies that our cosmology coincides with the rest frame of the fluid.

$$u_\mu = (1,\, 0,\, 0,\, 0)$$

- So the energy-momentum tensor has a relatively simple form. In Cartesian coordinates:

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Dynamics of FLRW universe

- For simplicity, let's assume flat FLRW metric with $K = 0$ and use Cartesian coordinates.
- Non-zero Christoffel symbols:

$$\Gamma^0_{ij} = \dot{a}\, a\, \delta_{ij}\,, \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$$

- Non-zero components of the Riemann tensor

$$R^0{}_{i0j} = a\, \ddot{a}\, \delta_{ij}\,, \qquad R^i{}_{jkl} = \dot{a}^2(\delta^i_k \delta_{jl} - \delta^i_l \delta_j k)\, \dot{a}^2$$

- Non-zero components of the Ricci tensor

$$R_{00} = -\frac{3\,\ddot{a}}{a}\,, \qquad R_{ij} = \left(\dot{a}^2 + a\,\ddot{a}\right)\delta_{ij}$$

- Ricci Scalar

$$R = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)$$

- Non-zero components of the Einstein Tensor

$$G_{00} = \frac{3\,\dot{a}^2}{a^2}\,, \qquad G_{ij} = -\left(\dot{a}^2 + 2\,a\,\ddot{a}\right)\delta_{ij}$$

- Generalisation to $K \neq 0$ is straightforward, especially if you use spherical coordinates.

Friedmann equations

- We now have all the tools to compute the *Einstein field equations*

$$G_{00} = \frac{3\,\dot{a}^2}{a^2}\,, \qquad G_{ij} = -\left(\dot{a}^2 + 2\,a\,\ddot{a}\right)\delta_{ij}\,; \qquad T_{00} = \sum_n \rho_n\,, \qquad T_{ij} = \delta_{ij} a^2 \sum_n P_n\,,$$

- Two independent equations, *Friedmann equations*:

Hubble rate, expansion rate

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G_N}{3}\sum_n \rho_n + \frac{\Lambda}{3} - \frac{K}{a^2}\,,$$

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G_N}{3}\sum_n \left(\rho_n + 3\,P_n\right) + \frac{\Lambda}{3}$$

Equivalent perfect fluids for Λ and curvature:

$$\rho_\Lambda \equiv \frac{\Lambda}{8\,\pi\,G_N}\,, \qquad \rho_\Lambda + 3\,P_\Lambda = -\frac{\Lambda}{4\,\pi\,G_N} \Rightarrow P_\Lambda = -\rho_\Lambda$$

$$\rho_K \equiv -\frac{3\,K}{8\,\pi\,G_N\,a^2}\,, \qquad \rho_K + 3\,P_K = 0 \Rightarrow P_K = -\frac{1}{3}\rho_K$$

Universe accelerates if dominant fluid has $\rho + 3\,P < 0$

- Warning: Curvature is pure geometry, therefore not an actual physical matter field.

Intermission: a bit of history

Einstein's Static Universe

- One of the first cosmological models. The aim is to avoid expansion/contraction by eliminating the attractive force of gravity. Einstein did not have access to Friedmann equations in 1917, but it is convenient to use them for this discussion.
- Non-relativistic matter: mass conservation $\rho\, a^3 = \rho_0 a_0^3 \Longrightarrow \rho = \rho_0 \left(a_0/a\right)^3$

$$\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G_N}{3}\frac{\rho_0 a_0^3}{a^3} - \frac{K}{a^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G_N}{3}\frac{\rho_0 a_0^3}{a^3} + \frac{\Lambda}{3}$$

- We can make $\dot{a} = 0$ by fixing the curvature, but cannot eliminate acceleration.
- Adding **cosmological constant**, we can have a static universe! However, any small deviation leads to an evolving universe, i.e. it is unstable.
- Einstein introduced the c.c. 81 years before its discovery. The model was dropped after Hubble's observational confirmation of expansion.

Steady State Universe

- Introduced by Bondi, Gold and Hoyle in 1948.
- In Big Bang cosmology, the early universe looks very different from the present day one. It was hot and the energy density was much higher.
- In steady state theory, the Universe looks roughly the same at any time, no beginning. There is expansion, but matter is not diluted. Instead, new matter is injected (very slowly) so that the density stays constant throughout the evolution.
- Hoyle formulated a *creation field* with negative pressure, so that energy conservation is satisfied. An old version of inflation in a completely different context!
- In tension with quasars and radio galaxies: they should be everywhere according to Steady State, but they are only observed at large redshift.
- The theory was eventually discarded by the majority of cosmologists after the discovery of the CMB, which shows that the universe was indeed denser in the past.

"In a sense, the disagreement is a credit to the model; alone among all cosmologies, the steady state model makes such definite predictions that it can be disproved even with the limited observational evidence at our disposal."

Weinberg, "Gravitation and Cosmology" 1972

Energy conservation

- By construction, the Einstein tensor is divergence-free

$$\nabla^\mu G_{\mu\nu} = 0$$

Geometric property:
contracted Bianchi identities

- From Einstein equations, energy-momentum tensor ***covariantly conserved*** $\nabla^\mu T_{\mu\nu} = 0$.
- For a perfect fluid in FLRW, this reduces to an ***energy conservation*** (or ***continuity***) equation.

$$\dot\rho + 3\,\frac{\dot a}{a}\,(\rho + P) = 0$$

Not independent! Can be derived from Friedmann eqs

- Consider fluids with constant ***equation of state*** $w = P/\rho$, then we can solve the above

$$\frac{\dot\rho}{\rho} + 3\,\frac{\dot a}{a}\,(1+w) = 0 \quad \Rightarrow \quad \rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$$

Normalisation at t_0:
a_0 scale factor today
ρ_0 density today

  - *Non-relativistic matter:* $w = 0 \Rightarrow \rho \propto a^{-3}$ (volume expands as a^3)
  - *Radiation (relativistic matter):* $w = \frac{1}{3} \Rightarrow \rho \propto a^{-4}$ (volume expansion+energy redshift)
  - *Vacuum energy (c.c.):* $w = -1 \Rightarrow \rho \propto a^{0}$ (oblivious to expansion: “constant”)
  - *Curvature:* $w = -\frac{1}{3} \Rightarrow \rho \propto a^{-2}$ (* positive curvature never dominates)

A rough expansion history based on how density decreases:
Radiation → (non-relativistic) matter → curvature* → vacuum energy

Time evolution of the scale factor

- We can now use the solution for density $\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$ in the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G_N}{3}\sum\rho \simeq \frac{8\,\pi\,G_N}{3}\rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$$

Assumed that a single fluid dominates the expansion

- Assuming expanding universe $\dot{a} > 0$ and $w \neq -1$ we can integrate it:

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \qquad \text{(for } w \neq -1\text{)}$$

Normalisation of the scale factor is unphysical. Typical choice: $a_0 = 1$ *(but <u>not</u> in this lecture)*

  - <u>*Matter dominated:*</u> $w = 0\,,\rho \propto a^{-3}$ ➡ $a(t) \propto t^{2/3}$
  - <u>*Radiation dominated:*</u> $w = 1/3\,,\rho \propto a^{-4}$ ➡ $a(t) \propto t^{1/2}$
  - <u>*Vacuum energy (c.c.) dominated:*</u> $w = -1\,,\rho \sim \text{constant}$ ➡ $a(t) \propto e^{\sqrt{\Lambda/3}t}$

Cosmological parameters

- Define the ***critical density*** as the total energy density in the absence of curvature

$$\rho_c \equiv \frac{3\,H^2}{8\,\pi\,G_N}$$

- We can use dimensionless ***density parameters***, defined relative to the critical density:

$$\Omega_n \equiv \frac{\rho_n}{\rho_c}$$

- Friedmann equation can be written in the simple form $\sum_n \Omega_n = 1$
- It is convenient to express density parameters with respect to their present value

$$\Omega_n \equiv \Omega_{n,0} \left(\frac{a_0}{a}\right)^{3(1+w_n)} \frac{H_0^2}{H^2}$$

- Friedmann equation is thus

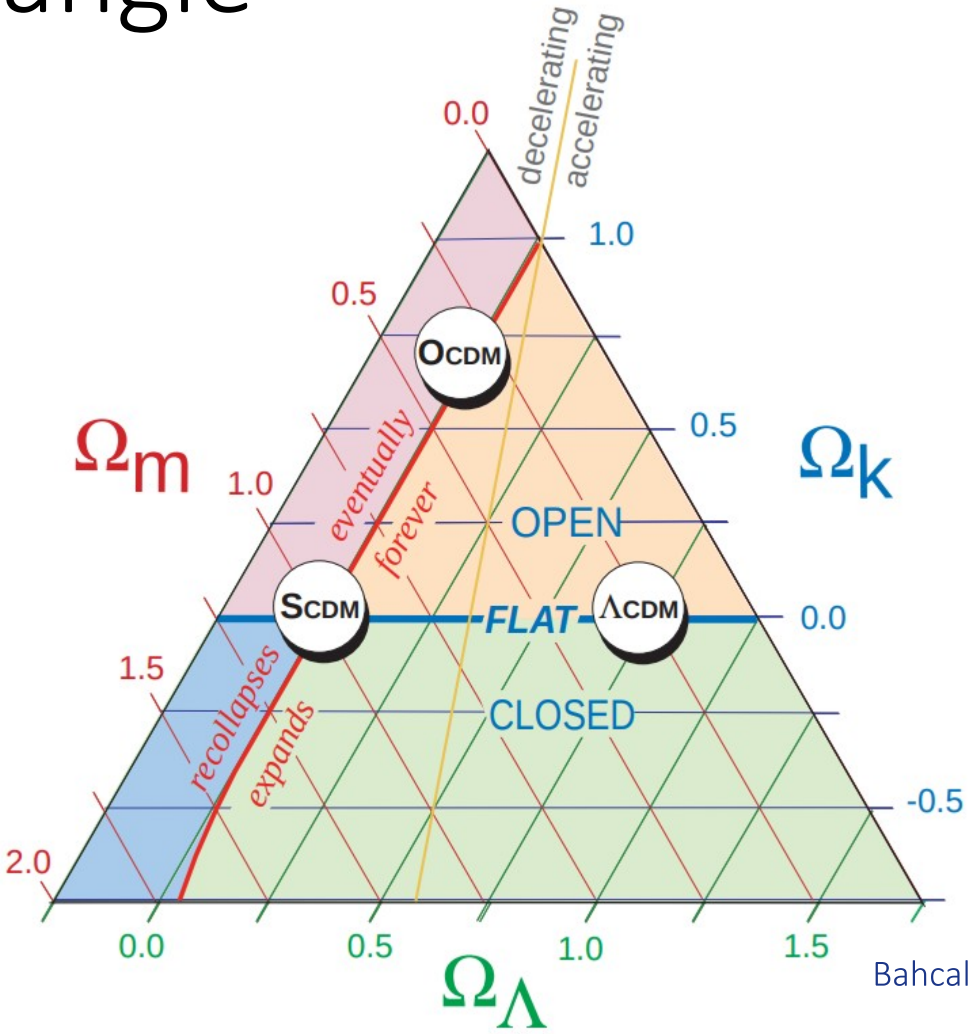
$$\frac{H^2(a)}{H_0^2} = \frac{a_0^4 \Omega_{r,0}}{a^4} + \frac{a_0^3 \Omega_{m,0}}{a^3} + \frac{a_0^2 \Omega_{K,0}}{a^2} + \Omega_{\Lambda,0} + \text{anything else}\ldots$$

- Free parameters $(H_0\,,\ \Omega_{n,0})$ determined by observations.

Cosmic triangle

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1$$

(for this plot $\Omega_r \sim 0$)



- OCDM (Open)

  $\Omega_\Lambda = 0$
- SCDM (Standard)

  $\Omega_\Lambda = \Omega_K = 0$

Bahcall, Ostriker, Perlmutter, Steinhardt '99 [arXiv:astro-ph/9906463]

Distance measures

- To determine the cosmological parameters, we need to make use of observations.
- Since the universe is expanding, redshift decreases with time.

$$z(t) = \frac{a_0}{a(t)} - 1$$

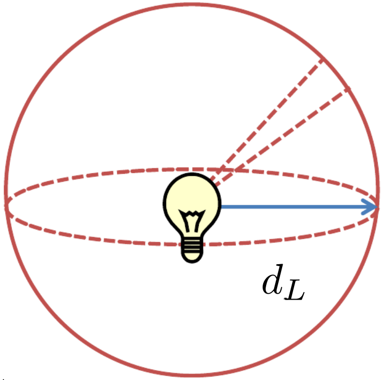
Measuring the redshift $\Rightarrow$ time of the event. $\boxed{t \Leftrightarrow a \Leftrightarrow z}$

- We also need to know ***where*** the events occur.

Measuring distances in an expanding background might be tricky.

1. *Proper distance*: time dependent; difficult to directly observe without a VERY LARGE ruler.
2. *Comoving distance:* time independent since expansion is factored out. Cannot observe.
3. *Luminosity distance:* Distance measure for objects with known luminosity
4. *Angular-diameter distance:* Distance measure for objects with known size

Luminosity distance



- Consider an object (e.g. a galaxy) with known ***luminosity*** L. This is ***energy emitted in unit time***.
- We observe the ***flux*** $\mathcal{F}$, i.e. ***energy observed per unit time per unit area***.
- In static spacetime, conservation of energy implies that a source at a distance d_L will lead to an observed flux of

$$\mathcal{F} = \frac{L}{4\,\pi\, d_L^2}$$

- *Generalisation to FLRW:* Luminosity of the source is corrected by two effects:
  - Energy redshift for each individual photon $\quad E_{obs} = \frac{E_e}{1+z}$
  - Time delay between two consecutive photons $\quad \delta t_{obs} = (1+z)\delta t_e$

$$\left.\begin{array}{l} E_{obs} = \frac{E_e}{1+z} \\ \delta t_{obs} = (1+z)\delta t_e \end{array}\right\} L_{obs} = \frac{L}{(1+z)^2}$$

- Light spread along a spherical surface whose area today is $4\,\pi\, f_K(\chi)^2 a_0^2$

$$ds^2 = -dt^2 + a(t)^2\left[d\chi^2 + f_K(\chi)^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

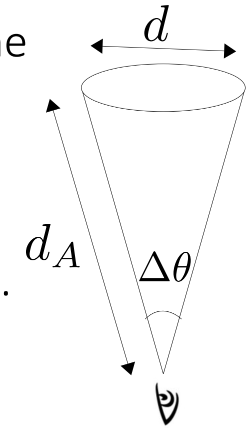
- Observed flux is thus

$$\mathcal{F} = \frac{L}{4\,\pi\, f_K(\chi)^2 a_0^2 (1+z)^2}$$

$f_K(\chi)$ also called *transverse comoving distance*

- Luminosity distance $\boxed{d_L = f_K(\chi)\, a_0 (1+z)}$

Angular-diameter distance



- Consider an object with known proper size d, at comoving coordinate χ . We observe the angular separation $\Delta\theta$. Angular diameter distance is defined as:

$$d_A = \left.\frac{d/2}{\tan(\Delta\theta/2)}\right|_{\Delta\theta\ll 1} = \frac{d}{\Delta\theta}$$

- Light emitted simultaneously from the two edges. All coordinates same except the angle. Proper size:

$$d^2 = ds^2 = f_K(\chi)^2 a(t_e)^2 (\Delta\theta)^2$$

- Thus, the observed angle is

$$\Delta\theta = \frac{d}{f_K(\chi)a(t_e)} = \frac{d\,(1+z)}{f_K(\chi)a_0}$$

- So the angular distance, in terms of comoving distance, is

$$\boxed{d_A = \frac{f_K(\chi)\,a_0}{(1+z)}}$$

- Distance measures summary:

$$d_A(1+z) = \frac{d_L}{1+z} = f_K(\chi)a_0$$

Distance relations in terms of observables

- The distance relations allow us to determine the comoving distance at a given redshift
$$d_A(1+z) = \frac{d_L}{1+z} = f_K(\chi)a_0$$
- We can start using cosmological models by considering a light moving on radial geodesics:
$$a(t)\,d\chi = -dt$$

Minus sign? *Incoming* photons.
As time↗, comoving distance↘

- Let us use redshift to measure time:
$$z = \frac{a_0}{a} - 1 \;\Rightarrow\; dz = -H\,\frac{a_0}{a}dt$$
- The comoving coordinate can then be expressed as
$$a_0 d\chi = \frac{dz}{H}$$
- Finally, we use the form of the Friedmann equation in terms of cosmological parameters
$$a_0\chi = \int_0^z \frac{dz}{H_0}\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{K,0}(1+z)^2 + \Omega_{\Lambda,0}\right)^{-1/2}$$
- We now have a distance-redshift relation in terms of model parameters. We can fit the model with observational data.

Hubble's law

- For small curvature, distance relations become

$$d_A(1+z) = \frac{d_L}{1+z} = a_0 f_K(\chi) \simeq a_0\chi$$

- For nearby objects with $z \ll 1$ and using that $a_0 d\chi = \frac{dz}{H}$, we find at leading order in z

$$d_A \simeq d_L \simeq a_0\chi \simeq \frac{z}{H_0}$$

  This is the ***Hubble's law*** $v = H_0 d$, where z above corresponds to the redshifted velocity in $c = 1$ units.

- The expansion rate H is not constant. To find first order corrections to the Hubble's law we expand H and find proper, luminosity and angular distances as

$$a_0\chi = \frac{z}{H_0}\left[1 - \frac{q_0}{2}\, z + \mathcal{O}(z^2)\right]$$

$$d_L = \frac{z}{H_0}\left[1 - \frac{1}{2}(q_0 - 1)\, z + \mathcal{O}(z^2)\right]$$

$$d_A = \frac{z}{H_0}\left[1 - \frac{1}{2}(q_0 + 3)\, z + \mathcal{O}(z^2)\right]$$

q_0 : deceleration parameter

$$q \equiv -\frac{\ddot{a}\, a}{\dot{a}^2} = \frac{z+1}{H}\frac{d\,H}{d\,z} - 1$$

Defined with a negative sign (hence the "deceleration") for historic reasons

The present Universe

- We reviewed the theoretical framework that provides a framework to interpret observations. Here are some fundamental observables:

    - ***<u>Supernovae:</u>*** The physics behind the peak luminosity of type 1a supernovae is well understood. They are *standard candles* and allow us to measure luminosity distance.

    - ***<u>Cosmic Microwave Background (CMB):</u>*** The sound horizon at the last scattering surface is a fixed scale that we use as a *standard ruler*.

    - ***<u>Baryon Acoustic Oscillations (BAO):</u>*** Regular fluctuations in the visible matter density provide an indirect measure of the sound horizon at the last scattering.

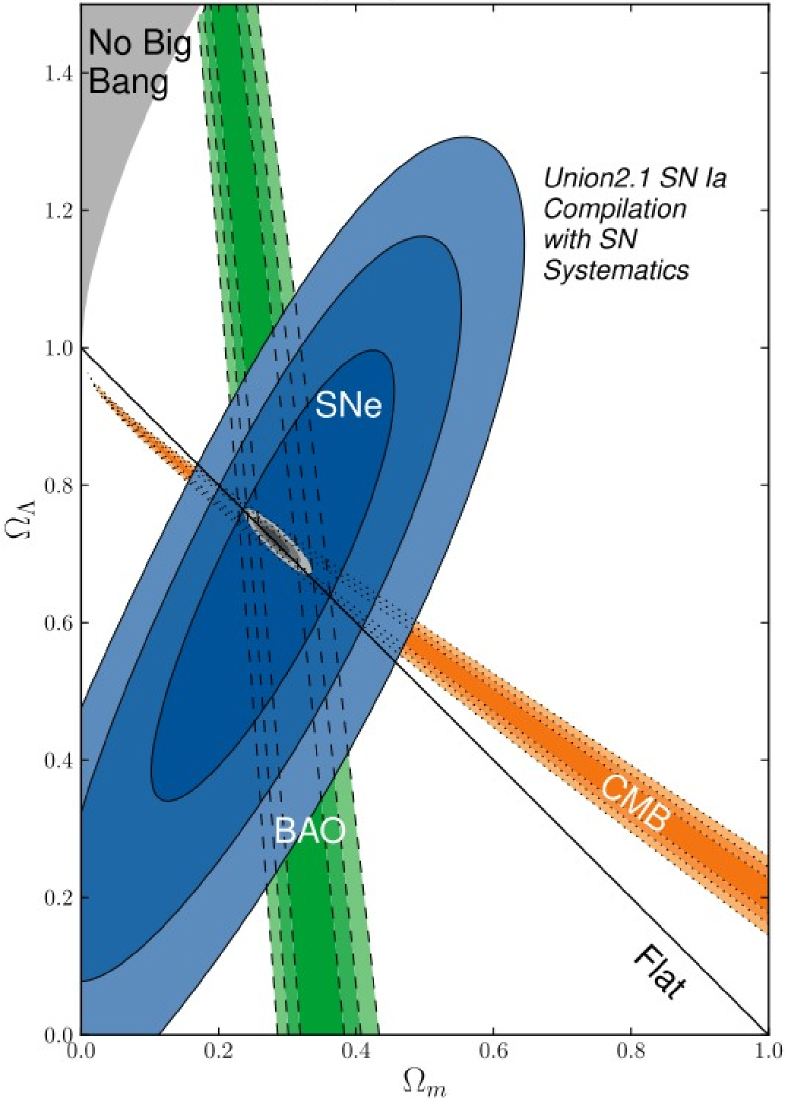
- The models are tested against the observations and best-fit values for the model parameters are determined.

- Independent observations should be consistent. If the best fit values contradict, we move on to another model.

The present Universe

- Different observations are remarkably consistent.
- The currently accepted standard (or concordance) model of cosmology is ***ΛCDM*** (Λ-Cold Dark Matter).
- Spatial geometry is consistent with ***flat*** FLRW.
- Non-relativisic (presureless) matter corresponds to **30%** of the total density. Only **5%** is visible (baryonic) matter. The rest is cold dark matter.
- The largest component is ***dark energy***, and it is consistent with a ***cosmological constant***. It covers **69%** of the density.
- Relativistic matter (CMB radiation) is diluted down to 5×10^{-5} of the total energy. Blackbody radiation with present-day temperature $T = 2.7\,K$.
- No sign of departure from ΛCDM. However:
  - Some tension among observations remain (e.g. H_0)
  - The nature of dark matter and dark energy remain unknown.

Supernova Cosmology Project
Suzuki, et al., *Ap.J.* (2011)



Age of the Universe

- Using the definition of redshift, we relate the cosmological time to expansion rate:

$$\int_0^{t_0} dt = -\int_\infty^0 \frac{dz}{H(z)(1+z)}$$

- Replacing the Hubble rate with the expression in terms of cosmological parameters

$$H_0 t_0 = \int_0^\infty \frac{dz}{(1+z)} \left[\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{K,0}(1+z)^2 + \Omega_{\Lambda,0}\right]^{-1/2}$$

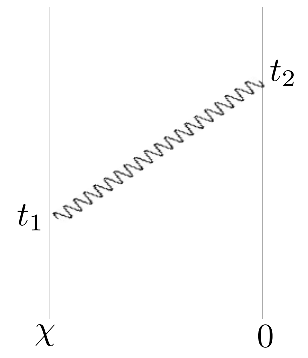
- Using the best fit values, we can compute the above numerically. A simple analytic solution is also possible if we neglect radiation and curvature (i.e., $\Omega_{m,0} = 1 - \Omega_{\Lambda,0}$).
In this case, we find

$$H_0 t_0 = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \log\left(\frac{1+\sqrt{\Omega_{\Lambda,0}}}{\sqrt{1-\Omega_{\Lambda,0}}}\right)$$

- Using $H_0 = 0.069\,\mathrm{GyR}$ and $\Omega_{\Lambda,0} = 0.69$

$$t_0 = 13.84 \times 10^9 \mathrm{years}$$

Particle (cosmological) horizon



- Let us first answer the question *"How far can a photon go in a given time interval?"*

Null geodesics ⟹

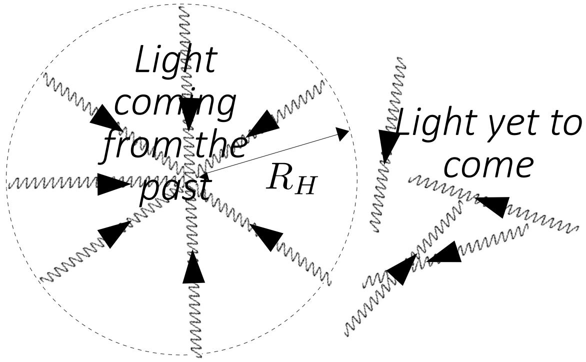
$$\chi = \int_{t_1}^{t_2} \frac{dt}{a} = \int_{a_1}^{a_2} \frac{da}{H\,a^2}$$

$$\dot{a} = da/dt$$
$$\Rightarrow dt = da/(a\,H)$$

- **Particle horizon:** What portion of the Universe can be observed at the origin at time t ?

Assumed single dominant fluid ⟹

$$\chi_H = \int_0^a \frac{da}{H\,a^2} = \sqrt{\frac{3}{8\,\pi\,G_N}} \int_0^a \frac{da}{\sqrt{\rho}\,a^2}$$



- The integral is convergent (i.e. $\exists$ horizon) if $\sqrt{\rho}a$ diverges as $a \to 0$
- If the dominant fluid in the early universe decays faster than curvature, i.e. decelerating, there is a finite distance beyond which we cannot observe.
- For big bang cosmology early Universe dominated by radiation $\rho \propto a^{-4}$ so we can only detect light coming from early Universe within a finite region.

! *Not to be confused with Event Horizon, which defines the region we will eventually probe. In other words, the event horizon is similar to the above, but the emission time t_1 is today, while the arrival time $t_2 \to \infty$.*

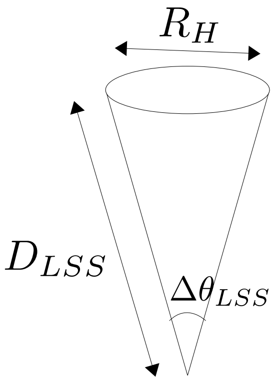
Horizon Problem

- Horizon size at redshift z, as observed today (assuming matter domination)

$$R_H = a_0 \chi_H \simeq \int_z^\infty \frac{dz}{H_0\sqrt{\Omega_{m,0}}(1+z)^{3/2}} \simeq \frac{2}{H_0\sqrt{\Omega_{m,0} z}} \qquad (z \gg 1)$$

- Horizon size at last scattering surface (LSS), when CMB was emitted has $z \simeq 1100$. As observed today, it is about $500\mathrm{Mpc}$.
- Observed today, our distance to LSS can be estimated at

$$D_{LSS} = a_0 \chi_{LSS} \simeq -\int_{z_{LSS}}^0 \frac{dz}{H_0\sqrt{\Omega_{m,0}}(1+z)^{3/2}} \simeq \frac{2}{H_0\sqrt{\Omega_{m,0}}}$$



- so the angle we observe LSS is

$$\Delta\theta_{LSS} \simeq \frac{R_H}{D_{LSS}} \simeq \frac{1}{\sqrt{z_{LSS}}} \sim 1^\circ$$

- 1° in the sky corresponds to a causally connected region. In standard big bang cosmology, there should be no correlation in individual 1° regions. However, CMB from all directions are isotropic to 1 part in 10^5. This is the ***horizon problem***.
- ***Inflation*** solves this problem with an early stage of acceleration.

Is that it?

- Not really. The FLRW universe is an approximation. It is a description of the universe averaged over small scales, i.e. assuming smooth and non-interacting matter.
- However, we see structure everywhere at short distances. Despite this, FLRW model is still remarkably successful.
- Good news: many phenomena can be described by considering small departures from FLRW, with cosmological perturbation theory. Where things become highly nonlinear, we can use numerical tools to simulate the Universe.
- In the next lecture, we will take into account the fundamental nature of the matter sector and see how the particle interactions play a role in the early Universe.

Next time on *Theoretical Cosmology*

- What does the “radiation fluid” actually consist of?
- How did the chemical elements form?
- Was there really a *bang*?

Stay tuned for the fight between particle physics processes and the expansion of the Universe.