7

Probability Distribution

7.1 WHAT IS PROBABILITY?

In our daily conversation, we generally use such statements; like perhaps it will rain today, it is a chance that on drawing two cards in succession from a pack of 52 cards, the king may come up first and the queen second; my friend has a fair chance of getting admission in an engineering college. In each case we are not certain of the outcome but we wish to assess the chances of our predictions coming true. For such assertions, the study of probability provides a mathematical framework. The study of probability is essential in every decision making process. Here we explain certain terms relating to probability which are used frequently in the study of probability.

(*a*) **Trial and events.** Let an experiment be repeated under essentially the same conditions and let it result in anyone of the several possible outcomes. Then, the experiment is called a trial and the possible outcomes are known as events or cases.

For example:

(*i*) Tossing of a coin is a trial and the turning up of head or tail is an event.

(*ii*) Throwing a die is a trial getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

(*b*) **Exhaustive events.** The total number of all possible outcomes in any trial is known as *exhaustive events* or *exhaustive cases. If $A_1, A_2, A_3, ..., A_n$ are exhaustive events then $A_1 \cup A_2 \cup A_3 ... \cup A_n = S$, the sample space.*

For example:

(*i*) In tossing a coin, there are two exhaustive cases, head and tail.

(*ii*) In throwing a die, there are 6 exhaustive cases, for anyone of the six faces may turn up.

(*iii*) In throwing two die, the exhaustive cases are $6 \times 6 = 6^2$ for any of the 6 numbers from 1 to 6 on one die can be associated with any of the 6 numbers on the other die.

In general, in throwing n dice, the exhaustive cases are 6^n.

(*c*) **Favourable events or cases.** The cases which entail the happening of an event are said to be *favourable* to the event. It is the total number of possible outcomes in which the specified event happens.

For example:

(*i*) In throwing a die, the number of cases favourable to the appearance of a multiple of 3 are two *viz.* 3 and 6 while the number of cases favourable to the appearance of an even number are three *viz.*, 2, 4 and 6.

(*ii*) In a throw of two dice, the number of cases favourable to getting a sum 6 is 5, *viz.*, (1, 5) ; (5, 1) ; (2, 4); (4, 2) ; (3, 3).

(*d*) **Mutually exclusive events.** Events are said to be *mutually exclusive* or *incompatible* if the happening of anyone of them precludes (*i.e.*, rules out) the happening of all others *i.e.*, if no two or more than two of them can happen simultaneously in the same trial. If $A_1, A_2, ..., A_n$ are mutually exclusive events then $A_1 \cap A_2 \cap A_3 ... \cap A_n = \phi$.

For example:

(*i*) In tossing a coin, the events head and tail are mutually exclusive, since if the outcome is head, the possibility of getting tail in the same trial is ruled out.

(*ii*) In throwing a die, all the six faces numbered, 1, 2, 3, 4, 5, 6 are mutually exclusive since any outcome rules out the possibility of getting any other.

(*e*) **Equally likely events.** Events are said to be *equally likely* if there is no reason to expect anyone in preference to any other.

For example:

(*i*) When a card is drawn from a well shuffled pack, any card may appear in the draw so that the 52 different cases are equally likely.

(*ii*) In throwing a die, all the six faces are equally likely to come.

(*f*) **Independent and dependent events.** Two or more events are said to be *independent* if the happening or non-happening of any one does not depend (or is not effected) by the happening or non-happening of any other. Otherwise they are said to be *dependent*.

For example : If a card is drawn from a pack of well shuffled cards and replaced before drawing the second card, the result of the second draw is independent of the first draw. However, if the first card drawn is not replaced, then, the second draw is dependent on the first draw.

7.2. MATHEMATICAL (OR CLASSICAL) DEFINITION OF PROBABILITY

If a trial results in n *exhaustive, mutually exclusive* and *equally likely cases* and m of them are favourable to the happening of an event E, then the probability of happening of E is given by

$$p \text{ or } P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}.$$

Note 1. Since the number of cases favourable to happening of E is m and the exhaustive number of cases is n, therefore, the number of cases unfavourable to happening of E are $n - m$.

Note 2. The probability that the event E will not happen is given by

$$q \text{ or } P(\bar{E}) = \frac{\text{Unfavourable number of cases}}{\text{Exhaustive number of cases}} = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

$$p + q = 1 \;\; i.e., \;\; P(E) + P(\bar{E}) = 1$$

obviously, p and q are non-negative and cannot exceed unity, *i.e.*, $0 \le p \le 1, 0 \le q \le 1$.

Note 3. If $P(E) = 1$, E is called a *certain event i.e.,* the chance of its happening is cent percent.

If $P(E) = 0$, then E is an *impossible event.*

Note 4. If n cases are favourable to E and m cases are favourable to $\bar{E}$ (*i.e.*, unfavourable to E), then exhaustive number of cases $= n + m$.

$$P(E) = \frac{n}{n+m} \text{ and } P(\bar{E}) = \frac{m}{n+m}$$

We say that "odds *in favour of* E" are $n : m$ and "odds *against* E" are $m : n$.

7.3(a). RANDOM VARIABLE

(P.T.U., May 2005)

Random variable is a real number associated with the outcomes of a random experiment. It can take anyone of the various possible values each with a definite probability. Thus the value of a random variable is known only when the outcome of a random experiment is known.

A random variable is also called 'chance variable' or 'stochastic variable'. Random variables are denoted by capital letters, usually, from the last part of the alphabets, for instance X, Y, Z, ...

Example. *Let X be a random variable which is the number of heads obtained in two independent tosses of an unbiased coin then S = {HH, HT, TH, TT}*

Now, $X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0$

$\therefore$ X can take values 0, 1, 2

Note. It should be clearly understood that the actual values which the event assumes is not a random variable. As in the above example the random variable can take the values 0, 1, or 2 as long as the coin is not tossed. But after it is tossed and we get 2 heads, then 2 is not a random variable. Random variables are of two types.

1. Discrete random variable
2. Continuous random variable

7.3(b). DISCRETE RANDOM VARIABLE

If the random variable X assumes only a finite or countable set of values, within its range and the value which the variable takes depends on chance is known as discrete random variable. For example marks obtained by a student in a test, number of defective apples in a basket, number of accidents taking place on a particular road, number of aces in a draw of 2 cards from a well shuffled deck, are all discrete random variables.

7.3(c). CONTINUOUS RANDOM VARIABLE

If the random variable X assumes infinite and uncountable set of values, in the interval of its range is said to be continuous random variable. For example, height, age, weight of the students in a class are all continuous random variables. In continuous random variable we usually talk of the value in a particular interval and not at a point. Generally discrete random variables represent counted data while continuous random variable represent measured data.

7.4. DISCRETE PROBABILITY DISTRIBUTION (P.T.U., Dec. 2005, Jan. 2009)

Let a random variable X assume values $x_1, x_2, x_3, \ldots\ldots, x_n$ with probabilities $p_1, p_2, p_3, \ldots\ldots, p_n$ respectively, where $P(X = x_i) = p_i \geq 0$ for each x_i and $p_1 + p_2 + p_3 + \ldots\ldots + p_n = \sum_{i=1}^{n} p_i = 1$. Then the table given below:

X:	x_1,	x_2,	$x_3, \ldots\ldots, x_n$
P(X):	p_1,	p_2,	$p_3, \ldots\ldots, p_n$

Shows the discrete Probability Distribution for X *i.e.*, the set of all possible ordered pairs $(x, p(x))$ is called Probability Distribution of X. It spells out how a total probability of 1 is distributed over several values of the random variable. For example the probability distribution of a pair of fair dice tossed is given by

X :	2	3	4	5	6	7	8	9	10	11	12
P(X) :	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

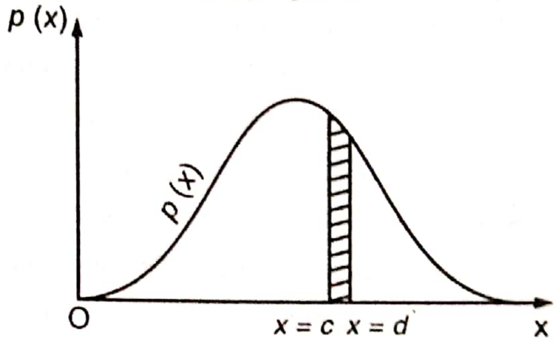
7.5. CONTINUOUS PROBABILITY DISTRIBUTION

When random variable X takes every value in an interval, it gives rise to continuous distribution of X. There is a major difference between discrete probability and continuous probability. In discrete terms, the probability associates with an event is meaningful but with continuous events (where the number of events is infinitely large) the probability that a specific event wiH occur is practically zero. It does not make any sense. For this reason, instead of tending the probability that x equals some value, we find probability of x falling in a small interval.

Thus the probability distribution of a continuous random variable X, taking values on $[a, b]$ is defined by a function $p(x)$, called the probability density function which satisfies the following properties

(i) $p(x) \geq 0 \ \forall \ x$ in $[a, b]$

(ii) For two distinct numbers c and d lying in the interval $[a, b]$; $P(c \leq x \leq d)$ represents the area under the probability curve between $x = c$ and $x = d$ (shown in the adjoining figure) [Area can be obtained by integration discussed in integral calculus $= \int_c^d p(x)\,dx$]



(iii) Total area under probability curve is 1. *i.e.*, $\int_{-\infty}^{\infty} p(x)\,dx = 1$

7.6. MEAN AND VARIANCE OF PROBABILITY DISTRIBUTION

Let

$X:$	$x_1,$	$x_2,$	$x_3, \ldots\ldots\ldots, x_n$
$P(X):$	$p_1,$	$p_2,$	$p_3, \ldots\ldots\ldots, p_n$

be a discrete Probability Distribution.

We denote the *mean* by μ and define $\mu = \dfrac{\Sigma p_i x_i}{\Sigma p_i} = \Sigma p_i x_i \quad \because \Sigma p_i = 1$

Other names for the mean are *average or expected value E(X).*

We denote the *variance* by σ^2 and define $\sigma^2 = \Sigma p_i (x_i - \mu)^2$

If μ is not a whole number, then $\sigma^2 = \Sigma p_i x_i^2 - \mu^2$

Since

$$
\begin{aligned}
\sigma^2 &= \Sigma p_i (x_i - \mu)^2 \\
&= \Sigma p_i (x_i^2 - 2\mu x_i + \mu^2) \\
&= \Sigma p_i x_i^2 - 2\mu \Sigma p_i x_i + \mu^2 \Sigma p_i \\
&= \Sigma p_i x_i^2 - 2\mu . \mu + \mu^2 . 1 \qquad \because \Sigma p_i x_i = \mu, \ \Sigma p_i = 1 \\
&= \Sigma p_i x_i^2 - \mu^2 .
\end{aligned}
$$

Standard deviation $\sigma = +\sqrt{\text{Variance}}$.

7.7. THEORETICAL PROBABILITY DISTRIBUTIONS

Distributions which are not obtained by actual observations or experiments but are mathematically deduced on certain assumptions are called theoretical distributions.

There are many types of theoretical probability distributions but at present we shall consider only three which are of great importance :

(i) Binomial Distribution (or Bernoulli's Distribution);

(ii) Poisson's Distribution;

(iii) Normal Distribution.

BINOMIAL (OR BERNOULLI'S) DISTRIBUTION

7.8. BINOMIAL PROBABILITY DISTRIBUTION

(P.T.U., May 2009)

Binomial distribution is concerned with trials of a repetitive nature in which only the occurrance or non-occurrance, success or failure acceptance or rejection, yes or no of a particular event is of interest.

Let there be n independent trials in an experiment. Let a random variable X denote the number of successes in these n trials. Let p be the probability of a success and q that of a failure in a single trial so that $p+q=1$. Let the trials be independent and p be constant for every trial.

Let us find the probability of r successes in n trials

r successes can be obtained in n trials in ^nC_r ways.

$$\therefore \quad P(X=r) = {}^nC_r \, P(\underbrace{S\,S\,S.....S}_{r \text{ times}}) \quad P(\underbrace{F\,F\,F\,.....\,F}_{(n-r) \text{ times}})$$

('S' Stands for success and 'F' stands for failure)

$$= {}^nC_r \underbrace{P(S)P(S).....P(S)}_{r \text{ factors}} \quad \underbrace{P(F)\,P(F)......P(F)}_{(n-r) \text{ factors}}$$

$$= {}^nC_r \underbrace{p\,p\,p.....\,p}_{r \text{ factors}} \quad \underbrace{q\,q\,q\,.....q}_{(n-r) \text{ factors}} \qquad ...(1)$$

Hence $\mathbf{P(X=r) = {}^nC_r\, q^{n-r} p^r}$, where $\mathbf{p+q=1 \text{ and } r=0, 1, 2,.....,n.}$

The distribution (1) is called the *Binomial Probability Distribution* and X is called the *binomial variate.*

Note 1. P(X = r) is usually written as P(r).

Note 2. The successive probabilities P(r) in (1) for r = 0, 1, 2, ...,n are

$$^nC_0q^n, {}^nC_1\, q^{n-1}\, p, {}^nC_2q^{n-2}\,p^2, ..., {}^nC_n\,p_n.$$

which are the successive terms of the binomial expansion of $(q+p)^n$. That is why this distribution is called "binomial" distribution.

Note 3. n and p occurring in the Binomial Distribution are called the *parameters* of the distribution.

Note 4. Binomial Distribution is a Discrete Distribution (P.T.U., Dec. 2004)

∵ We know that in a Binomial Distribution if the random variable X takes the values 0, 1, 2, ... n, then

$P(X=r) = {}^nC_r\, q^{n-r}\, p^r$; $r = 0, 1, 2, ..., n$ $p, q > 0$ and $p+q=1$

i.e., X	0	1	2	3	...	n
then P (X)	$^nC_0\,q^n$	$^nC_1 q^{n-1}.p$	$^nC_2\,q^{n-2}.p^2$	$^nC_3\,q^{n-3}.p^3$	...	$^nC_n\,p^n$

is a discrete probability distribution for X

Note 5. If n trials constitute an experiment and the experiment is repeated N times, then the frequencies of 0, 1, 2, 3, ..., n successes are given by N . P (X = 0), N . P (X = 1), N . P (X = 2) ... N . P (X = n).

where $\Sigma P(X) = {}^nC_0\, q^n + {}^nC_1\, q^{n-1}\, p + {}^nC_2\, q^{n-2}\, p^2 \,.....\, {}^nC_x\, p^n = (q+p)^n = 1$

Note 6. Binomial probability distribution is also known as Binomial frequency distribution.

7.9. CONDITIONS UNDER WHICH BINOMIAL DISTRIBUTION IS APPLICABLE

(P.T.U., May 2011, Dec. 2013)

Binomial distribution is $P(X=r) = {}^nC_r q^{n-r} p^r$, where $p+q=1$ and $r = 0, 1, 2, 3, ..., n$

Conditions:

(i) n, the number of trials is an experiment should be finite and fixed.

(ii) In every trial, there should be only two mutually exclusive and exhaustive outcomes—success or failure.

(iii) The trials should be independent. The out come of one trial does not affect the other trial.

(iv) p, the probability of success from trial to trial is fixed and q, the probability of failure $= 1-p$. This is same in all trials.

7.10. PROPERTIES OF A BINOMIAL DISTRIBUTION

(P.T.U., Dec. 2004)

Binomial Distribution (B.D) is $P(X=r) = {}^nC_r q^{n-r} p^r$ where $p+q=1$ and $r = 0, 1, 2, 3, ..., n$

Properties:

(i) B.D is a discrete Probability Distribution in which the random variable takes only the discrete values like 0, 1, 2, 3, ...

(ii) B.D spells out, how a total probability of 1 is distributed over several values of random variable.

(iii) B.D has two constants n and p, n- the number of trials and p- the probability of success is a single trial. Entire B.D can be determined if n and p are known because $q = 1-p$.

(iv) Mean and variance of B.D are np and npq respectively and variance is always less than the mean.

7.11. APPLICATIONS OF BINOMIAL PROBABILITY DISTRIBUTION

Binomial distribution is applied to problems concerning

(i) To find number of defectives in a sample from production line

(ii) To estimate the realibility of the system.

(iii) To find number of rounds fired from a gun hitting a target

(iv) For radar detection.

7.12. RECURRENCE FORMULA FOR THE BINOMIAL DISTRIBUTION

In a Binomial Distribution,

$$P(r) = {}^nC_r q^{n-r} p^r = \frac{n!}{(n-r)!r!} q^{n-r} p^r$$

$$P(r+1) = {}^nC_{r+1} q^{n-r-1} p^{r+1} = \frac{n!}{(n-r-1)!(r+1)!} q^{n-r-1} p^{r+1}$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{(n-r)!}{(n-r-1)!} \times \frac{r!}{(r+1)!} \times \frac{p}{q} = \frac{(n-r)\times(n-r-1)!}{(r-r-1)} \times \frac{r!}{(r+1)\times r!} \times \frac{p}{q} = \frac{n-r}{r+1} \cdot \frac{p}{q}$$

$$\Rightarrow \quad P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} P(r)$$

which is the required recurrence formula. Applying this formula successively, we can find P(1), P(2), P(3), ..., if P(0) is known.

7.13. MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION

For the Binomial Distribution, $P(r) = {}^nC_r q^{n-r} p^r$

$$\text{Mean } \mu = \sum_{r=0}^{n} rP(r) = \sum_{r=0}^{n} r.{}^nC_r q^{n-r} p^r$$

$$= 0 + 1.{}^nC_1 q^{n-1} p + 2.{}^nC_2 q^{n-2} p^2 + 3.{}^nC_3 q^{n-3} p^3 + ... + n.{}^nC_n p^n$$

$$= nq^{n-1} p + 2.\frac{n(n-1)}{2.1} q^{n-2} p^2 + 3.\frac{n(n-1)(n-2)}{3.2.1} q^{n-3} p^3 + ... + n.p^n$$

$$= nq^{n-1} p + n(n-1) q^{n-2} p^2 + \frac{n(n-1)(n-2)}{2.1} q^{n-3} p^3 + ... + n.p^n$$

$$= np\left[q^{n-1} + (n-1) q^{n-2} p + \frac{(n-1)(n-2)}{2.1} q^{n-3} p^2 + ... + p^{n-1}\right]$$

$$= np\left[{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2} p + {}^{n-1}C_2 q^{n-3} p^2 + ... + {}^{n-1}C_{n-1} p^{n-1}\right]$$

$$= np(q+p)^{n-1} = np \qquad (\because\ p+q=1)$$

Hence, the mean of the Binomial Distribution is np.

$$\textit{Variance } \sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2 = \sum_{r=0}^{n} [r + r(r-1)] P(r) - \mu^2$$

$$= \sum_{r=0}^{n} rP(r) + \sum_{r=0}^{n} r(r-1) P(r) - \mu^2 = \mu + \sum_{r=2}^{n} r(r-1)\, {}^nC_r q^{n-r} p^r - \mu^2$$

(Since, the contribution due to $r = 0$ and $r = 1$ is zero)

$$= \mu + \left[2.1.{}^nC_2 q^{n-2} p^2 + 3.2.{}^nC_3 q^{n-3} p^3 + ... + {}^nC_n p^n\right] - \mu^2$$

$$= \mu + \left[2.1.\frac{n(n-1)}{2.1} q^{n-2} p^2 + 3.2\frac{n(n-1)(n-2)}{3.2.1} q^{n-3} p^3 + ... + n(n-1) p^n\right] - \mu^2$$

$$= \mu + \left[n(n-1) q^{n-2} p^2 + n(n-1)(n-2) q^{n-3} p^3 + ... + n(n-1) p^n\right] - \mu^2$$

$$= \mu + n(n-1)p^2\left[q^{n-2} + (n-2) q^{n-3} p + ... + p^{n-2}\right] - \mu^2$$

$$= \mu + n(n-1)p^2\left[{}^{n-2}C_0 q^{n-2} + {}^{n-2}C_1 q^{n-3} p + ... + {}^{n-2}C_{n-2} p^{n-2}\right] - \mu^2$$

$$= \mu + n(n-1)p^2(q+p)^{n-2} - \mu^2 = \mu + n(n-1)p^2 - \mu^2$$

$$= np + n(n-1)p^2 - n^2p^2 \qquad [\because q+p=1]$$

$$= pn[1+(n-1)p - np] = np[1-p] = npq. \qquad [\because \mu = np]$$

Hence, the variance of the Binomial Distribution is npq.

Standard deviation of the Binomial Distribution is $\sqrt{npq}$.

Note. Variance of the Binomial Distribution is less than the mean of the Binomial Distribution since variance $= npq$ and as $q < 1$ $\therefore$ variance $< np$, the mean. Hence variance $<$ mean.

7.14. FITTING A BINOMIAL DISTRIBUTION

From the given data determine the following:

(i) n ; which is one less than the number of variates in the given data

(ii) N ; the total frequency *i.e.*, the sum of all the frequencies

(iii) μ ; the mean $= \dfrac{\Sigma f(x)}{\Sigma f}$

(iv) p ; the number of successes; p is obtained from $np = \mu$ *i.e.*, $p = \dfrac{\mu}{n}$.

Also, find $q = 1 - p$.

(v) To get expected frequencies, find the successive terms in the expansion of $N(q+p)^n$

(vi) Complete the table, showing variates 0, 1, 2, 3, ..., n in 1st column, expected or theoretical frequencies represented by $NP(r) = N\,{}^nC_r\, q^{n-r}\, p^r$ $(r = 0, 1, 2, 3, ..., n)$ in 2nd column and the expected frequencies in round figures in 3rd column.

Note. The sum total of expected frequencies should also be equal to N.

ILLUSTRATIVE EXAMPLES

Example 1. (a) *During war, 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely?* **(P.T.U., Dec. 2003)**

(b) *With the usual notation, find p for a binomial variate X if $n = 6$ and $9\,P(X = 4) = P(X = 2)$.* **(P.T.U., May 2010)**

Solution. (a) p, the probability of a ship arriving safely $= 1 - \dfrac{1}{9} = \dfrac{8}{9}$, $\quad q = \dfrac{1}{9}$; $\quad n = 6$

Binomial Distribution is $\left(\dfrac{1}{9} + \dfrac{8}{9}\right)^6$

The probability that exactly 3 ships arrive safely $= {}^6C_3 \left(\dfrac{1}{9}\right)^3 \left(\dfrac{8}{9}\right)^3 = \dfrac{10240}{9^6}$.

(b) The Binomial probability distribution for $n = 6$ and variate $X = r$ is

$$P(X = r) = {}^6C_r\, p^r q^{6-r};\ r = 0, 1, 2, 3, ..., 6$$

Given $\qquad 9P(X = 4) = P(X = 2)$

$$\Rightarrow \quad 9\,{}^6C_4\,p^4q^2 = {}^6C_2\,p^2q^4$$

$$\Rightarrow \quad 9\,p^4q^2 = p^2q^4 \qquad \because \quad {}^6C_4 = {}^6C_2 = 15$$

$$\Rightarrow \quad 9\,p^2 = q^2 = (1-p)^2 \qquad \because \quad p+q=1$$

$$\Rightarrow \quad 9\,p^2 = 1 + p^2 - 2p$$

or $$8\,p^2 + 2p - 1 = 0$$

or $$p = \frac{-2 \pm \sqrt{4+32}}{16} = \frac{-2 \pm 6}{16}$$

$$p = \frac{1}{4}, -\frac{1}{2} \quad \text{But } p \neq \frac{-1}{2} \quad \because \text{ probability cannot be negative}$$

$$\therefore \quad p = \frac{1}{4}.$$

Example 2. (*a*) *If on an average, one ship out of 10 is wrecked, find the probability that out of five ships expected to arrive the port, at least four will arrive safely.* **(P.T.U., Dec. 2004, Dec. 2005)**

(*b*) *A coin is tossed four times. What is the probability of getting more than two heads?* **(P.T.U., Dec. 2006)**

Solution. (*a*) p, the probability of a ship arriving safely $= 1 - \frac{1}{10} = \frac{9}{10}$

$$q = 1 - \frac{9}{10} = \frac{1}{10}$$

Binomial Distribution is $\left(\frac{1}{10} + \frac{9}{10}\right)^5$

Probability that at least four ships out of five arrive safely

$$= P(4) + P(5)$$

$$= {}^5C_4\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^4 + {}^5C_5\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^5$$

$$= 5\,.\,\frac{1}{10}\left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^5$$

$$= \left(\frac{9}{10}\right)^4\left(\frac{1}{2} + \frac{9}{10}\right) = \left(\frac{9}{10}\right)^4\frac{14}{10}$$

$$= \left(\frac{9}{10}\right)^4 . \frac{7}{5} = 0.91854.$$

(*b*) p, the probability of getting head $= \frac{1}{2}$

q, the probability of getting tail $= \frac{1}{2}$

$n = 4$; Binomial Distribution $= \left(\frac{1}{2} + \frac{1}{2}\right)^4$

Probability of getting more than two heads $= P(3) + P(4)$

$$= {}^4C_3\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4[4+1] = \frac{5}{16}.$$

Example 3. *Assume that on the average one telephone number out of fifteen called between 2 P.M and 3 P.M on week-days is busy. What is the probability that if 6 randomly selected telephone numbers are called (i) not more than three, (ii) at least three of them will be busy?*

Solution. p, the probability of a telephone number being busy between 2 P.M and 3 P.M on week-days $= \frac{1}{15}$

$$q = 1 - \frac{1}{15} = \frac{14}{15}, n = 6; \text{ Binomial Distribution is } \left(\frac{14}{15} + \frac{1}{15}\right)^6$$

The probability that not more than three will be busy

$$= P(0) + P(1) + P(2) + P(3)$$

$$= {}^6C_0\left(\frac{14}{15}\right)^6 + {}^6C_1\left(\frac{14}{15}\right)^5\left(\frac{1}{15}\right) + {}^6C_2\left(\frac{14}{15}\right)^4\left(\frac{1}{15}\right)^2 + {}^6C_3\left(\frac{14}{15}\right)^3\left(\frac{1}{15}\right)^3$$

$$= \frac{(14)^3}{(15)^6}[2744 + 1176 + 210 + 20] = \frac{2744 \times 4150}{(15)^6} = 0 \cdot 9997$$

The probability that at least three of them will be busy

$$= P(3) + P(4) + P(5) + P(6)$$

$$= {}^6C_3\left(\frac{14}{15}\right)^3\left(\frac{1}{15}\right)^3 + {}^6C_4\left(\frac{14}{15}\right)^2\left(\frac{1}{15}\right)^4 + {}^6C_5\left(\frac{14}{15}\right)\left(\frac{1}{15}\right)^5 + {}^6C_6\left(\frac{1}{15}\right)^6 = 0.005.$$

Example 4. *If the probability of a defective bolt is 0.1, find the standard deviation (S.D.) for the defective bolts in a total of 400.* **(P.T.U., May 2006)**

Solution. p = Probability of defective bolts $= 0.1 = \frac{1}{10}$

q = Probability of non-defective bolts $= 1 - \frac{1}{10} = \frac{9}{10}$

$n = 400$

$$\text{S.D.} = \sqrt{npq} = \sqrt{400 \times \frac{1}{10} \times \frac{9}{10}} = 6.$$

Example 5. *The mean and variance of Binomial variable X are 2 and 1 respectively. Find the probability that X takes a value > 1.* **(P.T.U., May 2014)**

Solution. Mean of Binomial variable $X = np = 2$

Variance of Binomial variable $X = npq = 1$

Dividing; $$q = \frac{1}{2}$$

Also we know that $p + q = 1$

$\therefore$ $$p = \frac{1}{2}$$

$$np = 2 \Rightarrow n = 4$$

The Binomial probability distribution for $n = 4$ and variate $X = r$ is

$$P(X = r) = {}^nC_r\, q^{n-r}\, p^r = {}^4C_r\left(\frac{1}{2}\right)^{4-r}\left(\frac{1}{2}\right)^r$$

$$= {}^4C_r \frac{1}{2^4} = \frac{1}{16}\, {}^4C_r$$

Now the probability that X takes *a* value > 1 is

$$P(X=2)+P(X=3)+P(X=4)$$

$$= \frac{1}{16}\left\{{}^4C_2 + {}^4C_3 + {}^4C_4\right\} = \frac{1}{16}(6+4+1) = \frac{11}{16}.$$

Example 6. *Six dice the thrown 729 times. How many times do you expect at least three dice to show a five or six?* (P.T.U., Jan 2009)

Solution. p = the chance of getting 5 or 6 with one die $= \frac{2}{6} = \frac{1}{3}$

$$q = 1 - \frac{1}{3} = \frac{2}{3},\ n = 6,\ N = 729$$

Since dice are in sets of 6 and there are 729 sets.

The Binomial Distribution is $N(q+p)^n = 729\left(\frac{2}{3}+\frac{1}{3}\right)^6$

The expected number of times *as least three* dice showing five or six

$$= 729\left[{}^6C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 + {}^6C_4\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4 + {}^6C_5\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5 + {}^6C_6\left(\frac{1}{3}\right)^6\right]$$

$$= \frac{729}{3^6}[160+60+12+1] = 233.$$

Example 7. *Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) at most two girls? Assume equal probabilities for boys and girls.*

Solution. Since probabilities for boys and girls are equal.

p = probability of having a boy $= \frac{1}{2}$; q = probability of having a girl $= \frac{1}{2}$

$n = 4$, $N = 800$ $\therefore$ The Binomial Distribution is $800\left(\frac{1}{2}+\frac{1}{2}\right)^4$.

(*i*) The expected number of families having 2 *boys* and 2 girls

$$= 800\, {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 800 \times 6 \times \frac{1}{16} = 300.$$

(*ii*) The expected number of the families having *at least one boy*

$$= 800\left[{}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= 800 \times \frac{1}{16}[4 + 6 + 4 + 1] = 750.$$

(*iii*) The expected number of families having no girl *i.e.*, having 4 boys $= 800.{}^4C_4\left(\frac{1}{2}\right)^4 = 50$.

(*iv*) The expected number of families having *at most two girls i.e.*, having *at least 2 boys.*

$$= 800\left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right] = 800 \times \frac{1}{16}[6 + 4 + 1] = 550.$$

Example 8. *What is the probability of getting the king of hearts from a pack of cards at least once in 52 cards?* **(P.T.U., Dec. 2005)**

Solution. p = the probability of getting the king of hearts from a pack of 52 cards $= \frac{1}{52}$

$$\therefore \qquad q = 1 - \frac{1}{52} = \frac{51}{52}$$

n = the number of trials = 52

Binomial Distribution forgetting king of hearts

$$= (q + p)^{52} = \left(\frac{51}{52} + \frac{1}{52}\right)^{52}$$

Probability Distribution for getting at least one king of hearts = 1 – probability of getting no king of hearts

$$= 1 - P(X = 0)$$

$$= 1 - {}^{52}C_0\left(\frac{51}{52}\right)^{52}\left(\frac{1}{52}\right)^0$$

$$= 1 - \left(\frac{51}{52}\right)^{52} = 1 - 0.3643 - 0.6357.$$

Example 9. *The probability that a pen manufactured by a company will be defective is* $\frac{1}{10}$. *If 12 such pens are manufactured, find the probability that*

(*i*) *exactly two will be defective*

(*ii*) *at least two will be defective*

(*iii*) *none will be defective.*

Solution. p = Probability of defective pens $= \frac{1}{10}$

q = Probability of non-defective pens $= 1 - \frac{1}{10} = \frac{9}{10}$

$$n = 12$$

Binomial Distribution for defective pens $= \left(\frac{9}{10}+\frac{1}{10}\right)^{12}$

(*i*) Probability that exactly two will be defective

$$= {}^{12}C_2\left(\frac{9}{10}\right)^{10}\left(\frac{1}{10}\right)^2 = \frac{12\times 11}{2}\ \frac{9^{10}}{10^{12}} = 0.2301.$$

(*ii*) Probability that at least two will be defective

$= 1 -$ Probability that at the most one is defective

$$= 1-[P(0)+P(1)] = 1-\left[{}^{12}C_0\left(\frac{9}{10}\right)^{12} + {}^{12}C_1\left(\frac{9}{10}\right)^{11}\left(\frac{1}{10}\right)\right]$$

$$= 1-\left(\frac{9}{10}\right)^{11}\left(\frac{9}{10}+\frac{12}{10}\right) = 1-\frac{(9)^{11}\cdot 21}{(10)^{12}} = 0.3412.$$

(*iii*) Probability that non will be defective

$$= {}^{12}C_0\left(\frac{9}{10}\right)^{12}\left(\frac{1}{10}\right)^0 = \left(\frac{9}{10}\right)^2 = (0.9)^{12} = 0.2833.$$

Example 10. *Fit a Binomial distribution to the following data and compare the theoretical freq with the actual ones*

x	*0*	*1*	*2*	*3*	*4*
f	*28*	*62*	*46*	*10*	*4*

Solution. For B.D, we have to find p which is not given

$\therefore$ We first find $\quad \mu = \dfrac{\Sigma f\,(x)}{\Sigma f}$

$$\therefore \quad \mu = \frac{0.28+1.62+2.46+3.10+4.4}{28+62+46+10+4} = \frac{200}{150} = \frac{4}{3}$$

$\mu = np \;\therefore\; p = \dfrac{4}{3n}$, where $n = 4$ (one less than number of variates in given table)

$$\therefore \quad p = \frac{1}{3};\; q = 1-\frac{1}{3} = \frac{2}{3}$$

N = Total frequency = 150

Binomial Distribution $= 150\left(\frac{2}{3}+\frac{1}{3}\right)^4$

x	$NP(r)$	Theoretical frequency	Actual frequency
0	$150\times {}^4C_0\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 = 150\times\frac{16}{81} = 29.6$	30	28
1	$150\times {}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1 = 150\times 4\times\frac{8}{81} = 59.2$	59	62
2	$150\times {}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = 150\times 6\times\frac{4}{81} = 44.4$	44	46
3	$150\times {}^4C_3\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^3 = 150\times 4\times\frac{2}{81} = 14.8$	15	10
4	$150\times {}^4C_4\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^4 = 150\times\frac{1}{81} = 1.9$	2	4
		Total = 150	150

Example 11. *The Probability of a man hitting a target is $\frac{1}{4}$. If he fires 7 times, then what is the probability of his hitting the target at least twice? Also find that how many times must he fire so that the probability of his hitting the target at least once is greater than $\frac{2}{3}$?* **(P.T.U., May 2010)**

Solution. $p =$ the probability of hitting the target $= \frac{1}{4}$

$$q = 1 - p = \frac{3}{4}$$

$$n = 7$$

Binomial Distribution for hitting the target

$$= (q+p)^n = \left(\frac{3}{4} + \frac{1}{4}\right)^7$$

Probability Distribution for hitting the target at least twice

$$= 1 - P(X=0) - P(X=1)$$

$$= 1 - {}^7C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^7 - {}^7C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6$$

$$= 1 - \frac{3^7}{4^7} - 7\frac{3^6}{4^7} = 1 - \frac{3^6}{4^7}(10) = \frac{4547}{8192}$$

Now let he fires the target n times so that the probability of his hitting at least once is $> \frac{2}{3}$

$$1 - P(X=0) > \frac{2}{3}$$

or $\quad 1 - {}^nC_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^n > \frac{2}{3}$ or $1 - \left(\frac{3}{4}\right)^n > \frac{2}{3}$ or $\frac{1}{3} > \left(\frac{3}{4}\right)^n$

or $\quad 4^n > 3^{n+1}$ which holds good for $n = 4$

$\because \quad$ for $n = 1, 2, 3, 4^n < 3^{n+1} \quad \therefore \quad n = 4$

TEST YOUR KNOWLEDGE

1. Ten coins are tossed simultaneously. Find the probability of getting at least seven heads.

   [**Hint:** $p = \frac{1}{2}, q = \frac{1}{2}, n = 10$; Required Probability = (P(7) + P(8) + P(9) + P (10))]

2. The probability of any ship of a company being destroyed on a certain voyage is 0·02. The company owns 6 ships for the voyage. What is the probability of

   (*i*) losing one ship (*ii*) losing at most two ships (*iii*) losing none?

   [**Hint:** $p = 0.02$, $q = 0.98$, $n = 6$; Required probability (*i*) P(1), (*ii*) P(0) + P(1) + P(2) (*iii*) P(0)]

3. The probability that a man aged 60 will live to be 70 is 0·65. What is the probability that out of ten men now 60, at least 7 would live to be 70?

   [**Hint:** $p = 0.65$, $q = 0.35$, $n = 10$, Required probability = P(7) + P(8) + P(9) + P(10)]

4. The incidence of occupational decrease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of six workers chosen at random, four or more will suffer from the disease? [**Hint:** p = Probability of suffering from disease $= \frac{20}{100} = \frac{1}{5}, q = \frac{4}{5}, n = 6$ **(P.T.U., Dec. 2003)**

   Required Probability = P(4) + P(5) + P(6)]

5. A pair of dice is thrown 200 times. If getting a sum of 9 is considered a success, find the mean and variance of number of successes. [**Hint:** p = Probability of number of successes $= \frac{4}{36} = \frac{1}{9}, q = \frac{8}{9}, n = 200$

   mean $= np = 200 \times \frac{1}{9}$, variance $= npq = \frac{200}{9} \cdot \frac{8}{9}$]

6. If the chance that one of ten telephone lines is busy at an instant is 0.2.

   (*i*) What is the chance that 5 of the lines are busy?

   (*ii*) What is the probability that all the lines are busy?

7. If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive at least 4 will arrive safely.

8. Determine the Binomial distribution whose mean is 9 and standard deviation is 3/2.

9. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn, one by one, with replacement, what is the probability that:

   (*i*) none is white (*ii*) all are white (*iii*) at least one is white (*iv*) only 2 are white ?

   [**Hint:** p = Prob. of white balls $= \frac{5}{20}, = \frac{1}{4}$, q = Prob. of non-white balls $1 - \frac{1}{4} = \frac{3}{4}$.

   Balls to be drawn = 4 *i.e.*, $n = 4$.