To find a formula for the mean of the log-normal distribution, we write

$$\mu = \frac{1}{\sqrt{2\pi}\beta}\int_0^{\infty} x \cdot x^{-1} e^{-(\ln x - \alpha)^2/2\beta^2}\, dx$$

and upon letting $y = \ln x$, this becomes

$$\mu = \frac{1}{\sqrt{2\pi}\beta}\int_{-\infty}^{\infty} e^y e^{-(y-\alpha)^2/2\beta^2}\, dy$$

This integral can be evaluated by completing the square on the exponent $y - (y-\alpha)^2/2\beta^2$, thus obtaining an integrand which has the form of a normal density. The final result, which the reader will be asked to verify in Exercise 4.48 on page 124, is

Mean of log-normal distribution

$$\mu = e^{\alpha + \beta^2/2}$$

Similar, but more lengthy, calculations yield

Variance of log-normal distribution

$$\sigma^2 = e^{2\alpha+\beta^2}(e^{\beta^2} - 1)$$

EXAMPLE With reference to the preceding example, find the mean and the variance of the distribution of the ratio of the output to the input current.

Solution Substituting $\alpha = 2$ and $\beta = 0.1$ into the above formulas, we get

$$\mu = e^{2+(0.1)^2/2} = 7.4$$

and

$$\sigma^2 = e^{4+(0.1)^2}(e^{(0.1)^2} - 1) = 0.56$$

4.7 THE GAMMA DISTRIBUTION

Several important probability densities whose applications will be discussed later are special cases of the **gamma distribution**. This distribution is given by

Gamma distribution

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0, \alpha > 0, \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\Gamma(\alpha)$ is a value of the **gamma function**, defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

Integration by parts shows that

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

for any $\alpha > 1$ and, hence, that $\Gamma(\alpha) = (\alpha - 1)!$ when α is a positive integer. Graphs of several gamma distributions are shown in Figure 4.6 and they exhibit the fact that these distributions are positively skewed. In fact, the skewness decreases as α increases for any fixed value of β.

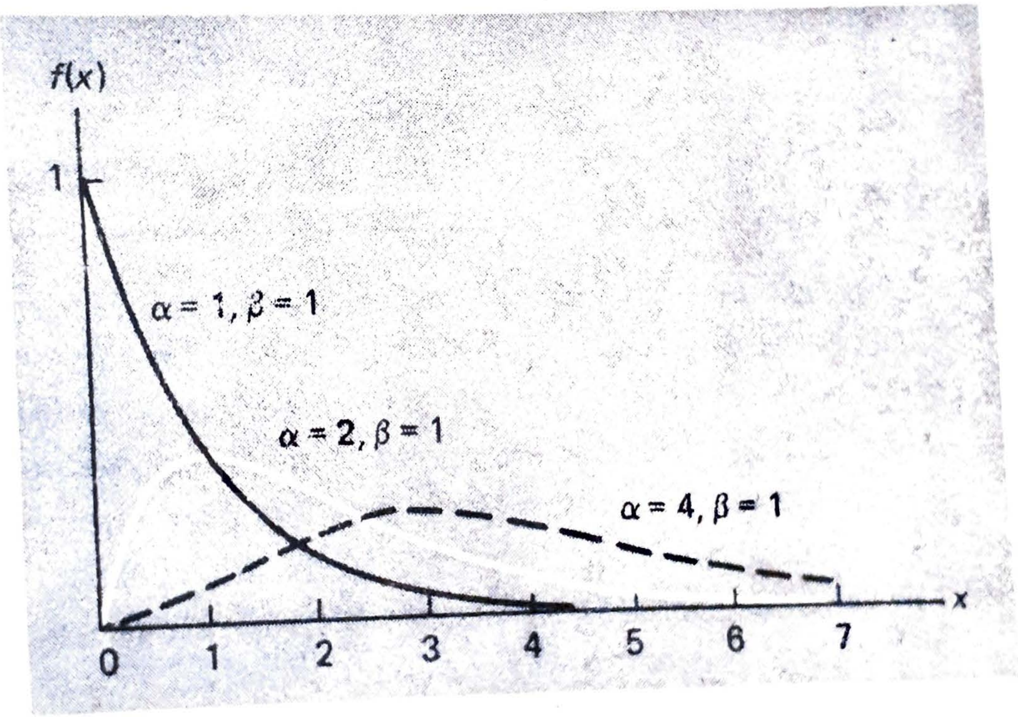

Figure 4.6 Graph of gamma distribution.

The mean and the variance of the gamma distribution may be obtained by making use of the gamma function and its special properties mentioned above. For the mean we have

$$\mu = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x \cdot x^{\alpha-1}e^{-x/\beta}\,dx$$

and after letting $y = x/\beta$, we get

$$\mu = \frac{\beta}{\Gamma(\alpha)}\int_0^\infty y^\alpha e^{-y}\,dy = \frac{\beta\Gamma(\alpha + 1)}{\Gamma(\alpha)}$$

... the resul

Mean of gamma distribution

$$\boxed{\mu = \alpha\beta}$$

Using similar methods, it can also be shown that the variance of the gamma distribution is given by

ariance f gamma distribution

$$\boxed{\sigma^2 = \alpha\beta^2}$$

In the special case where $\alpha \neq 1$, we get the **exponential distribution**, whose probability density is thus

Exponential distribution

$$f(x) = \begin{cases} \dfrac{1}{\beta} e^{-x/\beta} & \text{for } x > 0,\ \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and whose mean and variance are $\mu = \beta$ and $\sigma^2 = \beta^2$. Note that the distribution of the example on page 102 is an exponential distribution with $\beta = \frac{1}{2}$.

The exponential distribution has many important applications; for instance, it can be shown that in connection with Poisson processes (see Section 3.7) the **waiting time** between successive arrivals (successes) has an exponential distribution. More specifically, it can be shown that if in a Poisson process the mean arrival rate (average number of arrivals per unit time) is α, the time until the first arrival, or the waiting time between successive arrivals, has an exponential distribution with $\beta = \dfrac{1}{\alpha}$ (see Exercise 4.62 on page 126).

EXAMPLE With reference to the example on page 84, where on the average three trucks arrived per hour to be unloaded at a warehouse, what are the probabilities that the time between the arrival of successive trucks will be

(a) less than 5 minutes;
(b) at least 45 minutes.

Solution Since $\beta = \frac{1}{3}$, we get

$$\int_0^{1/12} 3e^{-3x}\,dx = 1 - e^{-1/4} = 0.221$$

for part (a), and

$$\int_{3/4}^{\infty} 3e^{-3x}\,dx = e^{-9/4} = 0.105$$

for part (b).

8 THE BETA DISTRIBUTION

In Chapter 9 we shall need a probability density for a random variable which takes on values on the interval from 0 to 1, and most appropriate for this purpose is the **beta distribution**, whose probability density is

ta tribution

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1, \alpha > 0, \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The mean and the variance of this distribution are given by

ean and riance of ta distribution

$$\mu = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Note that for $\alpha = 1$ and $\beta = 1$ we obtain as a special case the uniform distribution of Section 4.5 defined on the interval from 0 to 1. The following example, pertaining to a proportion, illustrates a typical application of the beta distribution:

EXAMPLE In a certain county, the proportion of highway sections requiring repairs in any given year is a random variable having the beta distribution with $\alpha = 3$ and $\beta = 2$ (shown in Figure 4.7). Find

(a) on the average what percentage of the highway sections require repairs in any given year;

(b) the probability that at most half of the highway sections will require repairs in any given year.

Solution (a) $\mu = \dfrac{2}{3+2} = 0.60$, which means that on the average 60% of the highway sections require repairs in any given year. (b) Substituting $\alpha = 3$ and $\beta = 2$ into the formula for the beta distribution and making use of the fact that $\Gamma(5) = 4! = 24$, $\Gamma(3) = 2! = 2$, and $\Gamma(2) = 1! = 1$, we get

$$f(x) = \begin{cases} 12x^2(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, the desired probability is given by

$$\int_0^{1/2} 12x^2(1-x)\,dx = \tfrac{5}{16}$$

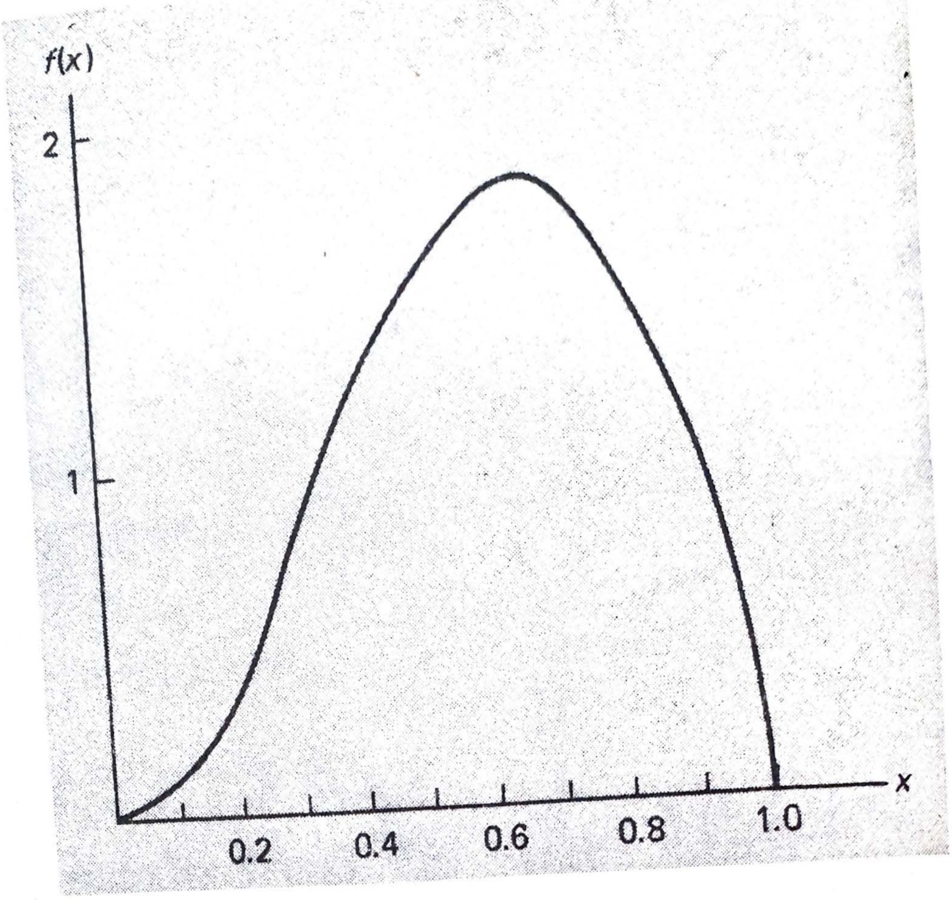

Figure 4.7 Graph of the beta distribution with $\alpha = 3$ and $\beta = 2$.

In most realistically complex situations, probabilities relating to gamma an beta distributions are obtained from special tables.

4.9 THE WEIBULL DISTRIBUTION

Closely related to the exponential distribution is the **Weibull distributio** whose probability density is given by

Weibull distribution

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}} & \text{for } x > 0, \alpha > 0, \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

To demonstrate this relationship, we evaluate the probability that a rand variable having the Weibull distribution will take on a value less tha namely, the integral

$$\int_0^a \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx$$

Making the change of variable $y = x^{\beta}$, we get

$$\int_0^{a^{\beta}} \alpha e^{-\alpha y}\,dy = 1 - e^{-\alpha a^{\beta}}$$

and it can be seen that y is a value of a random variable having an expone