Theoretical Cosmology

Part III: Inflation

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ICG PhD Lectures, November 2021

1. Introduction to Big-Bang cosmology	15 November
2. Hot thermal Universe	19 November
3. *Inflation*	*22 November*
4. Dark energy	26 November

Plan for today

1. *Problems of Hot Big Bang*
2. *(Single field) inflation, slow roll, reheating*
3. *Cosmological perturbations*
4. *Inflationary observables*

References

- *Inflation*, John Ellis and David Wands, in *Review of Particle Physics*.
- *Inflation and the theory of cosmological perturbations,* Antonio Riotto (hep-ph/0210162).
- Past lecture notes by Vincent Vennin and Matteo Fasiello.

- Inflation: Book suggestions
  - "Cosmological Inflation and Large-Scale Structure," Lyth and Liddle 2000
  - "Physical Foundations of Cosmology," Mukhanov 2005
  - "Primordial Cosmology," Peter and Uzan 2009
  - "Introduction to the Theory of the Early Universe," Gorbunov and Rubakov 2011

Big Bang in a nutshell

- Gravitation described by ***General Relativity***

$$G_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Reduced Planck mass

$$M_{Pl} \equiv \frac{1}{\sqrt{8\,\pi\,G_N}}$$

- At large distance scales the Universe is approximately homogeneous and isotropic

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - K\,r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]$$

- Friedmann equation

$$3\,M_{Pl}^2 H^2 = \sum_n \rho_n$$

Non-relativistic matter *(baryons, dark matter)*	$\rightarrow$	$P_m = 0\,,$	$\rho_m \propto a^{-3}\,,$	$a \propto t^{2/3}\,,$	$H = \frac{2}{3\,t}$
Relativistic matter and photons	$\rightarrow$	$P_r = \rho_r/3\,,$	$\rho_r \propto a^{-4}\,,$	$a \propto t^{1/2}\,,$	$H = \frac{1}{2\,t}$
Curvature	$\rightarrow$	$P_K = -\rho_K/3\,,$	$\rho_K \propto a^{-2}\,,$	$a \propto t\,,$	$H = \frac{1}{t}$
Cosmological constant	$\rightarrow$	$P_\Lambda = -\rho_\Lambda\,,$	$\rho_\Lambda \propto a^0\,,$	$a \propto \mathrm{e}^{Ht}\,,$	$H = \text{constant}$

Big Bang in a nutshell

- Hot Big Bang model predicts the cosmic microwave background and describes how elements are formed.
- However, there are several missing pieces. Initial singularity problem, cosmological constant problem, coincidence problem... Jury is still out.
- In this lecture, we will consider three other problems of Hot Big Bang that *can* be addressed by an early period of accelerated expansion:
  1. *Flatness problem*
  2. *Dangerous relics problem*
  3. *Horizon problem*

Problems of Hot Big Bang model

1. Flatness Problem

- Let's differentiate the definition of effective density function for curvature

$$\rho_K = -3\,K\,\frac{M_{Pl}^2}{a^2}$$
$$\rho_c = 3\,H^2\,M_{Pl}^2$$

$$\Omega_K \equiv \frac{\rho_K}{\rho_c} = -\frac{K}{\dot{a}^2} \xrightarrow{d/dt} \frac{d}{d\,t}\,\Omega_K = -2\,\frac{\ddot{a}}{\dot{a}}\Omega_K$$

- For a decelerating ($\ddot{a} < 0$) expansion ($\dot{a} > 0$), the curvature density always grows!
- Planck bound: $|\Omega_{K,0}| < 0.005$. To be within this small constraint today, curvature needs to be even smaller in the past.

$$|\Omega_{K,0}| < 0.005 \longleftarrow |\Omega_{K,LSS}| \lesssim 10^{-5} \longleftarrow |\Omega_{K,BBN}| \lesssim 10^{-19} \longleftarrow |\Omega_{K,Pl}| \lesssim 10^{-62}$$

- A wild coincidence! Way out? Universe exactly flat, i.e. $K = 0$? But how?
- This instability of curvature is interrupted if the Universe accelerates ($\ddot{a} > 0$).

Problems of Hot Big Bang model

2. Dangerous Relics Problem

- At early times, heavy exotic particles can be produced. If stable, they can spoil the thermal history. Eg. t*opological defects* from GUT*, *gravitinos* from SUGRA**...

*GUT: Grand Unified Theory, model that unifies Standard Model interactions at ~ 10^{15}GeV
**SUGRA: Supergravity, the gravitational version of supersymmetry.

- *Magnetic monopoles*: theorized by Dirac (1931), experimentally searched since 70s. Can be produced as topological defects in the early universe from phase transitions after symmetry breaking, via the Kibble mechanism (Kibble 1976).

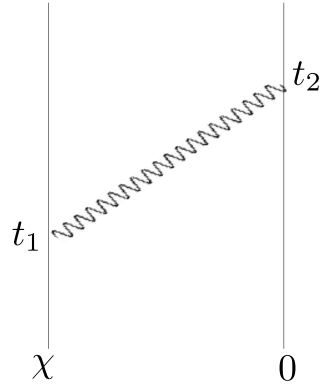
  SU(5) (a GUT) → SU(3)xSU(2)xU(1) (Standard Model)

  Prediction of their present-day abundance (Preskill 1979) many orders of magnitude larger than current bounds. Rough estimate: 1 monopole/nucleon. Experimental bounds 10^{-29} monopoles/nucleon (Jeon, Longo 1995).

- Way out: No grand unification? Universe never reaches the temperatures where the symmetry is restored?

Problems of Hot Big Bang model

3. Horizon Problem *[homogeneity problem]*



- First let us answer the question "*How far can a photon go in a given time?*"

Null geodesics

$$\chi = \int_{t_1}^{t_2} \frac{dt}{a}$$

- **Particle horizon:** What portion of the Universe can be observed at the origin at time t?

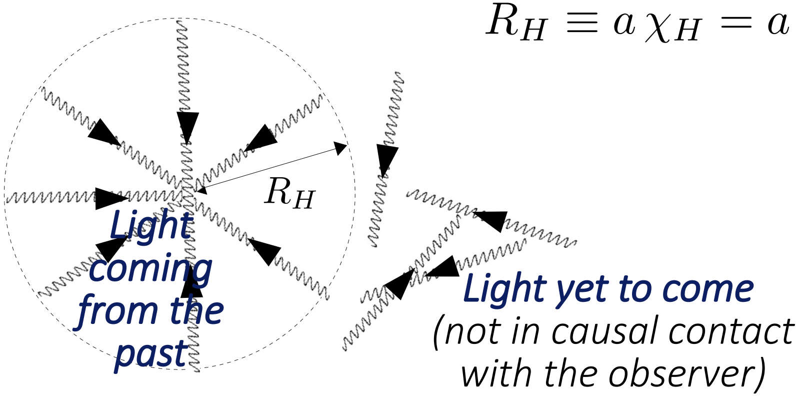
Assumed MD.
$a \propto t^{2/3}$
$H = \dfrac{2}{3\,t}$

$$R_H \equiv a\,\chi_H = a \int_0^t \frac{dt}{a} = t^{2/3} \int_0^t dt\, t^{-2/3}$$

$$= 3\,t$$

$$= \frac{2}{H}$$

Particle horizon at a given time (MD)



Problems of Hot Big Bang model

3. Horizon Problem

- The largest distance we can observe today is the particle horizon $R_0 = 2/H_0$.

  This is the furthest distance the light travelled to reach us. Regions beyond this has never been in causal contact with us.
- CMB was emitted at LSS, where the horizon size was $R_{LSS} = 2/H_{LSS}$ which today corresponds to

$$R_{LSS,0} = 2z_{LSS}/H_{LSS} \sim R_0/\sqrt{z_{LSS}}$$

Used: $H \propto a^{-3/2}$

- Can fit 30,000 causally disconnected spheres with radius $R_{LSS,0}$ into the observable universe today!
- How can *non-interacting free photons* from these regions can have the **same temperature that differs only by 1:100000?**
- No way out within the Hot Big Bang model!

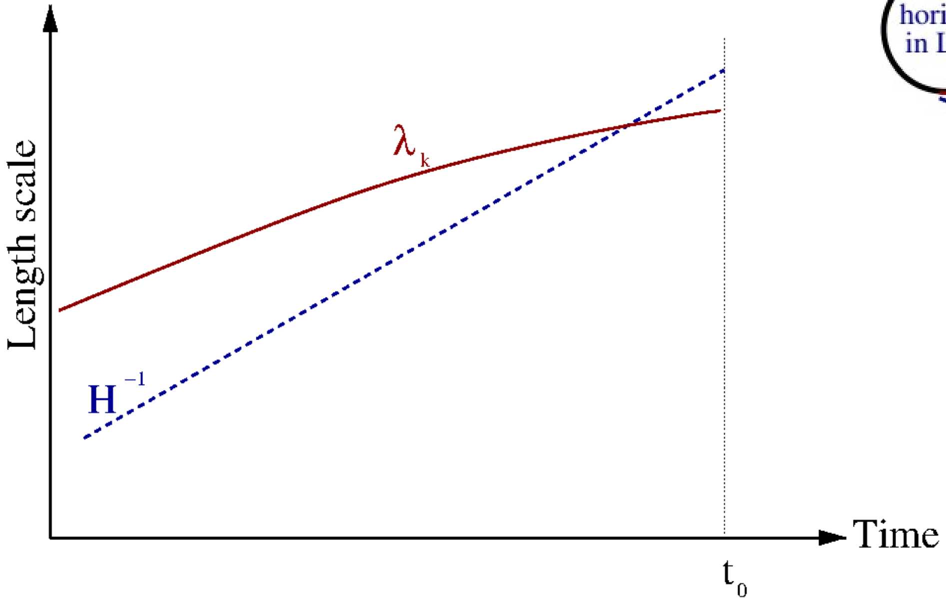
Problems of Hot Big Bang model

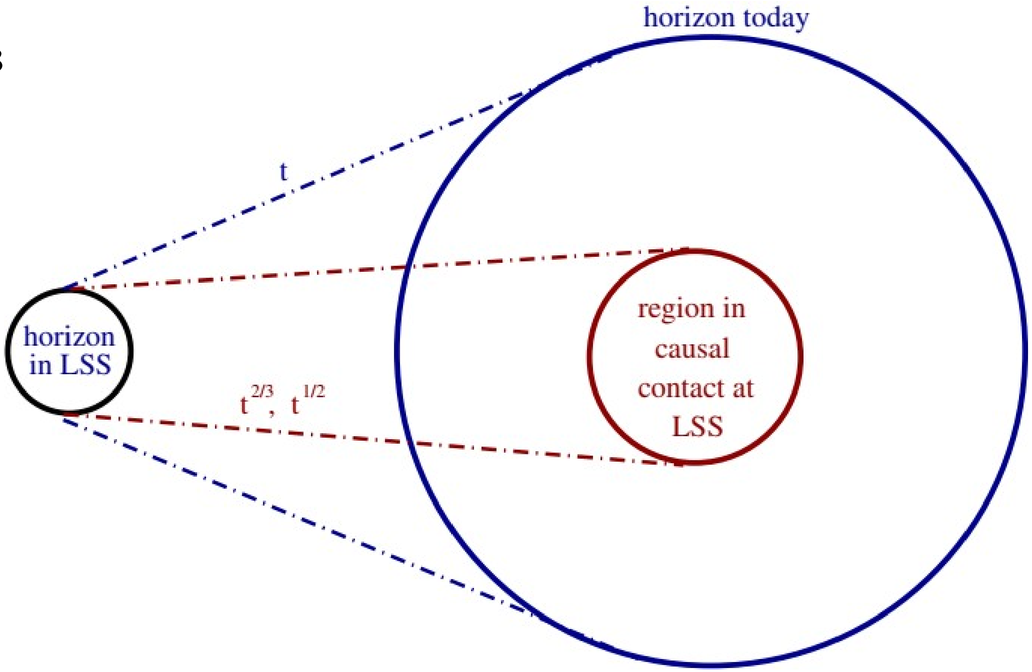
3. Horizon Problem

Physical distances grow as $a \propto t^{2/3}$

Horizon grows as $H^{-1} \propto t$

- The Universe expands slower than the horizon.





Problems of Hot Big Bang model

3. Horizon Problem

- If we have a new dominant fluid with equation of state w_i, the scale factor driven by it is

$$a \propto t^{\frac{2}{3(1+w_i)}}$$

- We need at least linear growth to solve the horizon problem

$$\frac{2}{3(1+w_i)} > 1 \quad \Rightarrow \quad w_i < -\frac{1}{3}$$

- For $w_i < -\frac{1}{3}$ we have $\ddot{a} > 0$. If there is an ***early stage of accelerated expansion*** before Hot Big Bang, we can explain how the CMB photons could be in causal contact at the time of last scattering!

INFLATION

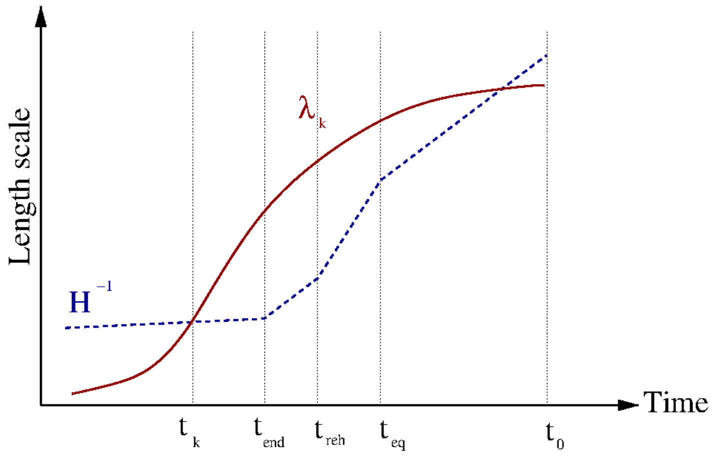
Problems of Hot Big Bang model

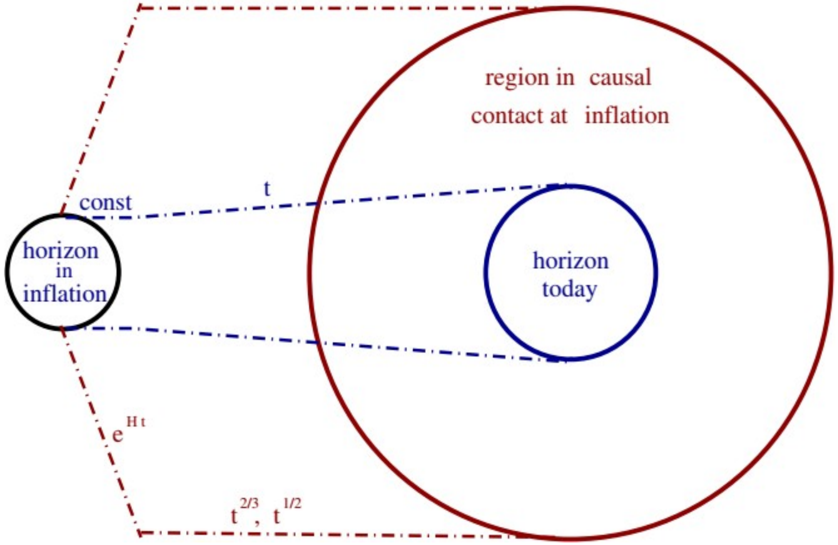
Resolution of BB Problems with Inflation:

During inflation:

distances grow as $a \propto \mathrm{e}^{H\,t}$

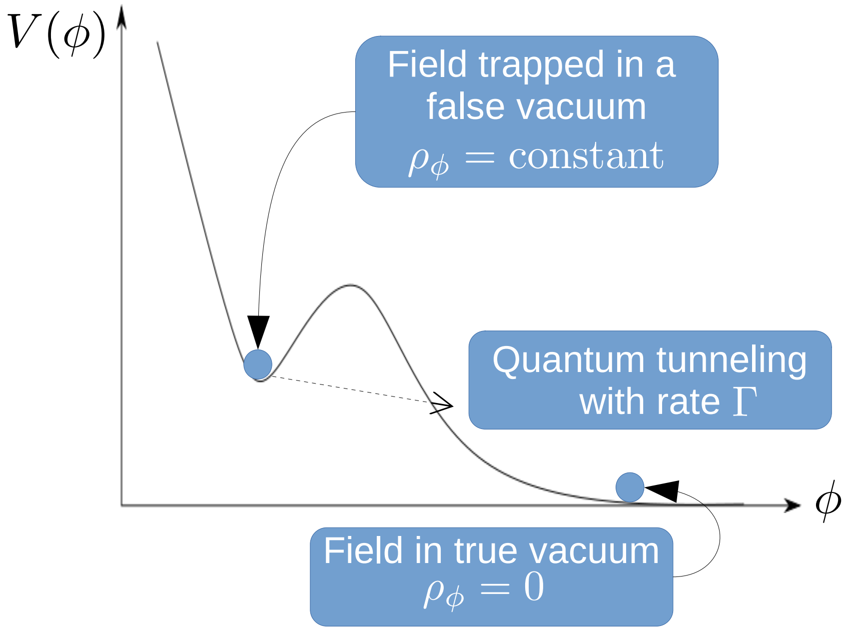
horizon is constant $H^{-1} \sim \mathrm{const}$

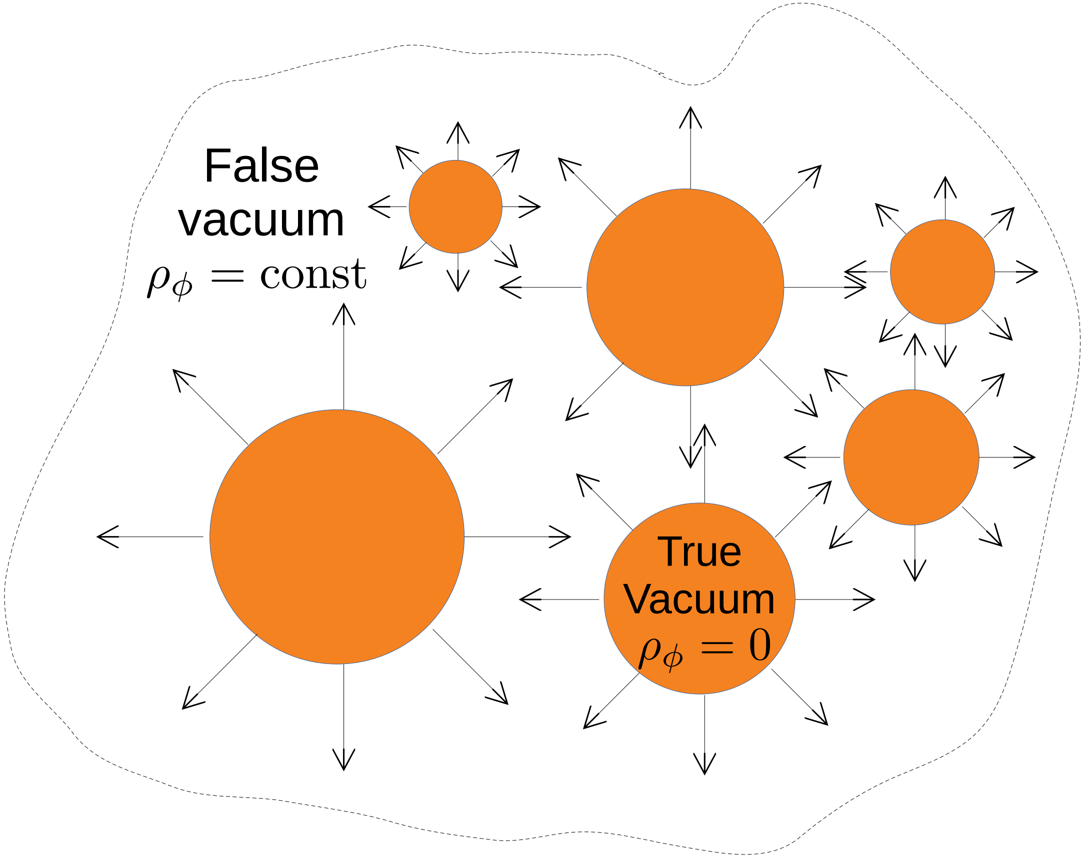




- $\Omega_K \propto \mathrm{e}^{-2\,H\,t}$ curvature quickly decays.
- $n_{\mathrm{relics}} \propto \mathrm{e}^{-3\,H\,t}$ pre-existing relics are quickly diluted. New ones not created if hot big bang starts at temperatures lower than the symmetry breaking energy.

Old Inflation





- In parts of the universe the field tunnels to true vacuum state.
- True vacuum energetically favourable, region expands.
- Many bubbles nucleate. End of inflation triggered by bubble collision. Particle production, thermalisation: ***reheating***. Then standard Big Bang cosmology.
- But there is a problem….

Graceful exit problem

- In order to solve horizon problem, inflation should last long enough to homogenise the universe. Tunnelling should not be too frequent:

$$\Gamma < H$$

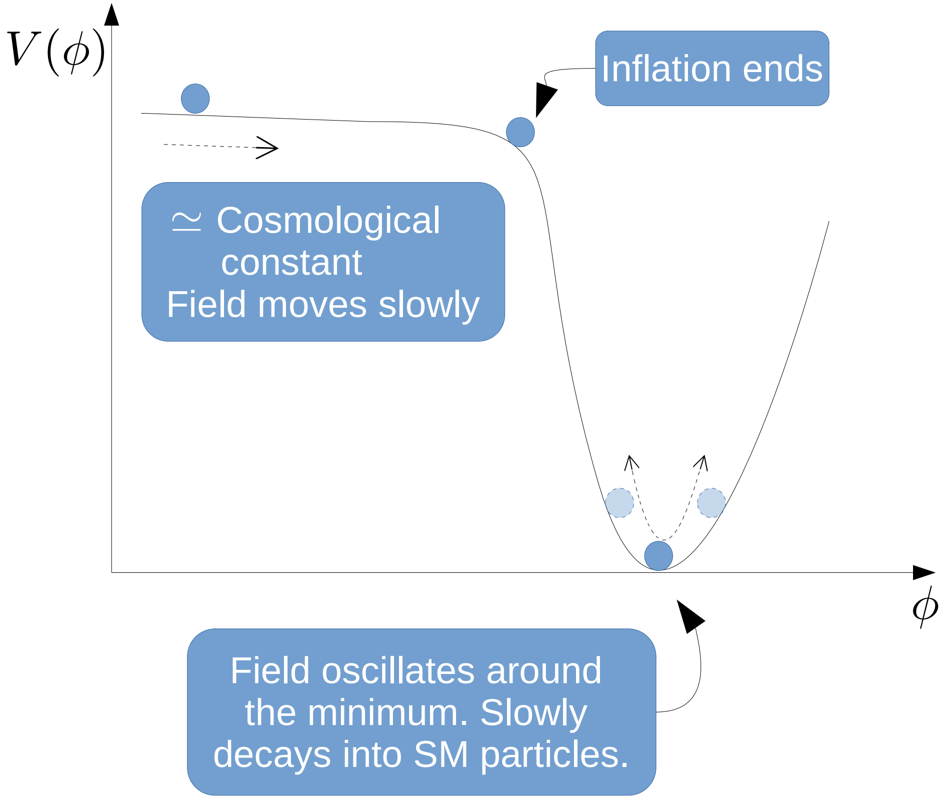
- But for $\Gamma < H$ bubbles diluted quickly since false vacuum continues to inflate. Cannot reheat enough. Inflation never ends!

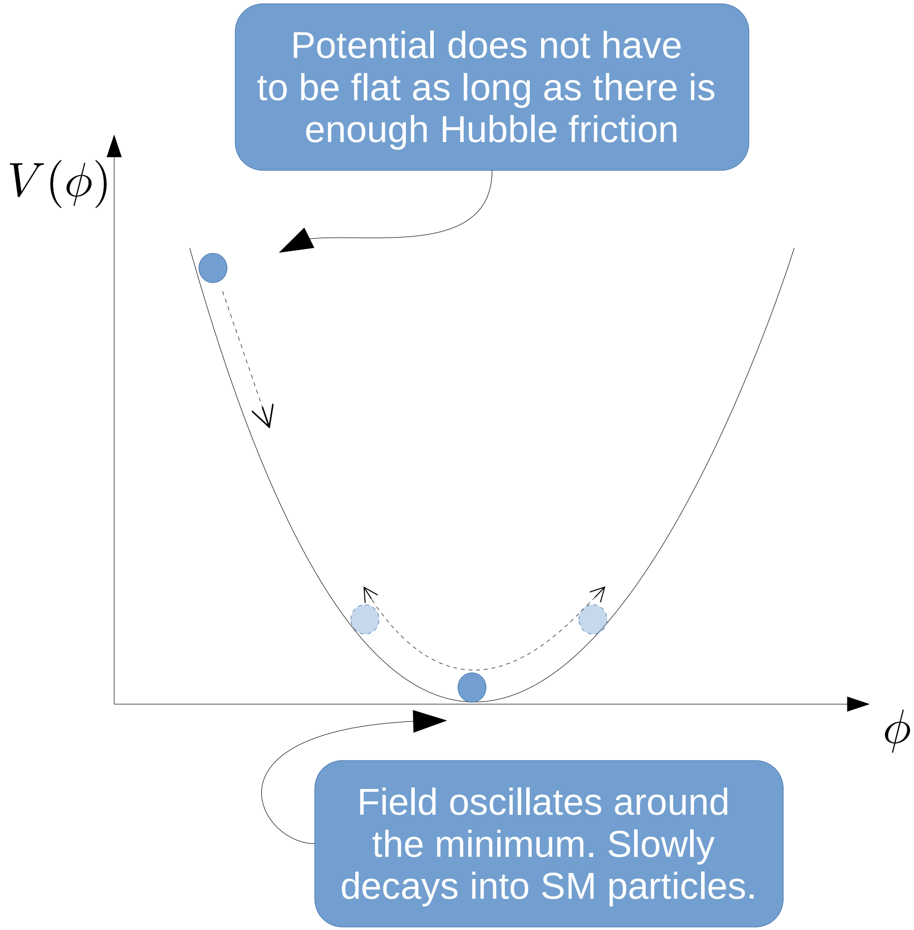
Guth, Weinberg 1983

- Resolution: Continuous transition via *slow roll*.

Linde 1982
Albrecht, Steinhardt 1982

Rolling down slowly





Scalar field cosmology

- Scalar action $$S_\phi = \int d^4x\,\sqrt{-g}\mathcal{L}_\phi = \int d^4x\,\sqrt{-g}\left[\underbrace{-\frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi}_{\textit{kinetic term}} - \underbrace{V(\phi)}_{\textit{potential}}\right]$$
- Scalar equation of motion $\delta S_\phi/\delta\phi = 0$ (*Klein-Gordon*) $$\nabla_\mu\nabla^\mu\phi + V'(\phi) = 0$$
- Energy-momentum tensor $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\,\delta S_\phi/\delta g^{\mu\nu}$ $$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi$$
- EOM as continuity: $$\nabla_\mu T^{\mu\nu} = 0$$
- For FLRW, uniform field $\phi = \phi(t)$ $$\rho_\phi = \frac{1}{2}\,\dot{\phi}^2 + V\,, \qquad P_\phi = \frac{1}{2}\,\dot{\phi}^2 - V$$

Analoguous to a fluid with
$$\rho = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi + V(\phi)$$
$$P = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$
$$u_\mu = \frac{\partial_\mu\phi}{\sqrt{-\partial_\alpha\phi\,\partial^\alpha\phi}}$$

For a slow moving field

$$w_\phi = \frac{P_\phi}{\rho_\phi} \simeq -1$$

Approx. cosmological constant

Scalar field cosmology

Curvature decays exponentially during inflation. We will neglect it from now on

- Einstein's equations

$$H^2 = \frac{1}{3\,M_{Pl}^2}\left(\frac{\dot\phi^2}{2}+V\right) - \cancel{\frac{K}{a^2}}$$

$$\dot H = -\frac{\dot\phi^2}{2\,M_{Pl}^2} + \cancel{\frac{K}{a^2}}$$

- Klein-Gordon equation

$$\ddot\phi + 3\,H\dot\phi + V'(\phi) = 0$$

V' drives the evolution

$3\,H\,\dot\phi$ causes damping from expansion

- *Hubble slow-roll parameter*

$$\epsilon_H \equiv -\frac{\dot H}{H^2}$$

WARNING! There are different conventions for slow roll parameters! See next slide.

- For acceleration, $\ddot a = a\,H^2\,(1-\epsilon_H) > 0$ ⇒ $\epsilon_H < 1$. From e.o.m:

$$\epsilon_H = \frac{3\,\dot\phi^2}{2\,V + \dot\phi^2}$$

For inflation, we need:
$V > \dot\phi^2$

Slow-roll approximation

$$\ddot{\phi} + 3\,H\dot{\phi} + V'(\phi) = 0$$

Friction and potential balance each other

- Assumption: $V' \simeq -3\,H\,\dot{\phi} \;\Longleftrightarrow\; \ddot{\phi} \ll H\,\dot{\phi}$

- *Potential slow-roll parameters*

$$\epsilon = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2 , \qquad \eta = M_{Pl}^2\,\frac{V''}{V}$$

- Slow-roll conditions: $\epsilon \ll 1\,,\;\; |\eta| \ll 1$

- Using equations of motion (for $V' > 0$)

$$\dot{\phi} \simeq -\sqrt{2\,\epsilon}\,H\,M_{Pl}\,, \quad \dot{H} \simeq -\epsilon\,H^2\,, \quad \ddot{\phi} \simeq \sqrt{2}\,H^2\,M_{Pl}\sqrt{\epsilon}(\eta-\epsilon)$$

- $\ddot{\phi}$ further suppressed by slow roll approximation.

- $\epsilon \simeq \epsilon_H$ at leading order in slow roll parameters.

- $H = H_* + \mathcal{O}(\epsilon)$ with constant H_* . Exponential expansion: $a \propto \mathrm{e}^{H_* t}$

Duration of inflation in e-foldings

- How much inflation do we need to solve the horizon problem? Start at t_*, end at t_{end}
  Distance travelled by photon during this time (as the proper distance today), is

$$R_{inf} = a_0 \int_{t_*}^{t_{end}} \frac{dt}{a} = \frac{a_0}{a_{end}} \int_{t_*}^{t_{end}} dt\, \mathrm{e}^{-H_*(t-t_{end})} = -\frac{a_0}{a_{end} H_*} \mathrm{e}^{-H_*(t-t_{end})} \Big|_{t_*}^{t_{end}}$$

Used: $a = a_{end} \mathrm{e}^{H_*(t-tend)}$

$$= \frac{a_0}{a_{end}\, H_*} \left(\mathrm{e}^{N_*} - 1\right)$$

e-foldings $N_* \equiv \log\left(a_{end}/a_*\right)$

Duration of inflation in terms of logarithmic expansion.

- This region should be larger than the horizon today:

$$\frac{a_0}{a_{end}\, H_*} \left(\mathrm{e}^{N_*} - 1\right) > \frac{1}{H_0}$$

- Duration of inflation depends on the scale of inflation. Rough estimate*:

$$N_* \gtrsim 34 + \ln\left(\frac{H_*}{\mathrm{TeV}}\right)$$

- $H_* = \text{few TeV}$ ⇒ $N_* \gtrsim 40$
- $H_* = \Lambda_{GUT} = 10^{15}\mathrm{GeV}$ ⇒ $N_* \gtrsim 60$

(*) assumed MD after inflation until today

$$a_0/a_{end} \sim \left(H_*/H_0\right)^{2/3}$$

Duration of inflation

- We can write the number of e-foldings as field displacement

$$N_* = \ln\left(\frac{a_{end}}{a_*}\right) = \int_{a_*}^{a_{end}} \frac{da}{a} = \int_{t_*}^{t_{end}} H\,dt$$

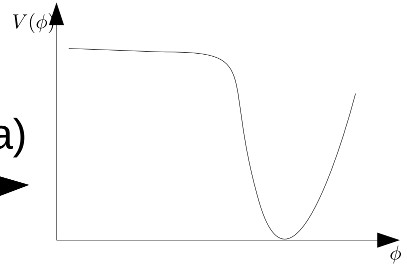
$$= \int_{\phi_*}^{\phi_{end}} \frac{H}{\dot\phi}\,d\phi \simeq -\int_{\phi_\star}^{\phi_{end}} \frac{1}{\sqrt{2\,\epsilon}}\,\frac{d\,\phi}{M_{Pl}}$$

Used:
$\dot\phi \simeq -\sqrt{2\,\epsilon}\,H\,M_{Pl}$
and $\dot\epsilon = \mathcal{O}(\epsilon^2)$

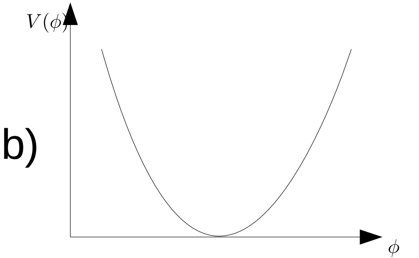
$$\boxed{N_* \simeq \frac{1}{\sqrt{2\,\epsilon}}\,\frac{\Delta\phi}{M_{Pl}}}$$

- To solve the problems of the Hot Big Bang, we need either
  a) very slow roll $\epsilon < 0.01$, *i.e.* very flat potential;
  b) a *large field displacement* $\Delta\phi > M_{Pl}$

a)



b)



End of inflation

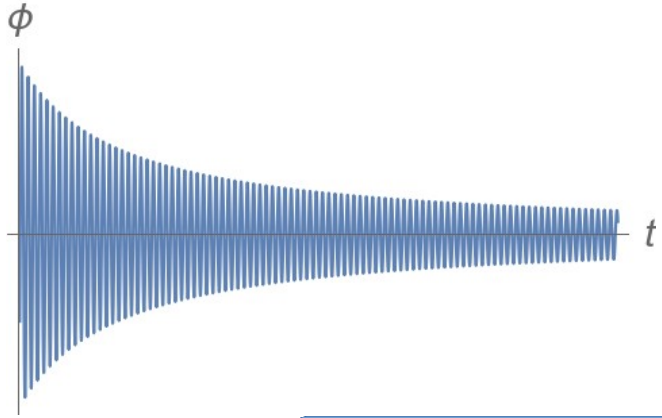
- Inflation ends when $\epsilon,\ \eta \sim \mathcal{O}(1)$. Field quickly loses its potential energy.
- The field still has kinetic energy, so it oscillates around the minimum of potential. For small values:

$$V(\phi) = V(0) + V'(0)\phi + \frac{1}{2}V''(0)\,\phi^2 + \mathcal{O}(\phi^3)$$

  This is a minimum point so $V'(0) = 0$. We take $V(0) = 0$ and $m^2 \equiv V''(0)$.

- After inflation $\phi < M_{Pl}$ and $H < m$, we have

$$\phi \simeq \frac{\sqrt{8}\,M_{Pl}}{\sqrt{3}}\,\frac{\sin m\,t}{m\,t}$$



- Averaging over the oscillations,

$$\left\langle \frac{\dot{\phi}^2}{2} \right\rangle \simeq \left\langle \frac{m^2\phi^2}{2} \right\rangle = \frac{2\,M_{Pl}^2}{3\,t^2} \Longrightarrow \langle\rho\rangle \simeq \frac{4\,M_{Pl}^2}{3\,t^2}\,,\ \langle P\rangle \simeq 0$$

Non-relativistic fluid behaviour on average

From inflation to Big Bang

Inflaton decays ➡ decay products thermalise (**REHEATING**) ➡ Hot Big Bang proceeds.

❶ Perturbative decay, e.g. through $\mathcal{L}_{int} = -M\,\phi\,\chi^2 - \lambda\,\phi\,\bar\psi\,\psi$

χ Boson, with $[M] = +1$
ψ Fermion, with $[\lambda] = 0$

Decay into light particles, with rates:

$$\Gamma_{\phi\to\chi\chi} = \frac{M^2}{8\,\pi\,m}\,,\quad \Gamma_{\phi\to\psi\bar\psi} = \frac{\lambda^2 m}{8\,\pi}$$

Decay products can reach thermal equilibrium after $\Gamma_{tot} > H$.

$$\Gamma_{tot} \sim \sqrt{\frac{g_{eff}\,\pi^2\,T_{rh}^4}{90\,M_{Pl}^2}} \quad \Longrightarrow \quad T_{rh} \sim 0.5\left(\frac{100}{g_{eff}}\right)^{1/4}\sqrt{\Gamma_{tot}\,M_{Pl}}$$

❷ ***Preheating:*** Non-perturbative decay during inflation oscillations, e.g. $\mathcal{L}_{int} = -\lambda^2\phi^2\chi^2$

Limiting case: decays at the end of inflation, instantly thermalise $\to \Gamma_{tot} \sim H_*$
Using upper bound on H_* from CMB normalisation (see later) ➡ $T_{rh} < 10^{16}\mathrm{GeV}$

❸ No direct decay channel? Gravitational decay ➡ $T_{rh} \sim 10^8 - 10^{10}\mathrm{GeV}$

Davidson & Sarkar 2000

Reheating after inflation

- Reheating is the least known stage in the Universe. Specifically, we do not know:
  - ➔ The energy scale of inflation
  - ➔ What particle(s) is (are) responsible for inflation
  - ➔ How inflaton interacts with the Standard Model fields

- Lower limit? Reheating should end before BBN $T_{rh} > 1\mathrm{MeV}$ (*very* conservative!)
- We can improve on the upper limit $T_{rh} < 10^{16}\mathrm{GeV}$ with new physics (e.g. in SUGRA, to prevent over-production of gravitinos that spoils BBN, $T_{rh} \lesssim 10^5 - 10^9\mathrm{GeV}$)

Kawasaki et al 2008

- Unknown physics of reheating leads to an uncertainty in N for a mode with wave number k

$$N(k) = 62 - \log\frac{k}{a_0 H_0} - \log\frac{10^{16}\mathrm{GeV}}{V_k^{1/4}} + \log\frac{V_k^{1/4}}{V_{end}^{1/4}} - \frac{1}{3}\log\frac{V_{end}^{1/4}}{\rho_{rh}^{1/4}}$$

Liddle, Lyth 2000
Kinney, Riotto 2005
…

$\mathcal{O}(10^{19})$ uncertainty in reheating translates into $\Delta N \sim 15$.

Signatures of inflation

Not part of this lecture

- A dramatic outcome of inflation: origin of large scale structure.
- Quantum fluctuations in the geometry+inflaton field are stretched to classical sizes.
- After inflation these perturbations are transferred to matter. After horizon re-entry, over-densities grow.
- Inflationary paradigm makes very precise predictions that can be probed with CMB observations.

Part of this lecture

Inflation pioneers

Alan Guth
American particle physicist. Developed *inflation* to address horizon and flatness problems (1981). The *old inflation model* is now abandoned due to the *graceful exit* problem.

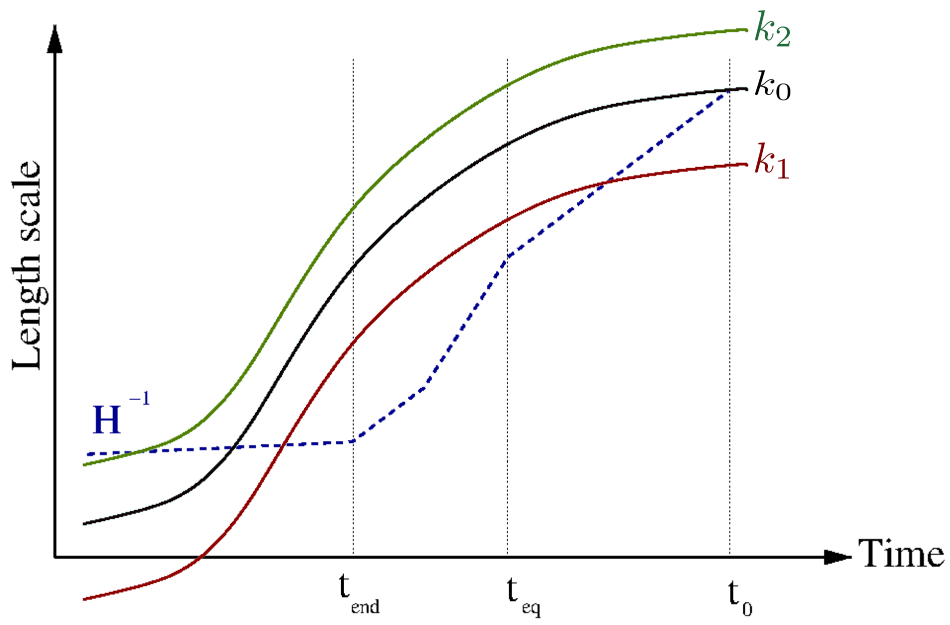
Alexei Starobinsky
Russian physicist. Independently developed the R^2 model of inflation (1980). The model is favoured by the current CMB data.

Andrei Linde
Russian-American physicist. Resolved the *graceful exit problem* by introducing *new inflation (1982).*

Paul Steinhardt & Andreas Albrecht
Independently proposed the *new inflation* model (1982).

Evolution of Scales

- Distances grow as scale factor, so physical wavelengths go as $\lambda \propto a$.
- For the discussion of perturbations, we will use the comoving wavenumber k to identify specific modes. Physical wavenumber, or *momentum* is thus $k/a \propto 1/\lambda$.
- Horizon crossing: $k/a = H$ ($\lambda \sim 1/H$). Evolution of modes is classified wrt horizon.



- $k_2 < k_0$: Mode yet to cross horizon *(longer length scale)*
- $k_0 = a_0 H_0$: Mode crossing horizon today
- $k_1 > k_0$: Mode crossed horizon in the past *(shorter length scale)*

→ ***Longer*** wavelengths (small *k*/large scale) leave the horizon ***early***, re-enter late.
→ ***Shorter*** wavelengths (large *k*/small scale) leave the horizon ***late***, re-enter early.

Cosmological Perturbations

$$x^\mu = (t, x^i)$$
$$i, j = 1, 2, 3$$

- Metric perturbations

$$g_{\mu\nu}(t, x^i) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, x^i)$$

- Inflaton perturbations

$$\phi(t, x^i) = \phi^{(0)}(t) + \delta\phi(t, x^i)$$

- Total of 11 perturbations:
  10 from metric (4×4 symmetric matrix) + 1 from inflaton.
- Perturbations small compared to background (classical) parts $\phi^{(0)}$ and $g^{(0)}_{\mu\nu}$

$$\delta\phi \ll \phi^{(0)}\,, \qquad \delta g_{\mu\nu} \ll \sqrt{g^{(0)}_{\mu\mu}\, g^{(0)}_{\nu\nu}} \qquad \left(\text{No sum over } \mu\,,\,\nu\right)$$

- We will consider corrections to equations of motion at linear order in perturbations.
- Inflaton acts like a clock, so it breaks time reparameterisation symmetry of GR. But spatial rotational symmetry remains intact. We will decompose the 11 perturbations wrt to 3-rotations.

Counting the degrees of freedom

- Metric perturbations decomposed wrt spatial rotations

$$\delta g_{\mu\nu} = \begin{pmatrix} -2\,\Phi & a\,(B_i + \partial_i B) \\ a\,(B_j + \partial_j B) & a^2(-2\,\psi\delta_{ij} + \partial_i E_j + \partial_j E_i + \partial_i\partial_j E + h_{ij}) \end{pmatrix}$$

$$\phi(t, x^i) = \phi^{(0)}(t) + \delta\phi(t, x^i)$$

- 5 Scalars: $\delta\phi\,,\ \Phi\,,\ B\,,\ \psi\,,\ E$ 5
- 2 Vectors: $B_i\,,\ E_i$ (transverse: $\partial^i B_i = \partial^i E_i = 0$) 4
- 1 Tensor: h_{ij} (transverse/traceless $\partial^i h_{ij} = g^{(0)\,i\,j} h_{ij} = 0$) 2

BG invariant under spatial rotations: Scalar, vector and tensor perturbations decouple in the linear equations of motion

- Invariance under gauge transformations $x^\mu \to x^\mu + \xi^\mu(t, x^i)$ with $\xi^\mu = (\xi^0, \xi^i + \partial^i \xi)$
  Can remove 2 scalars and 1 vector (2) with gauge transformations (unphysical)
- 00 and 0i Einstein equations→ Hamiltonian & momentum constraints:
  2 scalars and 1 vector (2) physical, but not independent.

Physical and independent perturbations:
1 scalar (1)
1 tensor (2)

Cosmological Perturbations

- GR in vacuum (no matter) only 2 degrees of freedom (d.o.f.): 1 tensor (GW).
- Single field inflation adds a scalar d.o.f., but the physical perturbation also contains metric perturbations. The physical (gauge invariant) variable:

Scalar field perturbation in spatially flat gauge

$$Q \equiv \delta\phi + \frac{\dot\phi^{(0)}}{H}\psi$$

The curvature of spatial metric g_{ij}

$${}^{(3)}R = \frac{4}{a^2}\nabla^2\psi \Longrightarrow \psi : \text{“curvature perturbation”}$$

- Other gauge invariant variables are available:

$$\mathcal{R} \equiv \psi + \frac{H}{\dot\phi^{(0)}}\delta\phi = \frac{H}{\dot\phi^{(0)}}Q$$

Comoving curvature perturbation

$$\zeta \equiv \psi + \frac{H}{\dot\rho^{(0)}}\delta\rho$$

Curvature perturbation on uniform density hypersurfaces

During slow roll and on super-horizon scales

$$\zeta \simeq \mathcal{R}$$

- Curvature perturbation and scalar matter perturbations are coupled! Inflation generates super-horizon ζ . After inflation, these perturbations are transferred into photons, baryons and matter in general.

Adiabatic and Entropy Perturbations

- *Adiabatic (curvature) perturbations:* Perturbs along the background trajectory. For any scalar perturbation q , we can write the following

$$H\delta t = H\,\frac{\delta q}{\dot q}$$

A time displacement δt causes the same change in all quantities democratically. Example:

$$\frac{\delta\rho}{\dot\rho} = \frac{\delta P}{\dot P}$$

- *Entropy (isocurvature) perturbations:* Perturbs away from the background trajectory.

$$\frac{\delta q_1}{\dot q_1} \neq \frac{\delta q_2}{\dot q_2}$$

- Single field inflation always generates only adiabatic perturbations. Primordial entropy perturbations are severely constrained by CMB data.

Important result: $\zeta = \psi + \dfrac{H\,\delta\rho}{\dot\rho^{(0)}}$ *remains conserved after leaving the horizon ($k \ll a\,H$)* ***if*** *perturbations are purely* ***adiabatic****.*
→ Consequence of energy conservation

Initial conditions and power spectrum

- Perturbations start off as quantum fluctuations. We expand them in terms of harmonics. For scalar fields in a flat Universe, this is a plane-wave expansion:

$$q(t,x^i) = \int \frac{d^3k}{(2\,\pi)^{3/2}}\, \mathrm{e}^{i\,\vec{k}\cdot\vec{x}} \left(q_k(t) a_{\vec{k}} + q_k^\star(t) a^\dagger_{-\vec{k}}\right)$$

- $\hat{a}_{\vec{k}},\ \hat{a}^\dagger_{\vec{k}} \Rightarrow$ creation/annihilation operators: $[\hat{a}_{\vec{k}}\,,\hat{a}^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k}-\vec{k}')\,,\quad \langle 0|\hat{a}_{\vec{k}}\,\hat{a}^\dagger_{\vec{k}'}|0\rangle = \delta^{(3)}(\vec{k}-\vec{k}')$
- Above the horizon, quantum to classical transition. *see e.g. Polarsi, Starobinsky 1996*

Quantum (sub-horizon)	*Classical (super-horizon)*
creation/annihilation operators	*Gaussian random variables*
Vacuum expectation value	*Ensemble average*

- Initial conditions set by quantum vacuum. We know how to quantize harmonic oscillators in Minkowski space-time: $\frac{d^2\,q_k}{d\tau^2} + k^2 q_k = 0 \Rightarrow q_k(\tau) = \frac{1}{\sqrt{2\,k}}\mathrm{e}^{-i\,k\,\tau}$

Bunch-Davies vacuum

We will bring the perturbation equations to this form.

- Measure of variance: Power spectrum

$$\langle 0|q(\vec{x})^2|0\rangle = \frac{1}{(2\,\pi)^3}\int d^3k|q_k|^2 = \int d\,\log k\, P_q(k)$$

Power in fluctuations per log interval

$$\boxed{P_q(k) \equiv \frac{k^3}{2\,\pi^2}\,|q_k|^2}$$

Scalar Perturbations

- We expand perturbations in terms of Fourier modes (plane waves). Isotropic background: No dependence on direction of $\vec{k}$, only on its magnitude $k \equiv |\vec{k}|$
- After using the Hamiltonian and momentum constraints, we reduce the scalar equation of motion for the gauge invariant variable $Q \equiv \delta\phi + \frac{\dot{\phi}^{(0)}}{H}\,\psi$

$$\ddot{Q}_k + 3\,H\,\dot{Q}_k + \left[\frac{k^2}{a^2} + V'' - \frac{1}{M_{Pl}^2 a^3}\,\frac{d}{dt}\left(\frac{a^3\,\dot{\phi}^2}{H}\right)\right] Q_k = 0$$

- We now make this look like a harmonic oscillator in Minkowski. We can remove the expansion contribution by choosing conformal time $d\tau = dt/a$ and defining a new variable $v_k \equiv a\,Q_k$. The equation is:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0\,, \quad \text{where } z \equiv \frac{a\,\dot{\phi}}{H}$$

- In terms of slow roll parameters:

$$\frac{z''}{z} \simeq (2 + 5\,\epsilon - 3\,\eta) a^2 H^2$$

Scalar Perturbations

- Easier to calculate in de Sitter (dS), i.e. cosmological constant dominated universe (zero order in slow roll approximation)
- Simple expression for conformal time in dS $\tau = -\frac{1}{a\,H}$

$\tau \in (-\infty, 0)$
In slow roll, τ gets corrections $\mathcal{O}(\epsilon)$

- In this simplified case (neglecting the motion of inflaton), scalar equation becomes

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k \simeq 0\,,$$

- For $|k\,\tau| \gg 1$, looks like harmonic oscillator in Minkowski. These are sub-horizon modes $k \gg a\,H$, corresponding to earliest times. Ideal for using Bunch-Davies vacuum for initial conditions. The solution:

$$v_k = \frac{i - k\,\tau}{\sqrt{2\,k^3(-\tau)}}\,\mathrm{e}^{-i\,k\,\tau} = \begin{cases} \dfrac{\mathrm{e}^{-i\,k\,\tau}}{\sqrt{2\,k}} & , \quad |k\,\tau| \gg 1 \\ \dfrac{i}{\sqrt{2\,k^3(-\tau)}} & , \quad |k\,\tau| \ll 1 \end{cases}$$

$$\mathcal{R}_k\Big|_{|k\tau|\ll 1} = \frac{H}{\dot{\phi}^{(0)}}\frac{v_k}{a}\Big|_{|k\tau|\ll 1} = \frac{i\,H^2}{\dot{\phi}^{(0)}\sqrt{2\,k^3}} \Rightarrow \boxed{P_\zeta \simeq P_{\mathcal{R}} = \left(\frac{H}{\dot{\phi}^{(0)}}\right)^2 \left(\frac{H}{2\,\pi}\right)^2}$$

Scale invariant spectrum

Tensor Perturbations

- We expand tensor modes in terms of plane waves and polarization tensor e_{ij}^{λ}

$$h_{ij}(t, x^i) = \sum_{\lambda} \int \frac{d^3k}{(2\,\pi)^{3/2}}\, \mathrm{e}^{i\,\vec{k}\cdot\vec{x}} \left(h_{\lambda,\,k}(t) e_{ij}^{\lambda} a_{\vec{k}} + h_{\lambda,\,k}^{\star}(t) e_{ij}^{\lambda} a_{-\vec{k}}^{\dagger} \right)$$

- Using conformal time and redefining the field $u_{\lambda,\,k} \equiv a\, M_{Pl} h_{\lambda,\,k}/2$

$$u_{\lambda,\,k}'' + \left(k^2 - \frac{a''}{a} \right) u_{\lambda,\,k} = 0\,,$$

- On de Sitter, $a \simeq -1/(\tau H)$ so $a''/a \simeq 2/\tau^2$. Same as scalar at zeroth order.

$$h_{\lambda,\,k}\Big|_{|k\tau|\ll 1} = \frac{\sqrt{2}\, i\, H}{M_{Pl}\, k^{3/2}} \quad \Rightarrow \quad \boxed{P_h = \sum_{\lambda} P_{h_\lambda} = \frac{8}{M_{Pl}^2} \left(\frac{H}{2\,\pi} \right)^2}$$

Scale invariant spectrum

- The *tensor-to-scalar ratio*

$$\boxed{r \equiv \frac{P_h}{P_\zeta} = \frac{8\,\dot{\phi}^{(0)\,2}}{H^2\, M_{Pl}^2} \simeq 16\,\epsilon}$$

The ratio suppressed by slow roll parameter

Spectral tilts

- The two power spectra, after horizon crossing, without the slow-roll corrections:

$$P_\zeta = \left(\frac{H}{\dot{\phi}^{(0)}}\right)^2 \left(\frac{H}{2\,\pi}\right)^2 \qquad\qquad P_h = \frac{8}{M_{Pl}^2}\left(\frac{H}{2\,\pi}\right)^2$$

- Slow-roll shifts the time horizon crossing. Including corrections:

$$P_\zeta = \left(\frac{H}{\dot{\phi}^{(0)}}\right)^2 \left(\frac{H}{2\,\pi}\right)^2 \left(\frac{k}{a\,H}\right)^{n_s-1} \qquad\qquad P_h = \frac{8}{M_{Pl}^2}\left(\frac{H}{2\,\pi}\right)^2 \left(\frac{k}{a\,H}\right)^{n_t}$$

$n_s - 1 < 0$ Red

$n_s - 1 > 0$ Blue

- Scale time of horizon crossing. For $k = a_\star\, H_\star$, evaluate at $t_\star$

$d\log k \simeq H_\star dt_\star$

$$n_s - 1 = \frac{1}{P_{\zeta_\star}}\frac{d\,P_{\zeta_\star}}{d\,\log k} \simeq \frac{1}{H_\star\,P_{\zeta_\star}}\frac{d\,P_{\zeta_\star}}{d\,t_\star} = 4\,\frac{\dot{H}_\star}{H_\star^2} - 2\,\frac{\ddot{\phi}_\star^{(0)}}{H_* \dot{\phi}_\star^{(0)}} \simeq -6\,\epsilon + 2\,\eta$$

$$n_t = \frac{1}{P_{h_\star}}\frac{d\,P_{h_\star}}{d\,\log k} \simeq \frac{1}{H_\star\,P_{h_\star}}\frac{d\,P_{h_\star}}{d\,t_\star} = 2\,\frac{\dot{H}_\star}{H_\star^2} \simeq -2\,\epsilon$$

$$n_s = 1 - 6\,\epsilon + 2\,\eta$$
$$n_t = -2\,\epsilon$$

Almost scale invariant spectra

Predictions vs. observations

- $\boxed{P_\zeta = \frac{V}{24\,\pi^2 M_{Pl}^4 \epsilon}\left(\frac{k}{a\,H}\right)^{n_s-1}}$ $\boxed{n_s = 1 - 6\,\epsilon + 2\,\eta}$ $\boxed{n_t = -2\,\epsilon}$ $\boxed{r = 16\,\epsilon = -8\,n_t}$

- CMB temperature anisotropies at large scales are imprinted at the LSS (via Sachs-Wolfe effect) by the gravitational potential from primordial perturbations

$$P_\zeta \sim 2.1 \times 10^{-9}\,, \qquad n_s = 0.96$$

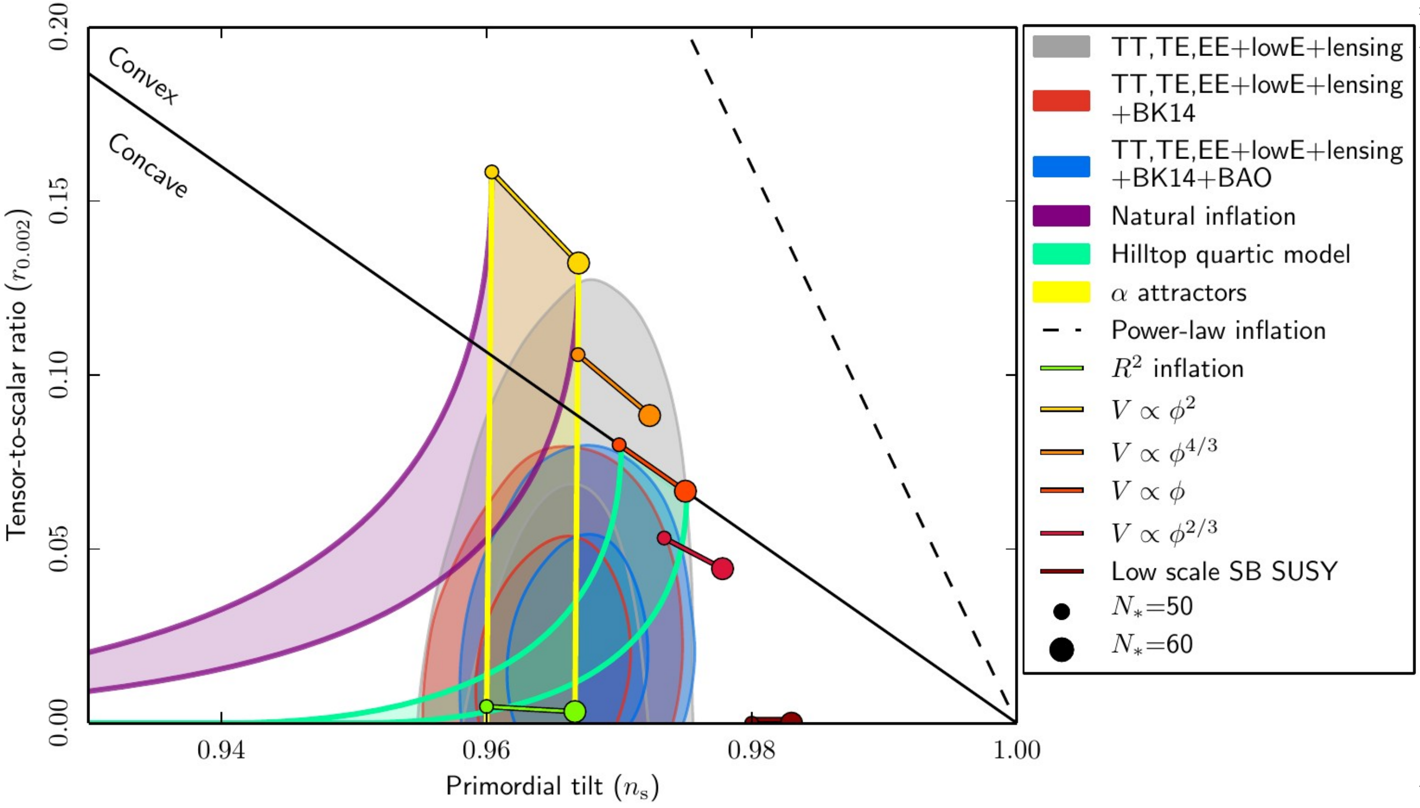
- Tensor modes weak, but can distinguish with CMB photons' B-mode polarization. No primordial B-mode has been observed. Bounds from Planck 2015 & BICEP2

$$r < 0.06$$

- With scalar amplitude, we can calculate $\left(\frac{V}{\epsilon}\right)^{1/4} \simeq 6.4 \times 10^{16}\mathrm{GeV}$. Without r, we do not know the scale of inflation. If observed, the relation $r = -8\,n_t$ would work as a consistency check.

- For models with small field displacement (i.e. small ϵ), this may be challenging.

Planck 2018 results



Inflationary model	Potential $V(\phi)$
$R + R^2/(6M^2)$	$\Lambda^4\left(1 - e^{-\sqrt{2/3}\phi/M_{\rm Pl}}\right)^2$
Power-law potential	$\lambda M_{\rm Pl}^{10/3}\phi^{2/3}$
Power-law potential	$\lambda M_{\rm Pl}^3\phi$
Power-law potential	$\lambda M_{\rm Pl}^{8/3}\phi^{4/3}$
Power-law potential	$\lambda M_{\rm Pl}^2\phi^2$
Power-law potential	$\lambda M_{\rm Pl}\phi^3$
Power-law potential	$\lambda\phi^4$
Non-minimal coupling	$\lambda^4\phi^4 + \xi\phi^2 R/2$
Natural inflation	$\Lambda^4\left[1 + \cos(\phi/f)\right]$
Hilltop quadratic model	$\Lambda^4\left(1 - \phi^2/\mu_2^2 + \ldots\right)$
Hilltop quartic model	$\Lambda^4\left(1 - \phi^4/\mu_4^4 + \ldots\right)$
D-brane inflation ($p = 2$)	$\Lambda^4\left(1 - \mu_{D2}^2/\phi^p + \ldots\right)$
D-brane inflation ($p = 4$)	$\Lambda^4\left(1 - \mu_{D4}^4/\phi^p + \ldots\right)$
Potential with exponential tails	$\Lambda^4\left[1 - \exp(-q\phi/M_{\rm Pl}) + \ldots\right]$
Spontaneously broken SUSY	$\Lambda^4\left[1 + \alpha_h \log(\phi/M_{\rm Pl}) + \ldots\right]$
E-model ($n = 1$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_1^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$
E-model ($n = 2$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_2^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$
T-model ($m = 1$)	$\Lambda^4 \tanh^{2m}\left[\phi\left(\sqrt{6\alpha_1^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$
T-model ($m = 2$)	$\Lambda^4 \tanh^{2m}\left[\phi\left(\sqrt{6\alpha_2^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$

Topics we did not have time for

- Gravity is unstable. Over-densities seeded by inflation that are within the particle horizon (but outside the sound horizon) grow thanks to Jeans instability during matter domination.
- Other observational probes from more contrived models (e.g. multi-field inflation): running of spectral index, non-Gaussian signatures...
- Problems of inflation:
  - Eternal inflation, initial conditions problem
  - Model building challenges: eta-problem, swampland conjecture, radiative corrections, trans-Planckian problem ...
- Alternatives to inflation, ekpyrotic/cyclic models, bouncing cosmologies...

Next time on *Theoretical Cosmology*

- What *is* the cosmological constant? Is it real or is it a convenient place-holder
- What assumptions does dark energy rely on? Which ones can we relax?
- What is modified gravity and its challenges?

Stay tuned for "the worst theoretical prediction in the history of physics" (Hobson et al 2006)