Economic Growth

Lecture 5: Extended Solow Model

İlhan Güner

University of Kent | EC569

Questions

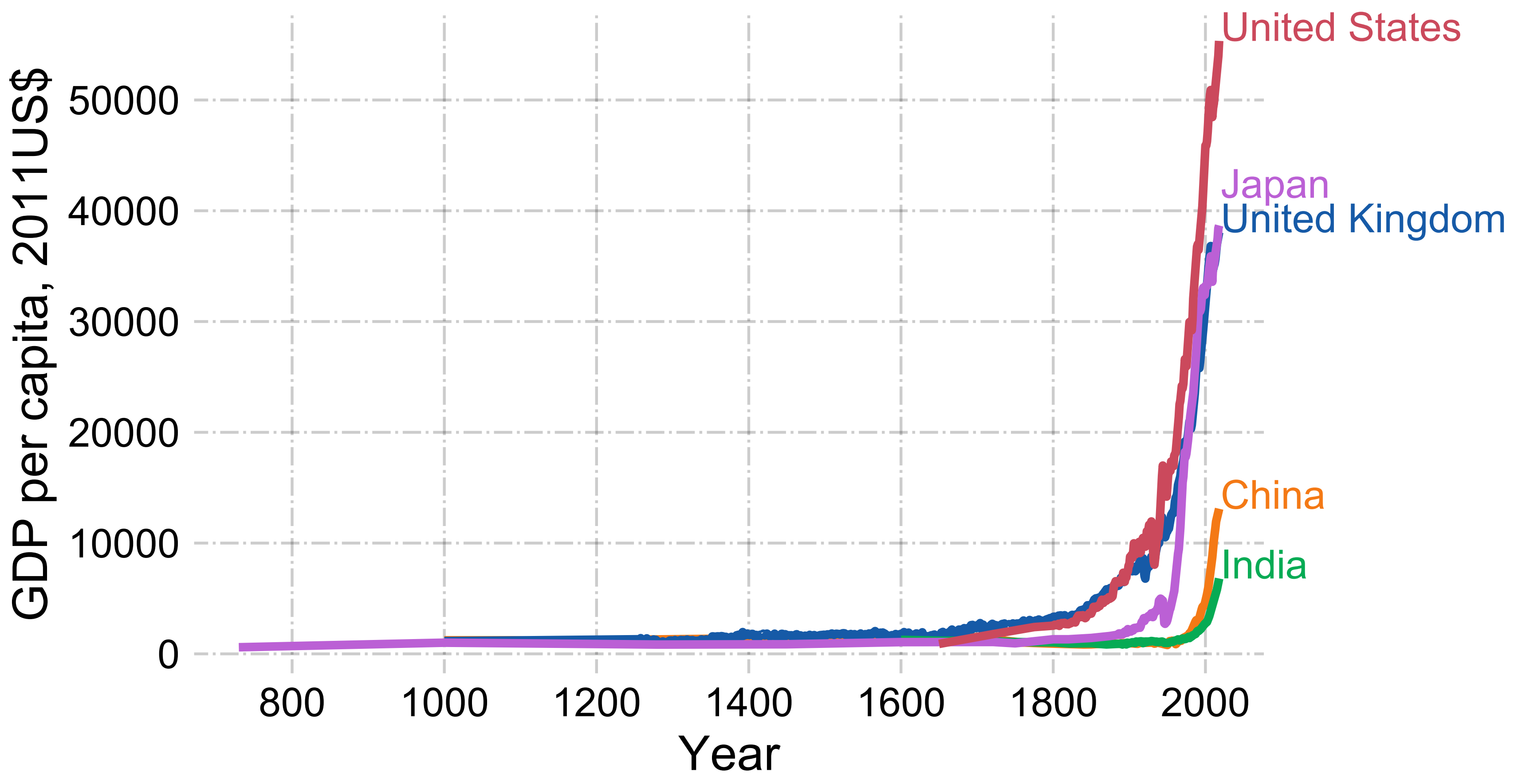
- What's the role of tehnological progress on economic growth?
- Do relatively low income countries grow faster than high income countries?

What's the long-run growth rate of income per capita in the Solow model?

What's the long-run growth rate of income per capita in the Solow model?

0

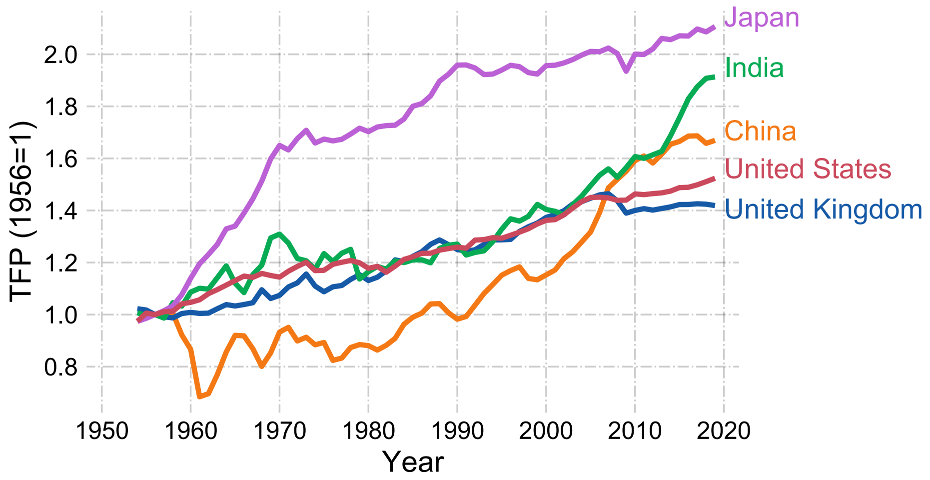
Economies do grow over time



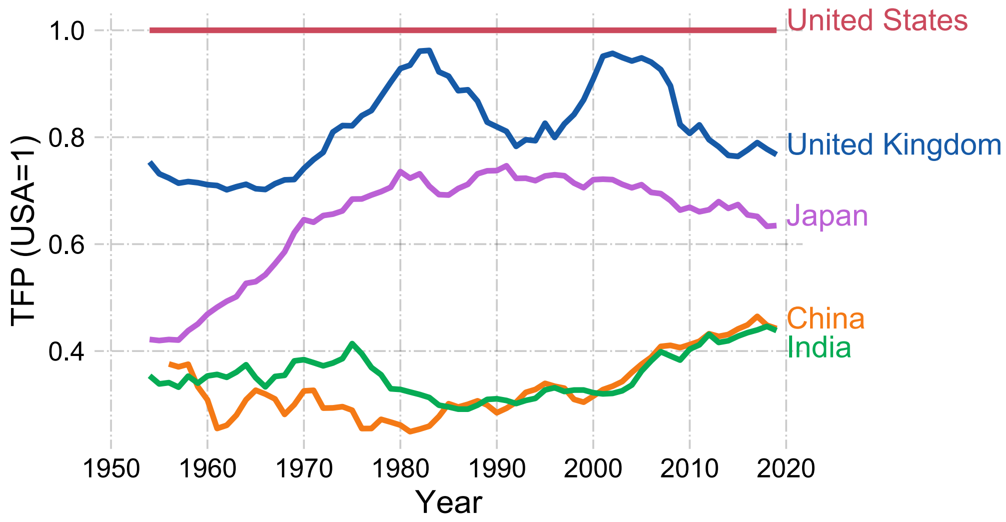
Data source: Maddison Project Database (MPD) 2020

Countries get more productive

- TFP stands for total factor productivity
- TFP = total real output / total inputs into production (capital and labor)



- It does not mean that India is more productive than the US
- It means the TFP of India has grown more than the US TFP since 1956



Data source: Penn World Tables, version 10.0

Is the Solow model wrong?

George E.P. Box:

> All models are wrong but some are useful.

- The Solow model is *very* useful in understanding the role of capital accumulation in short-run economic growth.

Extended Solow Model

Extended Solow Model

$$\frac{d \ln e_0 + gt}{dt} = g$$

- Also known as *the Solow model with technological progress*

- Retain assumptions:
  - Constant returns to scale production function: $F(K, ehL)$
  - A constant fraction, γ, of output is invested.
  - A constant fraction, δ, of physical capital stock depreciates.
  - Labor force participation rate is constant
  - Population grows at a constant rate, n.
  - Human capital, h, is constant.

- Differently, assume
  - There is labor augmenting technological progress, $F(K, ehL)$

    e
  - Labor augmenting technology, e, grows at a constant rate, g.

$$\frac{\dot{e}}{e} = g \Leftrightarrow e(t) = e_0 \exp(gt)$$

$$g = \frac{\frac{de(t)}{dt}}{e}$$

$$\ln(e_0 \exp(gt))$$

$$\ln e_0 + \ln(\exp(gt))$$

Clarification...

- Previously, we assumed production function is $AF(K, hL)$
  - Here, A is Hicks-neutral technology
- In the Extended Solow model, technology needs to be labor-augmenting: $F(K, ehL)$
  - Here, e is labor-augmenting or Harrod-neutral technology
- In Cobb-Douglas production, this distinction is not important.
  - $K^\alpha(ehL)^{1-\alpha} = AK^\alpha(hL)^{1-\alpha}$, where $A \equiv e^{1-\alpha}$

Accumulation of physical capital

goal: find change in capital per effective worker

- Change in capital stock = Investment - Depreciation

$$\dot{K} = I - D$$

- A constant fraction, γ, of output is invested

$$I = \gamma Y$$

- A constant fraction, δ, of capital depreciates

$$D = \delta K$$

- Change in capital stock:

$$\dot{K} = \gamma Y - \delta K$$

Accumulation of capital-technology ratio

- Goal: derive a formula for the accumulation of capital-technology ratio, $\tilde{\boldsymbol{k}} \equiv \frac{K}{eL}$
- Capital-technology ratio, $\tilde{\boldsymbol{k}}$, sometimes referred as *capital per effective worker*.
- Why do we need to convert physical capital accumulation equation into capital-technology units?
    - Because capital stock will grow as a result of increasing population and higher productivity.

Accumulation of capital-technology ratio, cont'd

Keep in mind

- Accumulation of capital

$$\boxed{\dot{K} = \gamma Y - \delta K}$$

1

- How do we transform $\dot{K}$ into $\dot{\tilde{k}}$?
  - make use of $\tilde{k} \equiv \frac{K}{eL}$
  - take log of $\tilde{k}(t) \equiv \frac{K(t)}{e(t)L(t)}$:

$$\ln(\tilde{k}(t)) = \ln\left(\frac{K(t)}{e(t)L(t)}\right) = \ln(K(t)) - \ln(e(t)) - \ln(L(t))$$

- Then differentiate with respect to time, t,

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{e}(t)}{e(t)} - \frac{\dot{L}(t)}{L(t)}, \quad \frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - g - n \Rightarrow \boxed{\frac{\dot{K}}{K} = \frac{\dot{\tilde{k}}}{\tilde{k}} + g + n}$$

2

- Remember chain-rule: $\frac{df(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$

Accumulation of capital-technology ratio, cont'd (2)

- Divide each side of $\dot{K} = \gamma Y - \delta K$ by K:

$$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta$$

- Substitute $\frac{\dot{K}}{K} = \frac{\dot{\tilde{k}}}{\tilde{k}} + g + n$ into above equation,

$$\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{\dot{K}}{K} = \frac{\gamma Y/eL}{K/eL} - \delta$$

- Divide both numerator and denominator of $\frac{\gamma Y}{K}$ with eL:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{\gamma Y/(eL)}{K/(eL)} - \delta$$

output per effective worker

or output-tech. ratio

- Substitute in $\tilde{y} \equiv \frac{Y}{eL}$ and $\tilde{k} \equiv \frac{K}{eL}$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{\gamma \tilde{y}}{\tilde{k}} - \delta, \text{ where } \tilde{y} \equiv \frac{Y}{eL}$$

- Take $g + n$ to the right hand side

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\gamma \tilde{y}}{\tilde{k}} - (\delta + g + n)$$

- Multiply both sides with $\tilde{k}$:

$$\boxed{\dot{\tilde{k}} = \gamma \tilde{y} - (\delta + g + n)\tilde{k}}$$

- Change in capital per effective workers is equal to investment per effective worker minus depreciation per effective worker minus capital dilution per effective worker due to population growth and technological progress.

Steady State Analysis

Steady State

$$\dot{\tilde{k}} = \gamma\tilde{y} - (\delta + g + n)\tilde{k}, \text{ or}$$

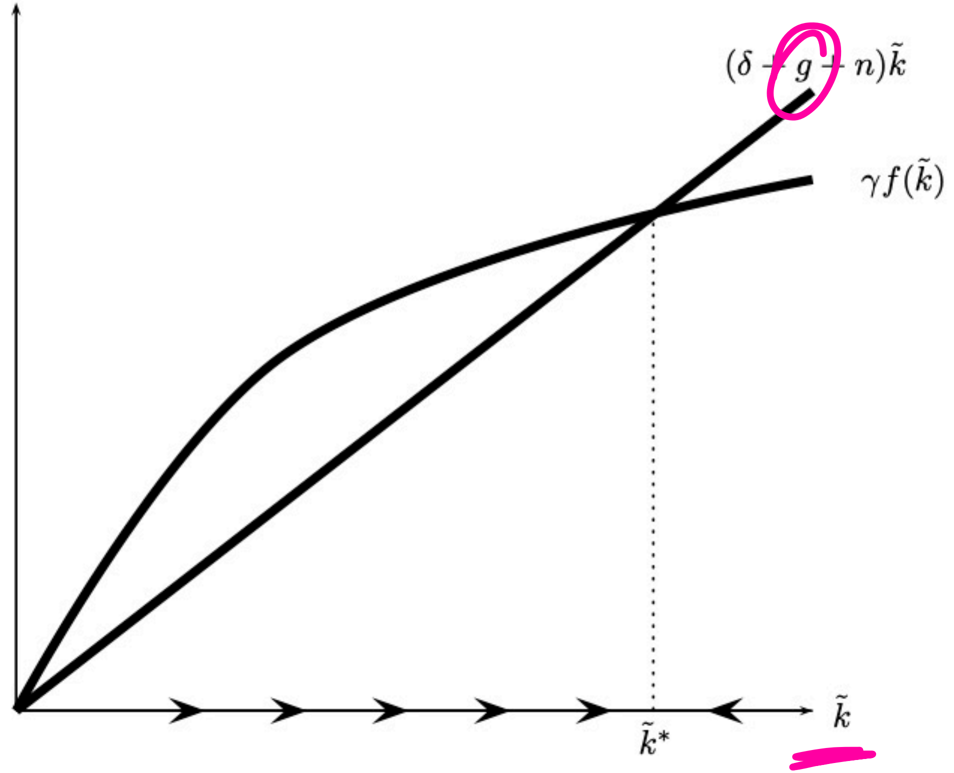
$$\dot{\tilde{k}} = \gamma f(\tilde{k}) - (\delta + g + n)\tilde{k}$$

where $\tilde{y} = f(\tilde{k}) = F(K, ehL)/(eL)$

Capital-technology ratio is constant at the steady state.

- if $\gamma f(\tilde{k}) > (\delta + g + n)\tilde{k}$, then $\dot{\tilde{k}} > 0$
- if $\gamma f(\tilde{k}) < (\delta + g + n)\tilde{k}$, then $\dot{\tilde{k}} < 0$
- if $\gamma f(\tilde{k}) = (\delta + g + n)\tilde{k}$, then $\dot{\tilde{k}} = 0$: **steady state**
- If $\tilde{k} < \tilde{k}^*$, capital-technology ratio will increase.
- If $\tilde{k} > \tilde{k}^*$, capital-technology ratio will decrease.

Solow Diagram



Graphic from Jones and Vollrath (2013)

Steady State, cont'd

- Cobb-Douglas production function

$$Y = K^\alpha (ehL)^{1-\alpha}$$

$$\frac{Y}{eL} = \frac{K^\alpha (ehL)^{1-\alpha}}{(eL)^\alpha (eL)^{1-\alpha}}$$

$$\tilde{y} = \tilde{k}^\alpha h^{1-\alpha}$$

- Law of motion of capital-technology ratio

$$\dot{\tilde{k}} = \gamma \tilde{k}^\alpha h^{1-\alpha} - (\delta + g + n)\tilde{k}$$

- No change of capital-technology ratio at the steady state

$$0 = \gamma(\tilde{k}^*)^\alpha h^{1-\alpha} - (\delta + g + n)\tilde{k}^*$$

holds only at the s.s.

$$\Downarrow$$

$$\gamma(\tilde{k}^*)^\alpha h^{1-\alpha} = (\delta + g + n)\tilde{k}^*$$

- Steady state capital-technology ratio:
    - capital-technology ratio = capital per effective worker

$$\tilde{k}^* = \left(\frac{\gamma}{\delta + g + n}\right)^{1/(1-\alpha)} h$$

- Steady steady output-technology ratio:
    - output-technology ratio = output per effective worker

$$\tilde{y}^* = (\tilde{k}^*)^\alpha h^{1-\alpha} = \left(\frac{\gamma}{\delta + g + n}\right)^{\alpha/(1-\alpha)} h$$

Per worker values

Output per worker:

$$y(t) = e(t)\tilde{y}(t)$$

- Recall that

$$y(t) = \frac{Y(t)}{L(t)} = e(t)\frac{Y(t)}{e(t)L(t)} = e(t)\tilde{y}(t)$$

Capital per worker:

$$k(t) = e(t)\tilde{k}(t)$$

- Recall that

$$k(t) = \frac{K(t)}{L(t)} = e(t)\frac{K(t)}{e(t)L(t)} = e(t)\tilde{k}(t)$$

Output per worker at the steady state:

$$y(t) = e(t)\tilde{y}^* = e(t)\left(\frac{\gamma}{\delta + g + n}\right)^{\alpha/(1-\alpha)} h$$

- Since $\tilde{y}^*$ is constant at the steady state, y at the steady state grows at the same rate as e: g.

Capital per worker:

$$k(t) = e(t)\tilde{k}^* = e(t)\left(\frac{\gamma}{\delta + g + n}\right)^{1/(1-\alpha)} h$$

- Since $\tilde{k}^*$ is constant at the steady state, k at the steady state grows at the same rate as e: g.

Comparative Statics

Comparative statics

Capital-technology ratio at the steady state:

$$\tilde{k}^* = \left(\frac{\gamma}{\delta + g + n}\right)^{1/(1-\alpha)} h$$

Output-technology ratio at the steady state:

$$\tilde{y}^* = \left(\frac{\gamma}{\delta + g + n}\right)^{\alpha/(1-\alpha)} h$$

- $\tilde{k}^*$ and $\tilde{y}^*$ are rising with investment rate γ, and human capital h,
- $\tilde{k}^*$ and $\tilde{y}^*$ are declining with depreciation rate, δ, population growth rate, n, and rate of technological progress, g.

In the long-run

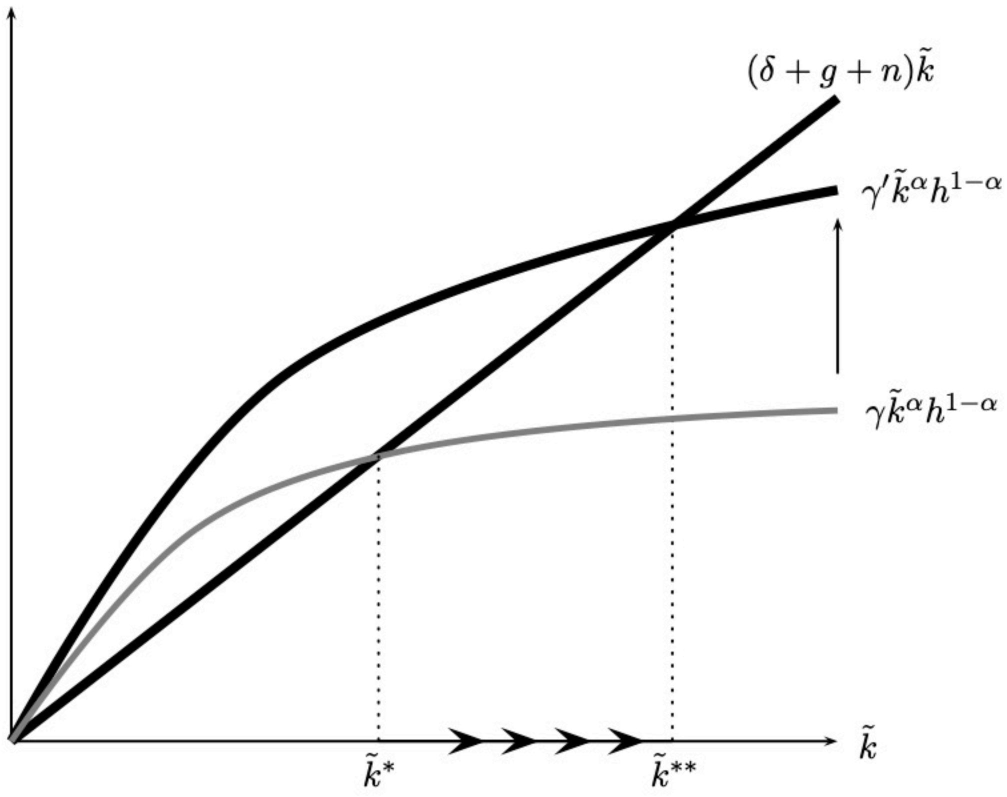
$k(t) = e(t)\tilde{k}^*$

$y(t) = e(t)\tilde{y}^*$

$e(t) = e_0 \exp(gt)$

Increasing the investment rate

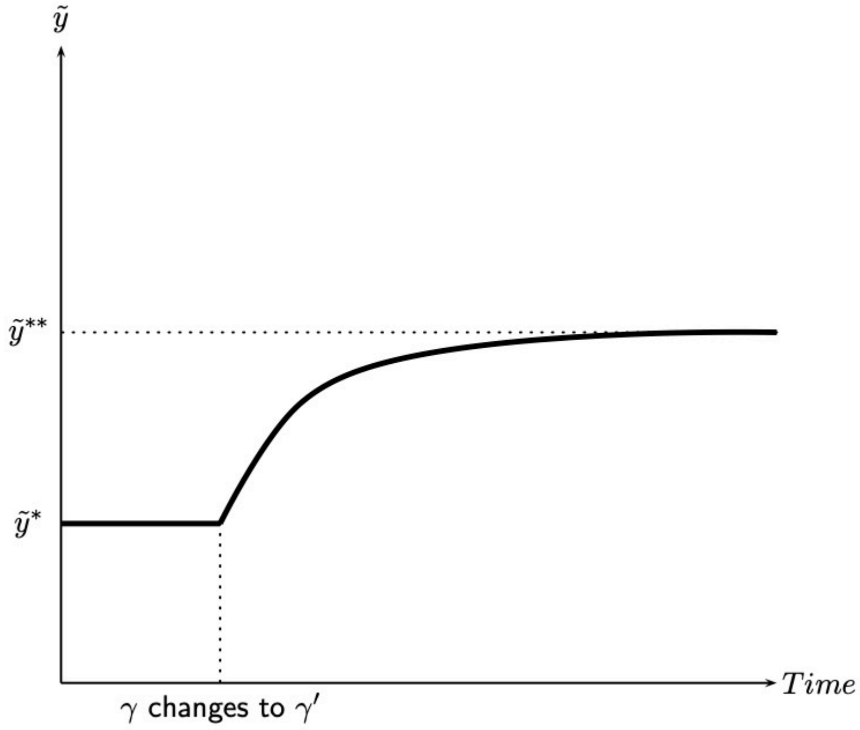
As $\gamma \uparrow$, $\tilde{k}^* = (\frac{\gamma}{\delta+g+n})^{1/(1-\alpha)} h \uparrow$, $\tilde{y}^* = (\frac{\gamma}{\delta+g+n})^{\alpha/(1-\alpha)} h \uparrow$



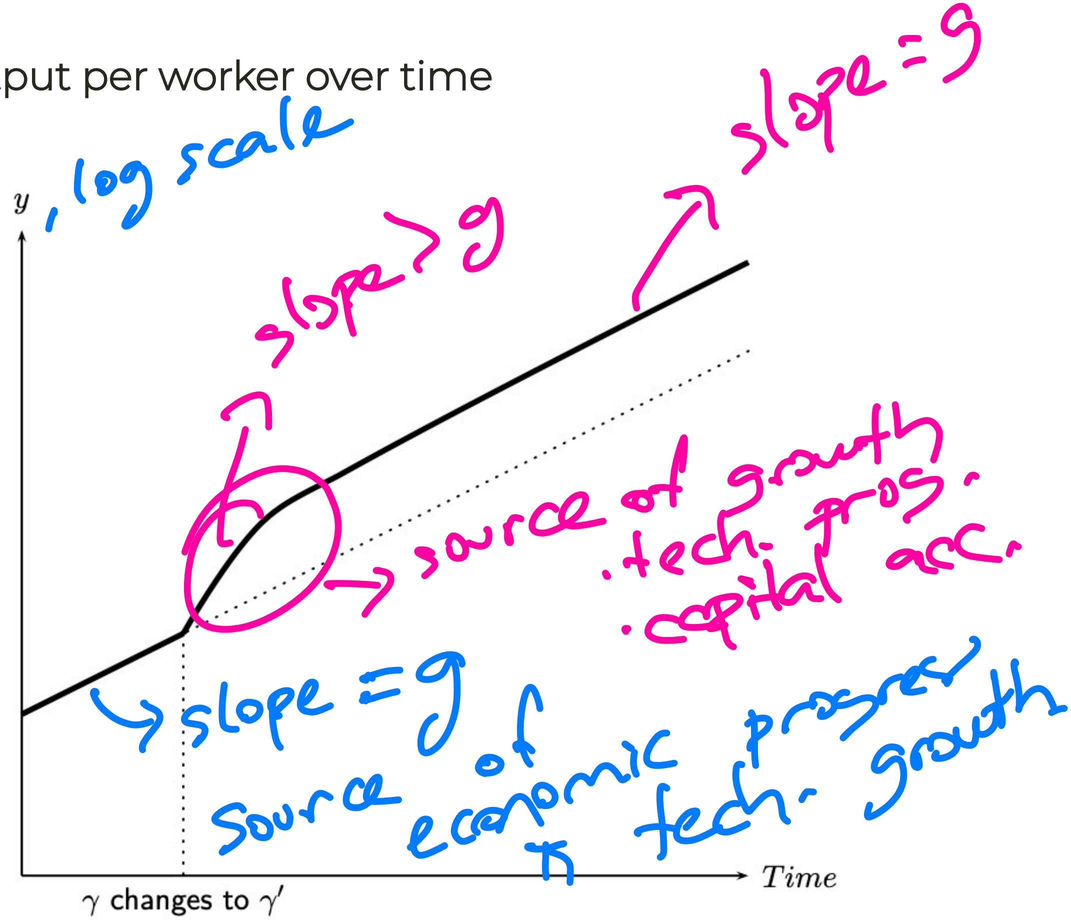
Graphic from Jones and Vollrath (2013)

Increasing the investment rate, cont'd

Output-technology ratio over time

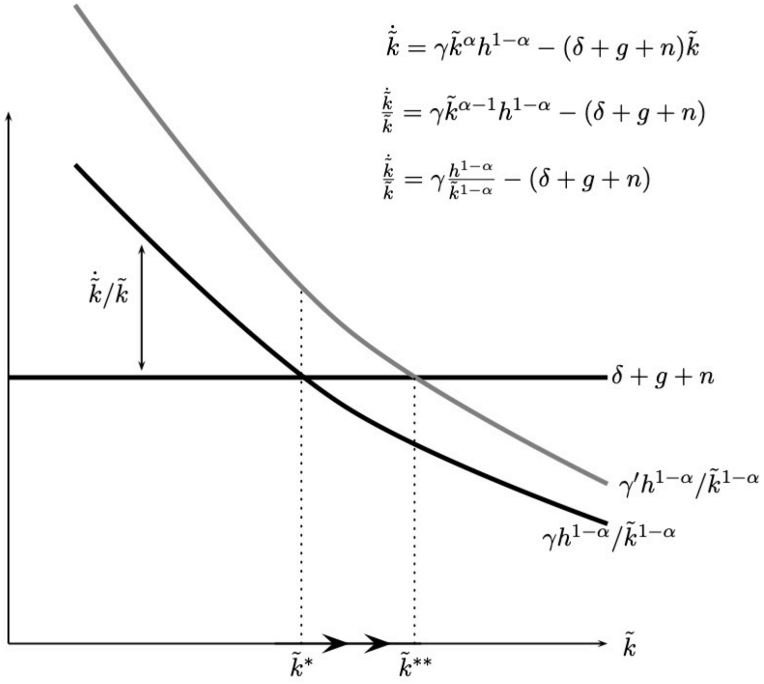


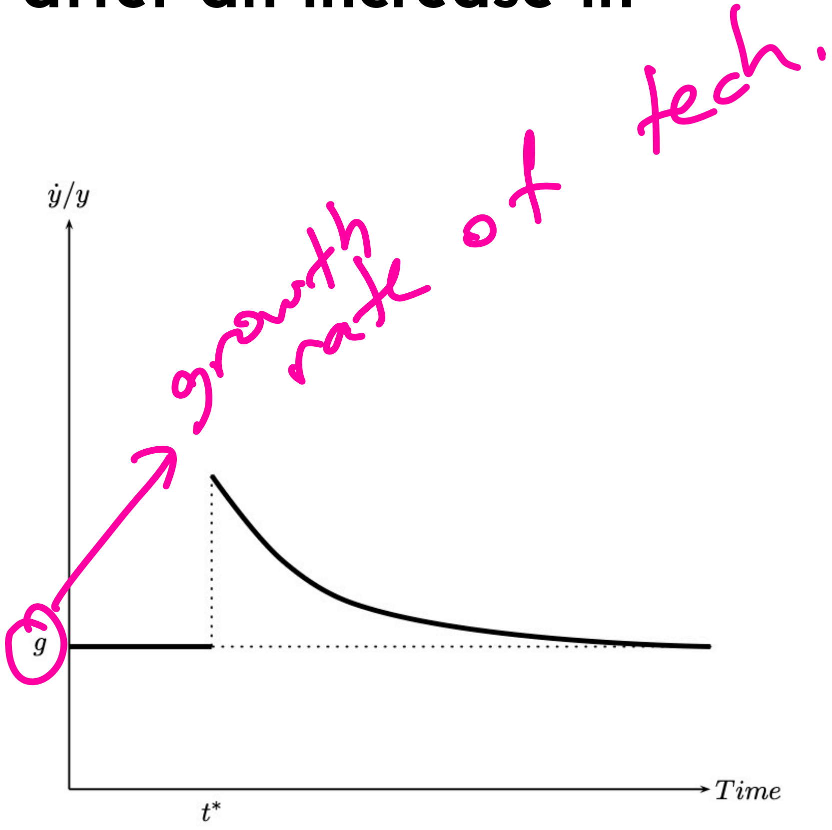
Output per worker over time



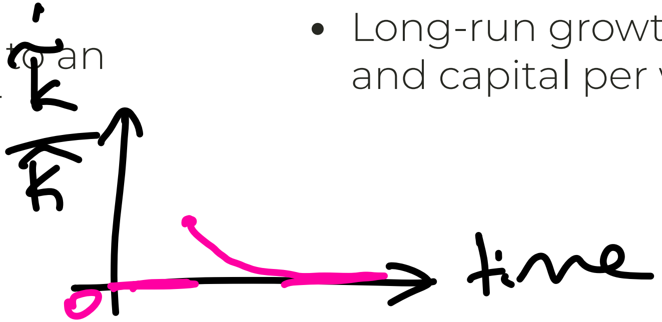
- Notice that $\tilde{k}$ has a similar trajectory as $\tilde{y}$
- Notice that k has a similar trajectory as y

Growth rate of capital-technology ratio after an increase in investment rate





- An increase in the investment rate leads to an increase in the growth rate of capital per worker in the short-run.



- Long-run growth rates of output per worker and capital per worker are not affected.

Exercise

Conduct comparative statics for changes in n, δ, g, and h.

Effects of an increase in investment rate

Short-run:

- Growth rates of capital per worker and income per worker increase.
- Capital per worker and income per worker increases.

Long-run (steady-state):

- Capital per worker and income per worker increase. → level increase
- Growth rate of capital per worker and income per worker do not change.

Growth Rates

Steady state growth rates

Variable	Notation	Definition	Growth rate as the s.s.
Capital-technology ratio	$\tilde{k}$	$\frac{K}{eL}$	?
Output-technology ratio	$\tilde{y}$	$\frac{Y}{eL}$	?
Consumption-technology ratio	$\tilde{c}$	$\frac{C}{eL}$	?

Steady state growth rates

Variable	Notation	Definition	Growth rate as the s.s.
Capital-technology ratio	$\tilde{k}$	$\frac{K}{eL}$	0
Output-technology ratio	$\tilde{y}$	$\frac{Y}{eL}$	0
Consumption-technology ratio	$\tilde{c}$	$\frac{C}{eL}$	0

Steady state growth rates

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Output-technology ratio	$\tilde{y}$	$\frac{Y}{eL}$	0
Consumption-technology ratio	$\tilde{c}$	$\frac{C}{eL}$	0
Capital per worker	k	$\frac{K}{L}$	?
Output per worker	y	$\frac{Y}{L}$	?
Consumption per worker	c	$\frac{C}{L}$	?

Steady state growth rates

Variable	Notation	Definition	Growth rate as the s.s.
Capital-technology ratio	$\tilde{k}$	$\frac{K}{eL}$	0
Output-technology ratio	$\tilde{y}$	$\frac{Y}{eL}$	0
Consumption-technology ratio	$\tilde{c}$	$\frac{C}{eL}$	0
Capital per worker	k	$\frac{K}{L}$	g
Output per worker	y	$\frac{Y}{L}$	g
Consumption per worker	c	$\frac{C}{L}$	g

$k = e(t)\tilde{k}$

$y = e(t)\tilde{y}$

$c = e(t)\tilde{c}$

Steady state growth rates

Variable	Notation	Definition	Growth rate as the s.s.
Capital-technology ratio	$\tilde{k}$	$\frac{K}{eL}$	0
Output-technology ratio	$\tilde{y}$	$\frac{Y}{eL}$	0
Consumption-technology ratio	$\tilde{c}$	$\frac{C}{eL}$	0
Capital per worker	k	$\frac{K}{L}$	g
Output per worker	y	$\frac{Y}{L}$	g
Consumption per worker	c	$\frac{C}{L}$	g
Capital	K	K	?
Output	Y	Y	?
Consumption	C	C	?

Steady state growth rates

Variable	Notation	Definition	Growth rate as the s.s.
Capital-technology ratio	$\tilde{k}$	$\frac{K}{eL}$	0
Output-technology ratio	$\tilde{y}$	$\frac{Y}{eL}$	0
Consumption-technology ratio	$\tilde{c}$	$\frac{C}{eL}$	0
Capital per worker	k	$\frac{K}{L}$	g
Output per worker	y	$\frac{Y}{L}$	g
Consumption per worker	c	$\frac{C}{L}$	g
Capital	K	K	$n+g$
Output	Y	Y	$n+g$
Consumption	C	C	$n+g$

$K = e(t)L(t)\tilde{k}^*$

$Y = e(t)L(t)\tilde{y}^*$

$C = e(t)L(t)\tilde{c}^*$

Convergence

Recall: Relative Growth Rates, Predictions

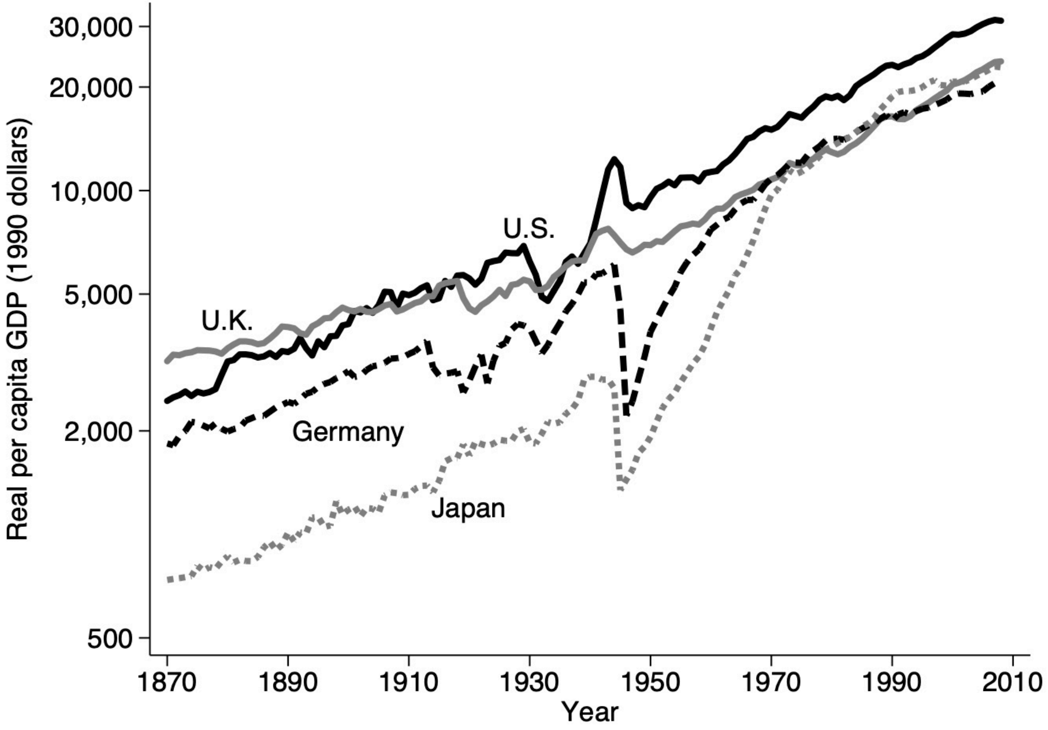
if γ, n, δ, h, A
They have same steady state.

- If two countries have the same rate of investment but different levels of income, the country with lower income will have higher growth.
- If two countries have the same level of income but different rates of investment, then the country with a higher rate of investment will have higher growth.
- A country that raises its level of investment will experience an increase in its rate of income growth.

Convergence

- Are *poor* countries growing faster than *rich* countries?
- Are *poor* countries *closing the gap*?
- Converge: The phenomenon of *poor* countries catching up with the *rich* countries.

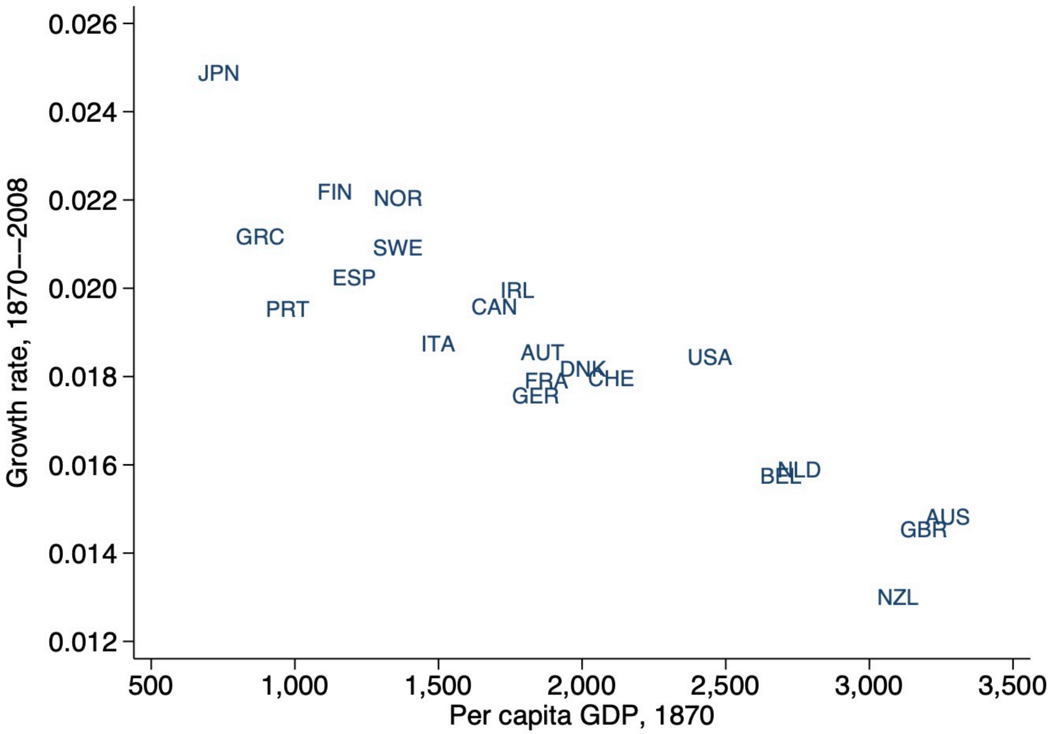
Convergence in a sample of industrialized countries, 1870-2008



Data source: Maddison (2010)

Graph from: Jones and Vollrath (2013)

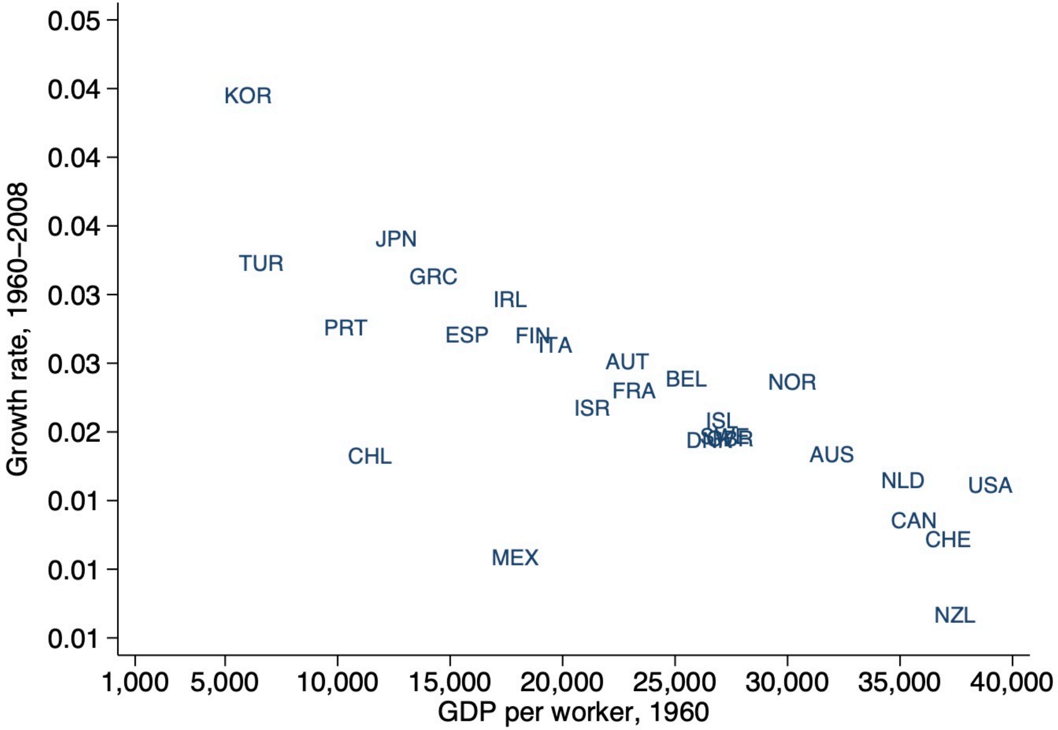
Convergence in a sample of industrialized countries, cont'd, 1870-2008



Data source: Maddison (2010)

Graph from: Jones and Vollrath (2013)

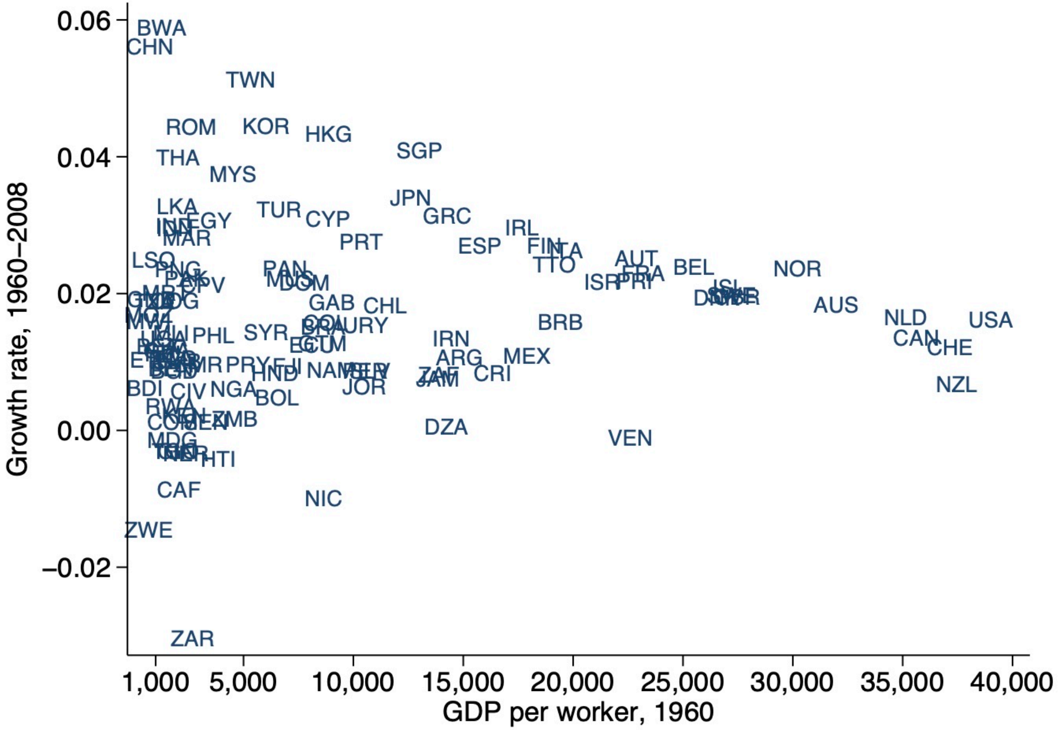
Convergence in OECD countries, 1960-2008



Data source: Penn World Tables Mark 7.0 and Summers and Heston (1991)

Graph from: Jones and Vollrath (2013)

The lack of convergence for the World, 1960-2008



Data source: Penn World Tables Mark 7.0 and Summers and Heston (1991)

Graph from: Jones and Vollrath (2013)

Convergence now

- The convergence results on Jones and Vollrath (2013) were based on the data of previous years.
- Dev Patel, Justin Sandefur and Arvind Subramanian show that countries do converge after 1995.
- Regress growth rate on initial level of GDP per capita

$$\frac{\ln y_t - \ln y_0}{t} = \alpha + \beta \ln y_0 + \epsilon$$

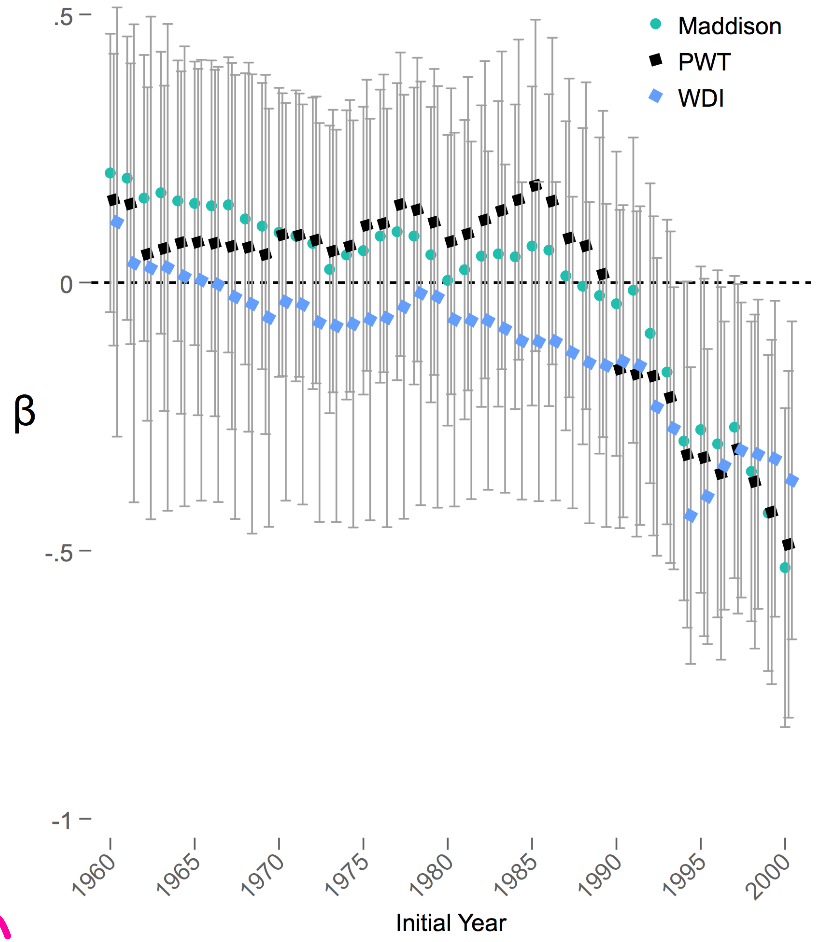
say GDP in $y_0 = 2010$

- Repeat regression with different intial years (results on the right)

if $\beta < 0$ statistically significant
→ evidence of convergence

β-coefficient of unconditional convergence

Regressing real per capita GDP growth to present day on the log of initial per capita GDP



Each point represents the coefficient from a separate, bivariate regression. The dependent variable is the annual real per capita growth rate from the year listed until the most recent data round. The independent variable is the log of real per capital GDP in the base year.
NB: Sample excludes oil-rich countries (i.e. 'Export Earnings: Fuel' in IMF DOTS), and countries with populations under 1 million.

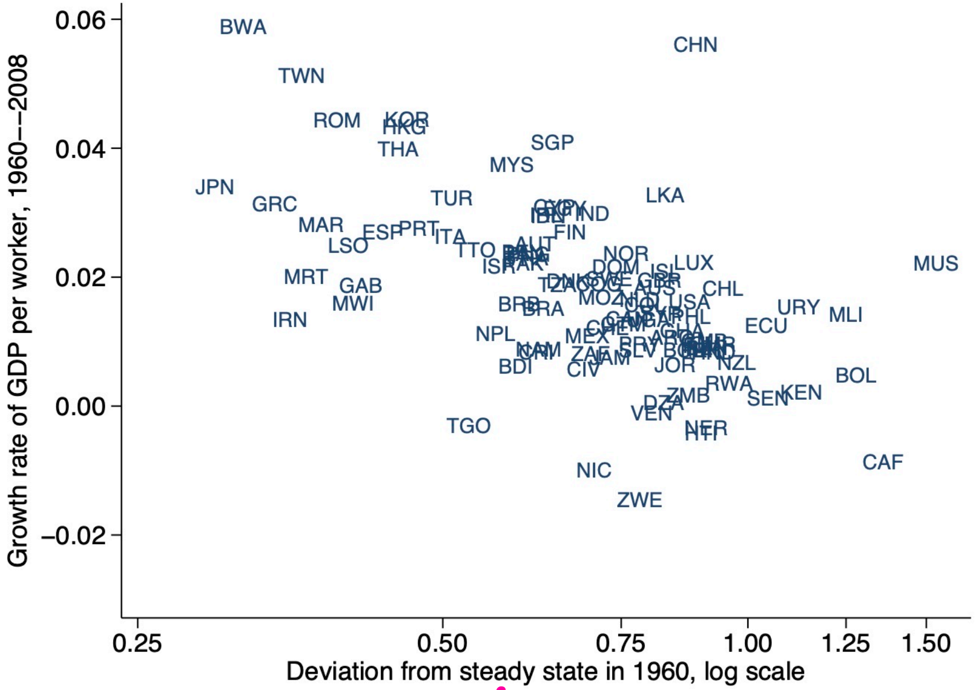
Convergence (?)

- How do we reconcile the converge in OECD but lack of convergence for the world?
- Prediction of the Solow model:
  - Among the countries with the same steady state, poor countries should grow faster than rich countries.
- Steady state depends on investment rate, population growth rate, technological progress rate
- OECD countries show similarities in investment rate, population growth rate, rate of technological progress.
- More variation in the World in these statistics.

Conditional vs. Unconditional convergence

Conditional convergence for the World, 1960-2008

- Mankiw, Romer, and Weil (1992), and Barro and Sala-i-Martin (1992):
- Convergence of countries "conditional on" their steady states
- Countries that are poor relative to their steady states tend to grow faster.



All relative to US

Data source: Author's calculations using Penn World Tables 7.0, update of Summers and Heston (1991).

Note: The variable on the x-axis is $\hat{y}_{60}/\hat{y}^*$. Estimates of A for 1970 are used to compute the steady state.

Summary

- There is no long-run economic growth in the Solow model
  - Capital accumulation in the short run generates transitional growth
  - GDP per capita is constant in the long run
- We developed a model in which technological progress is the only source of long-run growth
  - Economies grow at the same rate of technological progress in the long run
- We looked at the convergence of countries:
  - countries with similar steady states converge (conditional convergence)
  - but not all countries.
- Most recent data actually show (unconditional) convergence .

– To review, read Chapter 3.1 and 3.2 in Jones and Vollrath.

Exogenous Growth Model

- Technology is the only source of long-run growth.
- Technological progress is exogenous: not a result of interactions of agents in the model
- Hence, the extended Solow model is an example of *exogenous growth models*.
- It is also referred as *Neo-classical growth model*
- In upcoming lectures, we will analyze *endogenous growth models*
    - Technological progress as a result of actions of model agents.