Theoretical Cosmology

Part III: Inflation

Emir Gümrükçüoğlu

ICG PhD Lectures, November 2021

1. Introduction to Big-Bang cosmology	15 November
2. Hot thermal Universe	19 November
3. *Inflation*	*22 November*
4. Dark energy	26 November

Plan for today

1. *Problems of Hot Big Bang*
2. *(Single field) inflation, slow roll, reheating*
3. *Cosmological perturbations*
4. *Inflationary observables*

References

- *Inflation*, John Ellis and David Wands, in *Review of Particle Physics*.
- *Inflation and the theory of cosmological perturbations,* Antonio Riotto (hep-ph/0210162).
- Past lecture notes by Vincent Vennin and Matteo Fasiello.

- Inflation: Book suggestions
  - "Cosmological Inflation and Large-Scale Structure," Lyth and Liddle 2000
  - "Physical Foundations of Cosmology," Mukhanov 2005
  - "Primordial Cosmology," Peter and Uzan 2009
  - "Introduction to the Theory of the Early Universe," Gorbunov and Rubakov 2011

Big Bang in a nutshell

- Gravitation described by ***General Relativity***

$$G_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Reduced Planck mass

$$M_{Pl} \equiv \frac{1}{\sqrt{8\,\pi\,G_N}}$$

- At large distance scales the Universe is approximately homogeneous and isotropic

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - K\,r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]$$

- Friedmann equation

$$3\,M_{Pl}^2 H^2 = \sum_n \rho_n$$

Non-relativistic matter *(baryons, dark matter)*	$\rightarrow$	$P_m = 0\,,$	$\rho_m \propto a^{-3}\,,$	$a \propto t^{2/3}\,,$	$H = \frac{2}{3\,t}$
Relativistic matter and photons	$\rightarrow$	$P_r = \rho_r/3\,,$	$\rho_r \propto a^{-4}\,,$	$a \propto t^{1/2}\,,$	$H = \frac{1}{2\,t}$
Curvature	$\rightarrow$	$P_K = -\rho_K/3\,,$	$\rho_K \propto a^{-2}\,,$	$a \propto t\,,$	$H = \frac{1}{t}$
Cosmological constant	$\rightarrow$	$P_\Lambda = -\rho_\Lambda\,,$	$\rho_\Lambda \propto a^0\,,$	$a \propto e^{Ht}\,,$	$H = \text{constant}$

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- However, there are several missing pieces. Initial singularity problem, cosmological constant problem, coincidence problem... Jury is still out.
- In this lecture, we will consider three other problems of Hot Big Bang that *can* be addressed by an early period of accelerated expansion:
  1. *Flatness problem*
  2. *Dangerous relics problem*
  3. *Horizon problem*

Problems of Hot Big Bang model

1. Flatness Problem

- Let's differentiate the definition of effective density function for curvature

$$\boxed{\begin{aligned} \rho_K &= -3\,K\,\frac{M_{Pl}^2}{a^2} \\ \rho_c &= 3\,H^2\,M_{Pl}^2 \end{aligned}}$$

$$\Omega_K \equiv \frac{\rho_K}{\rho_c} = -\frac{K}{\dot{a}^2}$$

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- This instability of curvature is interrupted if the Universe accelerates ($\ddot{a} > 0$).

Problems of Hot Big Bang model

2. Dangerous Relics Problem

- At early times, heavy exotic particles can be produced. If stable, they can spoil the thermal history. Eg. t*opological defects* from GUT*, *gravitinos* from SUGRA**...

*GUT: Grand Unified Theory, model that unifies Standard Model interactions at ~ 10^{15}GeV
**SUGRA: Supergravity, the gravitational version of supersymmetry.

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SU(5) (a GUT) → SU(3)xSU(2)xU(1) (Standard Model)

Prediction of their present-day abundance (Preskill 1979) many orders of magnitude larger than current bounds. Rough estimate: 1 monopole/nucleon. Experimental bounds 10^{-29} monopoles/nucleon (Jeon, Longo 1995).

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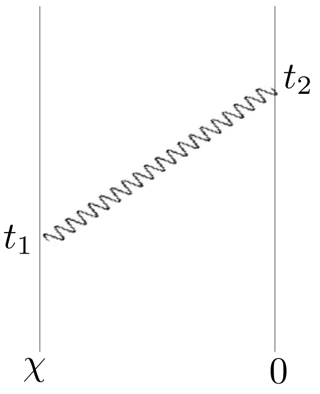
- Way out: No grand unification? Universe never reaches the temperatures where the symmetry is restored?

Problems of Hot Big Bang model

3. Horizon Problem *[homogeneity problem]*



- First let us answer the question *"How far can a photon go in a given time?"*

Null geodesics ➡ $$\chi = \int_{t_1}^{t_2} \frac{dt}{a}$$

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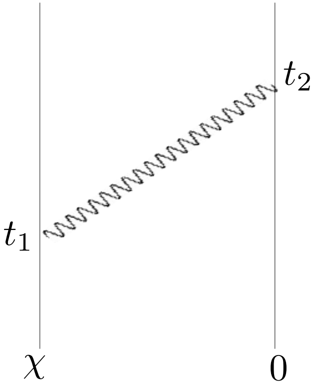

Null geodesics

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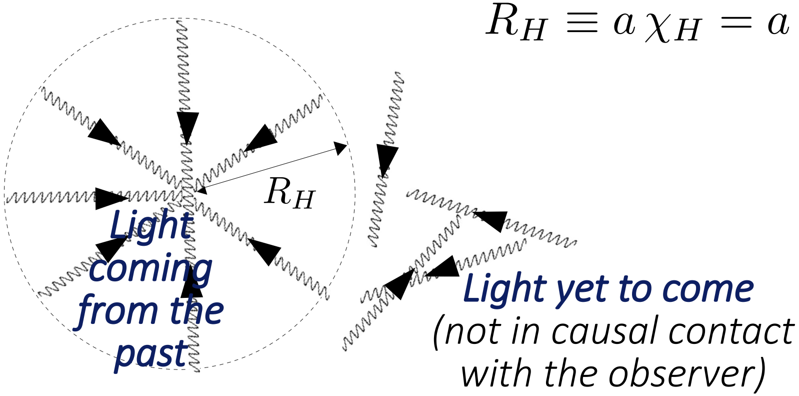

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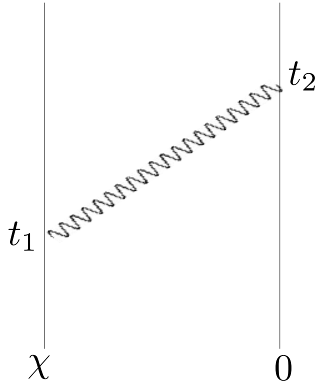

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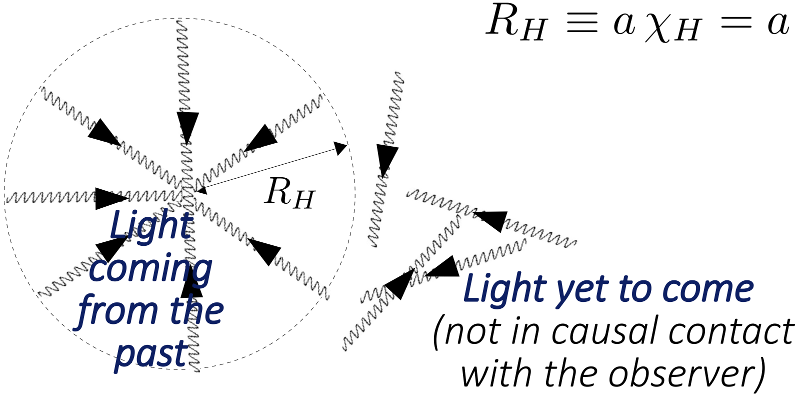

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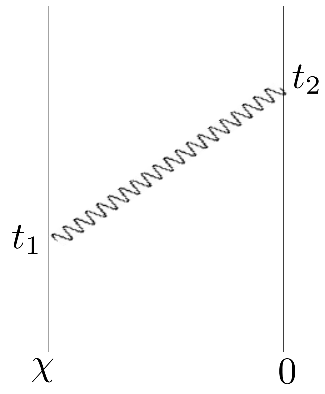

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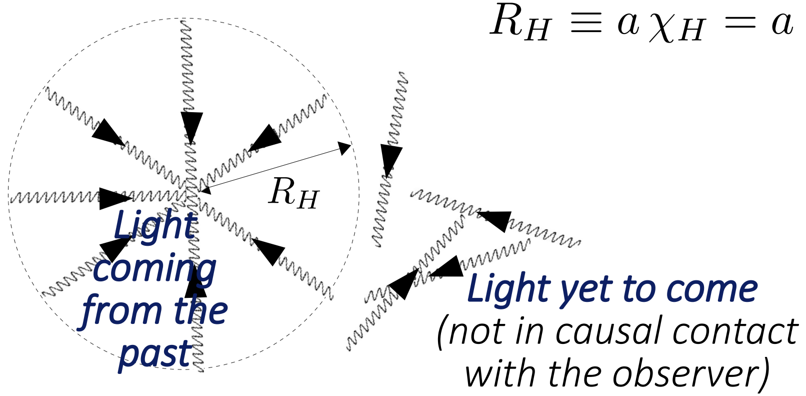

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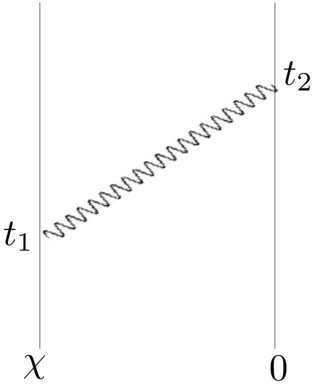

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Particle horizon at a given time (MD)

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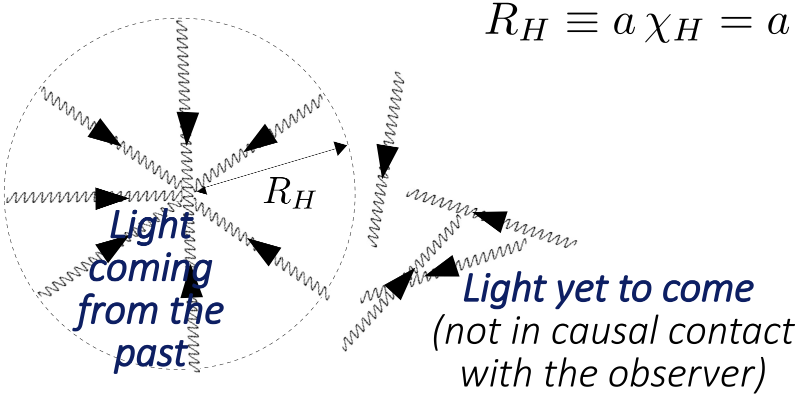

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- No way out within the Hot Big Bang model!

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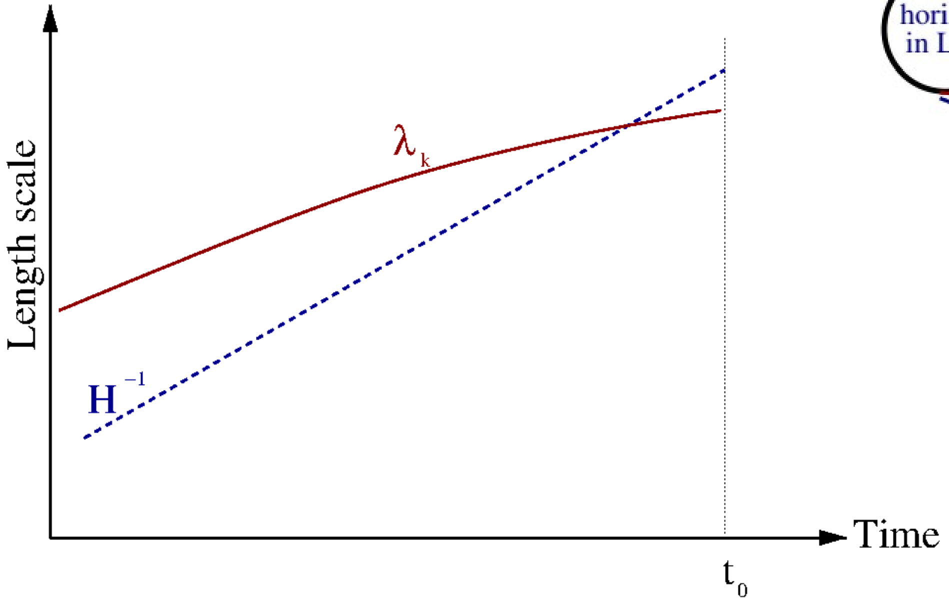
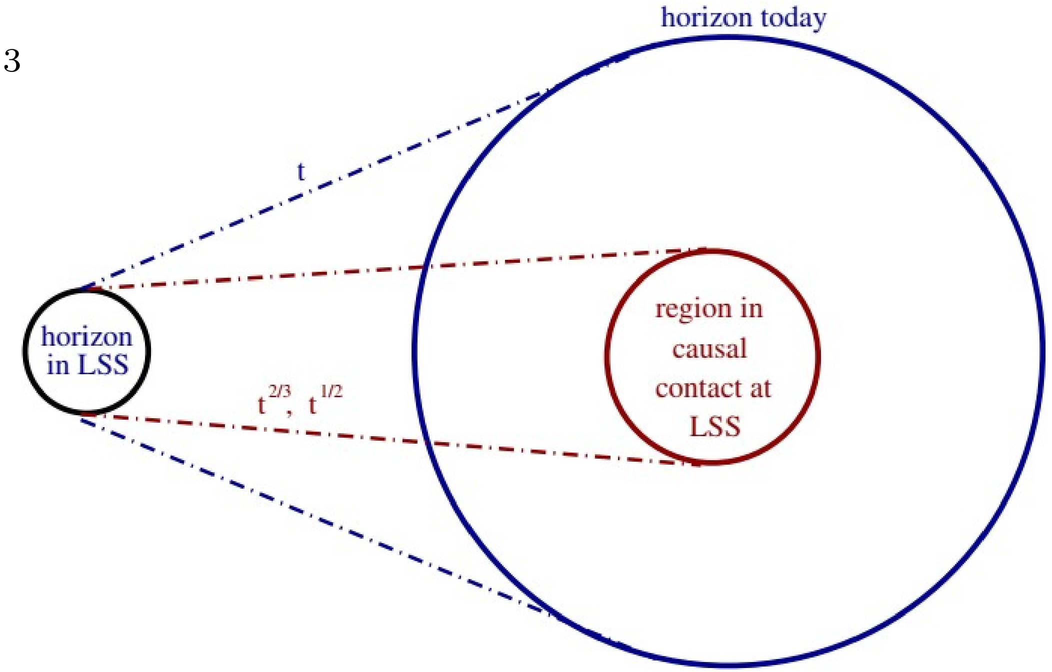
Physical distances grow as $a \propto t^{2/3}$

Horizon grows as $H^{-1} \propto t$

- The Universe expands slower than the horizon.





Problems of Hot Big Bang model

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INFLATION

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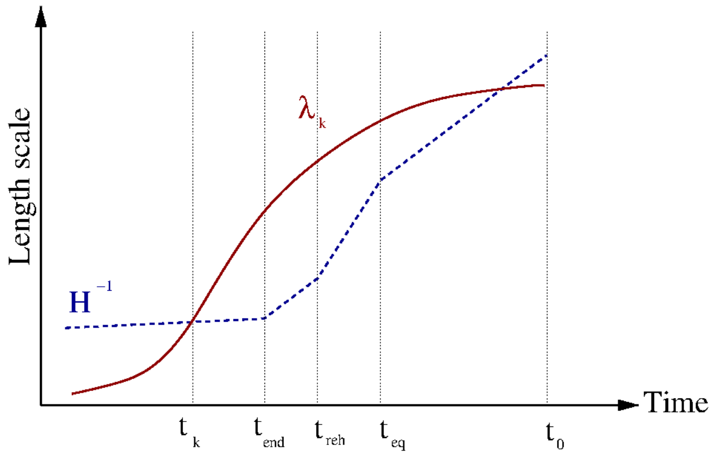
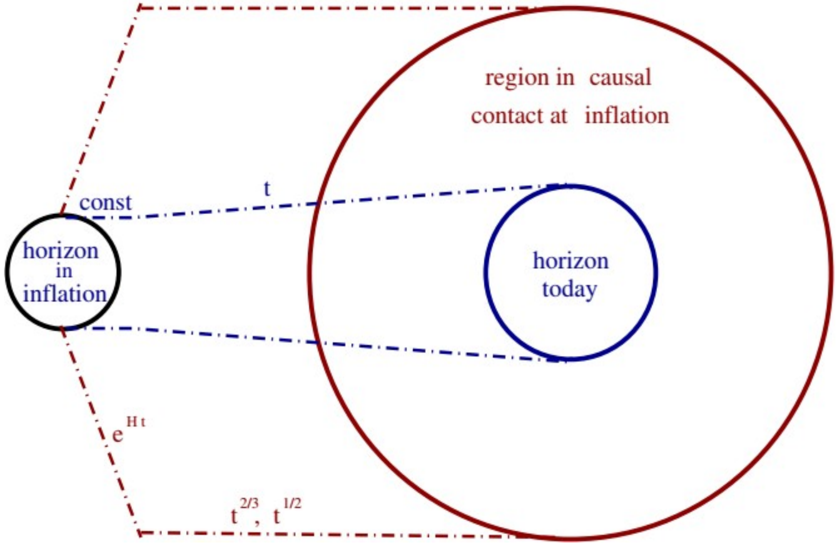
Resolution of BB Problems with Inflation:

During inflation:

distances grow as $a \propto \mathrm{e}^{H\,t}$

horizon is constant $H^{-1} \sim \mathrm{const}$





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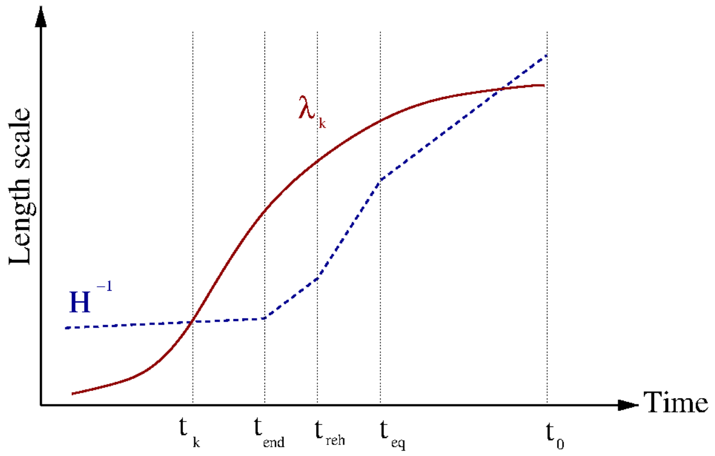

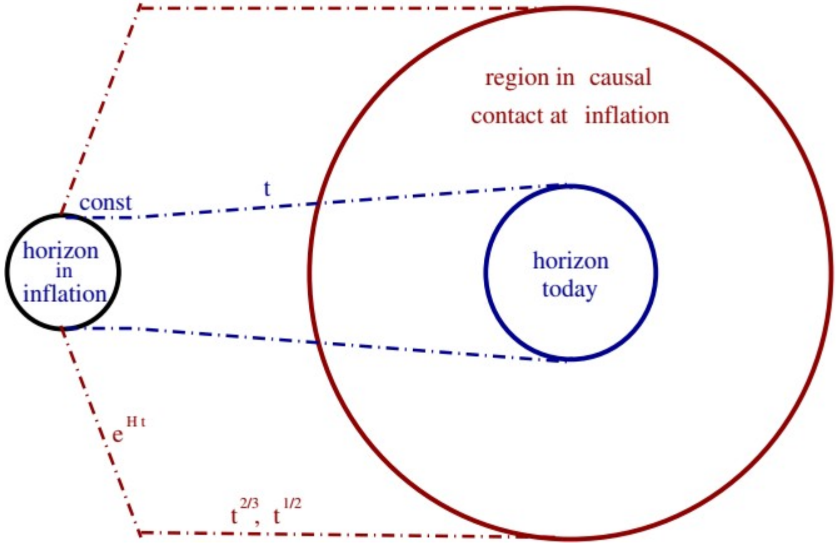

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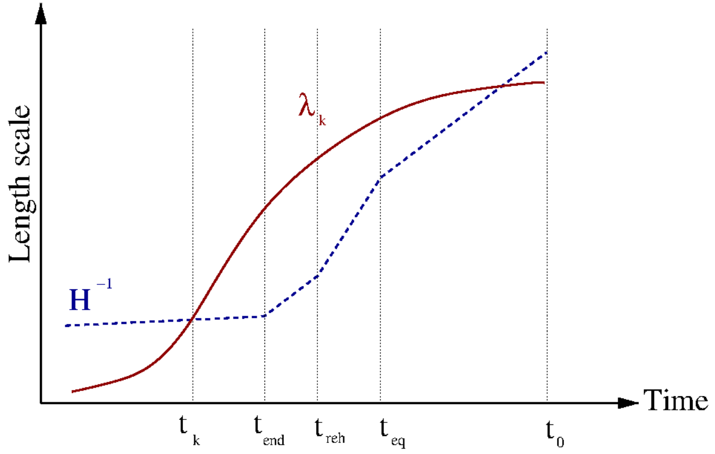

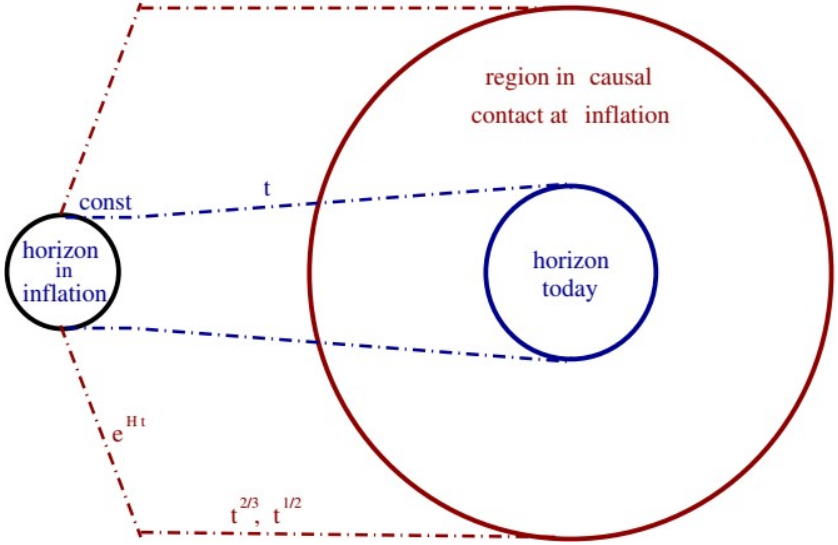

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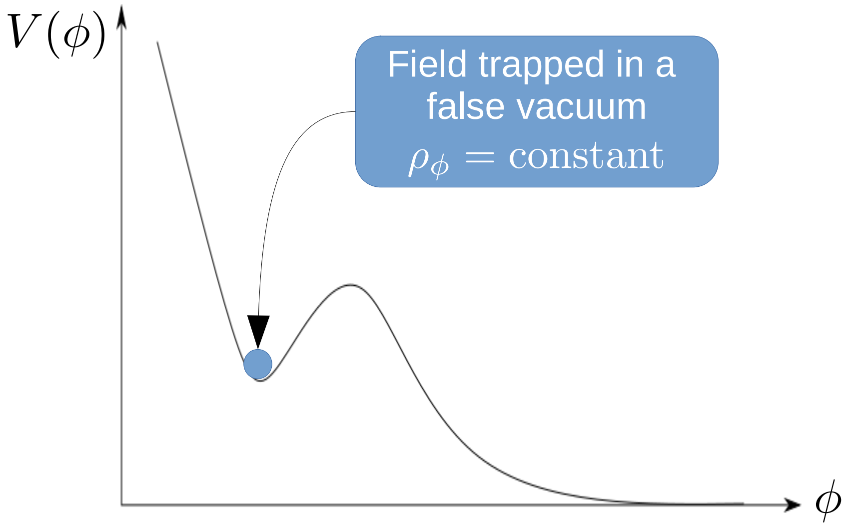
- $n_{\mathrm{relics}} \propto \mathrm{e}^{-3\,H\,t}$ pre-existing relics are quickly diluted. New ones not created if hot big bang starts at temperatures lower than the symmetry breaking energy.

Old Inflation



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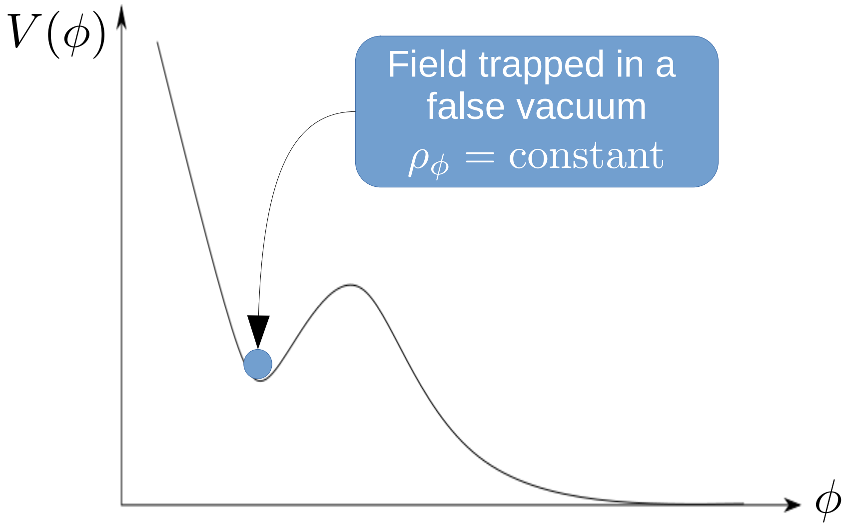

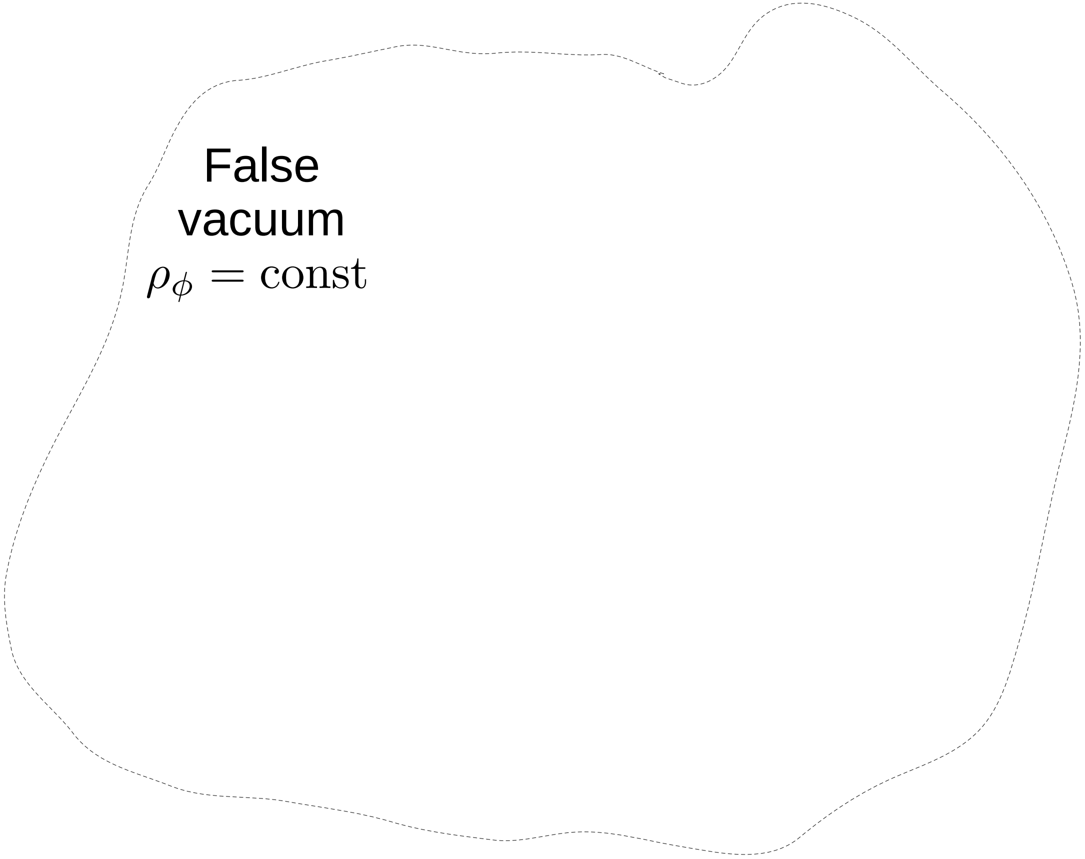

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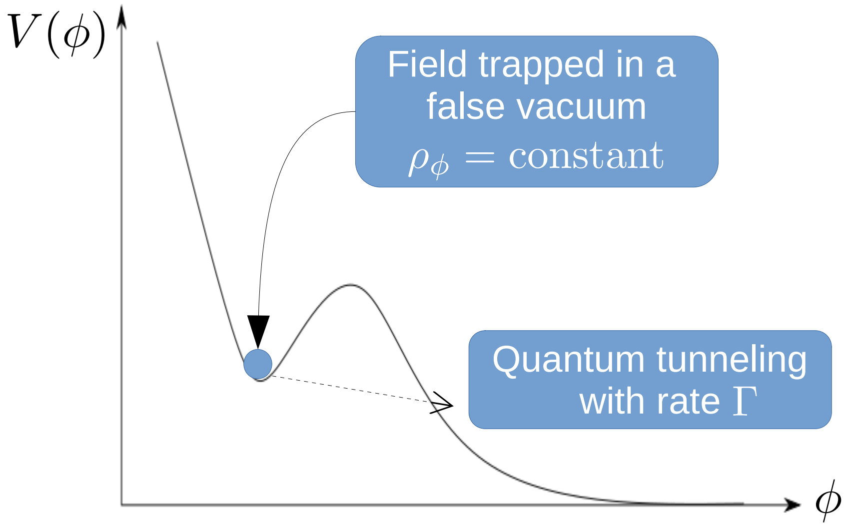

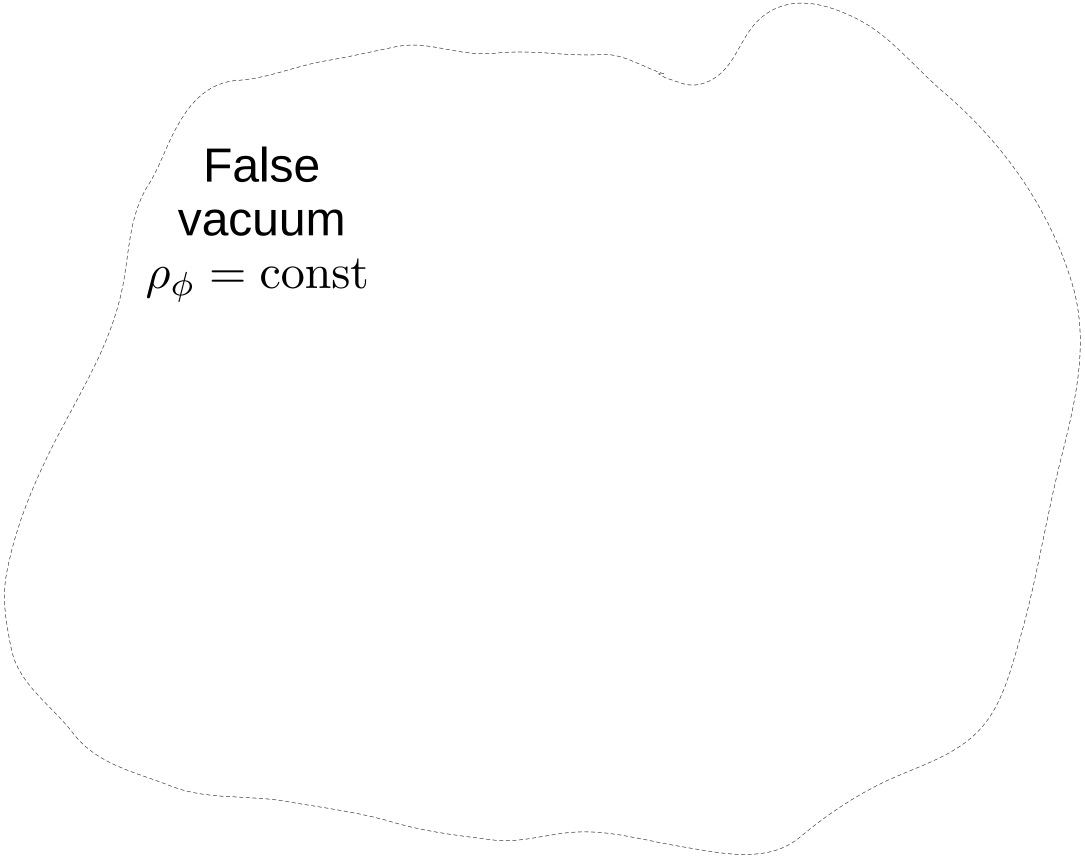

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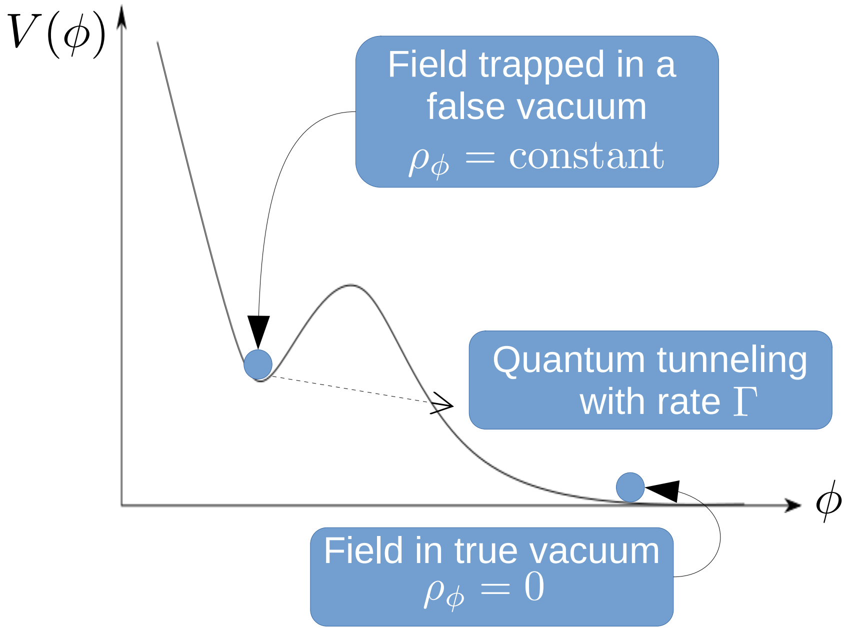

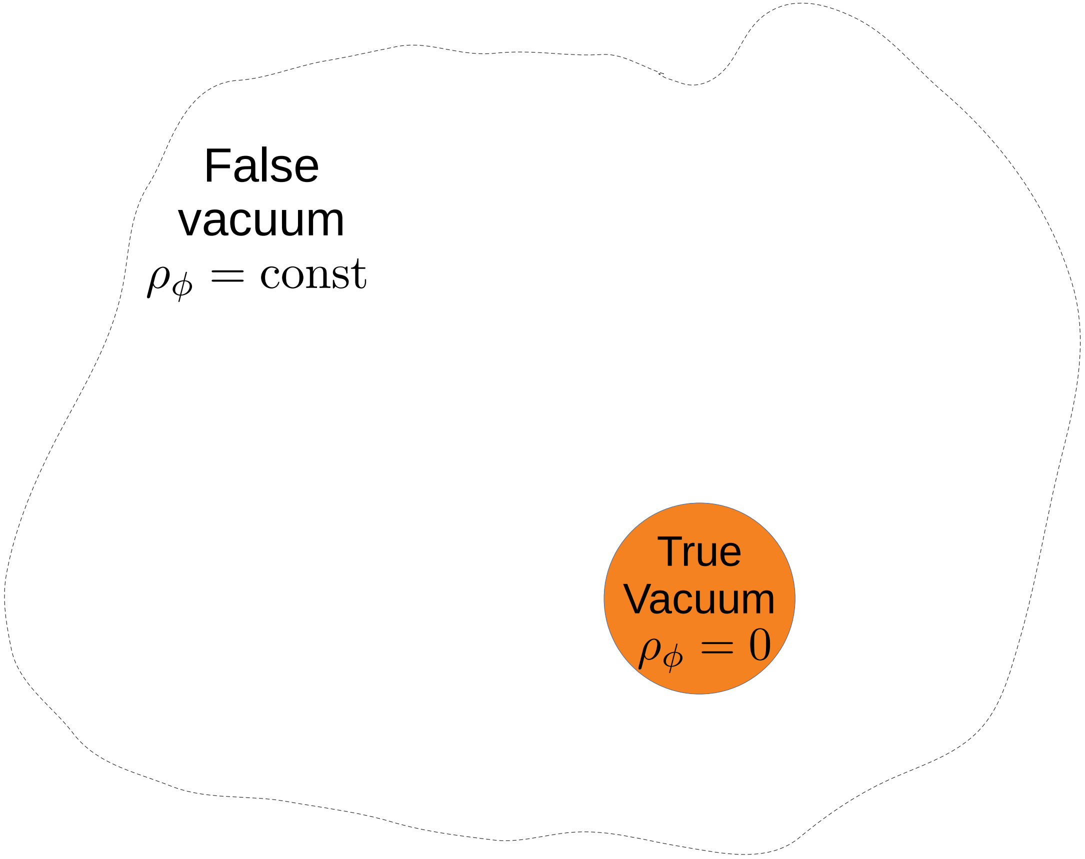

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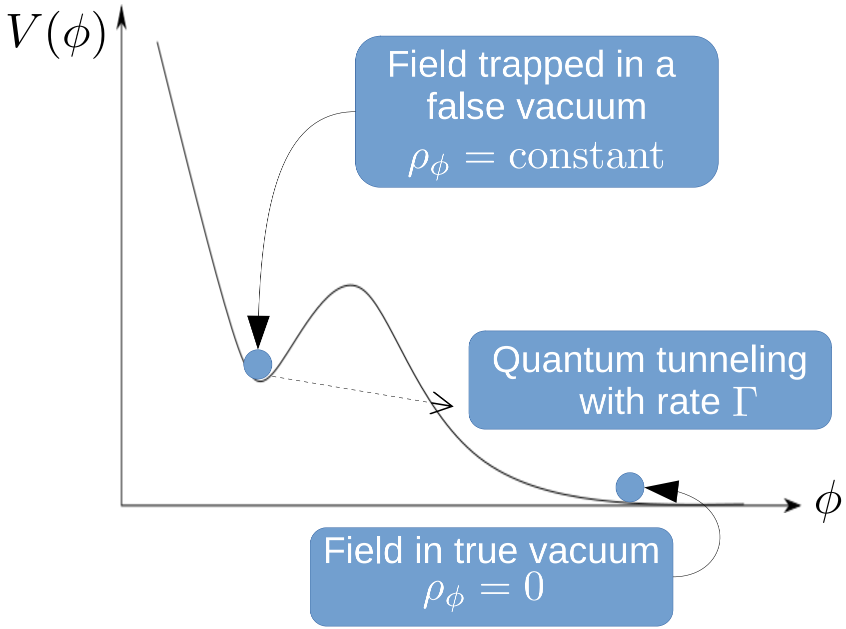

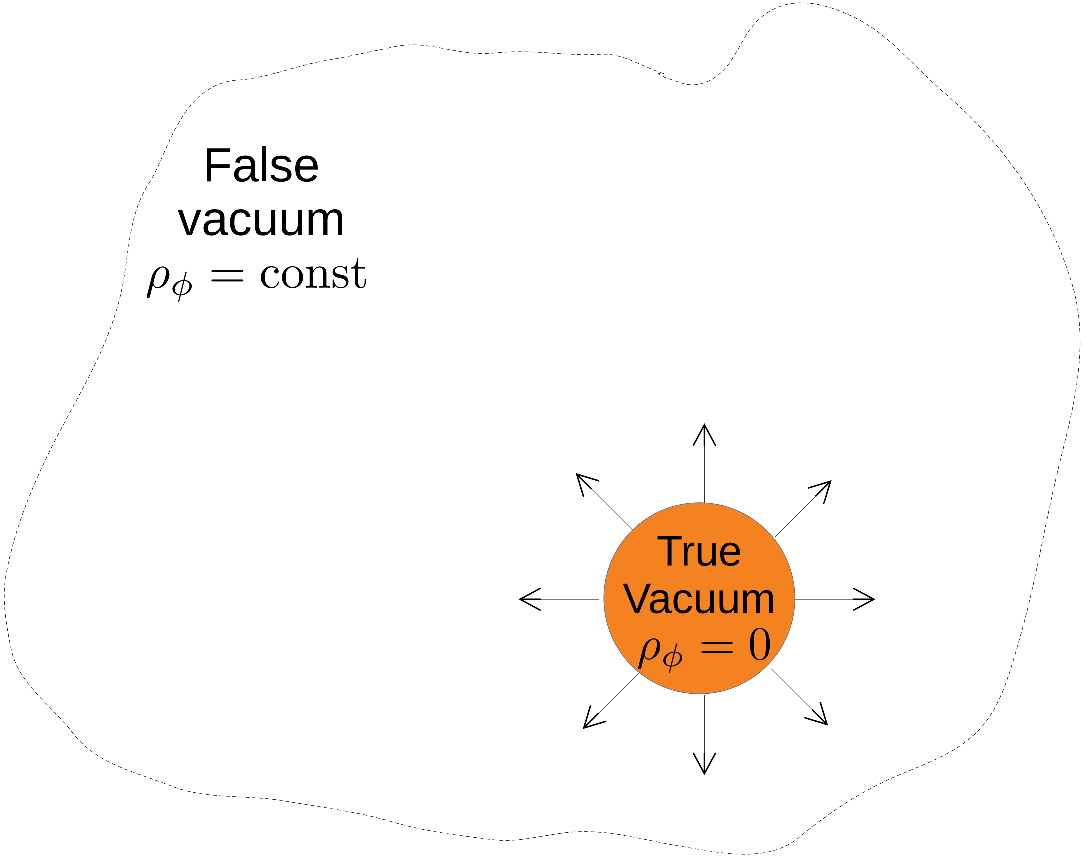

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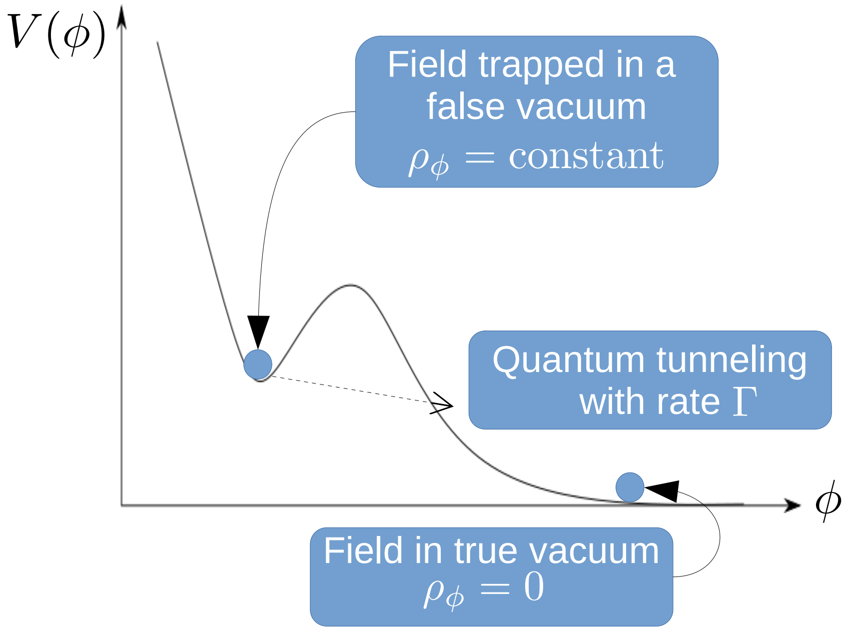

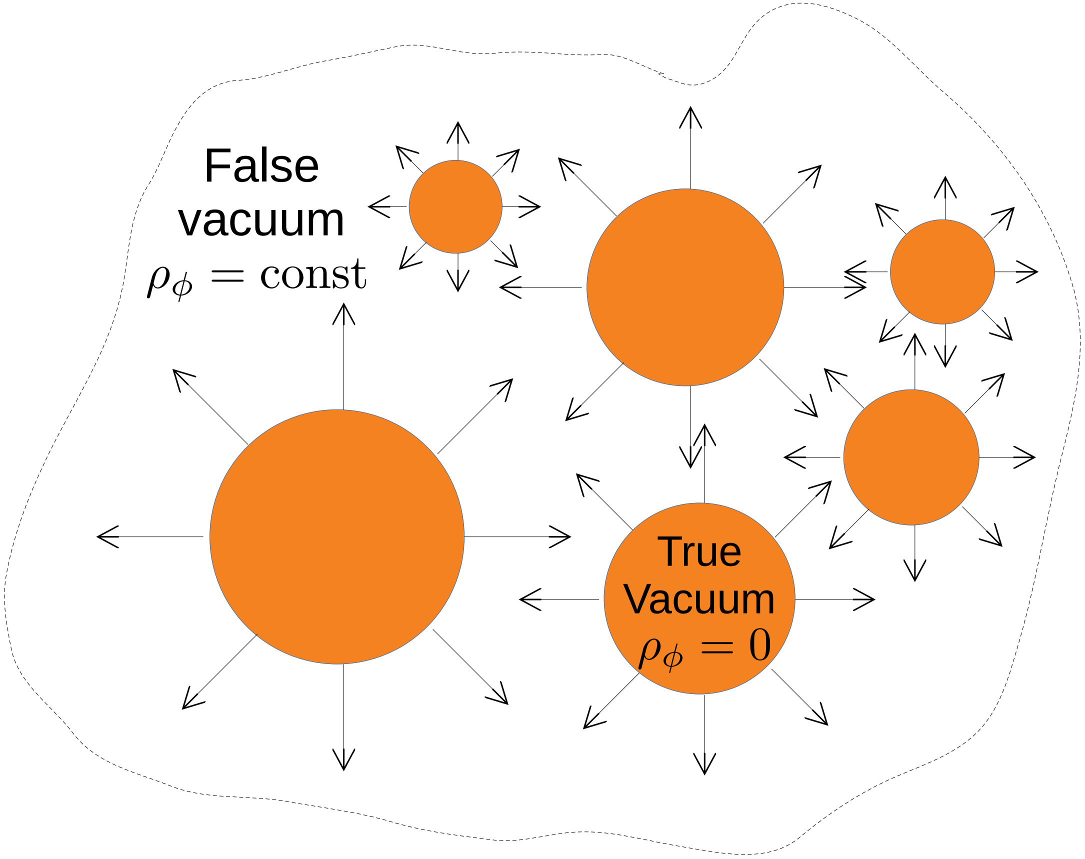

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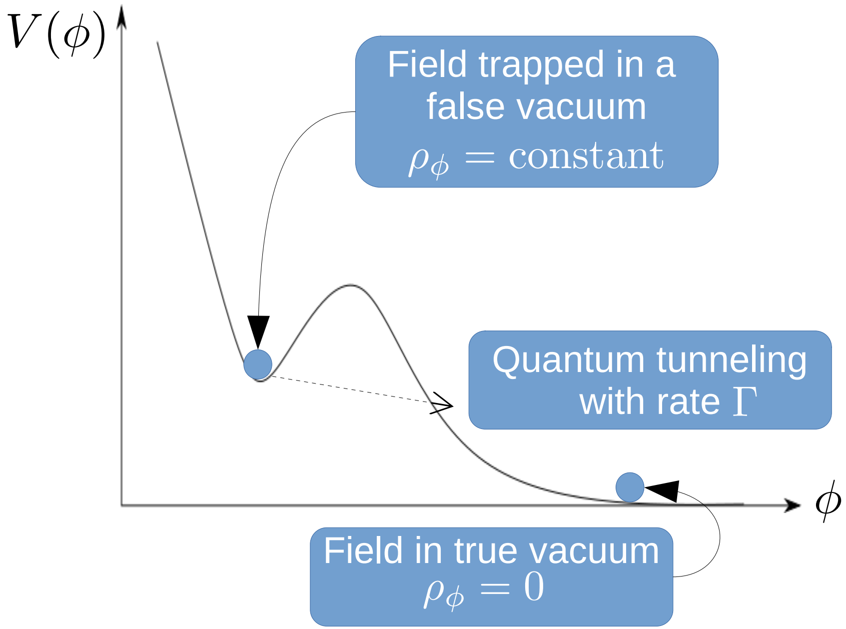

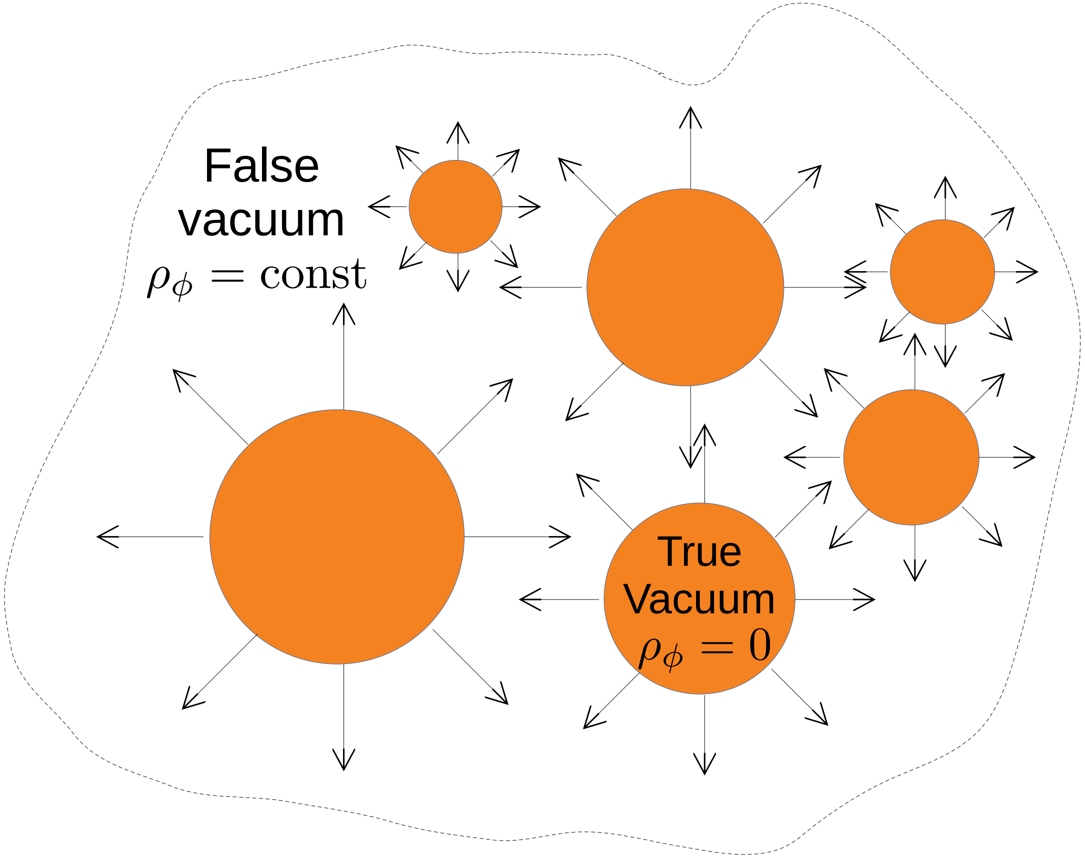

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- But there is a problem….

Graceful exit problem

- In order to solve horizon problem, inflation should last long enough to homogenise the universe. Tunnelling should not be too frequent:

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Guth, Weinberg 1983

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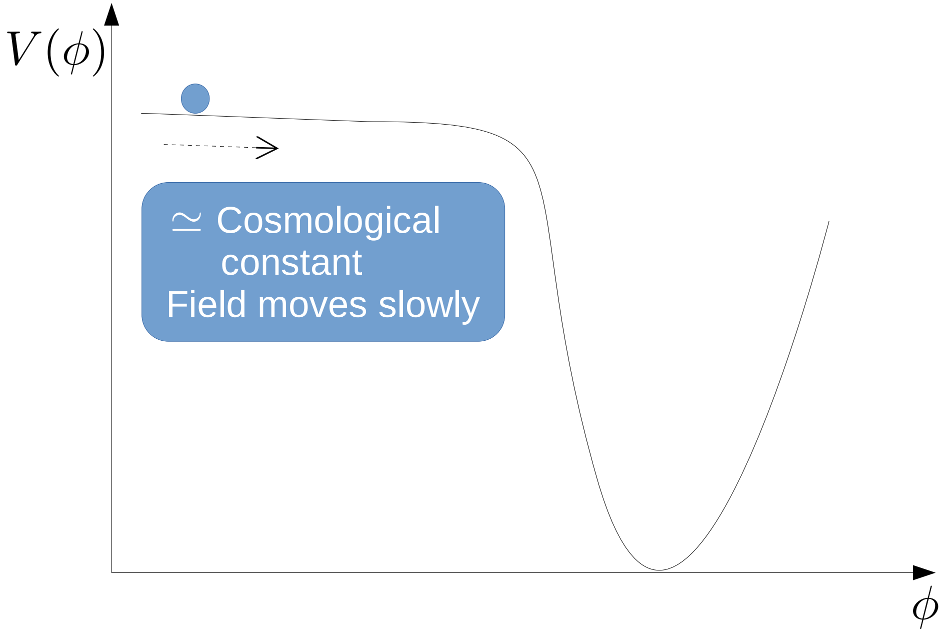
Guth, Weinberg 1983

- Resolution: Continuous transition via *slow roll*.

Linde 1982
Albrecht, Steinhardt 1982

Rolling down slowly



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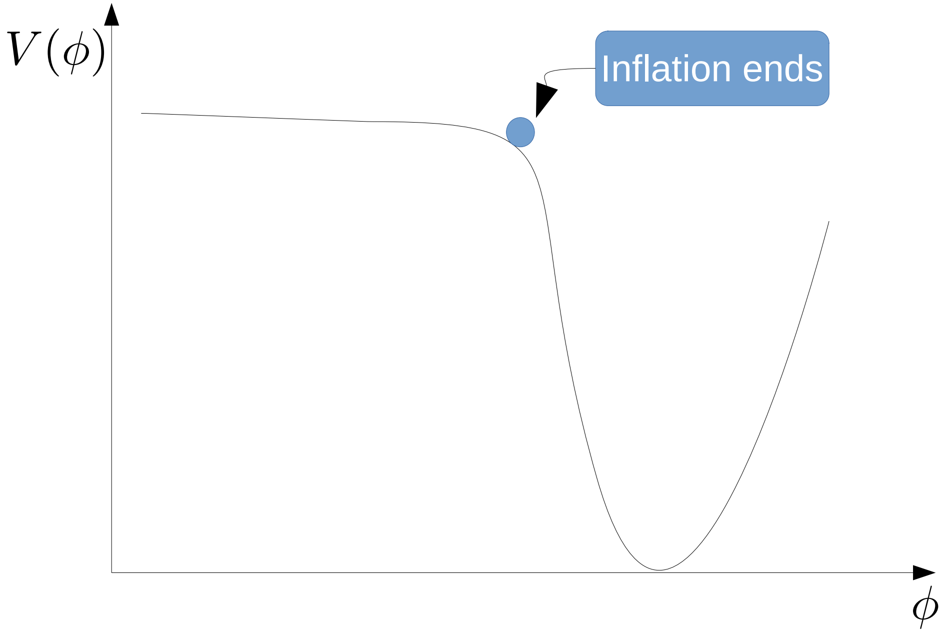

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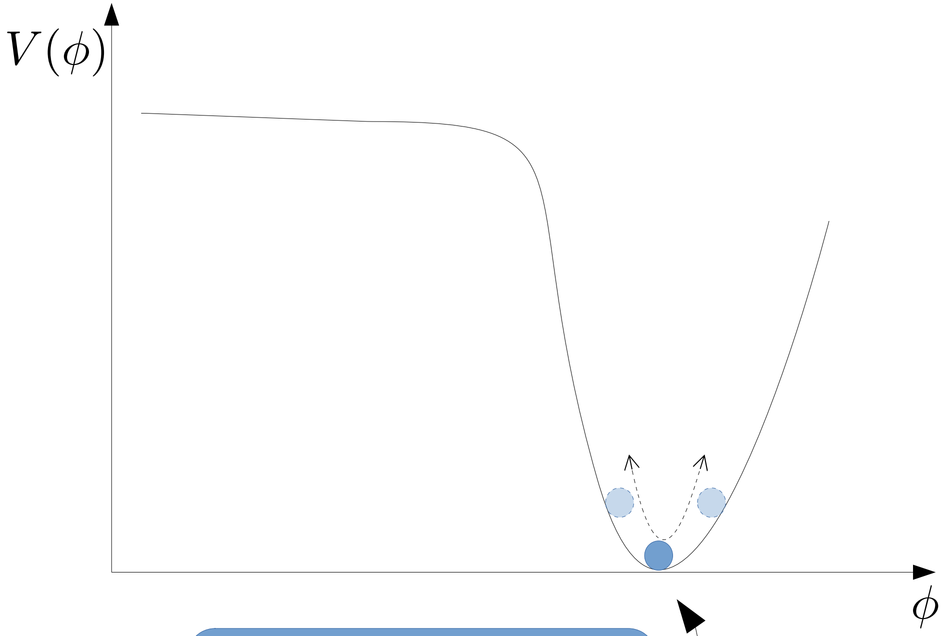
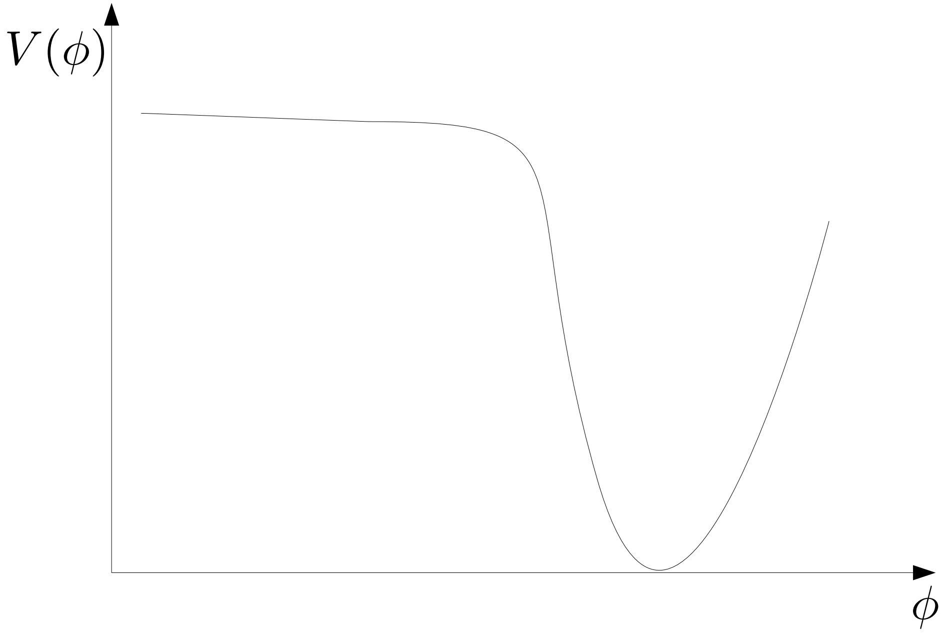
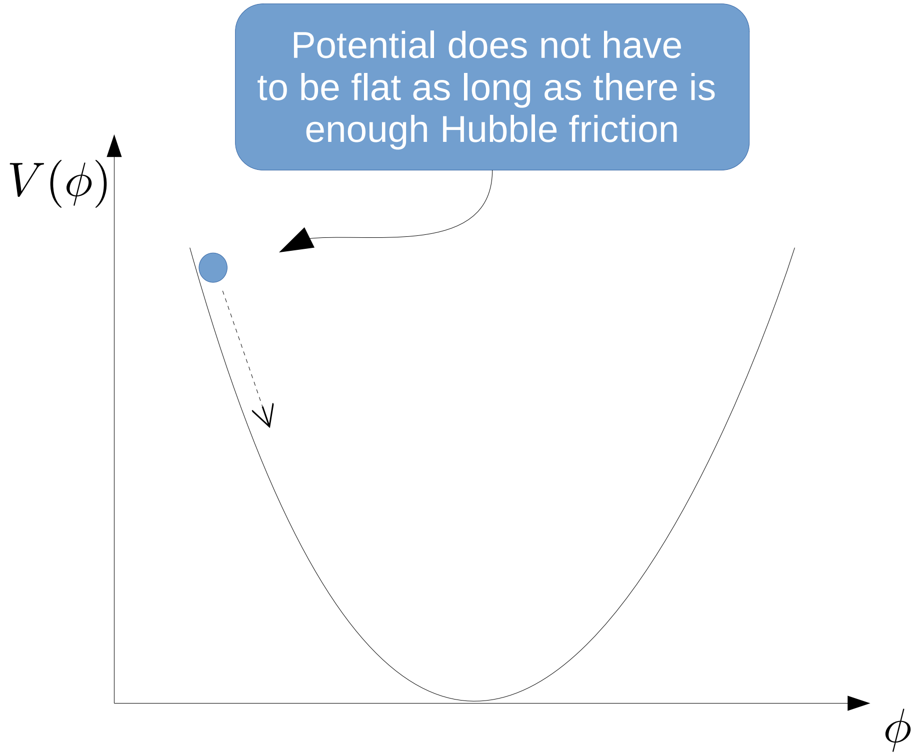


Field oscillates around the minimum. Slowly decays into SM particles.

Rolling down slowly





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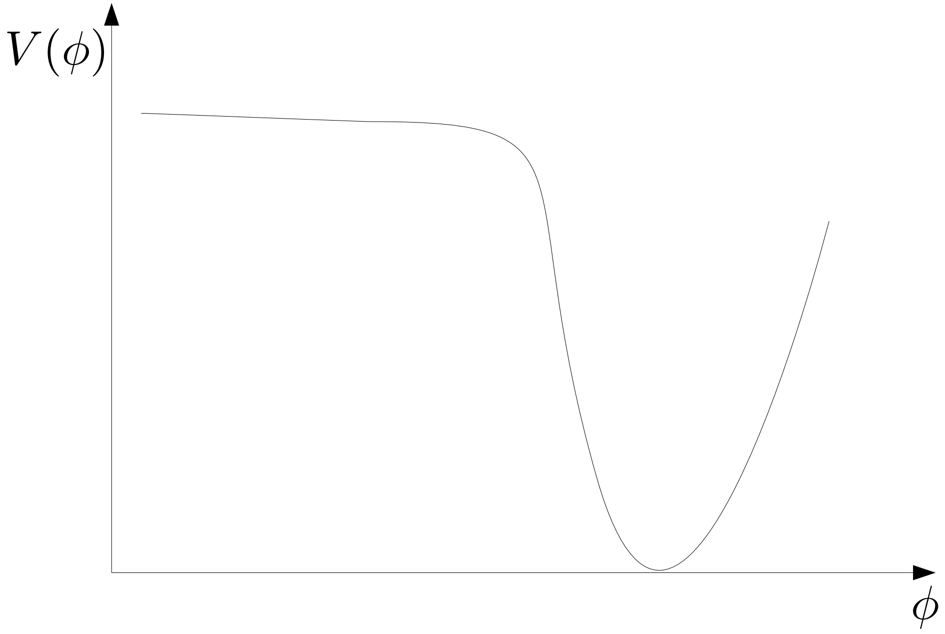

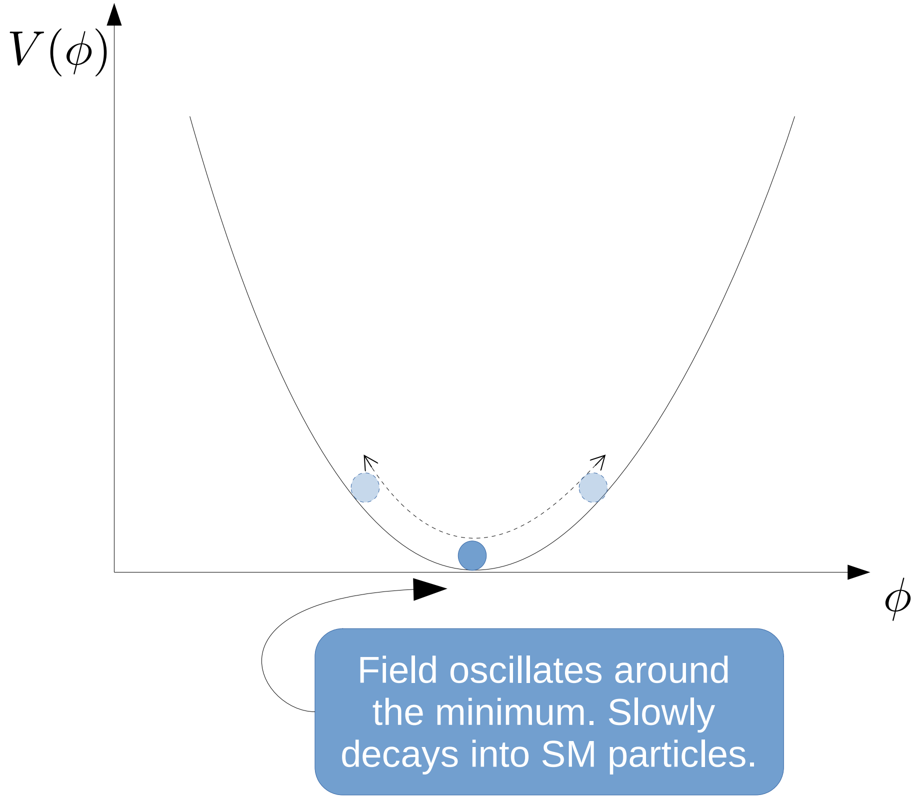

Scalar field cosmology

- Scalar action $$S_\phi = \int d^4x\,\sqrt{-g}\mathcal{L}_\phi = \int d^4x\,\sqrt{-g}\left[\underbrace{-\frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi}_{\textit{kinetic term}} - \underbrace{V(\phi)}_{\textit{potential}}\right]$$

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For a slow moving field

$$w_\phi = \frac{P_\phi}{\rho_\phi} \simeq -1$$

Approx. cosmological constant

Scalar field cosmology

- Einstein's equations

$$H^2 = \frac{1}{3\,M_{Pl}^2}\left(\frac{\dot\phi^2}{2} + V\right) - \frac{K}{a^2}$$

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WARNING! There are different conventions for slow roll parameters! See next slide.

- For acceleration, $\ddot a = a\,H^2\,(1-\epsilon_H) > 0$ ⇒ $\epsilon_H < 1$. From e.o.m:

$$\epsilon_H = \frac{3\,\dot\phi^2}{2\,V + \dot\phi^2}$$

For inflation, we need: $V > \dot\phi^2$

Slow-roll approximation

- Assumption: $V' \simeq -3\,H\,\dot{\phi}$ $\Longleftrightarrow$ $\ddot{\phi} \ll H\,\dot{\phi}$

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- $\epsilon \simeq \epsilon_H$ at leading order in slow roll parameters.
- $H = H_* + \mathcal{O}(\epsilon)$ with constant H_*. Exponential expansion: $a \propto \mathrm{e}^{H_* t}$

Duration of inflation in e-foldings

- How much inflation do we need to solve the horizon problem? Start at t_*, end at t_{end} Distance travelled by photon during this time (as the proper distance today), is

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Duration of inflation in terms of logarithmic expansion.

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- Duration of inflation depends on the scale of inflation. Rough estimate*:

$$N_* \gtrsim 34 + \ln\left(\frac{H_*}{\mathrm{TeV}}\right)$$

(*) assumed MD after inflation until today

$$a_0/a_{end} \sim \left(H_*/H_0\right)^{2/3}$$

Duration of inflation in e-foldings

- How much inflation do we need to solve the horizon problem? Start at t_*, end at t_{end} Distance travelled by photon during this time (as the proper distance today), is

$$R_{inf} = a_0 \int_{t_*}^{t_{end}} \frac{dt}{a} = \frac{a_0}{a_{end}} \int_{t_*}^{t_{end}} dt\, \mathrm{e}^{-H_*(t-t_{end})} = -\frac{a_0}{a_{end} H_*} \mathrm{e}^{-H_*(t-t_{end})} \Big|_{t_*}^{t_{end}}$$

Used: $a = a_{end} \mathrm{e}^{H_*(t-tend)}$

$$= \frac{a_0}{a_{end}\, H_*} \left(\mathrm{e}^{N_*} - 1\right)$$

e-foldings $N_* \equiv \log\left(a_{end}/a_*\right)$

Duration of inflation in terms of logarithmic expansion.

- This region should be larger than the horizon today:

$$\frac{a_0}{a_{end}\, H_*} \left(\mathrm{e}^{N_*} - 1\right) > \frac{1}{H_0}$$

- Duration of inflation depends on the scale of inflation. Rough estimate*:

$$N_* \gtrsim 34 + \ln\left(\frac{H_*}{\mathrm{TeV}}\right)$$

- $H_* = \text{few TeV}$ ⇒ $N_* \gtrsim 40$
- $H_* = \Lambda_{GUT} = 10^{15}\mathrm{GeV}$ ⇒ $N_* \gtrsim 60$

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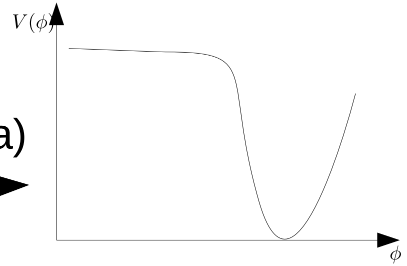

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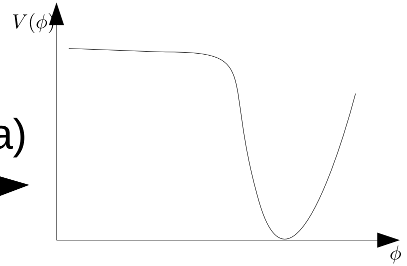

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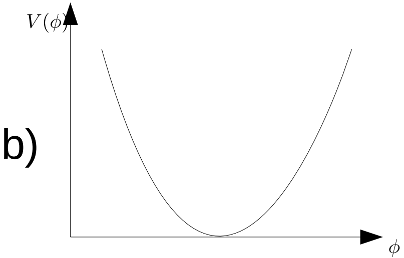

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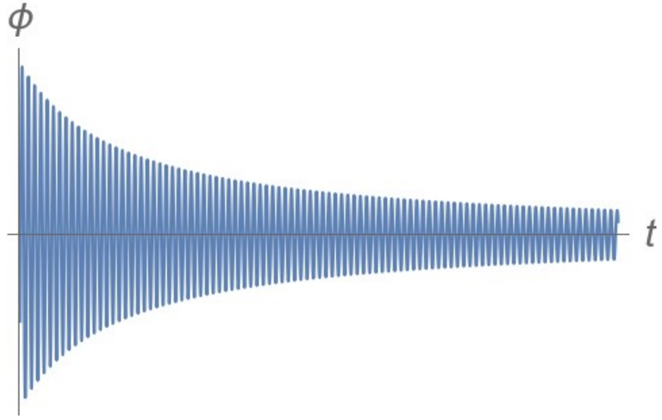

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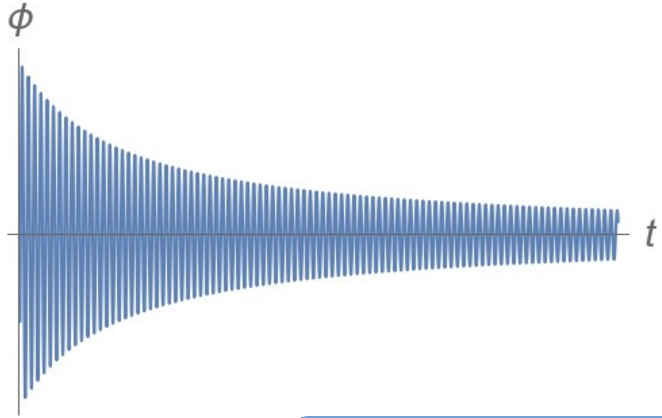

- Averaging over the oscillations,

$$\left\langle \frac{\dot\phi^2}{2} \right\rangle \simeq \left\langle \frac{m^2\phi^2}{2} \right\rangle = \frac{2\,M_{Pl}^2}{3\,t^2} \Longrightarrow \langle\rho\rangle \simeq \frac{4\,M_{Pl}^2}{3\,t^2}\,,\ \langle P\rangle \simeq 0$$

Non-relativistic fluid behaviour on average

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Inflaton decays → decay products thermalise (**REHEATING)** → Hot Big Bang proceeds.

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χ	Boson, with $[M] = +1$
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Decay into light particles, with rates:

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3. No direct decay channel? Gravitational decay ➡ $T_{rh} \sim 10^8 - 10^{10}\mathrm{GeV}$

Davidson & Sarkar 2000

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Kawasaki et al 2008

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  Kawasaki et al 2008
- Unknown physics of reheating leads to an uncertainty in N for a mode with wave number k

$$N(k) = 62 - \log\frac{k}{a_0 H_0} - \log\frac{10^{16}\mathrm{GeV}}{V_k^{1/4}} + \log\frac{V_k^{1/4}}{V_{end}^{1/4}} - \frac{1}{3}\log\frac{V_{end}^{1/4}}{\rho_{rh}^{1/4}}$$

Liddle, Lyth 2000
Kinney, Riotto 2005
…

$\mathcal{O}(10^{19})$ uncertainty in reheating translates into $\Delta N \sim 15$.

Signatures of inflation

Not part of this lecture

- A dramatic outcome of inflation: origin of large scale structure.
- Quantum fluctuations in the geometry+inflaton field are stretched to classical sizes.
- After inflation these perturbations are transferred to matter. After horizon re-entry, over-densities grow.
- Inflationary paradigm makes very precise predictions that can be probed with CMB observations.

Part of this lecture

Inflation pioneers

Alan Guth
American particle physicist. Developed *inflation* to address horizon and flatness problems (1981). The *old inflation model* is now abandoned due to the *graceful exit* problem.

Alexei Starobinsky
Russian physicist. Independently developed the R^2 model of inflation (1980). The model is favoured by the current CMB data.

Andrei Linde
Russian-American physicist. Resolved the *graceful exit problem* by introducing *new inflation (1982).*

Paul Steinhardt & Andreas Albrecht
Independently proposed the *new inflation* model (1982).

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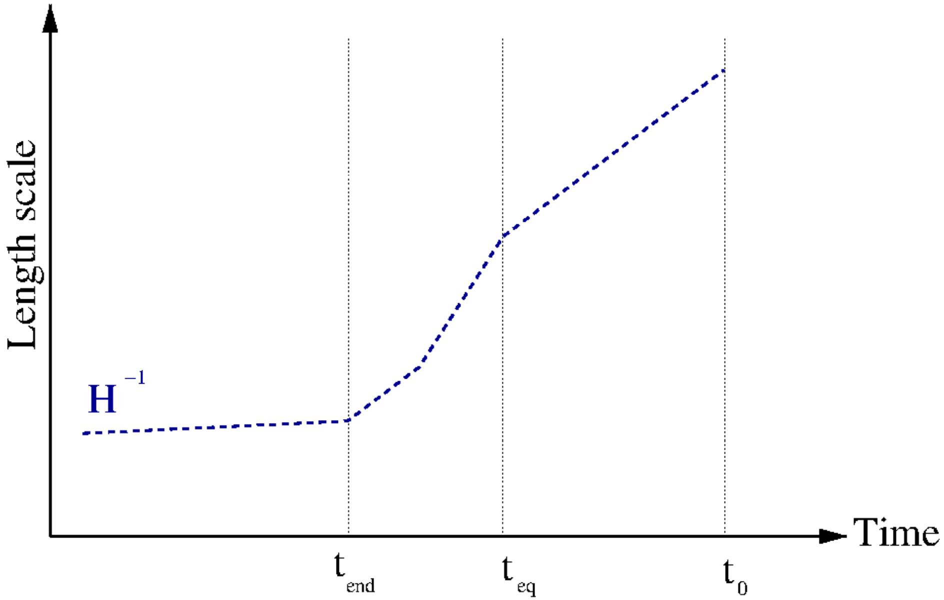

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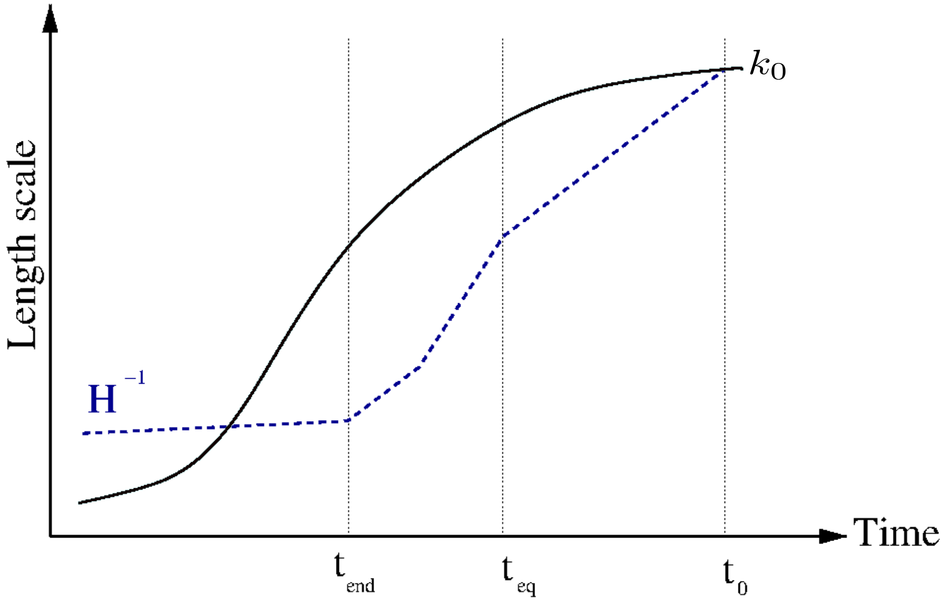

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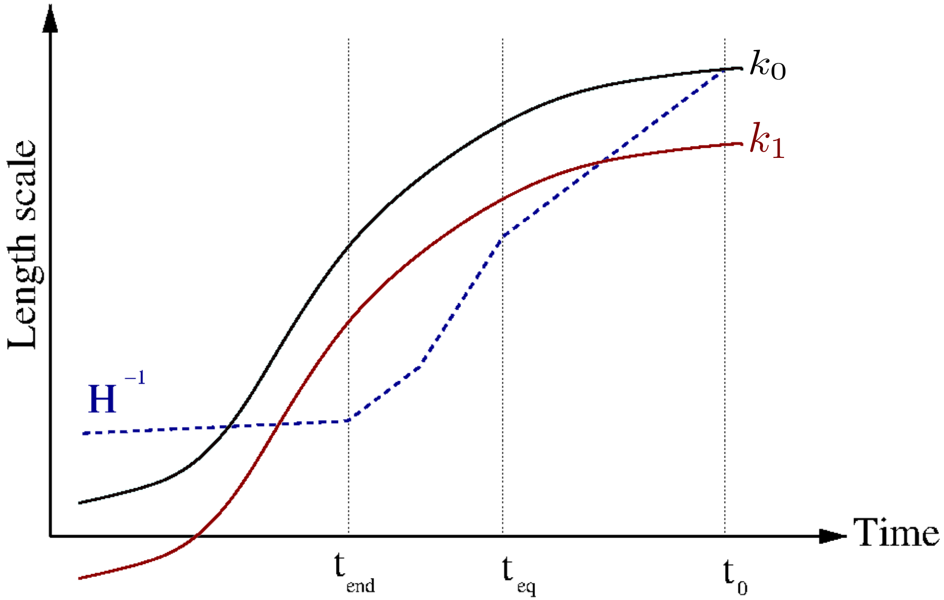

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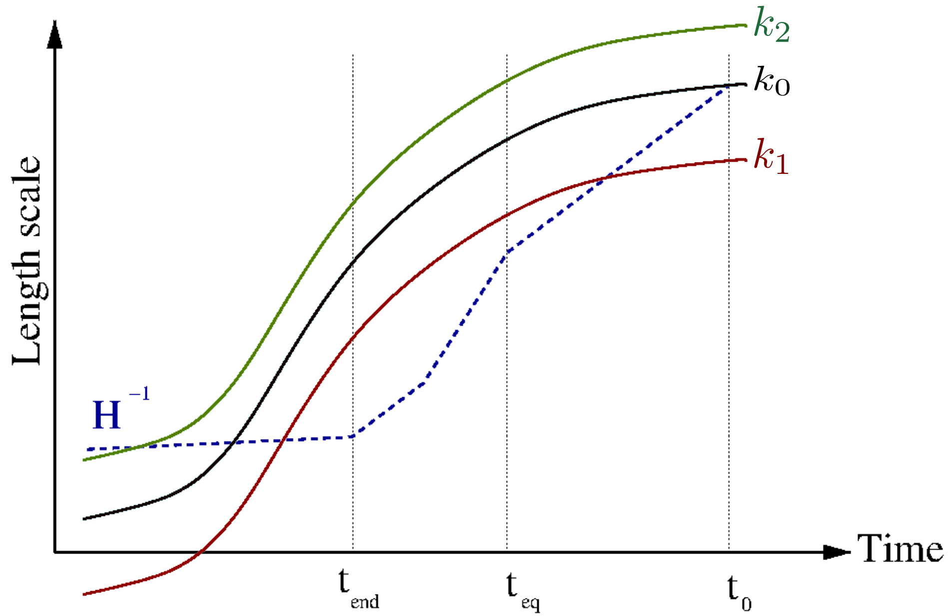

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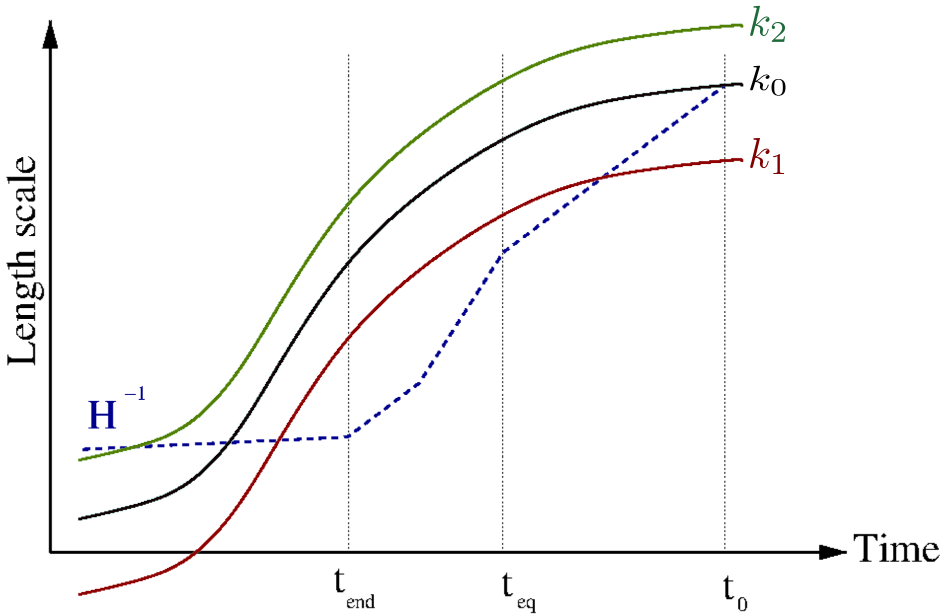

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- ***Longer*** wavelengths (small *k*/large scale) leave the horizon ***early***, re-enter late.
- ***Shorter*** wavelengths (large *k*/small scale) leave the horizon ***late***, re-enter early.

Cosmological Perturbations

- Metric perturbations

$$g_{\mu\nu}(t, x^i) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, x^i)$$

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- We will consider corrections to equations of motion at linear order in perturbations.

- Inflaton acts like a clock, so it breaks time reparameterisation symmetry of GR. But spatial rotational symmetry remains intact. We will decompose the 11 perturbations wrt to 3-rotations.

Counting the degrees of freedom

- Metric perturbations decomposed wrt spatial rotations

$$\delta g_{\mu\nu} = \left(\begin{array}{cc} -2\,\Phi & a\,(B_i + \partial_i B) \\ a\,(B_j + \partial_j B) & a^2(-2\,\psi\delta_{ij} + \partial_i E_j + \partial_j E_i + \partial_i\partial_j E + h_{ij}) \end{array} \right)$$

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- 5 Scalars: $\delta\phi\,,\ \Phi\,,\ B\,,\ \psi\,,\ E$

5

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Physical and independent perturbations:
1 scalar (1)
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- Curvature perturbation and scalar matter perturbations are coupled! Inflation generates super-horizon ζ. After inflation, these perturbations are transferred into photons, baryons and matter in general.

Adiabatic and Entropy Perturbations

- *Adiabatic (curvature) perturbations:* Perturbs along the background trajectory. For any scalar perturbation q , we can write the following

  $$H\delta t = H\,\frac{\delta q}{\dot{q}}$$

  A time displacement δt causes the same change in all quantities democratically. Example:

  $$\frac{\delta\rho}{\dot{\rho}} = \frac{\delta P}{\dot{P}}$$

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Important result: $\zeta = \psi + \dfrac{H\,\delta\rho}{\dot\rho^{(0)}}$ *remains conserved after leaving the horizon* $(k \ll a\,H)$ ***if** perturbations are purely **adiabatic**.*

→ *Consequence of energy conservation*

Initial conditions and power spectrum

- Perturbations start off as quantum fluctuations. We expand them in terms of harmonics. For scalar fields in a flat Universe, this is a plane-wave expansion:

$$q(t, x^i) = \int \frac{d^3k}{(2\,\pi)^{3/2}}\, \mathrm{e}^{i\,\vec{k}\cdot\vec{x}}\, \left(q_k(t) a_{\vec{k}} + q_k^\star(t) a^\dagger_{-\vec{k}}\right)$$

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Power in fluctuations per log interval

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- In terms of slow roll parameters:

$$\frac{z''}{z} \simeq (2 + 5\,\epsilon - 3\,\eta)a^2 H^2$$

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$$\mathcal{R}_k\Big|_{|k\tau|\ll 1} = \frac{H}{\dot\phi^{(0)}}\frac{v_k}{a}\Big|_{|k\tau|\ll 1} = \frac{i\,H^2}{\dot\phi^{(0)}\sqrt{2\,k^3}} \Rightarrow \boxed{P_\zeta \simeq P_{\mathcal{R}} = \left(\frac{H}{\dot\phi^{(0)}}\right)^2\left(\frac{H}{2\,\pi}\right)^2}$$

Scale invariant spectrum

Tensor Perturbations

- We expand tensor modes in terms of plane waves and polarization tensor e_{ij}^{λ}

$$h_{ij}(t,x^i) = \sum_{\lambda} \int \frac{d^3k}{(2\,\pi)^{3/2}}\, \mathrm{e}^{i\,\vec{k}\cdot\vec{x}}\, \left(h_{\lambda,\,k}(t) e_{ij}^{\lambda} a_{\vec{k}} + h_{\lambda,\,k}^{\star}(t) e_{ij}^{\lambda} a_{-\vec{k}}^{\dagger}\right)$$

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Scale invariant spectrum

- The *tensor-to-scalar ratio*

$$\boxed{r \equiv \frac{P_h}{P_\zeta} = \frac{8\,\dot{\phi}^{(0)\,2}}{H^2\, M_{Pl}^2} \simeq 16\,\epsilon}$$

The ratio suppressed by slow roll parameter

Spectral tilts

- The two power spectra, after horizon crossing, without the slow-roll corrections:

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$$n_s = 1 - 6\,\epsilon + 2\,\eta$$
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→ Almost scale invariant spectra

Predictions vs. observations

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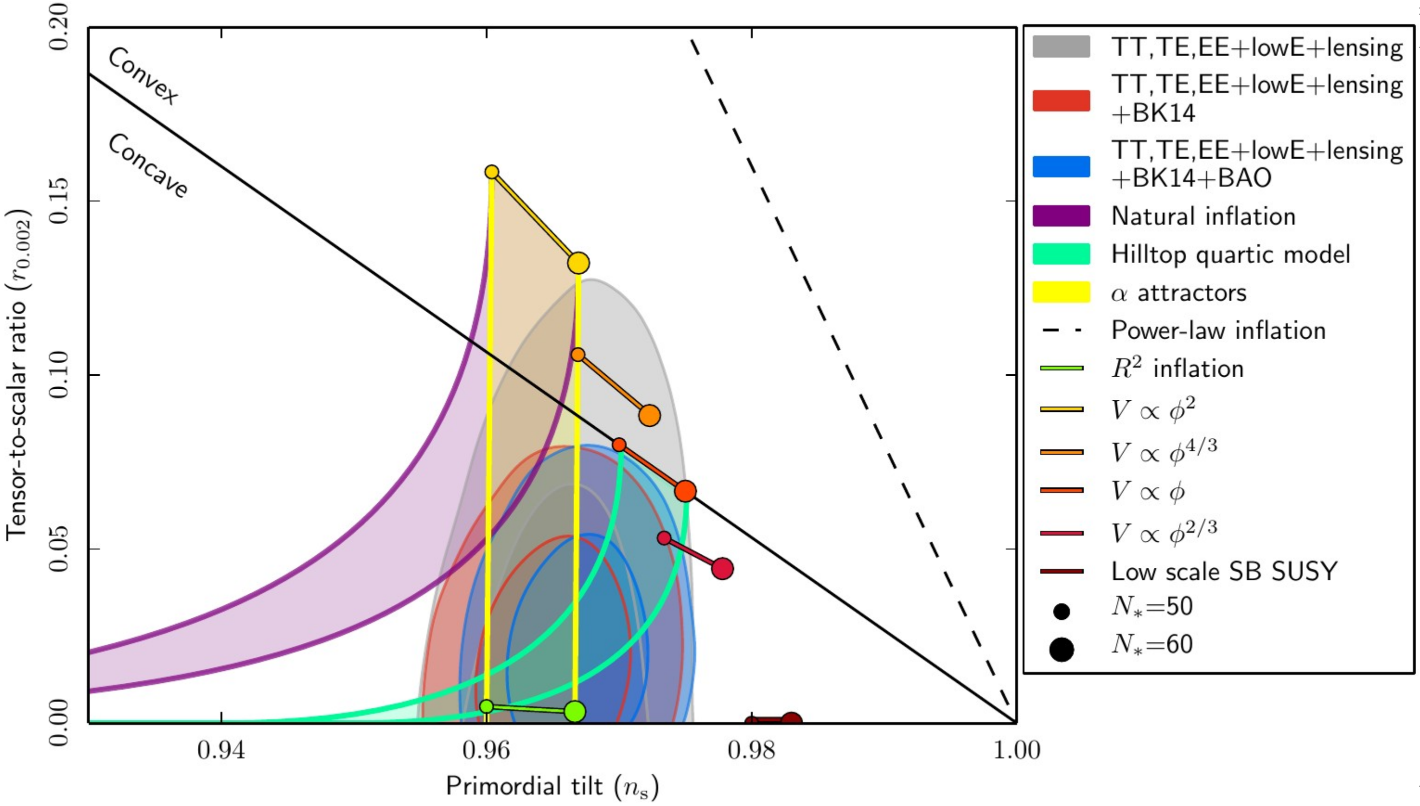
- For models with small field displacement (i.e. small ϵ), this may be challenging.

Planck 2018 results



Inflationary model	Potential $V(\phi)$
$R + R^2/(6M^2)$	$\Lambda^4\left(1 - e^{-\sqrt{2/3}\phi/M_{\rm Pl}}\right)^2$
Power-law potential	$\lambda M_{\rm Pl}^{10/3}\phi^{2/3}$
Power-law potential	$\lambda M_{\rm Pl}^{3}\phi$
Power-law potential	$\lambda M_{\rm Pl}^{8/3}\phi^{4/3}$
Power-law potential	$\lambda M_{\rm Pl}^{2}\phi^{2}$
Power-law potential	$\lambda M_{\rm Pl}\phi^{3}$
Power-law potential	$\lambda\phi^{4}$
Non-minimal coupling	$\lambda^4\phi^4 + \xi\phi^2 R/2$
Natural inflation	$\Lambda^4\left[1 + \cos\left(\phi/f\right)\right]$
Hilltop quadratic model	$\Lambda^4\left(1 - \phi^2/\mu_2^2 + \dots\right)$
Hilltop quartic model	$\Lambda^4\left(1 - \phi^4/\mu_4^4 + \dots\right)$
D-brane inflation ($p = 2$)	$\Lambda^4\left(1 - \mu_{\rm D\,2}^2/\phi^p + \dots\right)$
D-brane inflation ($p = 4$)	$\Lambda^4\left(1 - \mu_{\rm D\,4}^4/\phi^p + \dots\right)$
Potential with exponential tails	$\Lambda^4\left[1 - \exp\left(-q\phi/M_{\rm Pl}\right) + \dots\right]$
Spontaneously broken SUSY	$\Lambda^4\left[1 + \alpha_h \log\left(\phi/M_{\rm Pl}\right) + \dots\right]$
E-model ($n = 1$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_1^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$
E-model ($n = 2$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_2^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$
T-model ($m = 1$)	$\Lambda^4\tanh^{2m}\left[\phi\left(\sqrt{6\alpha_1^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$
T-model ($m = 2$)	$\Lambda^4\tanh^{2m}\left[\phi\left(\sqrt{6\alpha_2^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$

Topics we did not have time for

- Gravity is unstable. Over-densities seeded by inflation that are within the particle horizon (but outside the sound horizon) grow thanks to Jeans instability during matter domination.
- Other observational probes from more contrived models (e.g. multi-field inflation): running of spectral index, non-Gaussian signatures...
- Problems of inflation:

  Eternal inflation, initial conditions problem

  Model building challenges: eta-problem, swampland conjecture, radiative corrections, trans-Planckian problem ...
- Alternatives to inflation, ekpyrotic/cyclic models, bouncing cosmologies...

Next time on *Theoretical Cosmology*

- What *is* the cosmological constant? Is it real or is it a convenient place-holder
- What assumptions does dark energy rely on? Which ones can we relax?
- What is modified gravity and its challenges?

  Stay tuned for "the worst theoretical prediction in the history of physics" (Hobson et al 2006)