**Definition (Random variable)**

A **random variable** $X$ is a function defined on the sample space $S$ of an experiment. Its values are real numbers. For every number $a$ the probability $P(X = a)$ with which $X$ assumes $a$ is defined. Similarly, for any interval $I$ the probability $P(X \in I)$ with which $X$ assumes any value in $I$ is defined.

Although this definition is very general, practically only a very small number of distributions will occur over and over again in applications.

From (1) we obtain the fundamental formula for the probability corresponding to an interval $a < x \leqq b$,

$$P(a < X \leqq b) = F(b) - F(a). \tag{2}$$

This follows because $X \leqq a$ ("*$X$ assumes any value not exceeding $a$*") and $a < X \leqq b$ ("*$X$ assumes any value in the interval $a < x \leqq b$*") are mutually exclusive events, so that by (1) and Axiom 3,

$$F(b) = P(X \leqq b) = P(X \leqq a) + P(a < X \leqq b) = F(a) + P(a < X \leqq b)$$

and subtraction of $F(a)$ on both sides gives (2). ◀

## Discrete Random Variables and Distributions

By definition, a random variable $X$ and its distribution are **discrete** if $X$ assumes only finitely many or at most countably many values $x_1, x_2, x_3, \cdots$, called the **possible values** of $X$, with positive probabilities $p_1 = P(X = x_1)$, $p_2 = P(X = x_2)$, $p_3 = P(X = x_3), \cdots$, whereas the probability $P(X \in I)$ is zero for any interval $I$ containing no possible value.

Obviously, the discrete distribution is also determined by the **probability function** $f(x)$ of $X$, defined by

$$f(x) = \begin{cases} p_j & \text{if } x = x_j \\ 0 & \text{otherwise} \end{cases} \qquad (j = 1, 2, \cdots), \tag{3}$$

From this we get the values of the distribution function $F(x)$ by taking sums,

$$F(x) = \sum_{x_j \leqq x} f(x_j) = \sum_{x_j \leqq x} p_j \tag{4}$$

where for any given $x$ we sum all the probabilities $p_j$ for which $x_j$ is smaller than or equal to that $x$. This is a **step function** with upward jumps of size $p_j$ at the possible values $x_j$ of $X$ and constant in between.

**EXAMPLE 1 Probability function and distribution function**

Figure 481 shows the probability function $f(x)$ and the distribution function $F(x)$ of the discrete random variable

*$X$ = Number a fair die turns up.*

$X$ has the possible values $x = 1, 2, 3, 4, 5, 6$ with probability 1/6 each. At these $x$ the distribution function has upward jumps of magnitude 1/6. Hence from the graph of $f(x)$ we can construct the graph of $F(x)$, and conversely.

In Figure 481 (and the next one) at each jump the ***fat dot*** indicates the ***function value at the jump!*** ◀

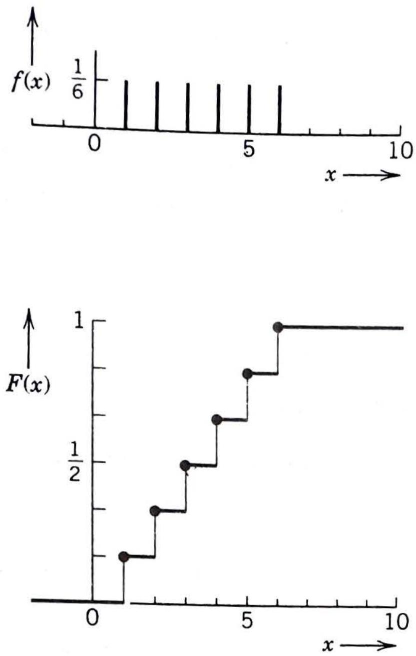


**Fig. 481.** Probability function $f(x)$ and distribution function $F(x)$ of the random variable $X$ = *Number obtained in tossing a fair die once*

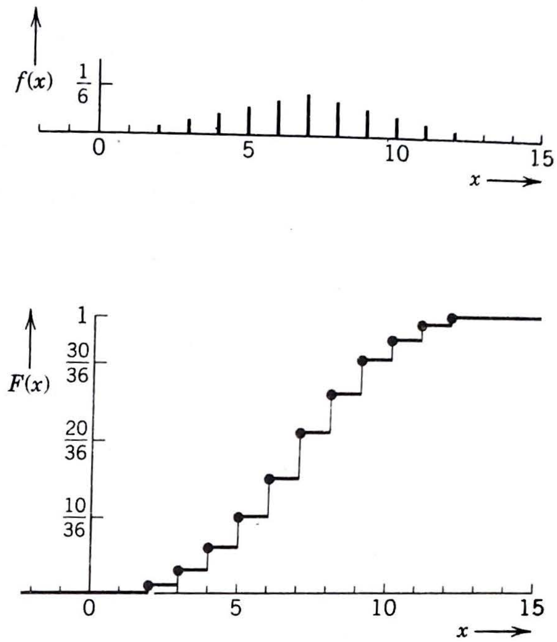


**Fig. 482.** Probability function $f(x)$ and distribution function $F(x)$ of the random variable $X$ = *Sum of the two numbers obtained in tossing two fair dice once*

**EXAMPLE 2 Probability function and distribution function**

The random variable $X$ = *Sum of the two numbers two fair dice turn up* is discrete and has the possible values $2\ (= 1 + 1), 3, 4, \cdots, 12\ (= 6 + 6)$. There are $6 \cdot 6 = 36$ equally likely outcomes $(1, 1), (1, 2), \cdots, (6, 6)$, where the first number is that shown on the first die and the second number that on the other die. Each such outcome has probability 1/36. Now $X = 2$ occurs in the case of the outcome $(1, 1)$; $X = 3$ in the case of the two outcomes $(1, 2)$ and $(2, 1)$; $X = 4$ in the case of the three outcomes $(1, 3), (2, 2), (3, 1)$; and so on. Hence $f(x) = P(X = x)$ and $F(x) = P(X \leqq x)$ have the values

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |
| $F(x)$ | 1/36 | 3/36 | 6/36 | 10/36 | 15/36 | 21/36 | 26/36 | 30/36 | 33/36 | 35/36 | 36/36 |

Figure 482 shows a bar chart of this function and the graph of the distribution function, which is again a step function, with jumps (of different height!) at the possible values of $X$. ◀

Two useful formulas for discrete distributions are readily obtained as follows. For the probability corresponding to intervals we have from (2) and (3)

$$\boxed{P(a < X \leqq b) = \sum_{a < x_j \leqq b} p_j} \qquad (X \text{ discrete}). \tag{5}$$

This is the sum of all probabilities $p_j$ for which $x_j$ satisfies $a < x_j \leqq b$. (Be careful about $<$ and $\leqq$!) From th[...] $) = 1$ (Sec. 22.3) we obtain the following formula.

(6) $$\boxed{\sum_j p_j = 1}$$ (sum of all probabilities).

**EXAMPLE 3 Illustration of formula (5)**

In Example 2, compute the probability of a sum of at least 4 and at most 8.

*Solution.* $P(3 < X \leqq 8) = F(8) - F(3) = \frac{26}{36} - \frac{3}{36} = \frac{23}{36}$. ◀

**EXAMPLE 4 Waiting problem. Countably infinite sample space**

In tossing a fair coin, let $X$ = *Number of trials until the first head appears.* Then, by independence of events (Sec. 22.3),

$$\begin{aligned} P(X = 1) &= P(H) &&= \tfrac{1}{2} && && (H = \text{Head}) \\ P(X = 2) &= P(TH) &&= \tfrac{1}{2}\cdot\tfrac{1}{2} &&= \tfrac{1}{4} && (T = \text{Tail}) \\ P(X = 3) &= P(TTH) &&= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} &&= \tfrac{1}{8}, && \text{etc.} \end{aligned}$$

and in general $P(X = n) = \left(\frac{1}{2}\right)^n$, $n = 1, 2, \cdots$. Also, (6) can be confirmed by the sum formula for the geometric series,

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = -1 + \frac{1}{1 - \frac{1}{2}} = -1 + 2 = 1.$$ ◀

## Continuous Random Variables and Distributions

Discrete random variables appear in experiments in which we ***count*** (defectives in a production, days of sunshine in Chicago, customers standing in a line, etc.). Continuous random variables appear in experiments in which we ***measure*** (lengths of screws, voltage in a power line, Brinell hardness of steel, etc.). By definition, a random variable $X$ and its distribution are *of continuous type* or, briefly, **continuous,** if its distribution function $F(x)$ [defined in (1)] can be given by an integral

(7) $$\boxed{F(x) = \int_{-\infty}^{x} f(v)\, dv}$$

(we write $v$ because $x$ is needed as the upper limit of the integral) whose integrand $f(x)$, called the **density** of the distribution, is nonnegative, and is continuous, perhaps except for finitely many $x$-values. Differentiation gives the relation of $f$ to $F$ as

(8) $$f(x) = F'(x)$$

for every $x$ at which $f(x)$ is continuous.

From (2) and (7) we obtain the very important formula for the probability corresponding to an interval:

(9) $$\boxed{P(a < X \leqq b) = F(b) - F(a) = \int_a^b f(v)\, dv.}$$

This is the analog of (5) for the discrete case.

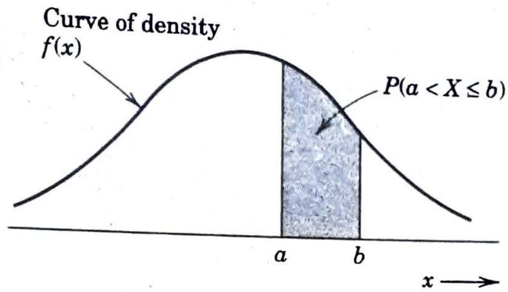


**Fig. 483.** Example illustrating formula (9)

From (7) and $P(S) = 1$ (Sec. 22.3) we also have the analog of (6):

$$\boxed{\int_{-\infty}^{\infty} f(v)\, dv = 1.} \qquad (10)$$

Continuous random variables are ***simpler than discrete ones*** with respect to intervals. Indeed, in the continuous case the four probabilities corresponding to $a < X \leqq b$, $a < X < b$, $a \leqq X < b$, and $a \leqq X \leqq b$ with any fixed $a$ and $b$ $(> a)$ are all the same. Can you see why? (*Answer.* This probability is the area under the density curve, as in Fig. 483, and does not change by adding or subtracting a single point in the interval of integration.) This is different from the discrete case! (Explain.)

The next example illustrates notations and typical applications of our present formulas.

**EXAMPLE 5** **A continuous distribution**

Let $X$ have the density function $f(x) = 0.75(1 - x^2)$ if $-1 \leqq x \leqq 1$ and zero otherwise. Find the distribution function. Find the probabilities $P(-\frac{1}{2} \leqq X \leqq \frac{1}{2})$ and $P(\frac{1}{4} \leqq X \leqq 2)$. Find $x$ such that $P(X \leqq x) = 0.95$.

***Solution.*** From (7) we obtain $F(x) = 0$ if $x \leqq -1$,

$$F(x) = 0.75 \int_{-1}^{x} (1 - v^2)\, dv = 0.5 + 0.75x - 0.25x^3 \qquad \text{if } -1 < x \leqq 1,$$

and $F(x) = 1$ if $x > 1$. From this and (9) we get

$$P(-\tfrac{1}{2} \leqq X \leqq \tfrac{1}{2}) = F(\tfrac{1}{2}) - F(-\tfrac{1}{2}) = 0.75 \int_{-1/2}^{1/2} (1 - v^2)\, dv = 68.75\%$$

(because $P(-\frac{1}{2} \leqq X \leqq \frac{1}{2}) = P(-\frac{1}{2} < X \leqq \frac{1}{2})$ for a continuous distribution) and

$$P(\tfrac{1}{4} \leqq X \leqq 2) = F(2) - F(\tfrac{1}{4}) = 0.75 \int_{1/4}^{1} (1 - v^2)\, dv = 31.64\%.$$

(Note that the upper limit of integration is 1, not 2. Why?) Finally,

$$P(X \leqq x) = F(x) = 0.5 + 0.75x - 0.25x^3 = 0.95.$$

Algebraic simplification gives $3x - x^3 = 1.8$. A solution is $x = 0.73$, approximately.

Sketch $f(x)$ and mark $x = -\frac{1}{2}, \frac{1}{2}, \frac{1}{4}$, and 0.73, so that you can see the results (the probabilities) as areas under the curve. Sketch also $F(x)$. ◀

Further examples of continuous distributions are included in the next problem set and in later sections.

## PROBLEM SET 22.3

1. Sketch the probability function $f(x) = x^2/14$ $(x = 1, 2, 3)$ and the distribution function.
2. Find $k$ in the probability function $f(x) = k\binom{3}{x}$, $x = 0, 1, 2, 3$, and sketch $f$ and the distribution function $F$.
3. Sketch $f$ and $F$ when $f(0) = f(3) = 1/6$, $f(1) = f(2) = 1/3$. Can $f$ have further positive values?
4. Let $X$ be the number of years before a certain kind of pump needs replacement. Let $X$ have the probability function $f(x) = kx^3$, $x = 0, 1, 2, 3, 4$. Find $k$. Sketch $f$ and $F$.
5. If $X$ has the probability function $f(x) = k/x!$ $(x = 0, 1, 2, \cdots)$, what are $k$ and $P(X \geqq 3)$?

6. Sketch the density $f(x) = \frac{1}{4}$ $(2 < x < 6)$ and the distribution function. Find $P(X \geqq 4)$, $P(X \leqq 3)$.
7. In Prob. 6 find $c$ such that (a) $P(X \leqq c) = 90\%$, (b) $P(X \geqq c) = \frac{1}{2}$, (c) $P(X \leqq c) = 5\%$.
8. Let $F(x) = 0$ if $x < 0$, $F(x) = 1 - e^{-0.1x}$ if $x > 0$. Sketch $F$ and the density $f$. Find $c$ such that $P(X \leqq c) = 95\%$.
9. Let $X$ [millimeters] be the thickness of washers a machine turns out. Assume that $X$ has the density $f(x) = kx$ if $0.9 < x < 1.1$ and 0 otherwise. Find $k$. What is the probability that a washer will have thickness between 0.95 mm and 1.05 mm?

10. Two screws are randomly drawn without replacement from a box containing 7 right-handed and 3 left-handed screws. Let $X$ be the number of left-handed screws drawn. Find $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, $P(1 < X < 2)$, $P(0.5 < X < 5)$.
11. Find the probability that none of three bulbs in a traffic signal will have to be replaced during the first 1500 hours of operation if the lifetime $X$ of a bulb is a random variable with the density $f(x) = 6[0.25 - (x - 1.5)^2]$ when $1 \leqq x \leqq 2$ and $f(x) = 0$ otherwise, where $x$ is measured in multiples of 1000 hours.
12. If the diameter $X$ of axles has the density $f(x) = k$ if $119.9 \leqq x \leqq 120.1$ and 0 otherwise, how many defectives will a lot of 500 axles approximately contain if defectives are axles slimmer than 119.91 or thicker than 120.09?
13. If the life of ball bearings has the density $f(x) = ke^{-0.2x}$ if $0 \leqq x \leqq 10$ and 0 otherwise, what is $k$? What is the probability $P(X \geqq 5)$?
14. Find the probability function of $X$ = *Number of times a fair die is rolled until the first Six appears* and show that it satisfies (6).
15. Suppose that certain bolts have length $L = 400 + X$ mm, where $X$ is a random variable with density $f(x) = \frac{3}{4}(1 - x^2)$ if $-1 \leqq x \leqq 1$ and 0 otherwise. Determine $c$ so that with a probability of 95% a bolt will have any length between $400 - c$ and $400 + c$.
16. Suppose that in an automatic process of filling oil into cans, the content of a can (in gallons) is $Y = 100 + X$, where $X$ is a random variable with density $f(x) = 1 - |x|$ when $|x| \leqq 1$ and 0 when $|x| > 1$. Sketch $f(x)$ and $F(x)$. In a lot of 1000 cans, about how many will contain 100 gallons or more? What is the probability that a can will contain less than 99.5 gallons? Less than 99 gallons?

17. Let $f(x) = kx^2$ if $0 \leqq x \leqq 2$ and 0 otherwise. Find $k$. Find $c_1$ and $c_2$ such that $P(X \leqq c_1) = 0.1$ and $P(X \leqq c_2) = 0.9$.
18. Let $X$ be the ratio of sales to profits of some firm. Assume that $X$ has the distribution function $F(x) = 0$ if $x < 2$, $F(x) = (x^2 - 4)/5$ if $2 \leqq x < 3$, $F(x) = 1$ if $x \geqq 3$. Find and sketch the density. What is the probability that $X$ is between 2.5 (40% profit) and 5 (20% profit)?
19. Let $X$ be a random variable that can assume every real value. What are the complements of the events $X \leqq b$, $X < b$, $X \geqq c$, $X > c$, $b \leqq X \leqq c$, $b < X \leqq c$?
20. Show that $b < c$ implies $P(X \leqq b) \leqq P(X \leqq c)$.

# 22.6 Mean and Variance of a Distribution

The mean $\mu$ and variance $\sigma^2$ of a random variable $X$ and of its distribution are the theoretical counterparts of the mean $\bar{x}$ and variance $s^2$ of a frequency distribution in Sec. 22.1 and serve a similar purpose. Indeed, the mean characterizes the central location and the variance the spread (the variability) of the distribution. The **mean** $\mu$ (mu) is defined by

(1)
$$\boxed{\begin{aligned} &\textbf{(a)} \quad \mu = \sum_j x_j f(x_j) \\ &\textbf{(b)} \quad \mu = \int_{-\infty}^{\infty} x f(x)\,dx \end{aligned}} \qquad \begin{aligned} &\text{(Discrete distribution)} \\ &\text{(Continuous distribution).} \end{aligned}$$

and the **variance** $\sigma^2$ (sigma square) by

(2)
$$\boxed{\begin{aligned} &\textbf{(a)} \quad \sigma^2 = \sum_j (x_j - \mu)^2 f(x_j) \\ &\textbf{(b)} \quad \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx \end{aligned}} \qquad \begin{aligned} &\text{(Discrete distribution)} \\ &\text{(Continuous distribution).} \end{aligned}$$

$\sigma$ (the positive square root of $\sigma^2$) is called the **standard deviation** of $X$ and its distribution. $f$ is the probability function or the density, respectively, in (a) and (b).

The mean $\mu$ is also denoted by $E(X)$ and is called the **expectation** *of* $X$ because it gives the average value of $X$ to be expected in many trials. Quantities such as $\mu$ and $\sigma^2$ that measure certain properties of a distribution are called **parameters.** $\mu$ and $\sigma^2$ are the two most important ones. From (2) we see that

(3)
$$\sigma^2 > 0$$

(except for a discrete "distribution" with only one possible value, so that $\sigma^2 = 0$). We assume that $\mu$ and $\sigma^2$ exist (are finite), as is the case for practically all distributions that are useful in applications.

**EXAMPLE 1 Mean and variance**

The random variable $X$ = *Number of heads in a single toss of a fair coin* has the possible values $X = 0$ and $X = 1$ with probabilities $P(X = 0) = \frac{1}{2}$ and $P(X = 1) = \frac{1}{2}$. From (1a) we thus obtain the mean $\mu = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$, and (3a) yields the variance

$$\sigma^2 = (0 - \tfrac{1}{2})^2 \cdot \tfrac{1}{2} + (1 - \tfrac{1}{2})^2 \cdot \tfrac{1}{2} = \tfrac{1}{4}.$$ ◀

**EXAMPLE 2 Uniform distribution. Variance measures spread**

The distribution with the density

$$f(x) = \frac{1}{b - a} \quad \text{if} \quad a < x < b$$

and $f = 0$ otherwise is called the **uniform distribution** on the interval $a < x < b$. From (1b) (or from Theorem 1, below) we find that $\mu = (a + b)/2$, and (2b) yields the variance

$$\sigma^2 = \int_a^b \left(x - \frac{a + b}{2}\right)^2 \frac{1}{b - a}\,dx = \frac{(b - a)^2}{12}.$$

Figure 484 on p. 1076 illustrates that the spread is large if and only if $\sigma^2$ is large. ◀

Note that in Fig. 484 the increase of the length of the interval by a factor 3 leads to a large increase of the variance (by what factor?).

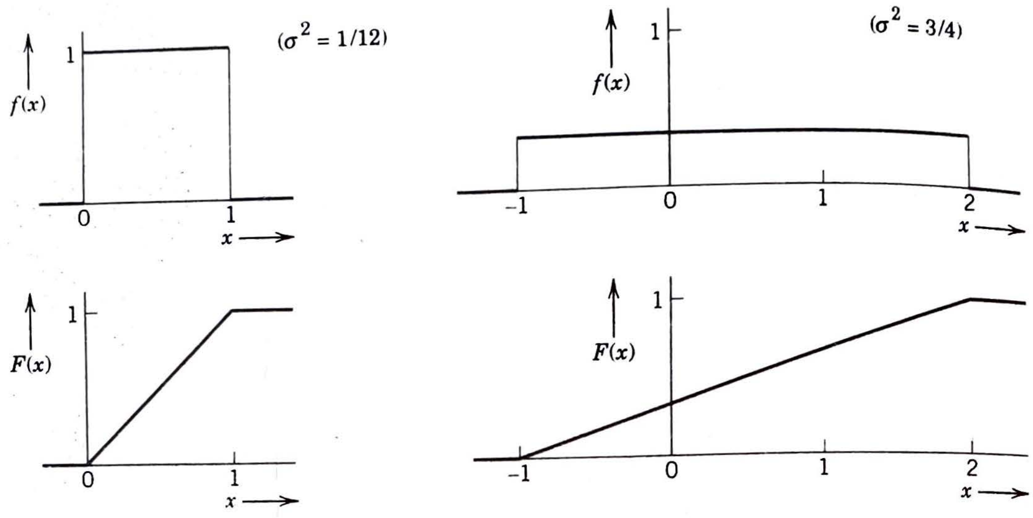


**Fig. 484.** Uniform distributions having the same mean (0.5) but different variances $\sigma^2$

**Symmetry.** We can obtain the mean $\mu$ without calculation if a distribution is symmetric. Indeed, you may prove

**THEOREM 1** **(Mean of a symmetric distribution)**

*If a distribution is* **symmetric** *with respect to* $x = c$, *that is,* $f(c - x) = f(c + x)$, *then* $\mu = c$. (Examples 1 and 2 illustrate this.)

## Transformation of Mean and Variance

Given a random variable $X$ with mean $\mu$ and variance $\sigma^2$, we want to calculate the unknown mean and variance of $X^* = a_1 + a_2X$, where $a_1$ and $a_2$ are given constants. This problem is important in statistics, where it appears often. Notably, we shall need it like our daily bread in connection with the "normal distribution," beginning in Sec. 22.8.

**THEOREM 2** **(Transformation of mean and variance)**

**(a)** *If a random variable $X$ has mean $\mu$ and variance $\sigma^2$, then the random variable*

$$X^* = a_1 + a_2X \qquad (a_2 > 0) \tag{4}$$

*has the mean $\mu^*$ and variance $\sigma^{*2}$, where*

$$\mu^* = a_1 + a_2\mu \quad \text{and} \quad \sigma^{*2} = a_2{}^2\sigma^2. \tag{5}$$

**(b)** *In particular, the* **standardized random variable** *$Z$ corresponding to $X$, given by*

$$\boxed{Z = \frac{X - \mu}{\sigma}} \tag{6}$$

*has the mean* 0 *and the variance* 1.

**PROOF.** We prove (5) for a continuous distribution. To a small interval $I$ of length $\Delta x$ on the $x$-axis there corresponds the probability $f(x)\,\Delta x$ [approximately; the area of a rectangle of base $\Delta x$ and height $f(x)$]. Then the probability $f(x)\,\Delta x$ must equal that for the corresponding interval on the $x^*$-axis, that is, $f^*(x^*)\,\Delta x^*$, where $f^*$ is the density of $X^*$ and $\Delta x^*$ is the length of the interval on the $x^*$-axis corresponding to $I$. Hence for differentials we have $f^*(x^*)\,dx^* = f(x)\,dx$. Also, $x^* = a_1 + a_2 x$ by (4), so that (1b) applied to $X^*$ gives

$$\mu^* = \int_{-\infty}^{\infty} x^* f^*(x^*)\,dx^* = \int_{-\infty}^{\infty} (a_1 + a_2 x) f(x)\,dx = a_1 \int_{-\infty}^{\infty} f(x)\,dx + a_2 \int_{-\infty}^{\infty} x f(x)\,dx.$$

On the right the first integral equals 1, by (10) in Sec. 22.5. The second integral is $\mu$. This proves (5) for $\mu^*$. It implies

$$x^* - \mu^* = (a_1 + a_2 x) - (a_1 + a_2 \mu) = a_2(x - \mu).$$

From this and (2) applied to $X^*$, again using $f^*(x^*)\,dx^* = f(x)\,dx$, we obtain the second formula in (5),

$$\sigma^{*2} = \int_{-\infty}^{\infty} (x^* - \mu^*)^2 f^*(x^*)\,dx^* = a_2^2 \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx = a_2^2 \sigma^2.$$

For a discrete distribution the proof of (5) is similar.

Choosing $a_1 = -\mu/\sigma$ and $a_2 = 1/\sigma$ we obtain (6) from (4), writing $X^* = Z$. For these $a_1, a_2$ formula (5) gives $\mu^* = 0$ and $\sigma^{*2} = 1$, as claimed in (b). ◀

## Expectation, Moments

Recall that (1) defines the expectation (the mean) of $X$, the value of $X$ to be expected on the average, written $\mu = E(X)$. More generally, if $g(x)$ is nonconstant and continuous for all $x$, then $g(X)$ is a random variable. Hence its *mathematical expectation* or, briefly, its **expectation** $E(g(X))$ is the value of $g(X)$ to be expected on the average, defined [similarly to (1)] by

$$\textbf{(7)} \qquad E(g(X)) = \sum_j g(x_j) f(x_j) \qquad \text{or} \qquad E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

In the first formula, $f$ is the probability function of the discrete random variable $X$. In the second formula, $f$ is the density of the continuous random variable $X$. Important special cases are the **$k$th moment** of $X$ ($k = 1, 2, \cdots$)

$$\textbf{(8)} \qquad E(X^k) = \sum_j x_j^k f(x_j) \qquad \text{or} \qquad \int_{-\infty}^{\infty} x^k f(x)\,dx$$

and the **$k$th central moment** of $X$ ($k = 1, 2, \cdots$)

$$\textbf{(9)} \qquad E([X - \mu]^k) = \sum_j (x_j - \mu)^k f(x_j) \qquad \text{or} \qquad \int_{-\infty}^{\infty} (x - \mu)^k f(x)\,dx.$$

This includes the first moment, the **mean** of $X$

$$\textbf{(10)} \qquad \boxed{\mu = E(X)} \qquad [(8) \text{ with } k = 1].$$

It also includes the second central moment, the **variance** of $X$

(11) $$\boxed{\sigma^2 = E([X - \mu]^2)}$$ [(9) with $k = 2$].

For later use you may prove

(12) $$E(1) = 1.$$

## PROBLEM SET 22.6

### Mean and Variance

Find the mean and the variance of the random variable $X$, where $f(x)$ is the probability function or density.

**1.** $f(x) = k\binom{3}{x}$, $x = 0, 1, 2, 3$

**2.** $X$ = *Number a fair die turns up*

**3.** $f(x) = 2x \quad (0 \leqq x \leqq 1)$

**4.** $f(x) = e^{-x} \quad (x > 0)$

**5.** Uniform distribution on [0, 10]

**6.** $Y = 4X - 2$ with $X$ as in Prob. 4

**7.** If the diameter $X$ [cm] of certain bolts has the density $f(x) = k(x - 0.9)(1.1 - x)$ for $0.9 < x < 1.1$ and 0 for other $x$, what are $k$, $\mu$, and $\sigma^2$? Sketch $f(x)$.

**8.** If in Prob. 7 a defective bolt is one that deviates from 1.00 cm by more than 0.06 cm, what percentage of defectives should we expect?

**9.** For what choice of the maximum possible deviation from 1.00 cm shall we obtain 10% defectives in Probs. 7 and 8?

**10.** What total sum can you expect in rolling a fair die 20 times? Do the experiment. Repeat it a number of times and record how the sum varies.

**11.** A small filling station is supplied with gasoline every Saturday afternoon. Assume that its volume $X$ of sales in ten thousands of gallons has the probability density $f(x) = 6x(1 - x)$ if $0 \leqq x \leqq 1$ and 0 otherwise. Determine the mean, the variance, and the standardized variable.

**12.** What capacity must the tank in Prob. 11 have in order that the probability that the tank will be emptied in a given week be 5%?

**13.** If the life of certain tires (in thousands of miles) has the density $f(x) = \theta e^{-\theta x}$ $(x > 0)$, what mileage can you expect to get on one of these tires? Let $\theta = 0.05$ and find the probability that a tire will last at least 30 000 miles.

**14.** If in rolling a fair die, Jack wins as many dimes as the die shows, how much per game should Jack pay to make the game fair?

**15.** What is the expected daily profit if a small grocery store sells $X$ turkeys per day with probabilities $f(5) = 0.1$, $f(6) = 0.3$, $f(7) = 0.4$, $f(8) = 0.2$ and the profit per turkey is \$3.50?

**16. TEAM PROJECT. Means, Variances, Expectations. (a)** Show that $E(X - \mu) = 0$, $\sigma^2 = E(X^2) - \mu^2$.

**(b)** Prove (10)–(12)

**(c)** Find all the moments of the uniform distribution on an interval $a \leqq x \leqq b$.

**(d)** The **skewness** $\gamma$ of a random variable $X$ is defined by

$$\boxed{\gamma = \frac{1}{\sigma^3} E([X - \mu]^3).}$$