# EC569 Economic Growth
# Lecture4

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# Economic Growth in the Solow model

What's the long-run growth rate of income per capita in the Solow Model?

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$$0$$

# The Extended Solow model

**Retain assumptions:**

- Constant returns to scale production function: $F(K, ehL)$
- A constant fraction ($\gamma$) of output is invested.
- A constant fraction ($\delta$) of physical capital stock depreciates.
- Labor force participation rate is constant.
- Population grows at a constant rate, $n$.
- Human capital, $h$, is constant.

**Differently, assume**

- Labor augmenting productivity, $e$, grows at a constant rate, $g$.

$$\frac{\dot{e}}{e} = g \Leftrightarrow e(t) = e_0 e^{gt}$$

# Clarification...

- Previously, we assumed production function is $AF(K, hL)$
  - Here, $A$ is Hicks-neutral technology
- In the Extended Solow model, technology needs to be labor-augmenting: $F(K, ehL)$
  - Here, $e$ is labor-augmenting or Harrod-neutral technology
- In Cobb-Douglas production, this distinction is not important.
  - $K^{\alpha}(ehL)^{1-\alpha} = AK^{\alpha}(hL)^{1-\alpha}$, where $A \equiv e^{1-\alpha}$

# Accumulation of physical capital

- Change in capital stock = Investment - Depreciation

$$\dot{K} = I - D$$

- A constant fraction, $\gamma$, of output is invested

$$I = \gamma Y$$

- A constant fraction, $\delta$, of capital depreciates

$$D = \delta K$$

$$\dot{K} = \gamma Y - \delta K$$

# Accumulation of capital-technology ratio

- Goal: write accumulation of capital-technology ratio, $\tilde{k} \equiv \frac{K}{eL}$
- Capital-technology ratio, $\tilde{k}$, sometimes referred as capital per effective labor.
- Why do we need to convert physical capital accumulation equation into capital-technology units?
- Because capital stock will grow as a result of increasing population and higher productivity.

## Accumulation of capital-technology ratio, cont'd

- Accumulation of capital

$$\dot{K} = \gamma Y - \delta K$$

- How do we transform $\dot{K}$ into $\dot{\tilde{k}}$?
    - make use of $\tilde{k} \equiv \frac{K}{eL}$
    - take log of $\tilde{k}(t) \equiv \frac{K(t)}{e(t)L(t)}$:

$$\ln(\tilde{k}(t)) = \ln\left(\frac{K(t)}{e(t)L(t)}\right) = \ln(K(t)) - \ln(e(t)) - \ln(L(t))$$

    - Then differentiate with respect to time, $t$,

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{e}(t)}{e(t)} - \frac{\dot{L}(t)}{L(t)}, \qquad \frac{\dot{\tilde{k}}}{k} = \frac{\dot{K}}{K} - g - n$$

    - Remember chain-rule:

$$\frac{df(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

# Accumulation of capital-technology ratio, cont'd (2)

- Divide each side of $\dot{K} = \gamma Y - \delta K$ by $K$:

$$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta$$

- Then,

$$\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{\gamma Y/(eL)}{K/(eL)} - \delta$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{\gamma \tilde{y}}{\tilde{k}} - \delta, \text{ where } \tilde{y} \equiv \frac{Y}{eL}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\gamma \tilde{y}}{\tilde{k}} + (\delta + g + n) \quad \Rightarrow \quad \dot{\tilde{k}} = \gamma \tilde{y} - (\delta + g + n)\tilde{k}$$

# Steady State

$$\dot{\tilde{k}} = \gamma\tilde{y} - (\delta + g + n)\tilde{k}, \text{ or}$$

$$\dot{\tilde{k}} = \gamma f(\tilde{k}) - (\delta + g + n)\tilde{k}, \text{ where } \tilde{y} = f(\tilde{k}) = F(K, ehL)/(eL)$$

Capital-technology ratio is constant at the steady state.

- if $\gamma f(\tilde{k}) > (\delta + g + n)\tilde{k}$, then $\dot{\tilde{k}} > 0$
- if $\gamma f(\tilde{k}) < (\delta + g + n)\tilde{k}$, then $\dot{\tilde{k}} < 0$
- if $\gamma f(\tilde{k}) = (\delta + g + n)\tilde{k}$, then $\dot{\tilde{k}} = 0$: **steady state**

# The Solow Diagram

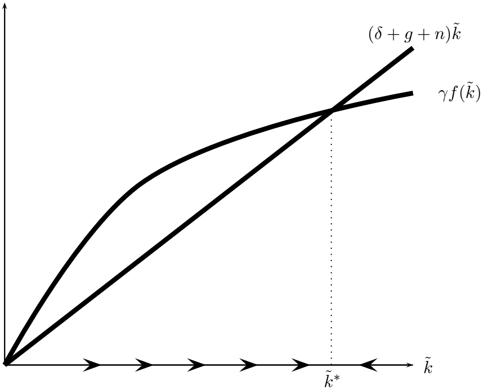



Diagram from Jones and Vollrath (2013)

If $\tilde{k} < \tilde{k}^*$, capital-technology ratio and output-technology ratio will increase.
If $\tilde{k} > \tilde{k}^*$, capital-technology ratio and output-technology ratio will decrease.

# Steady State, cont'd

Cobb-Douglas production function

$$Y = K^{\alpha}(ehL)^{1-\alpha}$$
$$\frac{Y}{eL} = \frac{K^{\alpha}(ehL)^{1-\alpha}}{(eL)^{\alpha}(eL)^{1-\alpha}}$$
$$\tilde{y} = \tilde{k}^{\alpha}h^{1-\alpha}$$

No change of capital-technology ratio at the steady state

$$\dot{\tilde{k}} = \gamma\tilde{k}^{\alpha}h^{1-\alpha} - (\delta + g + n)k$$
$$0 = \gamma(\tilde{k}^{*})^{\alpha}h^{1-\alpha} - (\delta + g + n)k^{*}$$
$$\gamma(\tilde{k}^{*})^{\alpha}h^{1-\alpha} = (\delta + g + n)\tilde{k}^{*}$$
$$\tilde{k}^{*} = \left(\frac{\gamma}{\delta + g + n}\right)^{1/(1-\alpha)} h$$

Steady steady output per worker:

$$\tilde{y}^{*} = (\tilde{k}^{*})^{\alpha}h^{1-\alpha} = \left(\frac{\gamma}{\delta + g + n}\right)^{\alpha/(1-\alpha)} h$$

# Per worker values

Output per worker:

$$y(t) = e(t)\tilde{y}(t)$$

Capital per worker:

$$k(t) = e(t)\tilde{k}(t)$$

Output per worker at the steady state:

$$y(t) = e(t)\tilde{y}^* = e(t)\left(\frac{\gamma}{\delta + g + n}\right)^{\alpha/(1-\alpha)} h$$

Capital per worker:

$$k(t) = e(t)\tilde{k}^* = e(t)\left(\frac{\gamma}{\delta + g + n}\right)^{1/(1-\alpha)} h$$

# Comparative Statics

Capital-technology ratio at the steady state:

$$\tilde{k}^* = \left(\frac{\gamma}{\delta + g + n}\right)^{1/(1-\alpha)} h$$

Output-technology ratio at the steady state:

$$\tilde{y}^* = \left(\frac{\gamma}{\delta + g + n}\right)^{\alpha/(1-\alpha)} h$$

- $\tilde{k}^*$ and $\tilde{y}^*$ are rising with investment rate $\gamma$, and human capital $h$,
- $\tilde{k}^*$ and $\tilde{y}^*$ are declining with depreciation rate, $\delta$, population growth rate, $n$, and rate of technological progress, $g$.

# Increasing the investment rate

As $\gamma \uparrow$, $\tilde{k}^* = \left(\frac{\gamma}{\delta+g+n}\right)^{1/(1-\alpha)} h \uparrow$, $\tilde{y}^* = \left(\frac{\gamma}{\delta+g+n}\right)^{\alpha/(1-\alpha)} h \uparrow$

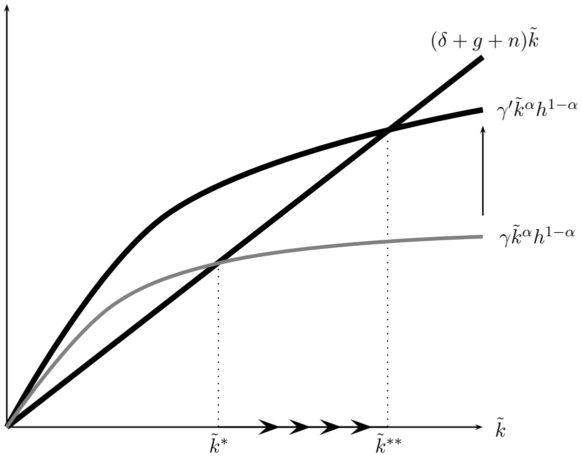



Diagram from Jones and Vollrath (2013)

## Output-technology ratio after an increase in investment rate

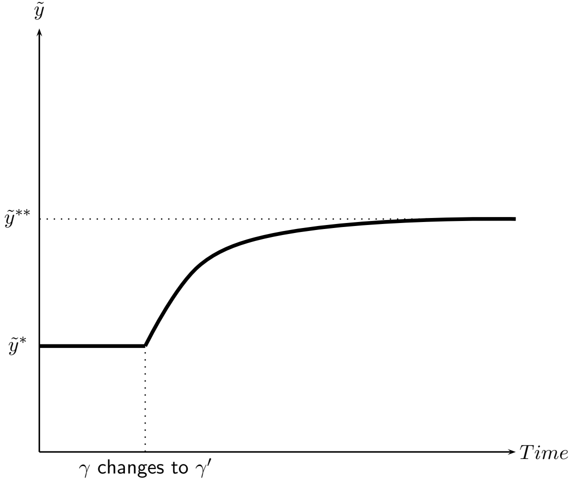



Diagram from Jones and Vollrath (2013)

# Output per worker after an increase in investment rate

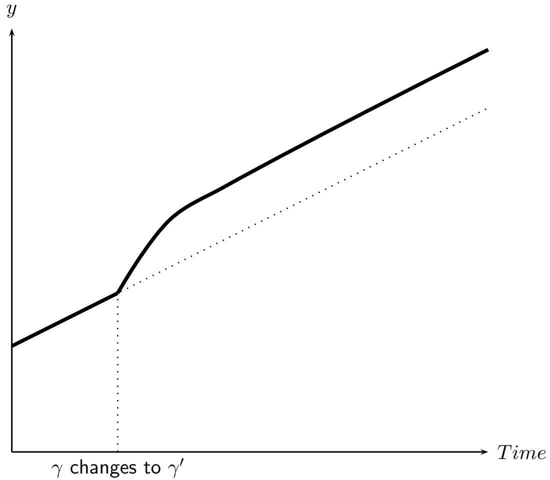


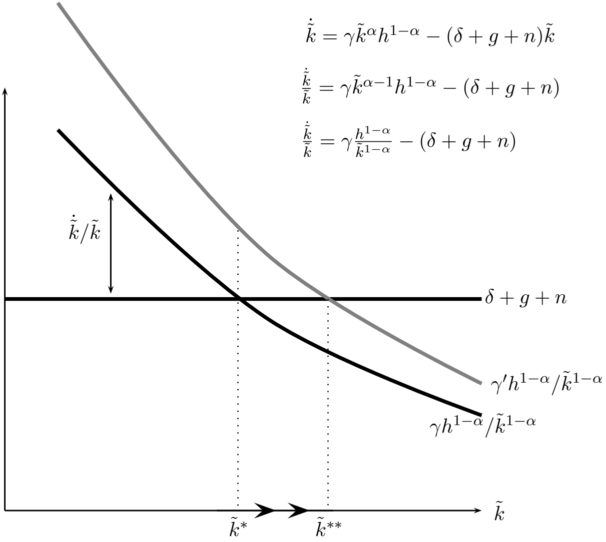

Diagram from Jones and Vollrath (2013)

# Growth rate of capital-technology ratio after an increase in investment rate

$$\dot{\tilde{k}} = \gamma \tilde{k}^{\alpha} h^{1-\alpha} - (\delta + g + n)\tilde{k}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \gamma \tilde{k}^{\alpha-1} h^{1-\alpha} - (\delta + g + n)$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \gamma \frac{h^{1-\alpha}}{\tilde{k}^{1-\alpha}} - (\delta + g + n)$$



# Growth rate of output per worker after an increase in investment rate

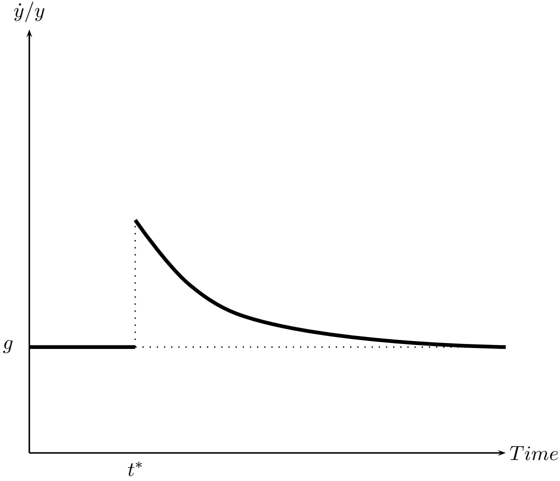



Diagram from Jones and Vollrath (2013)

# Exercise

Conduct comparative statics for changes in $n$, $\delta$, $g$, and $h$.

# Impact of an increase in investment rate

**Short-run:**

- Growth rates of capital per worker and income per worker increase.
- Capital per worker and income per worker increases.

**Long-run (steady-state):**

- Capital per worker and income per worker increases.
- Growth rate of capital per worker and income per worker do not change.

# Steady State Growth Rates

| | Growth rate at the s.s. |
|---|---|
| $\tilde{k}$ | |
| $\tilde{y}$ | |
| $\tilde{c}$ | |
| $k$ | |
| $y$ | |
| $c$ | |
| $K$ | |
| $Y$ | |
| $C$ | |

# Steady State Growth Rates

| | Growth rate at the s.s. |
|---|---|
| $\tilde{k}$ | 0 |
| $\tilde{y}$ | 0 |
| $\tilde{c}$ | 0 |
| $k$ | g |
| $y$ | g |
| $c$ | g |
| $K$ | g + n |
| $Y$ | g + n |
| $C$ | g + n |

# Exogenous Growth Model

- Technology is the only source of long-run growth.
- Technology is exogenous: not a results of interactions of agents in the model
- Hence, the extended Solow model is an example of "exogenous growth models."
- It is also referred as "Neo-classical growth model"
- In upcoming lectures, we will analyze "endogenous growth models"
  - Technological progress as a result of actions of model agents.

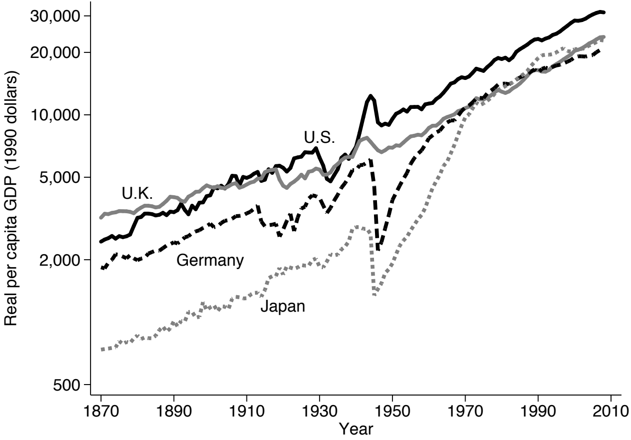
# Convergence

Are “poor” countries growing faster than “rich” countries?

Are “poor” countries “closing the gap”?

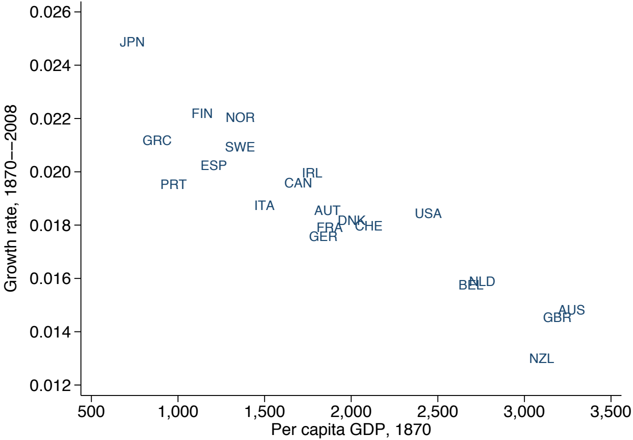
**Converge:** The phenomenon of “poor” countries catching up with the “rich” countries.

# Convergence in a sample of industrialized countries, 1870-2008



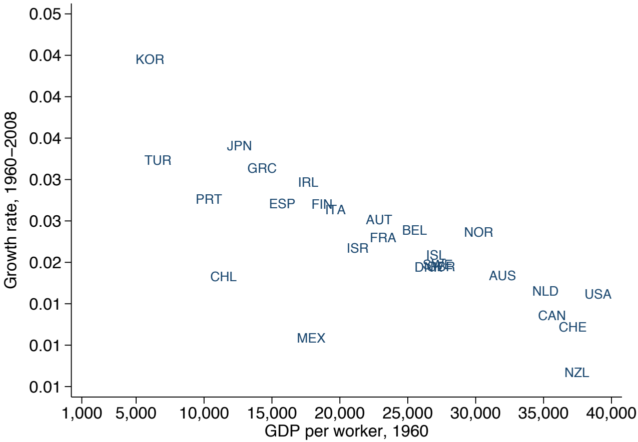

Data source: Maddison (2010)
Graph from: Jones and Vollrath (2013)

# Convergence in a sample of industrialized countries, cont'd, 1870-2008



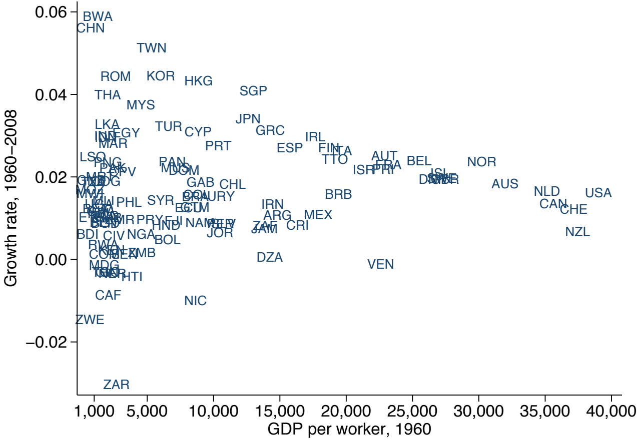

Data source: Maddison (2010)
Graph from: Jones and Vollrath (2013)

# Convergence in OECD countries, 1960-2008




Data source: Penn World Tables Mark 7.0 and Summers and Heston (1991)
Graph from: Jones and Vollrath (2013)

# The lack of convergence for the World, 1960-2008




Data source: Penn World Tables Mark 7.0 and Summers and Heston (1991)
Graph from: Jones and Vollrath (2013)

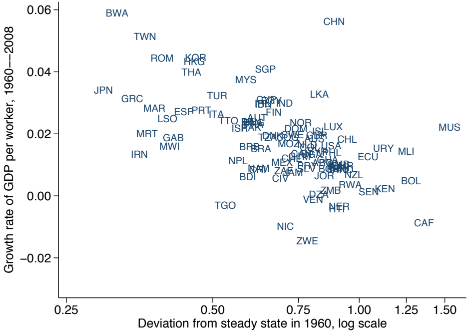
## Convergence (?)

- How do we reconcile the converge in OECD but lack of convergence for the world?
- Prediction of the Solow model:
  - Among the countries with the same steady state, poor countries should grow faster than rich countries.
  - Steady state depends on investment rate, population growth rate, technological progress rate
- OECD countries show similarities in investment rate, population growth rate, rate of technological progress.
- More variation in the World in these statistics.

# Conditional convergence

- Mankiw, Romer, and Weil (1992), and Barro and Sala-i-Martin (1992):
- Convergence of countries "conditional on" their steady states
- Countries that are poor relative to their steady states tend to grow faster.

# "Conditional" convergence for the World, 1960-2008




Data source: Author's calculations using Penn World Tables 7.0, update of Summers and Heston (1991).

Note: The variable on the x-axis is $\hat{y}_{60}/\hat{y}^*$. Estimates of A for 1970 are used to compute the steady state.

Graph from: Jones and Vollrath (2013)

## Summary

- We developed a model in which technological progress is the only source of long-run growth
- We looked at the convergence of countries: countries with similar steady states converge but not all countries.

Thank you!