Economic Growth

Lecture 2: The Solow Growth Model

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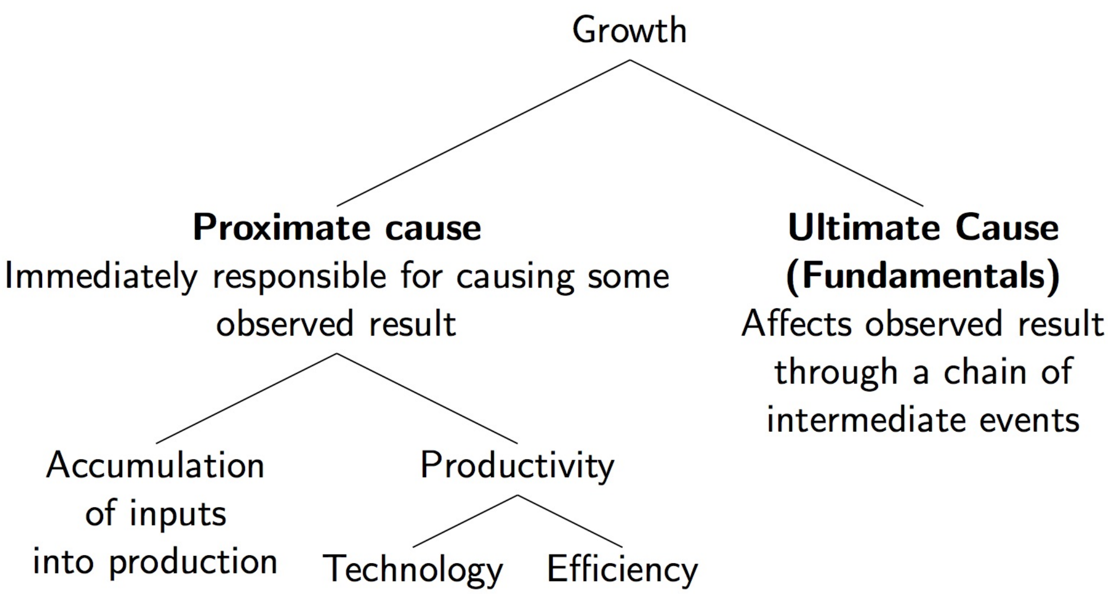
The Solow Growth Model

- A model to explain the role of factor accumulation in economic growth



About this lecture

- We will do some algebra
- Don't get scared
- Please do interrupt me if you cannot follow
- Make sure you read Chapter 2 of Jones and Vollrath
- Make sure you read my lecture slides
    - This weeks slides are intentially wordy to make it easier to follow after the class.

Physical capital

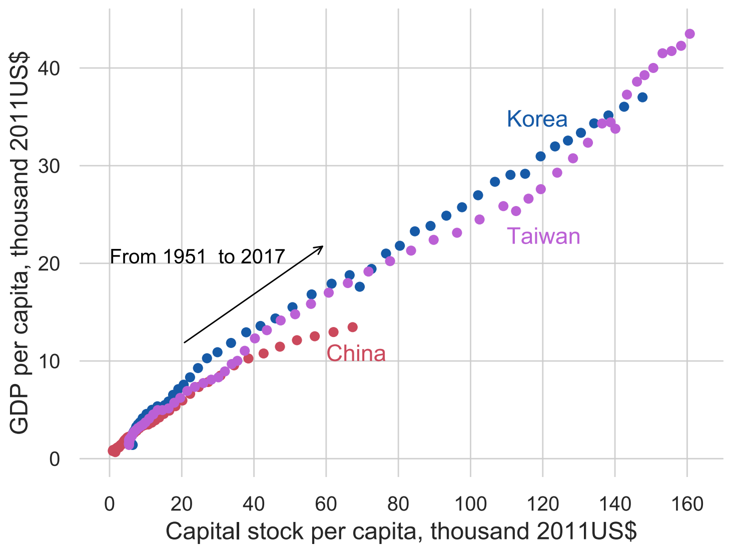
Capital: Tools – the physical objects that extend our ability or do work for us:

- machines
- buildings
- infrastructure
- vehicles
- computers

Key characteristics of capital

- It is productive
- It is produced (not like piece of land)
  - investment
- Its use is limited: limited number of people can use a given piece of capital at one time
  - unlike ideas
- It can earn a return
  - incentive for its creation
- It wears out
  - depreciation

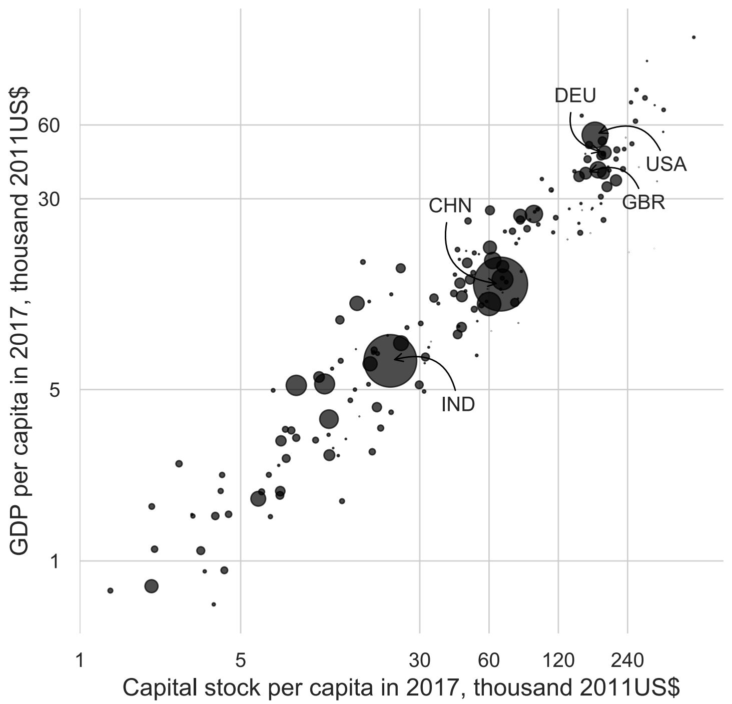
Role of capital accumulation on economic growth



Source: Penn World Tables, version 9.1

- Source of short-run economic growth in the Solow model is physical capital accumulation
- China, Taiwan and Korea are example countries that achieved high economic growth through capital accumulation.

Role of capital accumulation on economic growth



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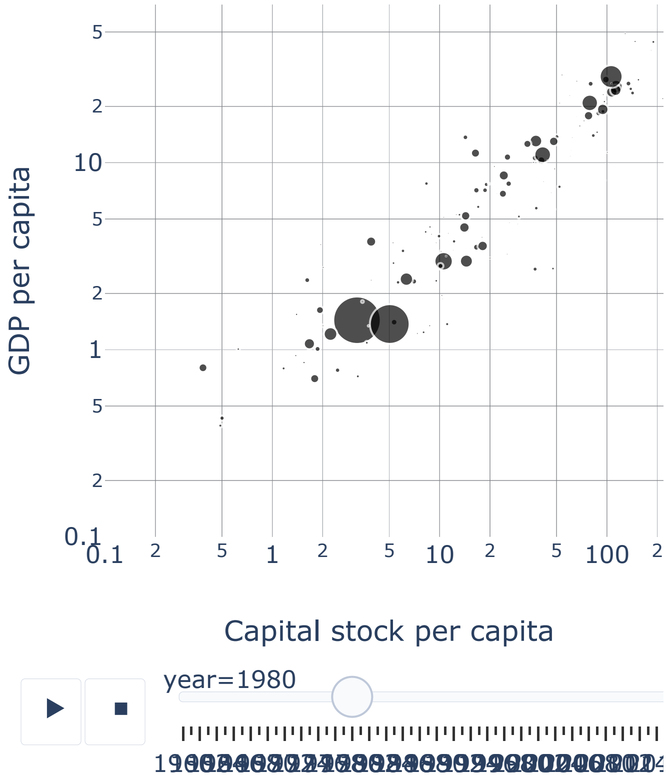
- Source of short-run economic growth in the Solow model is physical capital accumulation
- China, Taiwan and Korea are example countries that achieved high economic growth through capital accumulation.
- There is a positive correlation between GDP per capita and capital stock per capita.

Role of capital accumulation on economic growth



- GDP and capital stock over time values with 2011 thousand US$s.
- Data source: Penn World Tables, version 9.1

Model

The Solow model

Build around two equations:

- Production function
  - Constant returns to scale production function
- Capital accumulation equation
  - Households invest a constant share of income in capital stock

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Note: There are many varieties of the Solow model. Initially we will learn about the basic version with no technological change. In the later lectures, we will learn the Solow model with technological progress (extended Solow model).

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- Convert aggregate production function, $F(\cdot)$, into per worker production function, $f(\cdot)$, by multiplying $F(\cdot)$ with $1/L$:

$$\left(\frac{1}{L}\right) Y = \left(\frac{1}{L}\right) F(K, hL) = F\left(\frac{K}{L}, \frac{hL}{L}\right) = F\left(\frac{K}{L}, h\right)$$

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- Define $k \equiv K/L$, capital per labor, $y \equiv Y/L$, output per labor, and assume h is constant:

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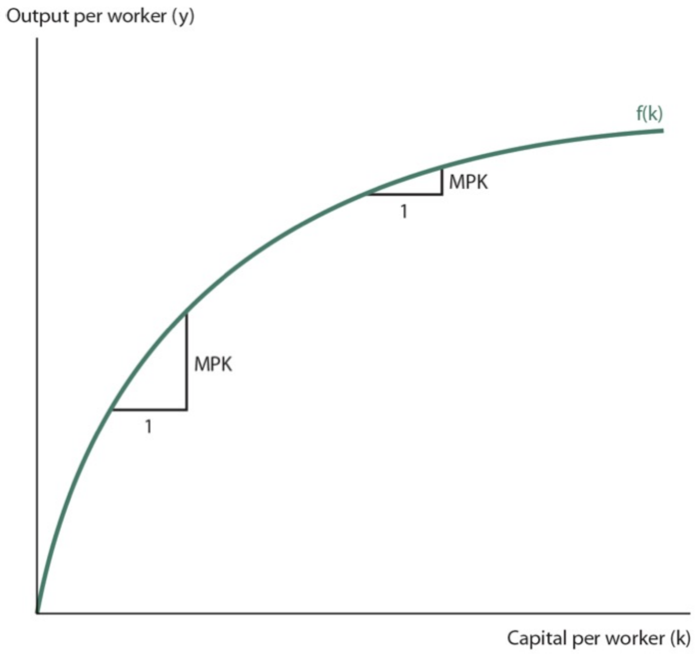
- $f(\cdot)$ displays diminishing marginal product:

  Marginal product of capital (MPK) $= \frac{\partial f(k)}{\partial k} \approx [f(k+1) - f(k)] \downarrow$ as $k \uparrow$

Production function, graphical representation



Source: Weil (2013)

An example production function displaying CRS: Cobb-Douglas Production Function

Production function: $Y = AK^{\alpha}(hL)^{1-\alpha}$,

where A is productivity, K is capital, h is human capital, and L is labor, $0 < \alpha < 1$.

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$$\frac{Y}{L} = \frac{AK^{\alpha}(hL)^{1-\alpha}}{L} = A\frac{K^{\alpha}}{L^{\alpha}}\frac{(hL)^{1-\alpha}}{L^{1-\alpha}} = A\left(\frac{K}{L}\right)^{\alpha}h^{1-\alpha}, \;\; y = Ak^{\alpha}h^{1-\alpha}$$

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- Marginal product of physical capital is positive

$$\mathrm{MPK} = \frac{\partial y}{\partial k} = \alpha A k^{\alpha-1} h^{1-\alpha} > 0$$

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Solow model assumptions, cnt'd

- Productivity, A, is constant. *Will be relaxed in later lectures*
- Human capital, h, is constant. *Will be relaxed in later lectures*
- Labor force participation rate is constant.

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$$L(t) = L_0 e^{nt}$$

- Trick: to find the growth rate of a variable, take log and differentiate w.r.t. time, t.

$$\ln L(t) = \ln L_0 + nt, \quad \frac{\dot{L(t)}}{L(t)} = n,$$

  where dot on top of a variable means derivative with respect to time: $\dot{L(t)} \equiv \frac{\partial L(t)}{\partial t}$

- E.g. population grows are a rate, $n = .01$ or 1% per year.
- $\dot{L} \approx L(t+1) - L(t)$

Accumulation of physical capital

- Second key equation of the Solow model
- Investment: a constant fraction of output, γ is invested:

$$I = \gamma Y$$

- Consumption: output minus investment.
- Depreciation: a constant fraction, δ, of capital wears out:

$$D = \delta K$$

- Change in physical capital stock = Investment - Depreciation

$$\dot{K} = I - D$$

- Subsituting investment and depreciation rules into above equation gives capital accumulation equation:

$$\dot{K} = \gamma Y - \delta K$$

The Solow Model assumptions, all together

- Production function displays constant returns to scale
- Production function displays diminishing marginal product of capital
- Everyone works in the economy: population = labor force
- Population and labor force grow at a constant rate n
- Society invests a constant fraction, γ, of output into capital stock
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Characterize the dynamics of output per worker over time

- To achieve this goal, we first need to characterize the dynamics of capital per worker

Accumulation of capital per worker

- Goal: write physical capital accumulation equation with per worker variables

- Start with the equation describing the evolution of the aggregate capital:

$$\dot{K} = \gamma Y - \delta K$$

- Divide each side by K:

$$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta$$

- Define capital per worker, $k \equiv K/L$.

- Our goal is to convert $\frac{\dot{K}}{K}$ into $\frac{\dot{k}}{k}$.

- take log (ln) of $k \equiv K/L$:

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- take log (ln) of $k \equiv K/L$:

- Differentiate $k = K/L$ with respect to time (remember $k = k(t)$ and $L = L(t)$)

  - Remember differentiation rules:

$$\frac{d\log(k(t))}{dt} = \frac{\frac{dk(t)}{dt}}{k} = \frac{\dot{k}}{k}$$

  - Then,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}, \quad \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n$$

Accumulation of capital per worker, cont'd

- Now combine these two formulas to get the accumulation of capital per worker

1. $$\frac{\dot{K}}{K} = \frac{\gamma Y}{K} - \delta \Rightarrow \frac{\dot{K}}{K} = \frac{\gamma Y/L}{K/K} - \delta$$

2. $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - n \Rightarrow \frac{\dot{k}}{k} + n = \frac{\dot{K}}{K}$$

- Substitute equation 2 into equation 1:

$$\frac{\dot{k}}{k} + n = \frac{\gamma Y/L}{K/L} - \delta$$

then take n to the right hand side

Accumulation of capital per worker, cont'd

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- Substitute equation 2 into equation 1:

$$\frac{\dot{k}}{k} + n = \frac{\gamma Y/L}{K/L} - \delta$$

then take n to the right hand side

replace Y/L with $y = Y/L$, and K/L with $k = K/L$

$$\frac{\dot{k}}{k} = \frac{\gamma y}{k} - \delta - n$$

multiply both sides of the equation with k to get capital per worker accumulation equation

$$\dot{k} = \gamma y - (\delta + n)k$$

which is equivalent to

$$\dot{k} = \gamma f(k) - (\delta + n)k$$

Change in capital per worker is equal to investment per worker, $\gamma f(k)$, minus capital dilution, $(\delta + n)k$, which is a result of capital

Numerical example

- $f(k) = Ak^{\alpha}h^{1-\alpha}$, $\alpha = 1/3$, $A = 1$, $k(0) = 1$, $h = 1$, $n = .01$, $\gamma = .2$, and $\delta = .05$, where $k(0)$ is capital per worker at time $t = 0$.
- Calculate $\dot{k}(0)$, $k(1)$, $y(1)$.

First write down capital per worker accumulation equation

$$\dot{k}(0) = \gamma f(k(0)) - (\delta + n)k(0)$$

Subsitute in the numerical values

$$\dot{k}(0) = .2 \times 1^{1/3} - (.05 + .01) \times 1 = .2 - .06 = .14$$

Notice that next periods capital per worker is approximately equal to current capital per worker plus change in capital per worker

$$k(1) \approx k(0) + \dot{k}(0) = 1 + .14 = 1.14$$

$$y(1) = f(k(1)) = 1.14^{1/3} \approx 1.04$$

What about $k(2)$ and $y(2)$?

Consumption

- Income-expenditure identity in a closed economy (no international trade) without the government

$$y = c + i,$$

  where c is consumption per worker.

- $c = y - i = y - \gamma y = (1 - \gamma)y$

- investment-savings equality

$$\begin{aligned} s &= y - c \\ s &= i = \gamma y, \end{aligned}$$

where s is savings per worker. Then γ is also saving rate, savings/income.

Steady State

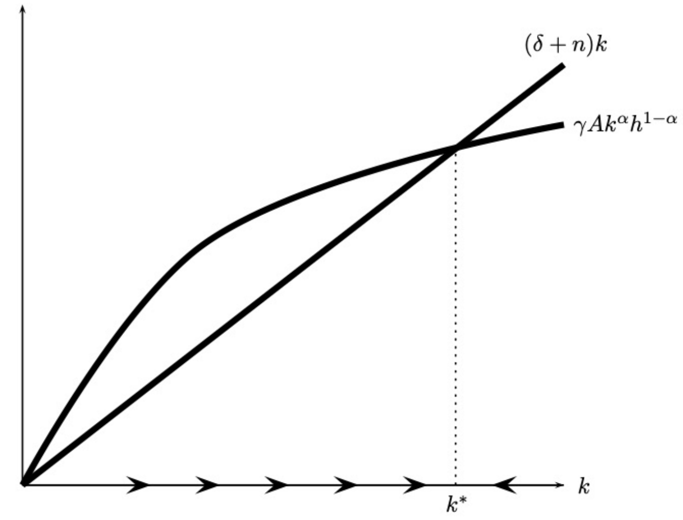
Accumulation of capital per worker:

$$\dot{k} = \gamma f(k) - (\delta + n)k$$

Capital per worker and output per worker are constant at the steady state.

- if $\gamma f(k) > (\delta + n)k$
    - then $\dot{k} > 0$
    - capital stock growing
- if $\gamma f(k) < (\delta + n)k$
    - then $\dot{k} < 0$
    - capital stock shrinking
- if $\gamma f(k) = (\delta + n)k$
    - then $\dot{k} = 0$

The Solow Diagram

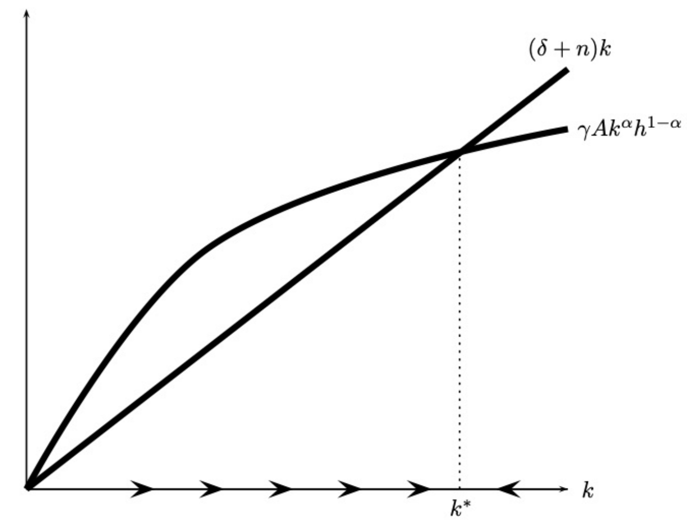


Source: Weil (2013)

k^* is the steady state capital per worker level.

Diminishing marginal product of capital per worker

- As capital per worker increases it requires more and more capital to make labor more productive.
    - Intuition: a computer makes a worker more productive. However if you give her another computer, it will make her more productive but the benefit of the second computer will not be as large as the first computer.
- Hence the concave shape of the production function
- Eventually capital dilution $(\delta + n)k$ catches up with investment $\gamma f(k)$, and capital per worker cannot grow more.



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Steady State, analytical solution

Assume production function is a Cobb-Douglas:

$$y = Ak^{\alpha}h^{1-\alpha}.$$

Then $\dot{k} = \gamma Ak^{\alpha}h^{1-\alpha} - (\delta + n)k$

No change of capital stock per worker at the steady state

$$0 = \gamma A(k^*)^{\alpha}h^{1-\alpha} - (\delta + n)k^*,$$

where k^* is steady state level of capital per worker.

$$\gamma A(k^*)^{\alpha}h^{1-\alpha} = (\delta + n)k^*$$

Steady state capital per worker:

$$k^* = h\left(\frac{\gamma A}{\delta + n}\right)^{1/(1-\alpha)}$$

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Steady state capital per worker:

$$k^* = h\left(\frac{\gamma A}{\delta + n}\right)^{1/(1-\alpha)}$$

Substitute steady state capital per worker into production function to find steady steady output per worker:

$$y^* = A(k^*)^{\alpha}h^{1-\alpha}$$

$$y^* = A[h\left(\frac{\gamma A}{\delta + n}\right)^{1/(1-\alpha)}]^{\alpha}h^{1-\alpha}$$

Rearrange to get the steady state output per worker:

$$y^* = A^{1/(1-\alpha)}\left(\frac{\gamma}{\delta + n}\right)^{\alpha/(1-\alpha)}h$$

Comparative Statics

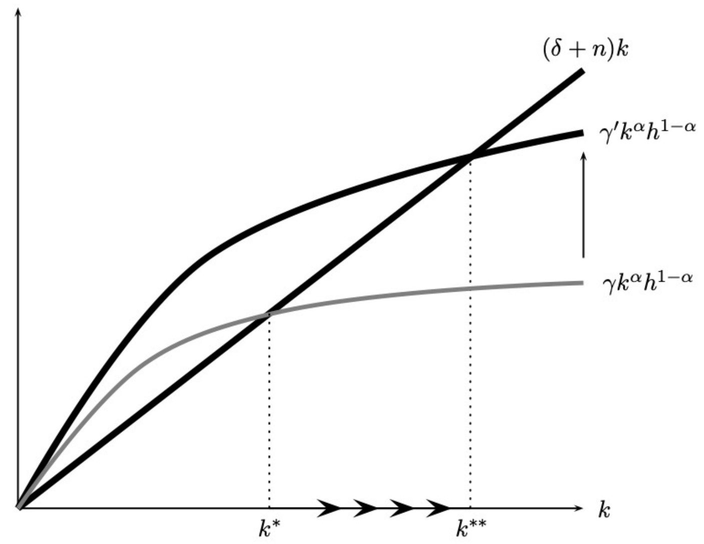
$$k^* = h\left(\frac{\gamma A}{\delta + n}\right)^{1/(1-\alpha)}$$

$$y^* = A^{1/(1-\alpha)}\left(\frac{\gamma}{\delta + n}\right)^{\alpha/(1-\alpha)} h$$

- k^* and y^* are rising with investment rate γ, technology A, human capital h,
- k^* and y^* are declining with depreciation rate, δ, and population growth rate, n.

Increasing the investment rate

- As $\gamma \uparrow$
  - $k^* = h(\frac{\gamma A}{\delta+n})^{1/(1-\alpha)} \uparrow$
  - $$y^* = A^{1/(1-\alpha)}(\frac{\gamma}{\delta+n})^{\alpha/(1-\alpha)} h \uparrow$$



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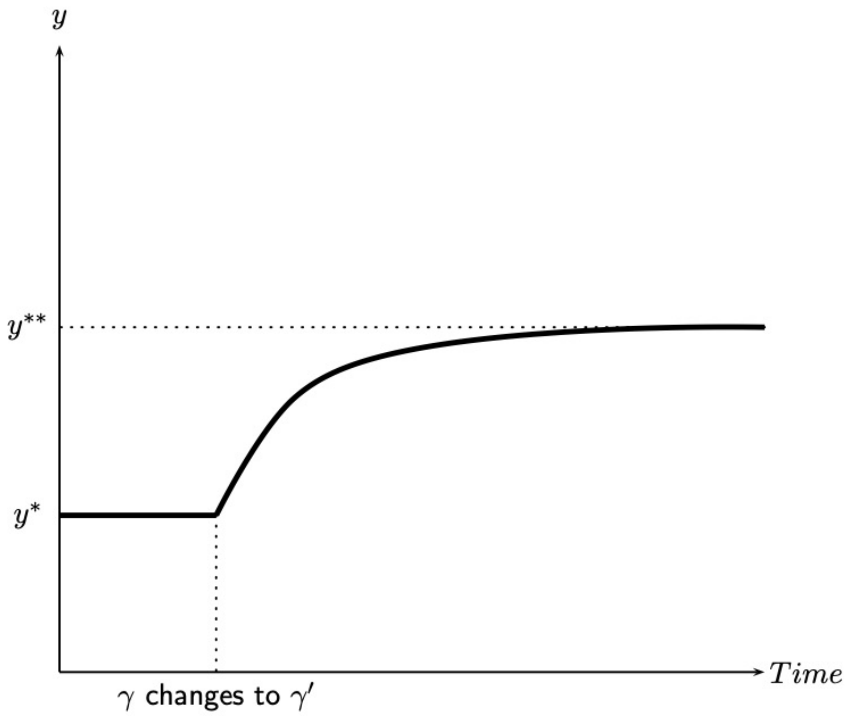
Income increase after an increase in investment rate



Source: Weil (2013)

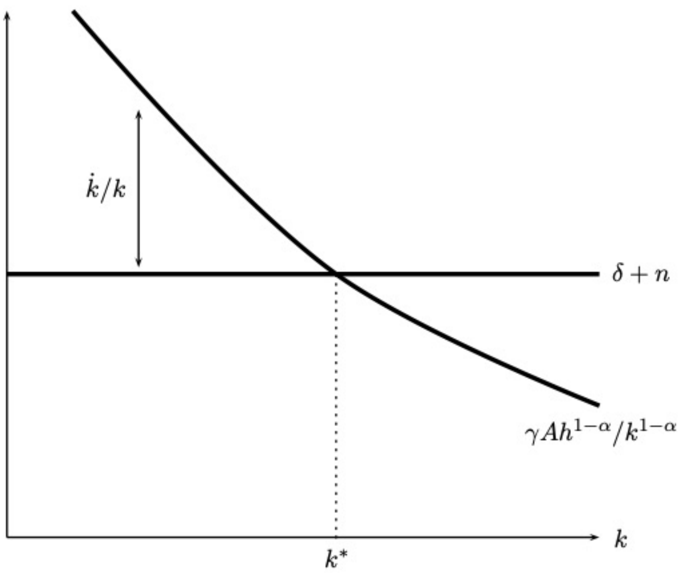
Growth Rates

- The Solow model predicts that growth is faster, the farther away from steady state is an economy.

- Look at the growth rate of k

$$\frac{\dot{k}}{k} = \frac{\gamma A h^{1-\alpha}}{k^{1-\alpha}} - (\delta + n).$$

- As k rises, the growth rate of k falls.



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Growth Rates, cont'd

In the Solow model (current version)

- No long-run growth of output per worker:

$$y^* = A(k^*)^\alpha h^{1-\alpha}, \quad \frac{\dot{y}^*}{y^*} = \alpha \frac{\dot{k}^*}{k^*} = 0$$

- Transitional growth
- Higher growth rate for lower capital stock countries
- Convergence toward the steady state:
    - A country's per-worker output will grow or shrink from some initial position toward the steady-state level determined by the investment rate.

Economics of Solow model

- What is the main source of economic growth in the Solow model?
    - Capital accumulation
- Why cannot countries grow forever?
    - Diminishing marginal product of physical capital.
- What are the growth rates of the following variables at the steady state?

Variable	Steady state growth rate
Capital per worker	
Output per worker	
Consumption per worker	
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Capital	n
Output	n
Consumption	n

Relative Growth Rates, Predictions

- If two countries have the same rate of investment but different levels of income, the country with lower income will have higher growth.
- If two countries have the same level of income but different rates of investment, then the country with a higher rate of investment will have higher growth.
- A country that raises its level of investment will experience an increase in its rate of income growth.

Next lecture

- Learn quantitative predictions of the Solow model
- Compare model predictions with the data