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Simulating the transits of planets, their satellites, and starspots

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ABSTRACT

The most popular method for detecting exoplanets is by using the transit method. Creating a simulation of a transit allows us to model the expected results from real observed transits; and from these results extrapolate a whole host of information about the exoplanet. This report outlines how, using some basic physics and the limb-darkening effect, we were able to create an accurate transit simulator. Beginning with a Jupiter-Sun orbit simulation, and modifying the parameters, we were able to create a simulation which models the flux change during the transit of any desired exoplanet system. In this report, we analysed data of the HD 209458b transit[20], and compared it with our simulated model. Our fit agreed with the data to a high degree of accuracy: obtaining χ_v^2 values ranging between $1.00 < \chi_v^2 < 1.77$. In doing so we were able to find a correlation between the strength of the limb-darkening and the wavelength. As well as this, the report outlines the effect that having an orbiting moon or sunspots has on the relative flux, and the method by which we were able to create them.

1 INTRODUCTION

A transit is a phenomenon that occurs when an interstellar object passes across the surface of its star. Typically in astronomy, the most widely studied transits are those of exoplanets. The most popular and frequently used method of discovering exoplanets is the transit method. Of all the exoplanets that have currently been discovered, over three-quarters[1] of them have been done so this way. Observing a transit however requires a great deal of good fortune; the sun must be down, the star in question must be above the horizon, and the planet must be passing in front of the star, all at the same time.

Exoplanet research is a relatively new topic of interest, with the first exoplanet only being discovered in 1995[2]; and out of a galaxy of approximately 100 billion stars[3], with just over 4300 confirmed exoplanet's[1], it is almost certain that there are many more left to discover. Collecting information on exoplanets is not only vital in understanding the structure of our universe, but it also allows us to grasp a better insight into whether life exists on other planets. By having a large data-set of exoplanets with different properties we are able to compare that data to the information we have of our own planet, and try to extrapolate the possibilities of exoplanets having the potential to harbour life.[4]

The Kepler mission's highest priority when it was launched in 2009[5] was to find terrestrial planets that are capable of holding liquid water on them, much like our own Earth. These Earth-like planets hold the answer to one of the oldest philosophical questions: are we alone in the universe? Liquid water is an essential requirement for life on Earth, and since we have not yet discovered any life outside of our planet, we assume that it is also a requirement of interstellar life. Suffice to say that any real confirmation of life outside our own planet will be Nobel Prize worthy. Not only is proof of extraterrestrial life important, but also, with a large enough data-set, we would be able to predict how probable life is; how frequently it occurs, and whether or not life really is that special at all.

A well-created simulation can be used to model data from different star systems. From these models we would be able to determine certain characteristics of an exoplanet, such as its orbital radius, planetary radius, mass, whether the planet has any satellites or rings, and if the planet has an atmosphere (and whether or not that atmosphere contains the chemical signatures of life).

This project began with the aim of creating an accurate simulation of planetary transits, using basic Newtonian mechanics. From there, the simulation was expanded to include multiple physical phenomena, such as the effect of limb-darkening, the orbit of moons, and starspots. In this report, I will be outlining the theoretical understanding necessary to create an accurate model of a transit, along with the step-by-step approach of devising the simulation. For the final part of our project, to analyse the accuracy of the simulation a comparison with the well-studied exoplanet HD 209458b was undertaken, to serve as a success criterion of our simulation.

2 THEORY OF EXOPLANETARY TRANSITS

Most of the theoretical framework for simulating transits comes from the chapters by Winn[6] and Murray & Correia[7]. This project closely follows the work by Winn, which has served as the foundation for our method throughout. In this section, I will outline the physics that has been used in creating our simulation.

2.1 Elliptical Orbit Mechanics

An essential aspect of modelling transits is accurately simulating the motion of a planet around a star. In a circular orbit approximation, the distance between the star and the planet is constant, and in most applications this is a very good approximation. However, for completeness and greater accuracy, we take the eccentricity of the orbit into account when finding r; given by equation (1)[6] and (20)[7] respectively:

$$r = \frac{a(1-e^2)}{1+e\cos f}, \tag{1}$$

where a refers to the semi-major axis of the orbit, e is the eccentricity, and f[1] is the true anomaly, which we will approximate as $f_{tra} = \frac{\pi}{2} - \omega$. We can note here that in the limit of a perfectly circular orbit, the eccentricity $e \to 0$, and thus $r \to a$. However all planetary orbits are elliptical[8] with $0 < e < 1$.

[1] f_{tra} refers to the true anomaly during *transit*, which is different to the true anomaly during *occultation* $f_{occ} = -\frac{\pi}{2} - \omega$.

We can take equation (1) and use it to extract 3D Cartesian coordinates, see equations (53-55)[7], which will be needed in tracking the exact position of the planet as it transits its star. We are approximating $\Omega = \pi$, where Ω is an orientation angle[7].

$$X = -r\cos(\omega + f), \tag{2}$$

$$Y = -r\sin(\omega + f)\cos i, \tag{3}$$

$$Z = r\sin(\omega + f)\sin i, \tag{4}$$

where ω and i are orientation angles. See Fig. 4[7] for a schematic of the geometry of the orientation angles.

Simulating how long the transit will take requires knowledge of the period of the planet's orbit around its star, which is given by equation (23)[7]:

$$T^2 = \frac{4\pi^2}{G(M_p + M_*)} a^{\frac{3}{2}}, \tag{5}$$

where G is Newton's Gravitational constant[2], M_p is the mass of the planet, and M_* is the mass of the star.

Along with the period, one needs to know the rate of change of position of the planet, so that one can determine the duration of the eclipse. The velocity is given by equation (33)[7]:

$$v^2 = \frac{n^2 a^2}{1 - e^2}(1 + 2e\cos f + e^2), \tag{6}$$

where $n = \frac{2\pi}{T}$. Using the basic equation that time $t = distance/v$ we can calculate the position of the planet relative to the star at any given time.

Using these equations, a simulation would calculate and show how long a planet would take to transit. In order to confirm whether the eclipse timings of the simulation is accurate, one can manually calculate the expected eclipse duration (known as the characteristic timescale T_0) using equation (19)[6]:

$$T_0 \equiv \frac{R_* P}{\pi a}, \tag{7}$$

where R_* is the radius of the star, P is the orbital period (5), and a is the semi-major axis.

Describing a satellite orbiting a planet can be likened to a planet orbiting a star. The physics is exactly the same, as there is no inherent difference between the orbital mechanics of a star-planet system compared to a planet-moon system, apart from the fact that masses and radii are much smaller. Theoretically one could even have an even smaller object orbiting the moon, it remains just as trivial. It should be noted however, that the smaller the satellite, the smaller the change of flux will be (9), and the harder it will be to observe.

2.2 Relative Flux

Exoplanets are indirectly detected by observing a drop in luminosity of its star as the planet passes across it. This drop in the luminosity occurs because, compared to their star, the planet is so cool that it can be considered completely opaque[6]. When the planet crosses the field of view of the observer and the star, the star appears dimmer[10]. It is the relative flux of the star that we use to compare how much the luminosity has fallen by. One normalises the luminosity of the star when there is no transit to get a relative flux of 1 at the maximum luminosity (assuming the maximum luminosity of the star is constant). The change in relative flux ΔF must be proportional to the ratio of the area of the star that has been eclipsed by the planet:

$$\Delta F = \frac{\pi R_p^2}{\pi R_*^2}, \tag{8}$$

$$\Delta F = \left(\frac{R_p}{R_*}\right)^2 = k^2, \tag{9}$$

where R_p is the radius of the planet, and k is by definition the ratio of the radius of the planet to the star's.

Using only data of the relative flux of the star during a transit, one can already determine the ratio of the radii of the planet and star.

2.3 Limb-darkening

As an elementary model, one may assume that the star has uniform brightness across its entire surface. However, we know that stars are actually brighter in the centre than they are on the outside due to an effect known as limb-darkening. Limb-darkening is "the gradual decrease in brightness of the disk of the Sun or of another star as observed from its centre to its edge"[11]. During a transit, the relative flux does not drop exactly proportional to k^2, as the luminosity of the star varies as a function of its radius. The greatest flux drop happens when the planet is positioned at the exact centre of the star (where the star's luminosity is greatest)[6]. The luminosity of the star is therefore inversely proportional to the distance from the centre of the star. Limb-darkening occurs because the temperature of the star increases with depth[11]. The effect of limb-darkening can be described in different ways. In this project we will use the quadratic law, and later the non-linear law (section 5). The simpler method, requiring only two coefficients, is the quadratic law, described by equation (24)[6]:

$$I = I_0\left(1 - u_1(1-\mu) - u_2(1-\mu)^2\right), \tag{10}$$

where I_0 is the maximum intensity, u_1, u_2 are constant coefficients that may be calculated[6], and

$$\mu = \sqrt{1 - X^2 - Y^2}. \tag{11}$$

The non-linear law is given by equation (6)[12]:

$$I = I_0\left(1 - \sum_{n=1}^{4} c_n(1 - \mu^{\frac{n}{2}})\right), \tag{12}$$

where c_n are all separate limb-darkening coefficients.

Limb-darkening serves as a fundamental requirement for the accurate simulation of exoplanet transits. As we will see, not incorporating limb-darkening in one form or another would result in a drastically different theoretical model when compared with the experimental data.

2.4 Starspots

Sunspots are regions on the sun's photosphere that are around 2000K cooler than the rest of the sun's surface[13]. Due to this temperature difference, these regions look much darker than the rest of the star in comparison. Sunspots are caused by magnetic interactions, and can produce spots "up to 50,000km in diameter"[13]. They are not, however, unique to just our sun. These dark regions also occur on other stars and are named'starspots'. They are caused by the same physical phenomenon as sunspots[14].

In order to incorporate starspots into the model, one simply needs to specify which regions of the sun's surface is to be kept dark. The spots can be irregular or shapely. In our simulation we created a circular spot, which also incorporated the effects of limb-darkening; utilising theory

[2] $^9G = 6.67430 \cdot 10^{-11} N \cdot m^2 \cdot kg^{-2}$

from sections 2.2 & 2.3. As starspots are not constant, to accurately simulate them one would first of all need to model the rotation of the star about its axis, and then have the starspots rotating as well. But the starspots can also move due to convection in the stellar atmosphere, which would be very difficult to model.

3 CREATING THE SIMULATION

3.1 Jupiter-Sun Model

To begin the project, a model of the transit of a Jupiter-like planet across a Sun-like star was created in python. The mass, planetary radius, and orbital radius are all the same as that of the Jupiter-Sun system (table 1), which has been well-studied. Detecting clear transits is easiest when k^2 (9) is large, and as Jupiter is the biggest planet in our solar system, it serves as the most appropriate transit to model.

Table 1. Variables used to create the simulation of the transit of Jupiter.[15] [16]

Jupiter-Sun Variables	
Variables	Values
Jupiter Mass (10^{24} kg)............	1,898.19
Solar Mass (10^{24} kg)................	1,988,500
Jupiter Mean Radius (km)........	69,911
Solar Mean Radius (km)..........	695,700
Semi-Major Axis (10^6 km)......	778.570
Orbital Eccentricity..................	0.0489

The approach taken was to model the system as a grid, and program functions onto it such that each grid-space is either the star, the planet or space. Using a grid-based approach, as opposed to simply modelling everything purely mathematically, allows the easy addition of extra features, which may not be trivial to represent by simply using equations. Starspots, for example, could have non-uniform shapes, which would be long and cumbersome to attempt to graph onto the star, but would only require changing the values of a few grid points to achieve the same effect.

Using equations (2), (3) and (4) we were able to plot basic graphs of the trajectory of the planet around the sun. This was achieved by identifying the angular range θ at which the transit took place, and had to be found manually. Once θ was found, we had to choose how many simulated data points we wanted to divide θ up into. The greater the number of simulated data points, the smoother and more precise the curve becomes; but also the more computational power is required to calculate them all. For our simulations, we chose to have 1000 points. When calculating the velocity (6) we obtained an array of 1000 different values. As we were not able to use an array when finding the distance[3], we instead had to settle for finding the average velocity, and use this value. As the eccentricity is small[8], the average velocity is very close to each of the 1000 different values calculated. If we were to simulate a vastly eccentric orbit, solely using the average velocity would completely eradicate the orbital timing accuracy. This is one of the limitations of our simulation, which we have not been able to rectify. However, the simulation of the orbit for the Jupiter-Sun system seems to work to a very high degree of accuracy despite this, as seen in Section 4. Changing variables such as orientation angles and radii is easy enough, and the simulation can thus be used to plot the trajectories of any desired orbit.

The orbits in Fig. 1 and Fig. 2 are represented in a coordinate system, and needed to be transferred onto a grid. To create our functioning grid

[3] A 1x1000 array cannot be broadcast with a single value (distance)

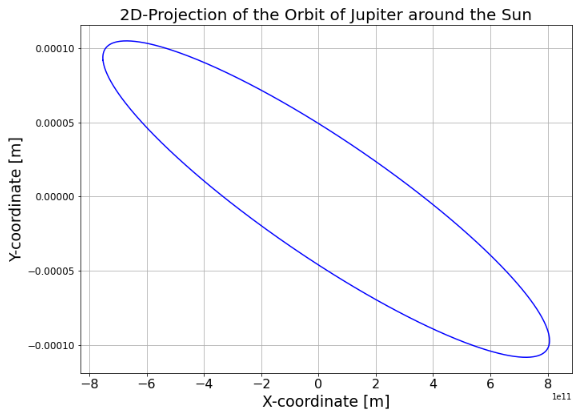

Fig. 1. 2D plot of the trajectory of a Jupiter-like planet around a Sun-like star, using solar parameters[15] and planetary parameters[16], with $\omega = \frac{\pi}{4}$ and $i = 0$.

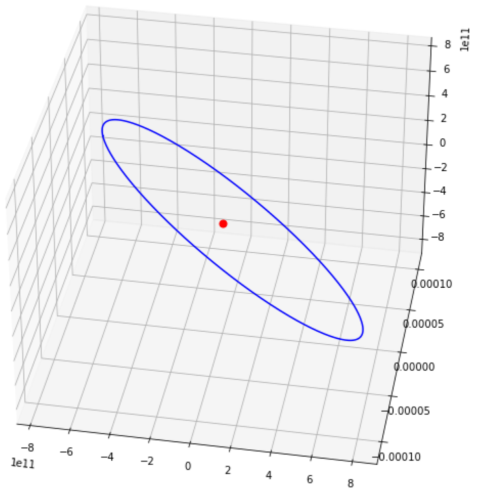

Fig. 2. 3D plot of the trajectory of a Jupiter-like planet around a Sun-like star, using solar parameters[15] and planetary parameters[16], with a red dot in the middle representing the star (star not to scale). $\omega = \frac{\pi}{4}$ & $i = 0$.

we first needed to make an NxN grid of 0's (representing empty space). Then we added a circle of 1's (representing the outline of the star). The circle was then filled with 1's (representing the area of the star). Finally, the orbital motion of the planet, shown in Fig. 1, was mapped onto the grid, with 2's acting as the planet (Appendix B). In order to make the whole thing look more like a planet transiting a star (as opposed to an array of 0's, 1's and 2's) we created a function which drew the star with some colour, and dark empty space (Appendix A). As the planet is so much cooler than the star, it can be considered completely opaque during the transit. Therefore, the planet appears to be the same colour as space. Using the quadratic limb-darkening law (10), we were easily able to incorporate the limb-darkening of the star into our grid, and create a star whose colour gradually fades radially outwards. This colour fading occurs because, due to equation (10), the value on the grid associated with the star (which was previously all 1's) now decreases. Essentially, the middle of the star is 1, and at the very outskirts of its circumference the value is a number between $0.60 < x < 1.00$. The range cannot drop as low as zero because there are still clear boundaries that define where there is a star and where there is space, limb-darkening simply decreases the outer intensity.

The code works by drawing a scaled version of the star and planet on the grid, using table 1. The scale-factor ζ used is:

$$\zeta = R_* + 2R_P, \tag{13}$$

where R_* is the stellar radius, and R_P is the planetary radius. This scale-factor is used to calibrate the stellar and planetary position, as well as the distance travelled, and is universal for whichever exoplanet system is being modelled.

In our simulation we are looking at the planet exactly along the X-Y plane[4]. Then, using the equations of motion (2), (3) and (4) we were able to scale how much the planet's position changes relative to the star. Increasing the size of the display in our code effectively increases the resolution of the image produced. For the highest resolution the display size is set to 2000px. The image generated in Fig. 3 works by setting the position of the planet γ to a value between $0 \leq \gamma < 2000$px. In the figure, the planet is plotted to a position of $\gamma = 1000$px, which is in the middle of the star.

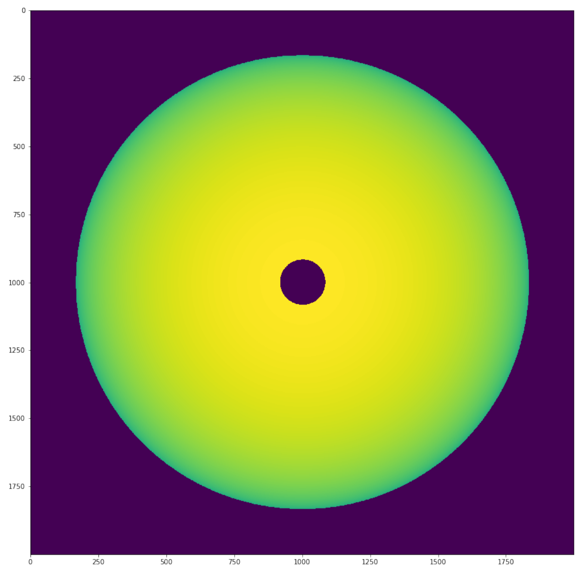

Fig. 3. Plot of the grid-based model of the star (yellow/green) and the planet (centre black circle), incorporating the quadratic limb-darkening effects: $u_1 = 0.3$ & $u_2 = 0.1$.

Having obtained an image of the planet transiting the star, a relation between the relative flux of the star and time was needed. In order to obtain a value for the relative flux two things are needed: the maximum flux and the current flux. The maximum flux can be calculated by summing the value of the display when we set $\gamma = 0$. At this planetary position the star is not eclipsed in any way, so the entire display is simply the star with empty space behind it. This is the maximum flux that we can possibly extract. In order to normalise this we divide this value by itself to get a maximum relative flux of 1 (by design). Finding the current flux works in much the same way, but instead we need to obtain values for all $0 \leq \gamma < 2000$px, before normalising. At this point we would be able to plot a graph of relative flux against γ. Noting that γ is simply a scaled distance, and that the true distance X is actually known to us (2), we are able to use this knowledge to calculate the total distance spanned between $0 \leq \gamma < 2000$px. With this distance we calculate the total time for the planet to move from $\gamma = 0$ to $\gamma = 2000$px using our equation for velocity (6).

The plot of relative flux produced can be seen in Fig. 4. We see the characteristic shape that is common in plots of relative flux. The graph has a curved shape due to limb-darkening. If this was not included in our model then the relative flux would decrease linearly to the minimum, where it would remain constant until increasing linearly again, represented by the dotted red line in Fig. 4.

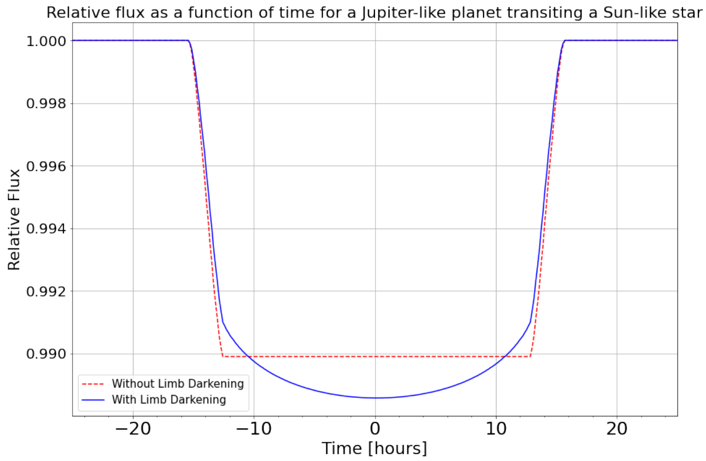

Fig. 4. A simulated graph of how relative flux varies with time as a Jupiter-like planet passes across the face of a Sun-like star, with and without limb-darkening.

This now serves as the basic template which transit simulations can be based off. By altering the variables one can just as easily simulate the transit of any other system.

3.2 Adding a moon

At the end of section 2.1 I mentioned that creating a moon orbiting a planet requires the exact same physics required to create a planet orbiting a star. Indeed all that was needed to simulate this was to create a separate function which draws the moon. Like the planet, the moon is also considered opaque in comparison to the sun. From there the moon needs to orbit around the planet, whilst tracking the planet's orbit around the star. The largest moon in the solar system belongs to Jupiter, and is called Ganymede [17], so that is the moon we decided to simulate. We were able to create a giff showing the orbit of the moon around the planet, whilst the planet orbited the sun. A snapshot of that giff can be seen in Fig. 5.

3.3 Creating Starspots

Although starspots can be irregular in shape, during this project we simulated a circular spot. The circular shape of the spot is a derivative of the circular shape of the planet. However, the spot incorporates a limb-darkening effect, resulting in a slightly brighter centre, and unlike the planet it does not move. A starspot with a brighter centre than edge seems intuitively unphysical, however this is how we decided to code the spot:

[4] Where the Y-axis can be imagined as going into the page.

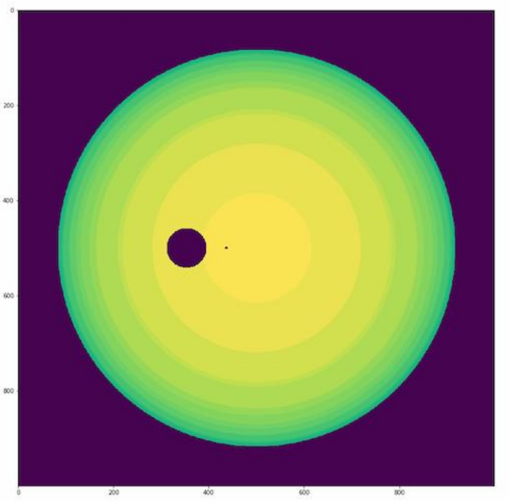

Fig. 5. Freeze-frame of the giff showing the moon Ganymede orbiting Jupiter, orbiting the Sun. Using the quadratic limb-darkening law (12).

to see what effect it would have on the star's luminosity. In a similar fashion to creating a graph of relative flux against time, we drew a graph of the stellar intensity as a function of distance from the centre of the star (Fig. 7).

4 INTERPRETING THE SIMULATION

Analysing the graph in Fig. 4, we can see that the minimum relative flux during the simulated transit is 0.989. This is equal to a luminosity drop of 1.13%. The expected luminosity drop of a Jupiter-like planet transiting a Sun-like star without any limb-darkening is approximately 1%[7], and that is exactly what can be seen in the dotted red line of Fig. 4. The difference of 0.13% in the minimum relative flux between the limb-darkened and non limb-darkened simulation is statistically significant, and emphasises the importance of including limb-darkening in our simulations. Limb-darkening does not, however, affect the eclipse duration. We see that both the red and blue lines describe an eclipse lasting approximately 30 hours. If we use equation (7) to calculate the theoretical eclipse duration we arrive at an answer of 29.6 hours. Thus we can conclude that our simulation has accurately produced an eclipse that lasts the expected length of time. As this is the foundation of our model, it was necessary that it produced accurate data in order for any new physics to be applied to it, or for the simulation to be used to model other star systems.

The simulation to add an orbiting moon to our planet does not produce any noticeable difference in the flux graph, apart from an unnoticeable (apparent) anomaly just when the transit is ending. If the moon-less simulation was used to model a planet with a moon, the simulation would not detect any anomalies. The error parameters for real data would actually be larger than the tiny difference in flux caused by the moon. The simulation however does indeed work, which is very clear when watching a giff of the orbit. The lack of the moon's presence in the graph is not a fault of the code but rather a consequence of the size of the moon compared with the size of Jupiter. Although Ganymede is Jupiter's largest moon, it is still only 7% the size of the planet[17]. Therefore, we would not expect to be able to detect a moon. Any slight discrepancy in the flux could simply be considered as noise from the exoplanet's transit, rather than be considered solid evidence for a moon. When considering how difficult finding Earth-sized exoplanets are, it should not come as a surprise that detecting a moon which is much smaller than the Earth would prove to be even more challenging. The difficulty is only enhanced due to the fact that a planetary transit is already resulting in a large drop in flux. In order for a moon to be detected, it must have a radius which is of the same order of magnitude as its planet. To demonstrate the affect that a larger moon would have on the flux graph, a comparison simulation was created alongside the simulation of Ganymede, seen in Fig. 6. In this simulation the moon is given a radius of $0.57R_{Jup}$. The large bulge seen around $T = 10$ is due to the moon coming out from behind the planet, and obscuring a larger area of the Sun. This results in a greater flux drop. When the moon has made a full orbit, the effect is no longer visible.

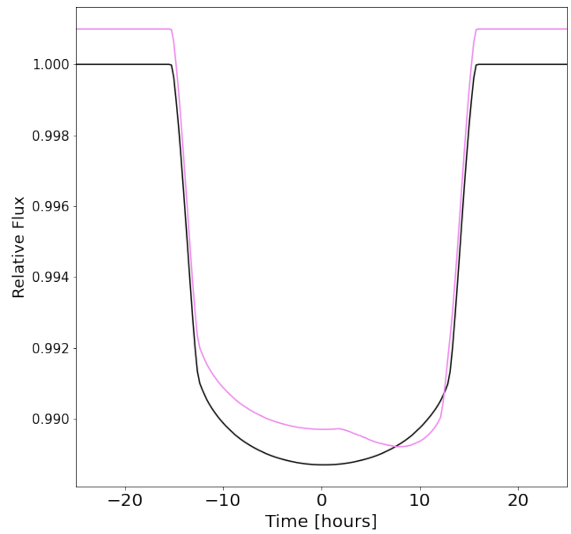

Fig. 6. Graphs showing the relative flux during the transit of Jupiter, orbited by a moon. The graph in black simulates the moon Ganymede orbiting Jupiter. In violet, the moon is given a radius of $0.57R_{Jup}$, and the graph is offset by 0.001.

Using this simulation to detect a moon is indeed possible, but only if the moon is large enough in comparison to the planet. As there are not many exomoons[18] that have been detected, it is difficult to conclude the accuracy of this model. As it is based on the simulation used to produce Fig. 4, it is reasonable to assume that the model probably produces accurate data, however this has not been confirmed. If raw exoplanet data was given it would be more sensible to assume that there was no moon, and only add a moon if a curve similar to that seen by the violet line in Fig. 6 is observed.

Finally, our simulation of the starspots produced a characteristic curve which would easily be identified with being caused by starspots. A graph of how the intensity of the star varies as a function of distance from its centre can be seen in Fig. 7. We notice two main characteristics of the star's intensity. First, the limb-darkening effect, which causes the brightness of the star to gradually increase as one moves from the limb to the centre. Second, the sharp drop in intensity, which is a result of the starspot. In the starspot itself one can clearly see the exact same limb darkening effect as used in the star, which causes the centre of the spot to be brighter than the edge. As the spot is completely stationary in our simulation, it is easy to correct for if we obtained data which suggested a stationary starspot. In reality, the star spins on its axis, which would cause the starspot to move. The simulation does not account for this, and so it cannot be considered a completely accurate model of a starspot. It does however highlight the difficulty in accounting for starspots, which unlike

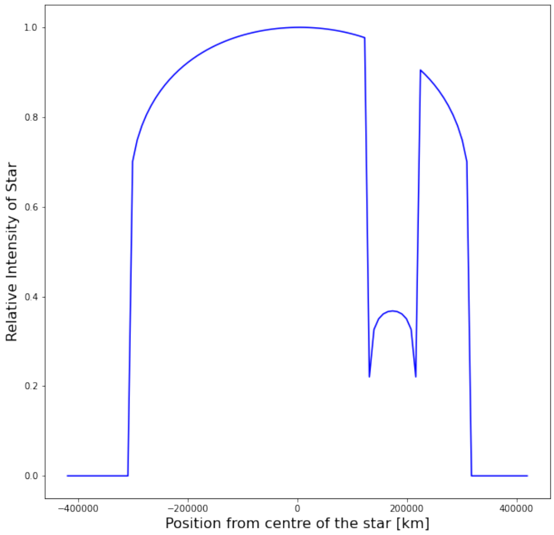

Fig. 7. Graph showing the intensity of the Sun with a starspot, as a function of distance from its centre, incorporating the quadratic limb-darkening effects: $u_1 = 0.3$ & $u_2 = 0.1$.

planets or moons do not always have a characteristic circular shape, and would require others to discern their shapes and properties in order to be incorporated by exoplanetologists.

On the whole, it can be concluded that we were able to successfully create an accurate simulation which incorporates a variety of physical phenomena based on the Jupiter-Sun system. Although our simulation of starspots could be further refined, our simulation of a planet with a moon, transiting a star, works to a suitable degree of accuracy. One aspect that we did not manage to incorporate is the simulation of occultations. Occultations occur when the exoplanet is orbiting the star at an angle relative to an observer on earth, such that we are able to see the planet when it is behind the star. To simulate this we must be able to vary the orientation angle Ω which we had, for simplicity, set as being equal to 180° (2,3 & 4). When the planet is behind the star, it can no longer be considered opaque, as it reflects the star's light back towards the observer, and so results in a relative flux greater than 1. However, the simulation we are able to create are suitable enough to be used when modelling exoplanet data.

5 MODELLING THE TRANSIT OF HD 209458B

One of the most well-studied exoplanets is HD 209458b, a planet 47pc away from Earth [19]. A key aspects of this project was to compare our simulation with real experimental data. To do this we used data taken from the paper by Knutson et al. [20]

After having simulated a Jupiter-Sun system, the procedure for modelling this exoplanet's transit was trivial, and only required changing the system's parameters. As this section of the project took inspiration from the work by Knutson, the limb-darkening function had to change from quadratic (as used in Fig. 4) to non-linear (described by equation 12). The orbital variables used are given in table 2, and the limb-darkening coefficients given in table 3.

Solely using the orbital variables used by Knutson [20] did not produce a sufficiently accurate simulation when compared with the data, and some trial and error was required to find the best fit. However, most variables used are either from Knutson's paper [20] or Winn's [6], with the exception of the inclination angle. Using an inclination angle 0.04° greater [20] gave me the best fit with her data, even though this angle does not lie within her own error parameters for the angle of inclination. The need for a different value for i may be due to the fact that our simulation uses an average velocity for the entire orbit of the planet, instead of incorporating the instantaneous velocity at every point. i is used when calculating the distance travelled (3 & 4), and as distance is used to calculate velocity it seems plausible that this may be the source of the inconsistency, which needed correcting for with a larger i. Nevertheless, the value for i is still remarkably close to the given value [20].

Table 2. Orbital variables used in creating a simulation of the HD 209458b transit.

HD 209458b Variables	
Variables	Values
M_*..............................	$1.101 M_\odot$ [20]
M_P............................	$0.69 M_{Jup}$ [19]
R_*...............................	$1.15 M_\odot$ [6]
R_P..............................	$1.38 M_{Jup}$ [6]
Orbital Period.............	3.52474859 days [20]
Inclination i................	86.969°
Orbital Eccentricity.....	0.0082 [19]
Semi-Major Axis.........	0.04747 AU [19]

The data provides information about the relative flux for ten different wavelengths, and was taken from two different days for each wavelength. By extracting the data from each day and selecting each individual wavelength, we were able to combine the relative flux to obtain a graph for each λ. It should be noted that there are gaps in the data. It does not cover every part of the transit, due to the fact that the Kepler space telescope orbits the Earth, and the light from HD 209458 is blocked by our planet. However, the data is sufficient enough to show the characteristic shape of the orbit.

When simulating the transit, separate limb-darkening coefficients are needed for each wavelength, which are given in table 3.

Table 3. Non-linear limb-darkening coefficients for the ten different wavelengths, as given by Knutson [20].

Non-linear Limb Darkening Coefficients				
$\lambda(nm)$	c_1	c_2	c_3	c_4
320.1	−0.1015	0.5547	0.6096	0.2814
390	0.0284	0.4248	0.6646	0.3450
430	0.20224	0.1101	0.9690	0.4801
484.9	0.3765	−0.0119	0.8863	0.4504
539.8	0.4957	−0.2057	0.9157	0.4340
580.2	0.5566	−0.2972	0.9190	0.4288
677.9	0.6239	−0.4176	0.8889	0.4026
775.5	0.6495	−0.4916	0.8722	0.3844
873.2	0.6623	−0.5338	0.8411	0.3661
970.8	0.6535	−0.5323	0.8142	0.3566

A graph showing how our simulation compares with the data can be seen in Fig. 8. We can see that the simulation produces a superficially good fit to the data; the modelled curves look as though they pass through most data-points. Both the timings of the eclipse and the flux at each point match to what appears to be a very high degree of accuracy. To assess the

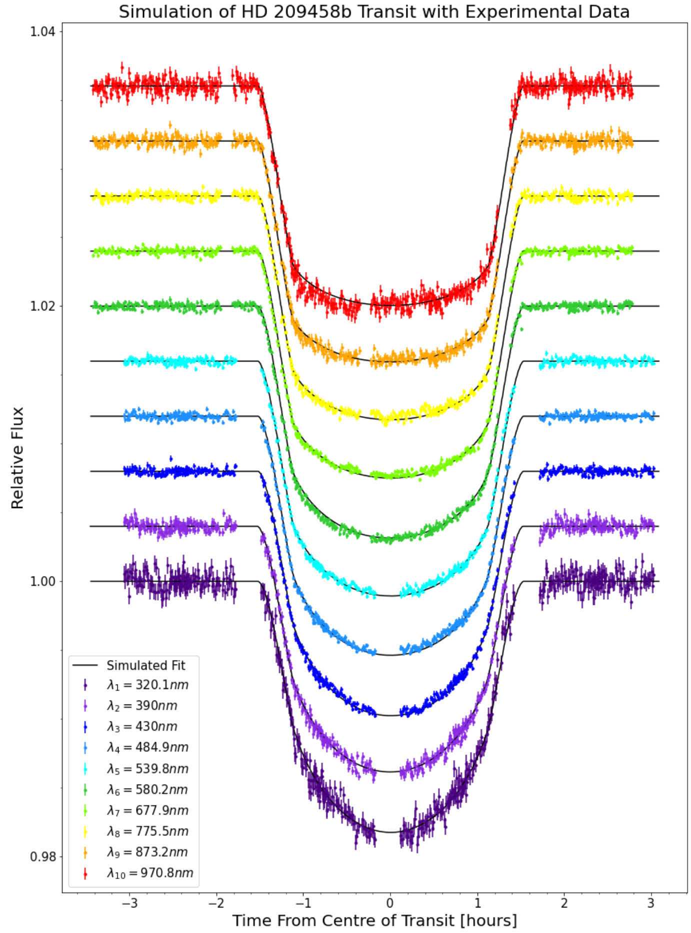

Fig. 8. Normalised data[20] for the ten wavelengths, fitted with the simulated graphs from Fig. 10. Each λ_n is offset by 0.004.

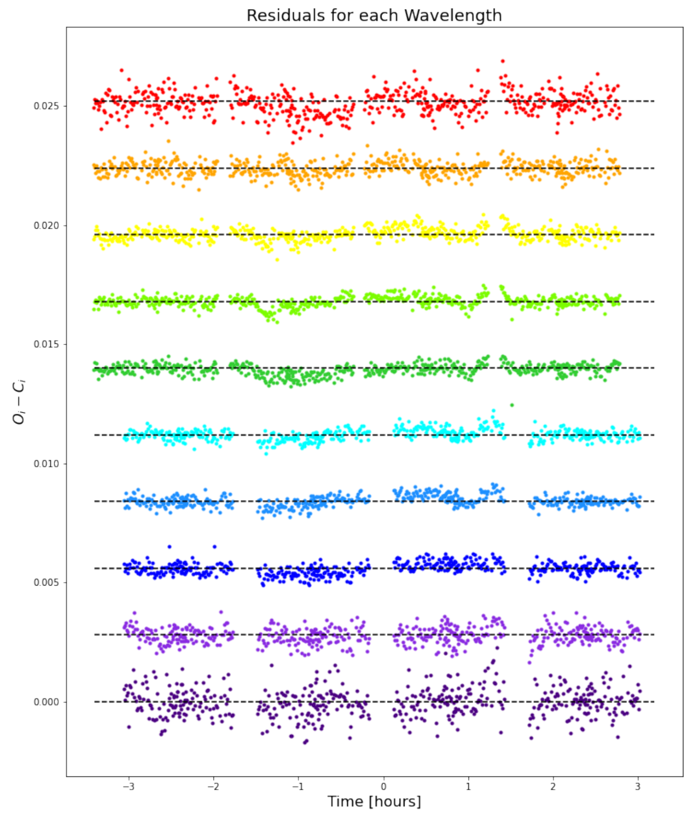

Fig. 9. Residuals of each wavelength from Fig. 8.

true accuracy of the model, residuals were plotted using:

$$r_i = O_i - C_i, \tag{14}$$

where r_i is the residual for each data point, O_i is the observed value (data), and C_i is the calculated value (simulation). The graph plotted of the residuals for each wavelength using this equation can be seen in Fig. 9. As we can see from a simple qualitative analysis, the residual values all show very small, randomly scattered deviations: as is expected. A random scatter is necessary to show that there are no systematic errors in the model. This suggests that our model is indeed a good fit with the data.

To quantitatively assess the accuracy of the simulation, a reduced chi-squared test was performed. Chi-squared tests the closeness of the fit to the data. A value of 1 suggests a perfect fit, $\chi_\upsilon > 1$ suggests a good fit, but some of the variables used in the fit require adjusting. $\chi_\upsilon \gg 1$ suggests a poor model fit, and $\chi_\upsilon < 1$ suggests that the model is "over-fitting" the data. The equation for calculating the chi-squared value for each wavelength is:

$$\chi^2 = \sum_i \frac{(O_i - C_i)^2}{\sigma_i^2}, \tag{15}$$

where σ_i is the error of each data value. The reduced chi-squared is simply a normalised value of the chi-squared:

$$\chi_\upsilon^2 = \frac{\chi^2}{\upsilon}, \tag{16}$$

where υ is the number of degrees of freedom; which in this case is given by the number of data-points.

The values for all chi-squared tests are given in table 4. All values lie between $1.0 < \chi_\upsilon^2 < 1.77$, which suggests that our model is a very good fit to the data. In the paper[20], the planet's radii was slightly adjusted for each wavelength. We decided not to do this, and instead tried to find variables that produced a universally strong fit. With the residuals and chi-squared tests as evidence, we can conclude that our simulation with our chosen variables (tables 2 & 3) accurately models the transit of HD 209458b.

To compare the difference made by the unique limb-darkening coefficients, a graph of all the simulations superimposed can be seen in Fig. 10. Here we can clearly see that the larger the wavelength, the smaller the limb-darkening effect. Limb-darkening results in a much steeper curve, as can be seen by the purple curve for λ_1, whereas having

Table 4. χ_v^2 values for each individual wavelength, as calculated using equations (15) & (16).

χ_v^2 Values for each λ	
Wavelength	χ_v^2
λ_1	1.1248
λ_2	1.1228
λ_3	1.7002
λ_4	1.7642
λ_5	1.5798
λ_6	1.5899
λ_7	1.6473
λ_8	1.4008
λ_9	1.0186
λ_{10}	1.1997

no limb-darkening causes a completely flat curve, as previously seen in Fig. 4. This suggests that 'redder' wavelengths are less prone to limb-darkening, due to the flatter nature of their curves. The limb-darkening also changes the value of the minimum flux. The implications of this is that, without accounting for limb-darkening, one would look at the data and assume that the minimum flux's value was equal to $1 - k^2$ (9). By calculating k, and with the knowledge of the star's radius, one would incorrectly calculate the radius of the planet. To grasp the seriousness of this error, we can simply look at the difference in minimum flux between λ_1 and λ_{10}. The difference $\Delta k^2 = 0.0023$ corresponds to a 4.8% discrepancy in the planet's radius; once again highlighting the importance of accounting for limb-darkening.

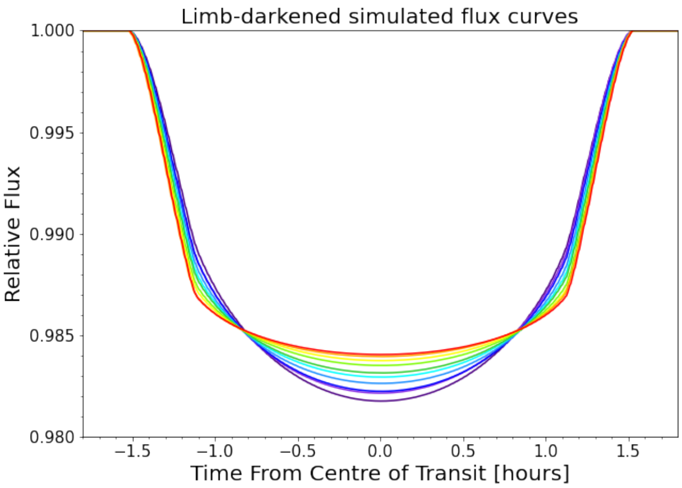

Fig. 10. Limb-darkened simulated transit curves for λ_{1-10}, using the variables in tables 2 and 3.

6 THE FUTURE OF EXOPLANET RESEARCH

There is much that could have still been incorporated into our simulation, alas time constraints limit the extent to which we can innovate. I have already mentioned the lack of occultations and the elementary starspot simulation, but there are more physical phenomenon that could still have been incorporated.

There are planets which, like Saturn, are orbited by a thin disk of ice, producing characteristic rings. Simulating these rings would have made for a nice addition to our model. The method to simulate them would be different from simulating moons, as this would require the creation of a disk with specific thickness. One could play around with varying the angle of the ring towards the observer and analyse the different effects that would have on flux.

Analysing the atmosphere of an exoplanet is something that we were not able to achieve, but it can be done using transmission spectroscopy. This would provide data which is instrumental in determining whether there are any chemical signatures of life. Simulating much more complicated systems, such as multi-planetary star systems or circumbinary planets, could give us a better insight into some of the more unfamiliar phenomena that occur in the universe. All life on earth derives its energy from the sun. A circumbinary planet, with two stars should, in theory, be able to host life of unimaginable complexity, due to receiving so much more energy from two stars. Studying these systems could lead to revolutionising our understanding of life itself.

7 CONCLUSION

Our project to simulate the transits of planets was a success. By starting with simple Newtonian mechanics, and incorporating limb-darkening, we were able to simulate the transit of a Jupiter-like planet across a Sun-like star. The simulated flux-time graph agrees with theoretical predictions, and shows that our simulation works to a high degree of accuracy. The importance of using limb-darkening in determining planetary radius from data was identified by comparing the curve of a non limb-darkened simulation to a limb-darkened one, and noticing the effect it has on both the shape and minimum flux.

Expanding the simulation by programming an orbiting moon added to the realism of the Jupiter orbit, but we can conclude that moons which are relatively small in comparison with their planet scarcely produce any effect on the relative flux graph. We were able to see how having a larger moon produces a noticeable trace in the flux graph.

Simulating starspots was an aspect which needed more time devoted to, but we were able to notice how having a starspot on a star affected the luminosity. Accounting for a stationary starspot in experimental data would not prove too challenging, because it produces a very characteristic shape.

The project's greatest achievement was successfully simulating the transit of HD 209458b. Using data from Knutson[20] we were able to splice and combine data to obtain ten different transit curves for the ten different wavelengths of light that the transits were recorded in. By taking variables from a few different sources, a simulation of the transit curve for each wavelength was plotted alongside the data. We were clearly able to see that our model went through many of the experimental data points. A graph of residuals and a table of chi-squared values provided a better qualitative and quantitative analysis of this respectively. The residual graph was comprable to that created by Knutson[20], and as all the chi-squared values lay between $1.0 < \chi_v^2 < 1.77$. We can safely conclude that the simulation was very accurate.

As a side-product of creating this simulation, we superimposed all of the simulations and noticed the correlation between wavelength and limb-darkening. Noticing that, the shorter the wavelength, the larger limb-darkening effect there is.

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APPENDIX A CODE

```python
import numpy as np
import matplotlib.pyplot as plt

def intensity (x,y): # Draws the star with limb darkening
    global r_star,centre_of_star
    r_squared= (x-centre_of_star[1])**2 + (y-centre_of_star[1])**2
    circle_mask = (x-centre_of_star[1])**2 + (y-centre_of_star[1])**2 <= r_star**2
    mu = np.sqrt(1 - r_squared/r_star**2,where=((x-centre_of_star[1])**2 + (y-centre_of_star[1])**2 <= r_star**2))

    return Intensity_max*(1-mu1*(1-mu)-mu2*(1 - mu)**2)*circle_mask

def planet_writer(x,y): # Draws the planet at the given centre
    global centre_of_planet,r_planet
    return (x-centre_of_planet[1])**2 + (y-centre_of_planet[0])**2 > (r_planet)**2


def draw_stage(i,background): # Adds together the planet and star onto the same canvas
    global centre_of_planet, display_dim
    centre_of_planet = [X_coord_on_grid[i],Z_coord_on_grid[i]]

    display_added_planet = 1 * np.fromfunction(planet_writer, (display_dim, display_dim), dtype=int)

    return display_added_planet * background
```

Fig. 11. Functions used to create and draw the planet, along with simulating the star with quadratic limb darkening

```python
display = np.zeros((display_dim, display_dim))
centre_of_display = display_dim/2

######################################## Planet
SF = (Rstar + 2*Rp) # Scaling display to fit star/planet

X_coord_on_grid = np.round(X_3D/SF * centre_of_display + centre_of_display) # Scaling X with respect to display
Z_coord_on_grid = np.round(Z_3D/SF * centre_of_display + centre_of_display) # Scaling Z with respect to display

r_planet = Rp/SF *  display_dim/2 # Scaling planet radius with respect to display
######################################## Star
centre_of_star = [centre_of_display,centre_of_display] # Centre of star
r_star = Rstar/SF * display_dim/2 # Scaling star centre with respect to display

display_with_star = np.fromfunction(intensity,(display_dim, display_dim), dtype=int) # Drawing Star
```

Fig. 12. Code used to scale the distances in order to fit the display size

```python
def moon_writer(z,x):
    global centre_of_moon,centre_of_planet,r_moon
    return (z-centre_of_planet[1]-centre_of_moon[1])**2 + (x-centre_of_planet[0]-centre_of_moon[0])**2 > r_moon**2

def planet_mask(x,y):
    global centre_of_planet,r_planet

    return (x-centre_of_planet[0])**2 + (y-centre_of_planet[1])**2 > r_planet**2
```

Fig. 13. Functions used to create and draw the moon.

```python
''' Day 1 '''
time1_1_1 = (first_transit['Heliocentric_Julian_Date'][0:252] - 2452763.183042)*24 # mi
time1_2_1 = (first_transit['Heliocentric_Julian_Date'][252:504] - 2452763.183042)*24
time1_3_1 = (first_transit['Heliocentric_Julian_Date'][504:756] - 2452763.183042)*24
time1_4_1 = (first_transit['Heliocentric_Julian_Date'][756:1008] - 2452763.183042)*24
time1_5_1 = (first_transit['Heliocentric_Julian_Date'][1008:1260] - 2452763.183042)*24

Rf1_1 = first_wavelength_1['Relative_flux']
Rf2_1 = second_wavelength_1['Relative_flux'] +0.004 # adding n*0.004 creates an offset
Rf3_1 = third_wavelength_1['Relative_flux'] +2*0.004
Rf4_1 = fourth_wavelength_1['Relative_flux'] +3*0.004
Rf5_1 = fifth_wavelength_1['Relative_flux'] +4*0.004

y_error1_1 = first_wavelength_1['Uncertainty_in_RFlux']
y_error2_1 = second_wavelength_1['Uncertainty_in_RFlux']
y_error3_1 = third_wavelength_1['Uncertainty_in_RFlux']
y_error4_1 = fourth_wavelength_1['Uncertainty_in_RFlux']
y_error5_1 = fifth_wavelength_1['Uncertainty_in_RFlux']
```

Fig. 14. An example of the code used to separate the data for each individual wavelength, so that they may be merged together to create a continuous flux graph.

APPENDIX B ADDITIONAL FIGURES

```
[[0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
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 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
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 [0. 0. 0. 0. 0. 0. 0. 0. 2. 2. 1. 1. 1. 1. 2. 0. 0. 0. 0. 0. 0. 0. 0.]
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 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]]
```

Fig. 15. Grid showing the Sun (1) with Jupiter orbiting around it (2).

```
 0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.
 0.   0.   0.   0.  60.  60.  60.  60.  60.   0.   0.   0.   0.
 0.   0.  60.  60.  84.  84.  84.  84.  84.  60.  60.   0.   0.
 0.  60.  84.  84.  91.  91.  91.  91.  91.  84.  84.  60.   0.
 0.  60.  84.  91.  95.  95.  95.  95.  95.  91.  84.  60.   0.
60.  84.  91.  95.  95.  98.  98.  98.  95.  95.  91.  84.  60.
60.  84.  91.  95.  98.  99.  99.  99.  98.  95.  91.  84.  60.
60.  84.  91.  95.  98.  99. 100.  99.  98.  95.  91.  84.  60.
60.  84.  91.  95.  98.  99.  99.  99.  98.  95.  91.  84.  60.
60.  84.  91.  95.  95.  98.  98.  98.  95.  95.  91.  84.  60.
 0.  60.  84.  91.  95.  95.  95.  95.  95.  91.  84.  60.   0.
 0.  60.  84.  84.  91.  91.  91.  91.  91.  84.  84.  60.   0.
 0.   0.  60.  60.  84.  84.  84.  84.  84.  60.  60.   0.   0.
 0.   0.   0.   0.  60.  60.  60.  60.  60.   0.   0.   0.   0.
 0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.
```

Fig. 16. Grid showing the limb darkening effect of the star, without graphics.

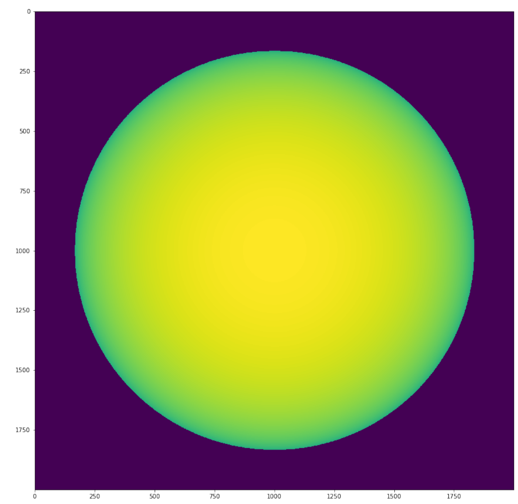

Fig. 17. Plot of the grid-based model of the star (yellow/green) with limb darkening effects incorporated, at a high resolution.

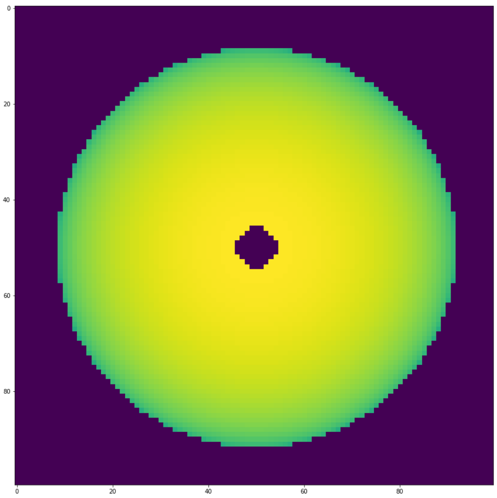

Fig. 18. Low resolution plot of the grid-based model of the star(yellow/green) and the planet (centre black) with limb darkening effects incorporated.

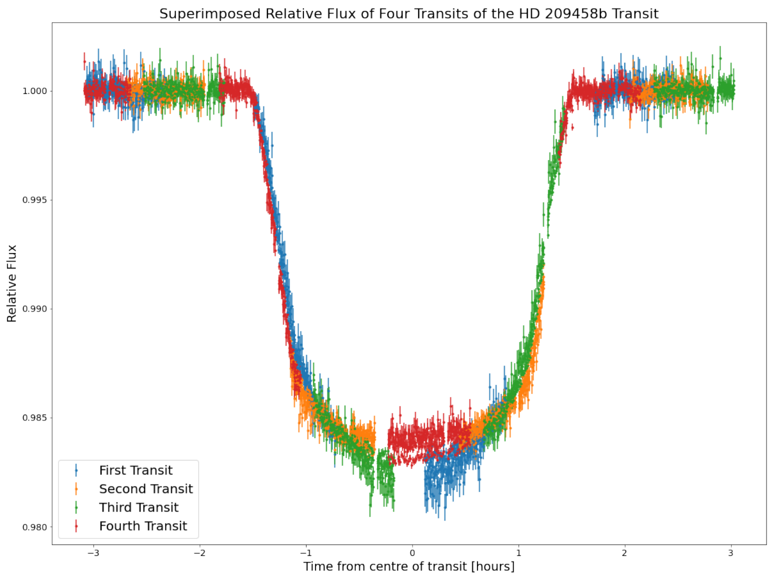

Fig. 19. Superimposed flux graph of all of the data, separated by colour based on when the transits occurred.