PS5

May 2, 2017

0.1 PS 5

```
In [12]: def theoretical_sigma(f,lower,upper,N,C):

             w=findw(f,C,lower,upper,True)

             flower=f(lower,C)
             fmiddle=f((lower+upper)/2,C)
             fupper=f(upper,C)

             wlower=w[0]*lower+w[1]
             wmiddle=w[0]*(lower+upper)/2+w[1]
             wupper=w[0]*upper+w[1]

             gupper=fupper/wupper
             gmiddle=fmiddle/wmiddle
             glower=flower/wlower


             sigma=np.abs(0.03727*(upper-lower)**3/np.sqrt(N)*(4*(gupper-2*gmiddle+


             #return  (upper-lower)*np.sqrt(var)
             return sigma

         low=4.6
         high=5.2
         C=np.arange(0,7,1)
         sigmas=[]
         for i in C:
             sigmas.append(theoretical_sigma(f,low,high,1000,10**i))


         plt.plot(C,sigmas,'d')
         plt.ylabel('Sigma')
         plt.xlabel('C')
         z = np.polyfit(C,sigmas,3)
         p = np.poly1d(z)
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plt.plot(C,p(C))
plt.show()

print(theoretical_sigma(f,low,high,1000,int(-f(low,0)/2-f(high,0)/2)))
#print(-f(low,0)/2-f(high,0)/2)
#print(np.argmin(sigmas),np.min(sigmas))
```

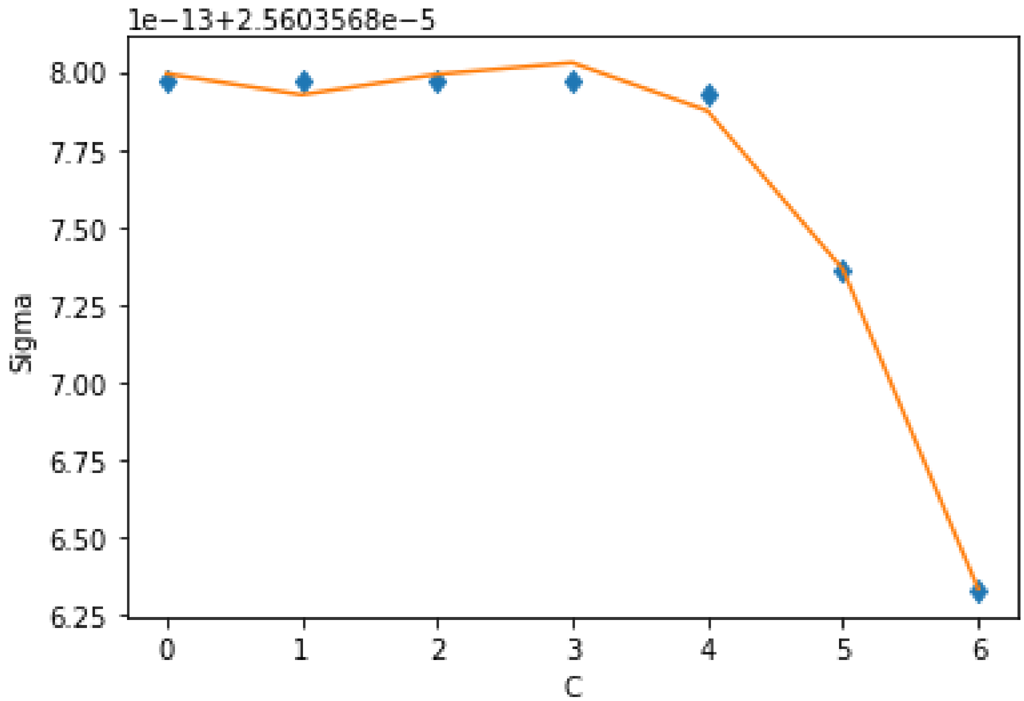


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2.56035687975e-05
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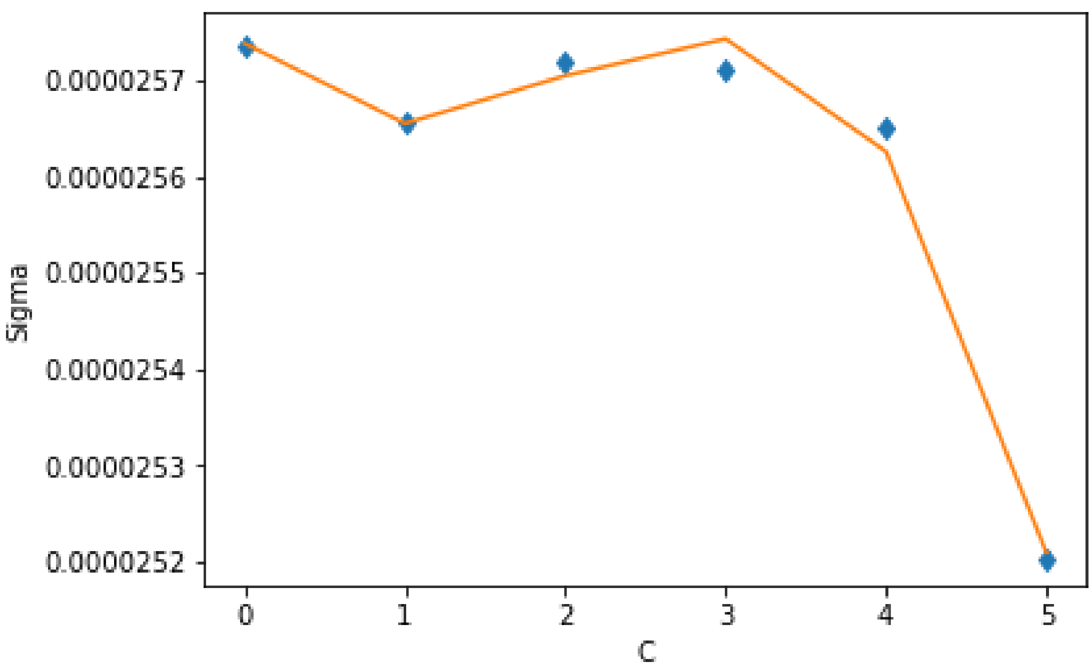
In [10]:
```
import numpy as np
from numpy.polynomial import Polynomial as P
#import plotly
#import plotly.plotly as py
#import plotly.figure_factory as ff
import matplotlib.pyplot as plt
#Integrand function
def f(x,H):
    #return (x-5)*np.exp(-(x/2-3))+H
    return np.sin(x)/x +H
#Calculates the coefficients of linear weight function.
def findw(f,H,lower,upper,normalize):
    #Find the linear function.
    slope=(f(upper,H)-f(lower,H))/(upper-lower)
```

```
    a=slope
    b=-slope*upper+f(upper,H)
    #Normalization.
    A=(a/2)*(upper**2)+b*upper-(a/2)*(lower**2)-b*lower
    if normalize:
        a/=A
        b/=A
    return [a,b]
#Performs integration.
def integrate(f,lower,upper,N,C):
    H=C
    w=findw(f,H,lower,upper,True)
    #Generate uniform random inputs.
    inputs=np.random.rand(N)
    a=w[0]/2
    b=w[1]
    c=-(a*lower**2+b*lower)

    SUM=0
    SUM2=0

    inverse_inputs=[]
    for i in inputs:
        p=[(-b-np.sqrt(b**2-4*a*(c-i)))/(2*a),(-b+np.sqrt(b**2-4*a*(c-i)))
        if p[0]>=lower and p[0]<=upper:
            inverse_inputs.append(p[0])
        else :
            inverse_inputs.append(p[1])

    inverse_inputs=np.array(inverse_inputs)
    #Calculate f(inverse(x))/w(inverse(x)).
    outputsF=f(inverse_inputs,H)
    outputsW=w[0]*(inverse_inputs)+w[1]
    outputs=outputsF/outputsW
    SUM=outputs.sum()
    SUM2=(outputs*outputs).sum()
    var=SUM2/N-(SUM/N)**2
    var=var/N
    #Store generated points for variance calculation.
    Vsum=outputs.sum()
    return Vsum/N-H*(upper-lower),(upper-lower)**2*var

sigmas=[]
sigma=0
I=0
l=[4.6,5.2]
C=np.arange(0,6,1)
for c in C:
```

```
    I=0
    sigma=0
    temp_sigmas=[]
    temp_results=[]
    for i in range (0,len(l)-1):
        for p in range (0,100):
            temp,temp2=integrate(f,l[i],l[i+1],1000,10**c)
            temp_sigmas.append(temp2)
            temp_results.append(temp)
        sigma+=np.mean(temp_sigmas)
        I+=np.mean(temp_results)
    sigmas.append(np.sqrt(sigma))

plt.plot(C,sigmas,'d')
plt.ylabel('Sigma')
plt.xlabel('C')
z = np.polyfit(C,sigmas,3)
p = np.poly1d(z)
plt.plot(C,p(C))
plt.show()
print(np.min(sigmas))
```

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2.52021248058e-05
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0.2 Minimum Sigma

We found that sigma is proportional to $g(a) + g(b) - 2g(\frac{a+b}{2})$. Taking derivative of this value with respect to C and equating to 0, we get

$$\frac{dg(a)}{dC} + \frac{dg(b)}{dC} - 2\frac{dg(\frac{a+b}{2})}{dC}$$

where

$$g(x) = \frac{f(x) + C}{kw}$$

. Note that in the previous derivation I denote k.w by simply w for simplification.

Also,

$$w(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + C$$

and

$$\frac{1}{k} = \frac{(f(b) - f(a))\,(b^2 - a^2)}{2(b - a)} + \left(f(b) - b\frac{f(b) - f(a)}{b - a} + C\right)(b - a)$$

Then,

$$g(x) = \frac{(f(x) + C)\left(\frac{(f(b)-f(a))(b^2-a^2)}{2(b-a)} + \left(f(b) - b\frac{f(b)-f(a)}{b-a} + C\right)(b - a)\right)}{f(a) + \frac{f(b)-f(a)}{b-a}(x - a) + C}$$

From here,

$$g(a) = \frac{(f(b) - f(a)(b^2 - a^2)}{2(b - a)} + \left(f(b) - b\frac{f(b) - f(a)}{b - a} + C\right)(b - a)$$

$$g(b) = \frac{(f(b) - f(a)(b^2 - a^2)}{2(b - a)} + \left(f(b) - b\frac{f(b) - f(a)}{b - a} + C\right)(b - a)$$

$$g(\frac{a + b}{2}) = \frac{(f(\frac{a+b}{2}) + C)\left(\frac{(f(b)-f(a))(b^2-a^2)}{2(b-a)} + \left(f(b) - b\frac{f(b)-f(a)}{b-a} + C\right)(b - a)\right)}{\frac{f(a)+f(b)}{2} + C}$$

$$\frac{dg(a)}{dC} = b - a$$

$$\frac{dg(b)}{dC} = b - a$$

It follows that $\frac{dg(\frac{a+b}{2})}{dC}$ should be equal to $b - a$.