CH5440

Assignment-1

CH18B035

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Problem :-1

a) a -

WTLS estimates of a, b

$$\min_{a,b,\hat{x}_i} (y_i - a\hat{x}_i - b)^2/\sigma_\epsilon^2 + (x_i - \hat{x}_i)^2/\sigma_\delta^2$$

$S \equiv \Downarrow \times \sigma_\epsilon^2$

$$S \equiv \min_{a,b,\hat{x}_i} (y_i - a\hat{x}_i - b)^2 + \alpha(x_i - \hat{x}_i)^2$$

we know $\frac{\partial S}{\partial b} = \frac{\partial S}{\partial a} = \frac{\partial S}{\partial \hat{x}_i} = 0$

$$\frac{\partial S}{\partial b} = 2(y_i - a\hat{x}_i - b)(-1) = 0$$

$$\Sigma y_i - a\Sigma\hat{x}_i - Nb = 0 \quad \text{— (i)}$$

$$\frac{\partial S}{\partial a} = 2(y_i - a\hat{x}_i - b)(-\hat{x}_i) = 0$$

$$\Sigma y_i\hat{x}_i - a\Sigma\hat{x}_i^2 - b\Sigma\hat{x}_i = 0 \quad \text{— (ii)}$$

$$\frac{\partial S}{\partial \hat{x}_i} = 0 \qquad y_i \cancel{x} - a^2\hat{x}_i - ab + \alpha(x_i) - \alpha\hat{x}_i = 0$$

$$\Sigma\hat{x}_i = \frac{\Sigma y_i a + \Sigma\alpha x_i - abN}{a^2 + \alpha} \quad \text{— (iii)}$$

$$a\Sigma y_i - a^2\Sigma\hat{x}_i - Nab + \alpha\Sigma x_i - \alpha\Sigma\hat{x}_i = 0$$

Sub. (iii) in (i)

$$\Sigma y_i(\cancel{a^2} + \alpha) - (\cancel{a^2\Sigma y_i} + \Sigma a\alpha x_i - \cancel{a^2 b N}) - Nb\cancel{(a^2} + \alpha) = 0$$

$$\Sigma y_i + (-a\Sigma x_i - Nb) = 0$$

$$\Rightarrow \boxed{\bar{y} = a\bar{x} + b} \quad \text{— (iv)}$$

from iii $\quad \hat{x}_i = \dfrac{a y_i - ab + \alpha x_i}{a^2 + \alpha}$

$$y_i \hat{x}_i = \frac{a y_i^2 - ab y_i + \alpha x_i y_i}{a^2 + \alpha} \quad \text{— (v)}$$

$$\hat{x}_i^{\,2} = \frac{a^2 y_i^2 + a^2 b^2 + \alpha^2 x_i^2 + 2a\alpha x_i y_i - 2a^2 b y_i - 2ab\alpha x_i}{(a^2+\alpha)^2} \quad \text{—(vi)}$$

Substitute v & vi & iii in ii

$$\Rightarrow \sum \Big(a^3\cancel{y_i}^2 - a^3\cancel{b}y_i + a^2\alpha\cancel{x_i}y_i + a\alpha y_i^2 - ab\alpha y_i + \alpha^2 x_i y_i$$
$$- a^3\cancel{b}y_i + a^3\cancel{b}^2 - \alpha b\cancel{a}^2 x_i - ab\alpha y_i + ab^2\alpha - b\alpha^2 x_i$$
$$- a^3\cancel{y_i}^2 - a^3\cancel{b}^2 - a\alpha^2 x_i^2 + 2a^3\cancel{b}y_i + \cancel{2}a^2 b\alpha x_i \cancel{+2}a^2\alpha x_i y_i = 0$$

$$\Rightarrow a y_i^2 - 2ab y_i + \alpha x_i y_i + ab^2 - b\alpha x_i - a\alpha x_i^2 + a^2 b x_i$$
$$- a^2 x_i y_i = 0$$

$$\Rightarrow \sum \Big(b(2a y_i + \alpha x_i - a^2 x_i) - ab^2 - a y_i^2 - \alpha x_i y_i + a\alpha x_i^2 + a^2 x_i y_i = 0 \Big) \quad \text{└(vii)}$$

Sub. iv in above eq. vii

$$\Rightarrow \left(\frac{\Sigma y_i - a\Sigma x_i}{N}\right)(2a\Sigma y_i + \alpha\Sigma x_i - a^2 \Sigma x_i) - Na\left(\frac{\Sigma y_i - a\Sigma x_i}{N}\right)^2$$
$$- a\Sigma y_i^2 - \alpha\Sigma x_i y_i + a\alpha\Sigma x_i^2 + a^2\Sigma(x_i y_i) = 0$$

$$\Rightarrow a^2\Big[\underbrace{\Sigma x_i y_i - \frac{\Sigma x_i \Sigma y_i}{N}}_{S_{xy}}\Big] + a\Big[\alpha\Big(\underbrace{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{N}}_{S_{xx}}\Big) - \Big(\underbrace{\Sigma y_i^2 - \frac{(\Sigma y_i)^2}{N}}_{S_{yy}}\Big)\Big]$$
$$- \alpha\left(\Sigma x_i y_i - \frac{\Sigma x_i \Sigma y_i}{N}\right) = 0$$

$$a^2 S_{xy} + a(\alpha S_{xx} - S_{yy}) - \alpha S_{xy} = 0 \quad \text{— (viii)}$$

$$\Rightarrow \boxed{\hat{a}_{WTLS} = \frac{S_{yy} - \alpha S_{xx} \pm \sqrt{(S_{yy} - \alpha S_{xx})^2 + 4\alpha S_{xy}^2}}{2 S_{xy}}}$$

from iv

$$\hat{b}_0 \quad \boxed{\hat{b}_{WTLS} = \bar{y} - \hat{a}_{WTLS} \cdot \bar{x}}$$

b - if $b = 0$

$$\hat{b}_{WTLS} = 0$$

$\Rightarrow$ from vii

$$a^2 \underbrace{\sum x_i y_i}_{N S_{xy}} + a \left(\alpha \underbrace{\sum x_i^2}_{N S_{xx}} - \underbrace{\sum y_i^2}_{N S_{yy}} - \cancel{N b^2} \right) - \alpha \sum x_i y_i = 0$$

$$a^2 S_{xy} + a (\alpha S_{xx} - S_{yy}) - \alpha S_{xy} = 0$$

$$\hat{a}_{WTLS} = \frac{S_{yy} - \alpha S_{xx} \pm \sqrt{(S_{yy} - S_{xx})^2 + 4 \alpha S_{xy}^2}}{2 S_{xy}}$$

where $S_{xy} = \frac{\sum x_i y_i}{N}$ $\quad S_{xx} = \frac{\sum x_i^2}{N}$ $\quad S_{yy} = \frac{\sum y_i^2}{N}$

b) from a)

$\alpha \to 0$

IOLS

$$\hat{a}_{IOLS} = \frac{S_{yy} - 0 \pm S_{yy}}{2 S_{xy}} = \frac{S_{yy}}{S_{xy}}$$

$$\boxed{\hat{a}_{IOLS} = S_{yy} / S_{xy}}$$

$$\boxed{\hat{b}_{IOLS} = -\hat{a}_{IOLS} \, \bar{x} \cdot + \bar{y}}$$

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from viii

OLS $\alpha \to \infty$

$$\frac{a^2 S_{xy}}{\alpha} + a\left(S_{xx} - \frac{S_{yy}}{\alpha}\right) - S_{xy} = 0$$

$\alpha \to \infty$

$$\Rightarrow \quad a\, S_{xx} = S_{xy}$$

$$\hat{a}_{OLS} = S_{xy}/S_{xx} \qquad \hat{b}_{OLS} = \bar{y} - \hat{a}_{OLS}\bar{x}$$

Solutions for estimates. from iii

$$\hat{x}_i = \frac{y_i a + \alpha x_i - ab}{a^2 + \alpha}$$

for OLS $\alpha \to \infty$

$$\hat{x}_i = \frac{\frac{y_i a}{\alpha} + x_i - \frac{ab}{\alpha}}{\frac{a^2+\alpha}{\alpha}} \Rightarrow \hat{x}_i = x_i \qquad \hat{y}_i = a\hat{x}_i + b$$

$\alpha \to \infty$

$$\hat{y} = a x_i + b$$

for IOLS $\alpha \to 0$

$$\hat{x}_i = \frac{y_i}{a} - \frac{b}{a}. \qquad \hat{y}_i = a\hat{x}_i + b$$

$$\hat{y}_i = y_i$$

for TLS

$\alpha = 1$

$$\hat{x}_i = \frac{y_i a + x_i - ab}{a^2+1} \qquad \hat{y}_i = a\hat{x}_i + b$$

$$\hat{y}_i = \frac{a^2 y_i + a x_i - \cancel{a^2 b} + \cancel{a^2 b} + b}{a^2 + 1}$$

$$\hat{y}_i = \frac{a^2 y_i + a x_i + b}{a^2+1}$$

Problem 2:-

let EP method be x

CF method be y

Given data using matlab

$\bar{x} = 2.0155$ $\bar{y} = 1.9505$ $S_{xx} = 0.9311$ $S_{yy} = 0.9232$

$S_{xy} = 0.9241$

method:-1

Using OLS

$\hat{a}_{OLS} = \frac{S_{xy}}{S_{xx}} = 0.992$ $\hat{b}_{OLS} = \bar{y} - \hat{a}_{OLS}\bar{x}$

$= -0.05$

Confidence Interval

of $\hat{a}_{OLS}$: $\sigma^2_{\epsilon_y} = \frac{\Sigma(y_i - \hat{y})^2}{N-2}$

$\sigma^2_{\epsilon_y} = 0.00687$

$\sigma^2_{\hat{a}_{OLS}} = \frac{\sigma^2_{\epsilon_y}}{NS_{xx}} = 0.000369$

95% confidence

interval $\Rightarrow 0.992 \pm t_{0.975,18} \cdot \sigma^2_{\hat{a}_{OLS}}$

$\Rightarrow 0.992 \pm 0.04$

$(0.952, 1.032)$

$\hat{a}_{OLS} = 1$ is an ideal case of best assumption

where 1 lies in the $\hat{a}_{OLS}$ CI

$\Rightarrow$ CF is good estimate.

method:-2

Using IOLS

$\hat{a}_{IOLS} = \frac{S_{yy}}{S_{xy}} = 0.999$ $\hat{b}_{IOLS} = +0.0669$

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Confidence interval of $\hat{a}_{IOLS}$:

$$\sigma^2_{\epsilon x} = \frac{1}{N-2}\Sigma(x_i - \bar{x})^2$$

$$= 0.006927$$

$$\sigma^2_{\hat{\alpha}_{IOLS}} = \frac{\sigma^2_{\epsilon x}}{NS_{yy}}$$

$$= 1.8757 \times 10^{-5}$$

95% CI $\Rightarrow$ $0.999 \pm t_{0.975, 18} \cdot \sigma_{\hat{a}_{IOLS}}$

$= 0.999 \pm 0.041$

$= (0.958, 1.04)$ //

as '1' lies in the interval $\hat{a}_{IOLS}$, IOLS CF is good estimate of EP

Method :3

Using TLS

$$\hat{a}_{TLS} = \frac{S_{yy} - S_{xx} + \sqrt{(S_{yy}-S_{xx})^2 + 4S_{xy}^2}}{2S_{xy}} = 0.996 //$$

$$\hat{b}_{TLS} = \bar{y} - \hat{a}_{TLS}\bar{x} = -0.0564 //$$

$\hat{a}_{TLS} \approx 1$ $\Rightarrow$ CF is approximately same as EP

$\Rightarrow$ a good estimate

b) EP measurment is 2.31mg/l

$\sigma_{\epsilon y} = 0.08288$ $\qquad$ $\sigma_{\epsilon x} = 0.08323$

$t_{18, 0.975} = 2.10092$

Using OLS:-

we can predict CF measurement

$$\hat{y} = 0.992x - 0.05 = 0.992 \times 2.31 - 0.05$$

$$= 2.42 \text{mg/L}$$

95% C.I $\Rightarrow$ $2.42 \pm (2.10092) \times \sigma_{\epsilon y}$

$(2.067, 2.416)$ → Level of phytic acid

Using IOLS:-

we can predict EP measurement.

$$\hat{x} = 0.999y + 0.0669 = 0.999 \times 2.2 + 0.0669$$

$$= 2.2647 \text{mg/L}$$

95% C.I $\Rightarrow$ $2.2647 \pm (2.10092) \times \sigma_{\epsilon x}$

$\Rightarrow (2.090, 2.44)$ mg/L → level of phytic acid

Using TLS

$$\hat{x}_i = \frac{y_i a + x_i - ab}{a^2+1} = 2.2876$$

$$\hat{y}_i = \frac{b + a x_i + a^2 y_i}{a^2+1} = 2.222$$

Level of phytic acid Using TLS $\Rightarrow \frac{2.2876 + 2.222}{2}$

$\Rightarrow 2.255 \text{mg/L}$

Problem :- 3

Maximum permissible level of CO_2 recommended that global temperature increase should be kept below

$1.5°C \approx 2.7°F$

Cause $\Rightarrow$ effect

CO_2 (x) $\Rightarrow$ Temp (y)

Method one

Using OLS :-

$$\hat{a}_{OLS} = S_{xy}/S_{xx}$$

$$= 0.01639$$

$$\hat{b}_{OLS} = -\hat{a}_{OLS}\bar{x} + \bar{y}$$

$$= -5.2048$$

$\therefore$ $y = 2.7$ $x = ?$

$$x = \frac{-\hat{b}_{OLS} + \bar{y}}{\hat{a}_{OLS}} = 482.294 \text{ ppm}$$

Using matlab

$$S_{xx} = \frac{\Sigma(x_i - \bar{x})^2}{N} = 255.7407$$

$$S_{yy} = \frac{\Sigma(y_i - \bar{y})^2}{N} = 0.0875$$

$$S_{xy} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{N} = 4.1922$$

$\bar{x} = 369.42$ $\bar{y} = 0.8512$

Method 2 :-

Using IOLS :-

$$\hat{a}_{IOLS} = S_{yy}/S_{xy} = 0.02087$$

$$\hat{b}_{IOS} = \bar{y} - \hat{a}_{OLS}\bar{x}$$

$$= -6.8594$$

$\therefore$ $y = 2.7$ $x = ?$

$$x = \frac{y - \hat{b}_{IOLS}}{\hat{a}_{IOLS}} = 458.045 \text{ ppm}$$

Method 3:-

Using TLS:

$$\hat{a}_{TLS} = \frac{S_{yy} - S_{xx} \pm \sqrt{(S_{yy} - S_{xx})^2 + 4S_{xy}^2}}{2S_{xy}}$$

$$= 0.01639 \quad (\text{or}) \quad -60.999$$

✓ ✗

$$\hat{b}_{TLS} = \bar{y} - \hat{a}_{TLS}\bar{x} = -5.2049$$

$\therefore \quad y = 2.7 \qquad x = ?$

$$x = \frac{y - \hat{b}_{TLS}}{\hat{a}_{TLS}} = 482.301 \text{ ppm}$$

Given estimate to be conservative

$\Rightarrow$ IOLS method gives $\underline{\underline{458.045\text{ppm}}}$ of CO_2 as maximum permissible level of CO_2