EC569 Economic Growth
The Solow Growth Model
Lecture 2

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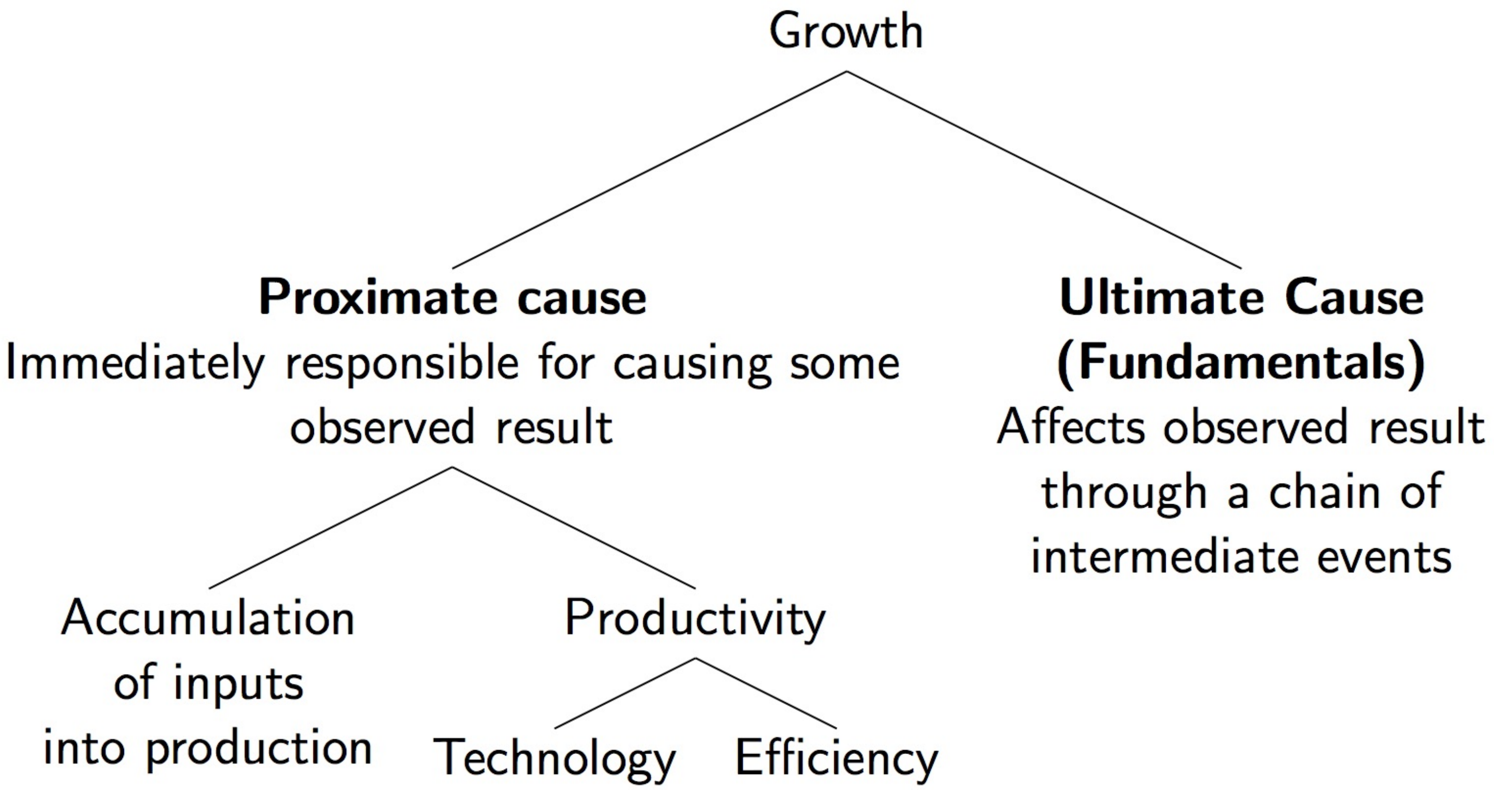
The Solow Growth Model

- A model to explain the role of factor accumulation in economic growth



Physical capital

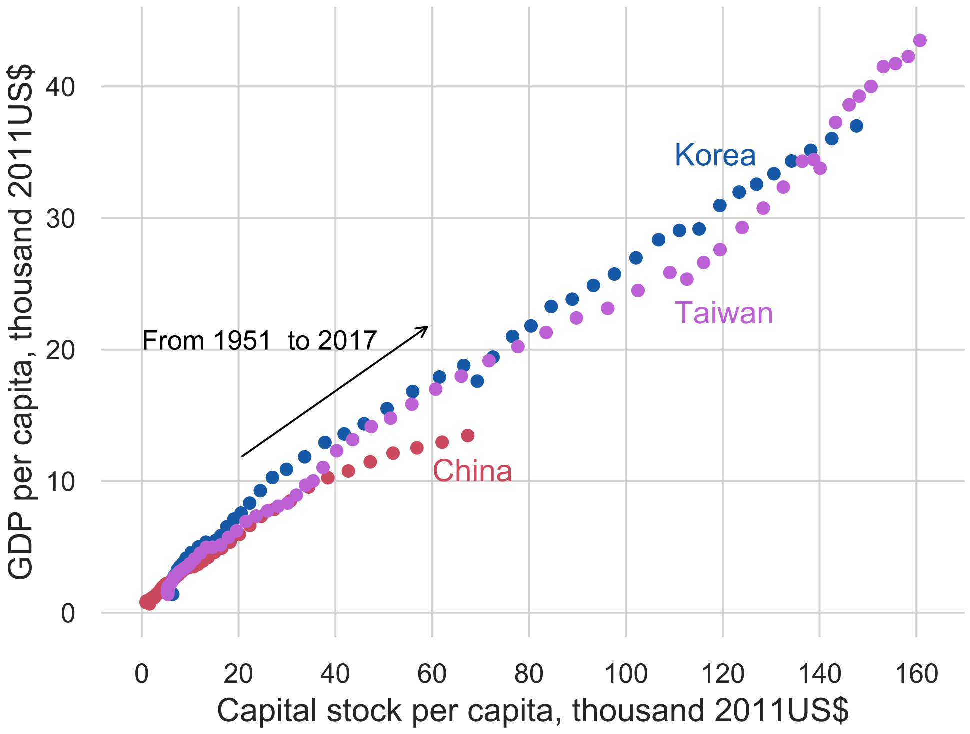
Capital: Tools – the physical objects that extend our ability or do work for us:

- machines
- buildings
- infrastructure
- vehicles
- computers

Key characteristics of capital

- It is productive
- It is produced (not like piece of land)
    - investment
- Its use is limited: limited number of people can use a given piece of capital at one time
    - unlike ideas
- It can earn a return
    - incentive for its creation
- It wears out
    - depreciation

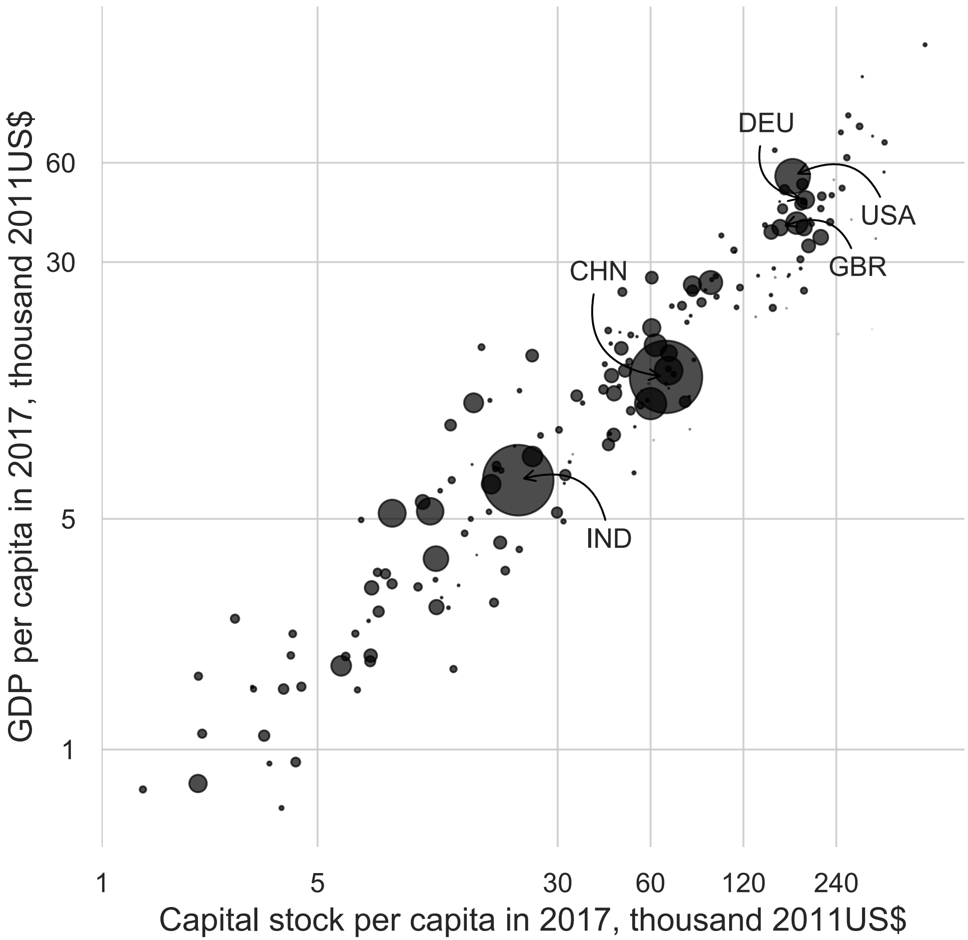
Role of capital accumulation on economic growth



Source: Penn World Tables, version 9.1

- Source of short-run economic growth in the Solow model is physical capital accumulation
- China, Taiwan and Korea are example countries that achieved high economic growth through capital accumulation.

Role of capital accumulation on economic growth



- Source of short-run economic growth in the Solow model is physical capital accumulation
- China, Taiwan and Korea are example countries that achieved high economic growth through capital accumulation.
- There is a positive correlation between GDP per capita and capital stock per capita.

Source: Penn World Tables, version 9.1

The Solow model

Build around two equations:

- Production function
  - Constant returns to scale production function
- Capital accumulation equation
  - Households invest a constant share of income in capital stock

Note: There are many varieties of the Solow model. Initially we will learn about the basic version with no technological change. In the later lectures, we will learn the Solow model with technological progress (extended Solow model).

Production Function

$Y = AF(K, hL)$, where Y is output, K is physical capital, L is labor, A is productivity, and h is human capital.

- Assume $F(\cdot)$ displays **constant returns to scale (CRS):** $F(zK, zhL) = zF(K, hL)$, where z is a constant.
- Convert aggregate production function, $F(\cdot)$, into per worker production function, $f(\cdot)$, by multiplying $F(\cdot)$ with $1/L$:

$$\left(\frac{1}{L}\right) Y = \left(\frac{1}{L}\right) F(K, hL) = F\left(\frac{K}{L}, \frac{hL}{L}\right) = F\left(\frac{K}{L}, h\right)$$

- Define $k \equiv K/L$, capital per labor, $y \equiv Y/L$, output per labor, and assume h is constant:

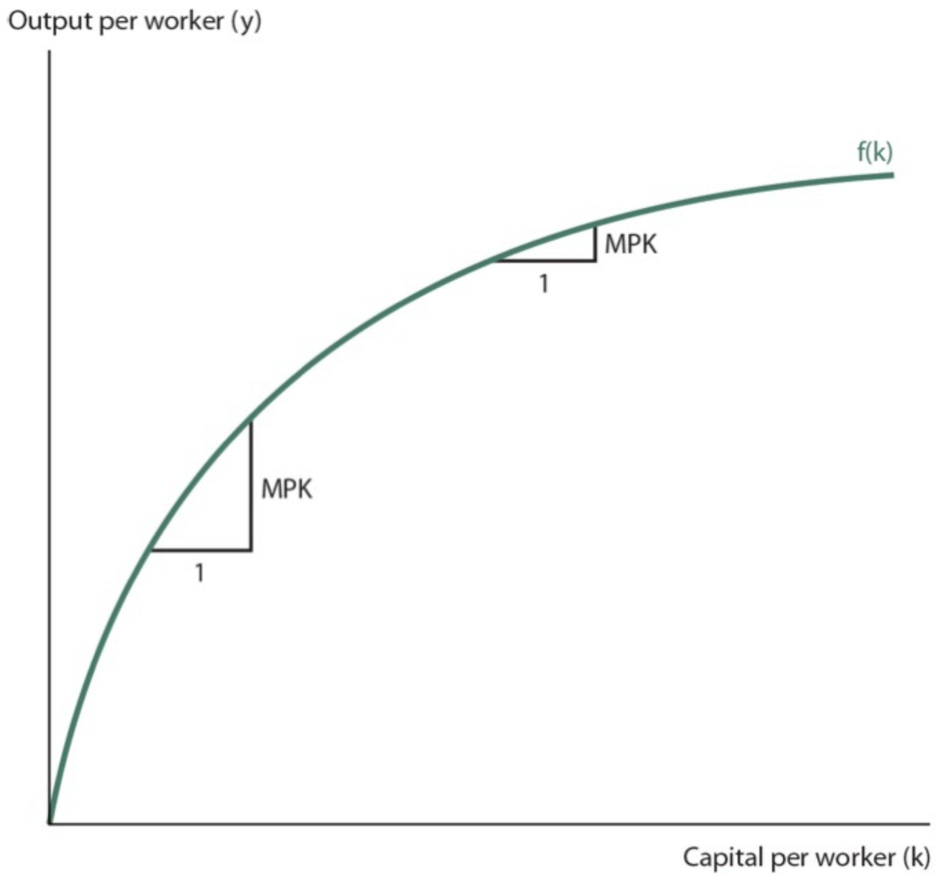
$$y = F(k, h) \equiv f(k)$$

- $f(\cdot)$ displays **diminishing marginal product:**

  Marginal product of capital (MPK) $= \frac{\partial f(k)}{\partial k} \approx [f(k+1) - f(k)] \downarrow$ as $k \uparrow$

- Diminishing MPK will be the key factor for not having long run economic growth without technological progress.
  - More on this point later

Production function, graphical representation



Source: Weil (2013)

An example production function displaying CRS: Cobb-Douglas Production Function

Production function: $Y = AK^\alpha (hL)^{1-\alpha}$,

where A is productivity, K is capital, h is human capital, and L is labor, $0 < \alpha < 1$.

- It satisfies CRS assumption. Check yourself!
- Therefore

$$\frac{Y}{L} = \frac{AK^\alpha (hL)^{1-\alpha}}{L} = A\frac{K^\alpha}{L^\alpha}\frac{(hL)^{1-\alpha}}{L^{1-\alpha}} = A\left(\frac{K}{L}\right)^\alpha h^{1-\alpha}, \ \ y = Ak^\alpha h^{1-\alpha}$$

- Marginal product of physical capital is positive

$$\mathrm{MPK} = \frac{\partial y}{\partial k} = \alpha A k^{\alpha-1} h^{1-\alpha} > 0$$

- Diminishing marginal product of capital

$$\frac{\partial^2 y}{\partial k^2} = (\alpha - 1)\alpha A k^{\alpha-2} h^{1-\alpha} < 0$$

- Hence, Cobb-Douglas function satisfies the Solow model assumptions on the production function.

Solow model assumptions, cnt'd

- Productivity, A, is constant. *Will be relaxed in later lectures*
- Human capital, h, is constant. *Will be relaxed in later lectures*
- Labor force participation rate is constant.
- Population grows at a rate n, and so does labor force, L.

$$L(t) = L_0 e^{nt}$$

- Trick: to find the growth rate of a variable, take log and differentiate w.r.t. time, t.

$$\ln L(t) = \ln L_0 + nt, \quad \frac{\dot{L(t)}}{L(t)} = n,$$

where dot on top of a variable means derivative with respect to time: $\dot{L(t)} \equiv \frac{\partial L(t)}{\partial t}$

- E.g. population grows are a rate, $n = .01$ or 1% per year.
- $\dot{L} \approx L(t+1) - L(t)$

Accumulation of physical capital

- Second key equation of the Solow model
- Investment: **a constant fraction of output, γ is invested:**

$$I = \gamma Y$$

- Consumption: output minus investment.
- Depreciation: **a constant fraction, δ, of capital wears out:**

$$D = \delta K$$

- Change in physical capital stock = Investment - Depreciation

$$\dot{K} = I - D$$

- Subsituting investment and depreciation rules into above equation gives capital accumulation equation:

$$\dot{K} = \gamma Y - \delta K$$

The Solow Model assumptions, all together

- Production function displays constant returns to scale
- Production function displays diminishing marginal product of capital
- Everyone works in the economy: population = labor force
- Population and labor force grow at a constant rate n
- Society invests a constant fraction, γ, of output into capital stock
- A constant fraction, δ, of capital depreciates

Characterize the dynamics of output per worker over time

- To achieve this goal, we first need to characterize the dynamics of capital per worker

Accumulation of capital per worker

- Goal: write physical capital accumulation equation with per worker variables
- Start with the equation describing the evolution of the aggregate capital:

$$\dot{K} = \gamma Y - \delta K$$

- Divide each side by K:

$$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta$$

- Define capital per worker, $k \equiv K/L$.
- Our goal is to convert $\frac{\dot{K}}{K}$ into $\frac{\dot{k}}{k}$.
- take log (ln) of $k \equiv K/L$:

$$\log(k) = \log(K/L) = \log(K) - \log(L)$$

- Differentiate $k = K/L$ with respect to time (remember $k = k(t)$ and $L = L(t)$)
  - Remember differentiation rules:

$$\frac{d\log(k(t))}{dt} = \frac{\frac{dk(t)}{dt}}{k} = \frac{\dot{k}}{k}$$

  - Then,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}, \quad \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n$$

Accumulation of capital per worker, cont'd

- Now combine these two formulas to get the accumulation of capital per worker

1. $$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta \Rightarrow \frac{\dot{K}}{K} = \frac{\gamma Y/L}{K/L} - \delta$$

2. $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n \Rightarrow \frac{\dot{k}}{k} + n = \frac{\dot{K}}{K}$$

- Substitute equation 2 into equation 1:

$$\frac{\dot{k}}{k} + n = \frac{\gamma Y/L}{K/L} - \delta$$

then take n to the right hand side

$$\frac{\dot{k}}{k} = \frac{\gamma Y/L}{K/L} - \delta - n$$

replace Y/L with $y = Y/L$, and K/L with $k = K/L$

$$\frac{\dot{k}}{k} = \frac{\gamma y}{k} - \delta - n$$

multiply both sides of the equation with k to get capital per worker accumulation equation

$$\dot{k} = \gamma y - (\delta + n)k$$

which is equivalent to

$$\dot{k} = \gamma f(k) - (\delta + n)k$$

Change in capital per worker is equal to investment per worker, $\gamma f(k)$, minus capital dilution, $(\delta + n)k$, which is a result of capital depreciation and growing labor force.