Economic Growth

Lecture 2: The Solow Growth Model

İlhan Güner

University of Kent | EC569

About this lecture

- We will do some algebra
- Don't get scared
- Make sure you read Chapter 2 of Jones and Vollrath, 2013
- Make sure you read my lecture slides
    - This weeks slides are intentially wordy to make it easier to follow on your own.

Capital accumulation and economic growth

Accumulation of physical capital

- Growth
  - **Proximate cause**
    Immediately responsible for causing some observed result
    - Accumulation of inputs into production
      - physical capital
    - Productivity
      - Technology
      - Efficiency
  - **Ultimate Cause (Fundamentals)**
    Affects observed result through a chain of intermediate events

Physical capital

Capital: the physical objects that extend our ability or do work for us (tools):

- machines
- buildings
- infrastructure
- vehicles
- computers

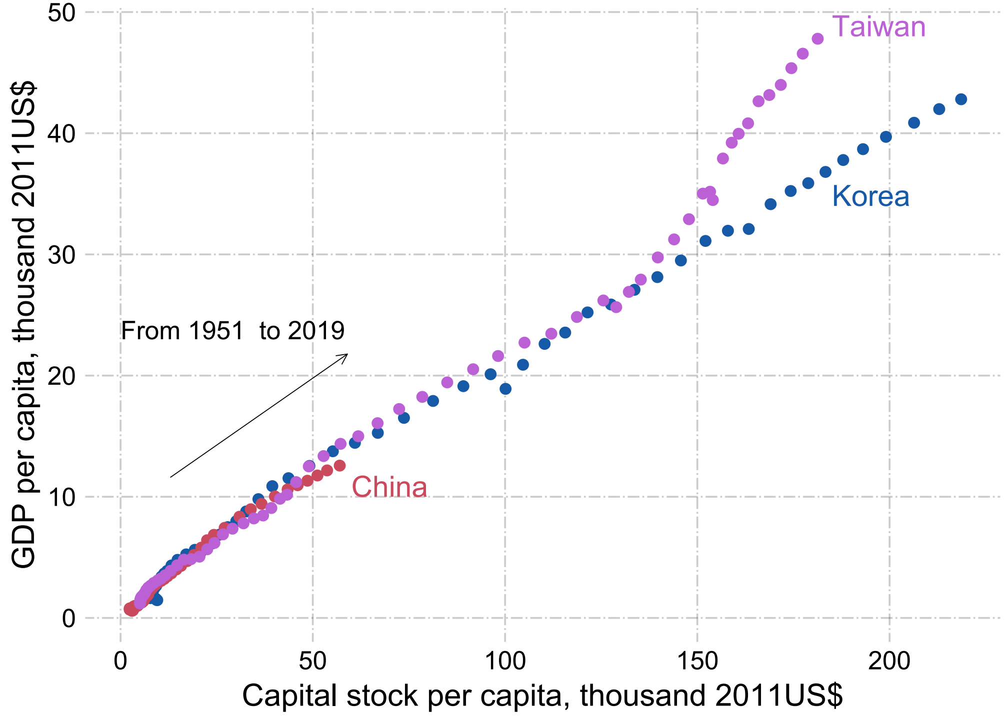
Key characteristics of physical capital

- It is productive
- It is produced (not like piece of land)
    - investment
- Its use is limited: limited number of people can use a given piece of capital at one time
    - unlike ideas

relevant when we learn long-run economic growth

- It can earn a return
    - incentive for its creation
- It wears out
    - depreciation

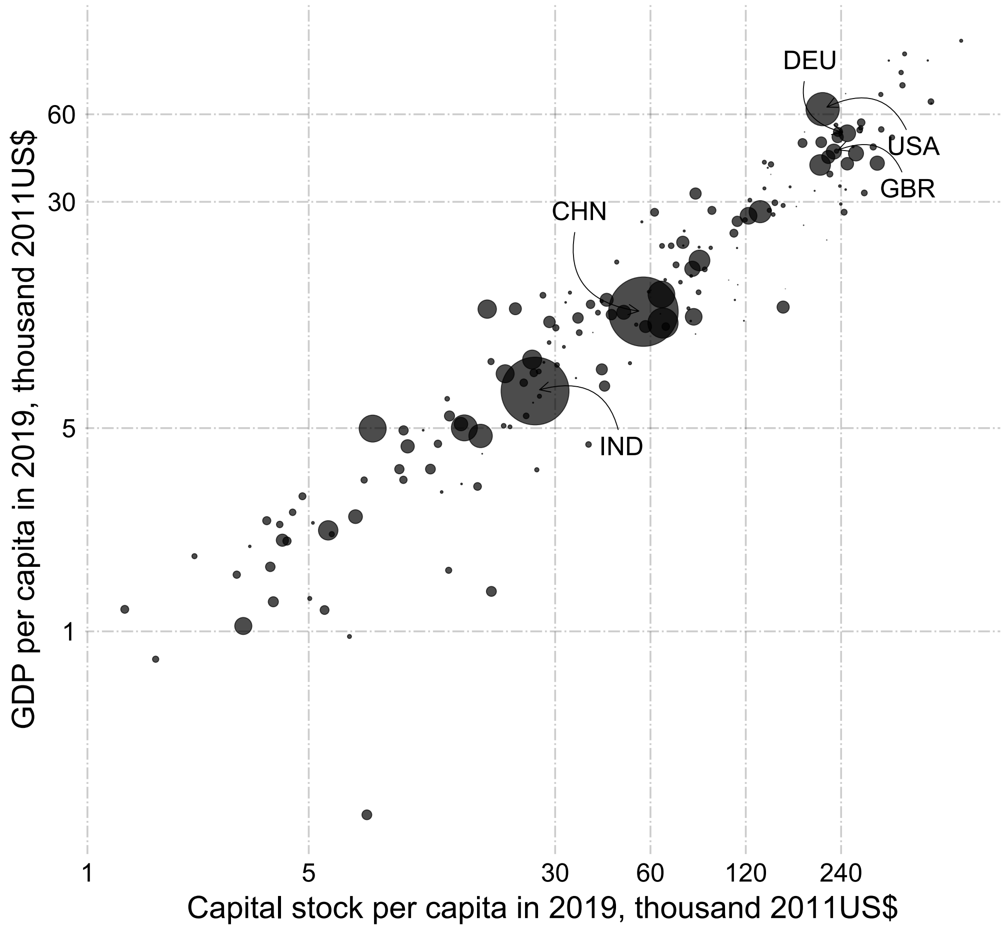
Role of capital accumulation in economic growth



Source: Penn World Tables, version 10.0

- Source of short-run economic growth in the Solow model is physical capital accumulation
- China, Taiwan and Korea are example countries that achieved high economic growth through capital accumulation.

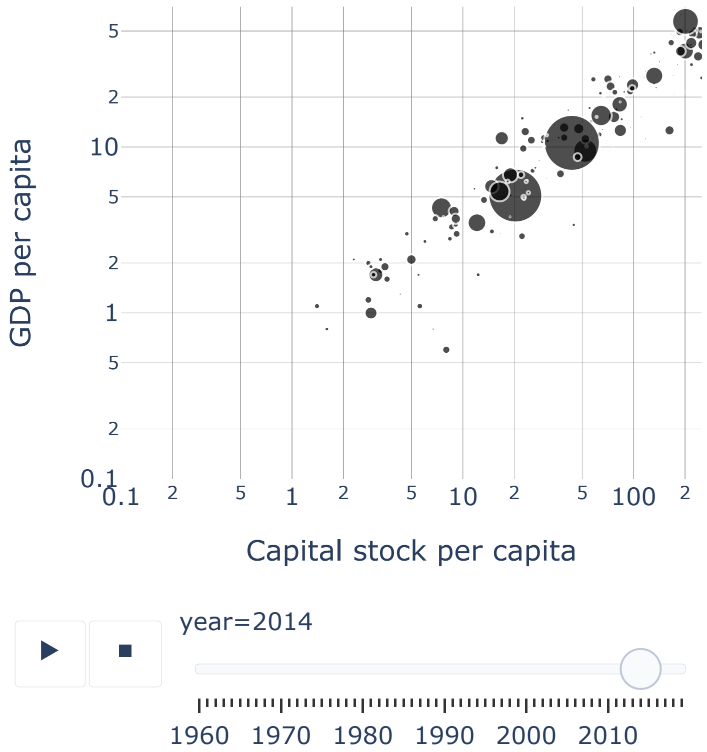
Role of capital accumulation in economic growth



- Source of short-run economic growth in the Solow model is physical capital accumulation
- China, Taiwan and Korea are example countries that achieved high economic growth through capital accumulation.
- There is a positive correlation between GDP per capita and capital stock per capita.

Source: Penn World Tables, version 10.0

Role of capital accumulation in economic growth



- GDP and capital stock over time values with 2011 thousand US$s.
- Data source: Penn World Tables, version 9.1

The Solow Model

The Solow model

short-run

A model to explain the role of factor accumulation in economic growth

Build around two equations:

- Production function
    - Constant returns to scale production function
- Capital accumulation equation
    - Households invest a constant share of income in capital stock

Note: There are many varieties of the Solow model. Initially we will learn about the basic version with no technological change. In the later lectures, we will learn the Solow model with technological progress (extended Solow model).

The Solow Model assumptions, all together

- Time is continuous
- Production function displays constant returns to scale
- Production function displays diminishing marginal product of capital → crucial in lack of long-run economic growth in the Solow model
- Everyone works in the economy: population = labor force
- Population and labor force grow at a constant rate n
- Society invests a constant fraction, γ, of output into capital stock
- A constant fraction, δ, of capital depreciates (wears out)

Notation convention: capital letters signify aggregate values

Production Function

$cM = cN$

$Y = F(K, e \cdot h \cdot L)$, where Y is output, K is physical capital, L is labor, e is labor-augmenting technology, and h is human capital. → constant in this lecture

- Assume $F(\cdot)$ displays **constant returns to scale** (CRS): $F(zK, zehL) = zF(K, ehL)$, where z is a constant.
- Convert aggregate production function, $F(\cdot)$, into per worker production function, $f(\cdot)$, by multiplying $F(\cdot)$ with $1/L$:

$$\left(\frac{1}{L}\right) Y = \left(\frac{1}{L}\right) F(K, ehL) \overset{\text{CRS}}{=} F\left(\frac{K}{L}, \frac{eh\cancel{L}}{\cancel{L}}\right) = F\left(\frac{K}{L}, eh\right)$$

- Define $k \equiv K/L$, capital per labor, $y \equiv Y/L$, output per labor, and assume h, e are constant:

$$y = F(k, eh) \equiv f(k)$$

↳ define

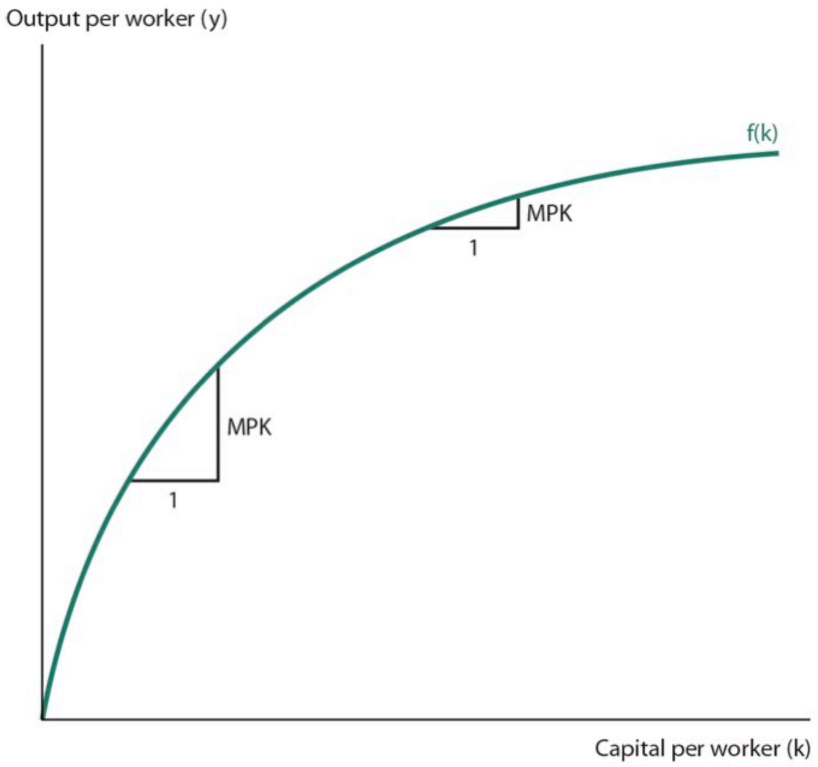
- Convert aggregate variables into per worker variables.
- Notation convention: small letter vars. represent per worker values.

Production function, cont'd

- $f(\cdot)$ displays **diminishing marginal product:**

  Marginal product of capital
  $MPK = \frac{\partial f(k)}{\partial k} \approx [f(k+1) - f(k)] \downarrow$ as $k \uparrow$

- Diminishing MPK will be the key factor for not having long run economic growth without technological progress.
  - More on this point later



Source: Weil (2013)

$Y = F(K, ehL)$

Example production function displaying CRS: Cobb-Douglas Production Function

Production function:
$Y = K^{\alpha}(ehL)^{1-\alpha} = AK^{\alpha}(hL)^{1-\alpha}$,

where $A \equiv e^{1-\alpha}$ is productivity, K is capital, h is human capital, and L is labor, $0 < \alpha < 1$.

- It satisfies CRS assumption. Check yourself!
- Therefore

$L = L^{\alpha} L^{1-\alpha} = L^{\alpha+1-\alpha}$

$$\frac{Y}{L} = \frac{AK^{\alpha}(hL)^{1-\alpha}}{L} = A\frac{K^{\alpha}}{L^{\alpha}}\frac{(h\cancel{L})^{1-\alpha}}{\cancel{L^{1-\alpha}}}$$

$$\frac{Y}{L} = A\left(\frac{K}{L}\right)^{\alpha} h^{1-\alpha}$$

$$\boxed{y = Ak^{\alpha}h^{1-\alpha}}$$

- Marginal product of physical capital is positive

$$\text{MPK} = \frac{\partial y}{\partial k} = \alpha Ak^{\alpha-1}h^{1-\alpha} > 0$$

- Diminishing marginal product of capital

$$\frac{\partial^2 y}{\partial k^2} = \underbrace{(\alpha - 1)}_{-}\underbrace{\alpha Ak^{\alpha-2}h^{1-\alpha}}_{+} < 0$$

- Hence, Cobb-Douglas function satisfies the Solow model assumptions on the production function.

Productivity

$= \dfrac{\text{output}}{\text{input}}$

Labor productivity

$= \dfrac{\text{output}}{\text{labor input}}$

output → real GDP

labor input → worker hours, # of employees

Total factor productivity

$= \dfrac{\text{output}}{\text{index of all input}}$

$$Y = A K^{\alpha} (hL)^{1-\alpha}$$

$$A = \frac{Y}{K^{\alpha}(hL)^{1-\alpha}}$$

Y → output

$K^{\alpha}(hL)^{1-\alpha}$ → index of all inputs

Solow model assumptions, cnt'd

$$\frac{d \ln(f(x))}{dx} = \frac{1}{f(x)} \frac{df}{dx}$$

- Productivity, A, is constant. *Will be relaxed in later lectures*
- Human capital, h, is constant. *Will be relaxed in later lectures*
- Labor force participation rate is constant.
- Population grows at a rate n, and so does labor force, L.

$$\frac{\dot{L}}{L} = n = \frac{dL_t/dt}{L_t}$$

$$L(t) = L_0 e^{nt} \rightarrow \text{exponent} = L_0 \exp(nt)$$

- Trick: to find the growth rate of a variable, take log and differentiate w.r.t. time, t.

$$\ln L(t) = \ln L_0 + nt, \quad \frac{\dot{L}(t)}{L(t)} = n,$$

$$\ln(\exp(nt)) = nt$$

where dot on top of a variable means derivative with respect to time $\boxed{\dot{L}(t) \equiv \frac{\partial L(t)}{\partial t}}$

- E.g. population grows are a rate, $n = .01$ or 1% per year.
- $\dot{L} \approx L(t+1) - L(t)$

L_t

$L_{t+1} \approx (1+0.01) L_t$

Growth rates

y_{t+1} : GDP per capita at year $t+1$
y_t : GDP per capita at year t

g_y: growth rate : $\frac{\Delta y_t}{y_t}$: $\frac{\text{change in GDP per capita}}{\text{GDP per capita}} = \frac{y_{t+1} - y_t}{y_t}$

time is discrete

y_t : GDP per worker at a point in time t → time is continuous

$\rightarrow$ instantaneous change

$$g_y \equiv \frac{dy_t / dt}{y_t} = \frac{\Delta y_t}{y_t}$$

$$\rightarrow = \frac{\dot{y}}{y}$$

Notation convention: dot on a variable means derivative w.r.t. time

$$\dot{y} = \frac{dy_t}{dt}$$

Accumulation of physical capital

Dot on a var. means derivatative w.r.t. t
≈ instantaneous change in a variable

- Second key equation of the Solow model
- Investment: a constant fraction of output, γ is invested:

$$I = \gamma Y$$

- Consumption: output minus investment. $C = Y - I$
- Depreciation: a constant fraction, δ, of capital wears out:

$$D = \delta K$$

- Change in physical capital stock = Investment - Depreciation

$$\boxed{\dot{K} = I - D}$$

- Subsituting investment and depreciation rules into above equation gives capital accumulation equation:

$$\boxed{\dot{K} = \gamma Y - \delta K}$$

The Solow Model assumptions, all together

- Production function displays constant returns to scale
- Production function displays diminishing marginal product of capital
- Everyone works in the economy: population = labor force
- Population and labor force grow at a constant rate n
- Society invests a constant fraction, γ, of output into capital stock
- A constant fraction, δ, of capital depreciates

Characterize the dynamics of output per worker over time

- To achieve this goal, we first need to characterize the dynamics of capital per worker

Dynamics of the Solow Model

Accumulation of capital per worker

- Goal: write physical capital accumulation equation with per worker variables
- Start with the equation describing the evolution of the aggregate capital:

$$\frac{\dot{K}}{K} = \frac{\gamma Y - \delta K}{K} = \frac{\gamma Y}{K} - \frac{\delta \cancel{K}}{\cancel{K}}$$

- Divide each side by K:

1

$$\boxed{\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta}$$

← let this equation stay here

- Define capital per worker, $k \equiv K/L$.
- Our goal is to convert $\frac{\dot{K}}{K}$ into $\frac{\dot{k}}{k}$.
- take log (ln) of $k \equiv K/L$:

$$\log(k) = \log(K/L) = \log(K) - \log(L)$$

Accumulation of capital per worker

$$\frac{d\,f(g(x))}{dx} = \frac{df}{dg} \times \frac{dg}{dx}$$

- Goal: write physical capital accumulation equation with per worker variables
- Start with the equation describing the evolution of the aggregate capital:

$$\dot{K}^{(t)} = \gamma Y^{(t)} - \delta K(t)$$

- Divide each side by K:

$$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta$$

- Define capital per worker, $k \equiv K/L$.
- Our goal is to convert $\frac{\dot{K}}{K}$ into $\frac{\dot{k}}{k}$. → growth rate of capital per worker
- take log (ln) of $k \equiv K/L$:

$$\log(k) = \log(K/L) = \log(K) - \log(L)$$

$= \ln$

$$\log(k(t)) = \log(K(t)) - \log(L(t))$$

- Differentiate $k = K/L$ with respect to time (remember $k = k(t)$ and $L = L(t)$)
  - Remember differentiation rules:

$$\frac{d\log(k(t))}{dt} = \frac{\frac{dk(t)}{dt}}{k} = \frac{\dot{k}}{k}$$

  - Then,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}, \quad \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n$$

2

$$\boxed{g_k = \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n}$$

$$\frac{M}{N} = \frac{M/c}{N/c}$$

Accumulation of capital per worker, cont'd

- Now combine these two formulas to get the accumulation of capital per worker

1. $$\frac{\dot{K}}{K} = \frac{\gamma Y/L}{K/L} - \delta \Rightarrow \frac{\dot{K}}{K} = \frac{\gamma Y/L}{K/L} - \delta$$

2. $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n \Rightarrow \frac{\dot{k}}{k} + n = \frac{\dot{K}}{K}$$

- Substitute equation 2 into equation 1:

$$\frac{\dot{k}}{k} + n = \frac{\gamma Y/L}{K/L} - \delta$$

then take n to the right hand side

$$\frac{\dot{k}}{k} = \frac{\gamma Y/L}{K/L} - \delta - n$$

Accumulation of capital per worker, cont'd

- Now combine these two formulas to get the accumulation of capital per worker

1. $$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta \Rightarrow \frac{\dot{K}}{K} = \frac{\gamma Y/L}{K/L} - \delta$$

2. $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n \Rightarrow \frac{\dot{k}}{k} + n = \frac{\dot{K}}{K}$$

- Substitute equation 2 into equation 1:

$$\frac{\dot{k}}{k} + n = \frac{\gamma Y/L}{K/L} - \delta$$

then take n to the right hand side

$$\frac{\dot{k}}{k} = \frac{\gamma Y/L}{K/L} - \delta - n$$

replace Y/L with $y = Y/L$, and K/L with $k = K/L$

$$\boxed{\frac{\dot{k}}{k} = \frac{\gamma y}{k} - \delta - n}$$

multiply both sides of the equation with k to get capital per worker accumulation equation

$$\dot{k} = \gamma y - (\delta + n)k$$

which is equivalent to

$$\boxed{\dot{k} = \gamma f(k) - (\delta + n)k}$$

Change in capital per worker is equal to investment per worker, $\gamma f(k)$, minus capital dilution, $(\delta + n)k$, which is a result of capital depreciation and growing labor force.

Summary

- Positive correlation between physical capital stock per capita and GDP per capita
- Learned the main assumptions of the Solow model
- Derived the law of motion for capital per worker

solow model

To review this lecture and to prepare for the next lecture

- Read Chapter 2.1 of Jones and Vollrath
- Read Chapter 3 of Weil
- Next lecture, we will stay on the Solow Model