PS4

May 2, 2017

0.1 Estimation Of Variance For Single Segment

Let's try to estimate σ_I analytically for one segment.

$$\sigma_I^2 = \left\langle \frac{f^2}{w^2} \right\rangle - \left\langle \frac{f}{w} \right\rangle^2$$

Let $g = \frac{f}{w}$, then,

$$\sigma_{f/w}^2 = \langle g^2 \rangle - \langle g \rangle^2$$

We can expand g using Taylor Series around $\frac{b+a}{2}$,

$$g(x) = g\left(\frac{b+a}{2}\right) + g'\left(\frac{b+a}{2}\right)\left(x - \frac{b+a}{2}\right) + \frac{1}{2}g''\left(\frac{b+a}{2}\right)\left(x - \frac{b+a}{2}\right)^2$$

We can also aprrroximate first and second derivative very accurately for a small region,

$$g'\left(\frac{b+a}{2}\right) = \frac{g(b) - g(a)}{b - a}$$

$$g''\left(\frac{b+a}{2}\right) = \frac{g(b) - 2g(\frac{b+a}{2}) + g(a)}{(\frac{b-a}{2})^2}$$

To simplify equations we can define the following variables,
$p = g(\frac{b+a}{2})$
$q = \frac{g(b)-g(a)}{b-a}$
$r = \frac{g(b)-2g(\frac{b+a}{2})+g(a)}{2(\frac{b-a}{2})^2}$
$u = (x - \frac{b+a}{2})$
Then,

$$g(x) = ru^2 + qu + p$$

Since we forced w to be equal to f at the boundaries, q is 0.
Our job reduces to calculate

$$\left\langle (ru^2 + p)^2 \right\rangle - \left\langle ru^2 + p \right\rangle^2$$

Expanding the terms,

$$\sigma^2_{f/w} = \left\langle r^2u^4 + 2rpu^2 + p^2 \right\rangle - \left\langle ru^2 + p \right\rangle^2$$

$$= (r^2 \left\langle u^4 \right\rangle + 2rp \left\langle u^2 \right\rangle + p^2) - (r^2 \left\langle u^2 \right\rangle^2 + 2rp \left\langle u^2 \right\rangle + p^2)$$

$$= r^2 \left\langle u^4 \right\rangle - r^2 \left\langle u^2 \right\rangle^2$$

$$= r^2(\left\langle u^4 \right\rangle - \left\langle u^2 \right\rangle^2)$$

This could further be simplified as,

$$\sigma^2_{f/w} = r^2\sigma^2_{u^2}$$

Finally σ_I can be estimated as,

$$\sigma_I = \frac{(b-a)}{\sqrt{N}} r\sigma_{u^2} = \frac{(b-a)}{\sqrt{N}} \frac{g(b) - 2g(\frac{b+a}{2}) + g(a)}{2(\frac{b-a}{2})^2} \sqrt{\left(\frac{2(b-a)^4}{5 \cdot 2^5} - \frac{4(b-a)^4}{9 \cdot 2^6}\right)}$$

$$\sigma_I = \sqrt{\left(\frac{2(b-a)^2}{5 \cdot 2^5} - \frac{4(b-a)^2}{9 \cdot 2^6}\right)} \frac{2}{\sqrt{N}}(g(b) - 2g(\frac{b+a}{2}) + g(a))$$

or

$$\sigma_I = \frac{0.03727(b-a)^3}{\sqrt{N}} g''\left(\frac{a+b}{2}\right)$$

In [23]:
```python
import numpy as np
from numpy.polynomial import Polynomial as P
#import plotly
#import plotly.plotly as py
#import plotly.figure_factory as ff
import matplotlib.pyplot as plt
#Integrand function
def f(x,H):
    return (x-5)*np.exp(-(x/2-3))+H

#Calculates the coefficients of linear weight function.
def findw(f,H,lower,upper,normalize):
    #Find the linear function.
    slope=(f(upper,H)-f(lower,H))/(upper-lower)
    a=slope
    b=-slope*upper+f(upper,H)
    #Normalization.
    A=(a/2)*(upper**2)+b*upper-(a/2)*(lower**2)-b*lower
    if normalize:
        a/=A
        b/=A
    return [a,b]
```

```
#Performs integration.
def integrate(f,lower,upper,N,C):
    H=C
    w=findw(f,H,lower,upper,True)
    #Generate uniform random inputs.
    inputs=np.random.rand(N)
    a=w[0]/2
    b=w[1]
    c=-(a*lower**2+b*lower)

    SUM=0
    SUM2=0

    inverse_inputs=[]
    for i in inputs:
        p=[(-b-np.sqrt(b**2-4*a*(c-i)))/(2*a),(-b+np.sqrt(b**2-4*a*(c-i)))
        if p[0]>=lower and p[0]<=upper:
            inverse_inputs.append(p[0])
        else :
            inverse_inputs.append(p[1])

    inverse_inputs=np.array(inverse_inputs)
    #Calculate f(inverse(x))/w(inverse(x)).
    outputsF=f(inverse_inputs,H)
    outputsW=w[0]*(inverse_inputs)+w[1]
    outputs=outputsF/outputsW
    SUM=outputs.sum()
    SUM2=(outputs*outputs).sum()
    var=SUM2/N-(SUM/N)**2
    var=var/N
    #Store generated points for variance calculation.
    Vsum=outputs.sum()
    return Vsum/N-H*(upper-lower),(upper-lower)**2*var




def theoretical_sigma(f,lower,upper,N,C):

    w=findw(f,C,lower,upper,True)

    flower=f(lower,C)
    fmiddle=f((lower+upper)/2,C)
    fupper=f(upper,C)

    wlower=w[0]*lower+w[1]
    wmiddle=w[0]*(lower+upper)/2+w[1]
    wupper=w[0]*upper+w[1]
```

```
        gupper=fupper/wupper
        gmiddle=fmiddle/wmiddle
        glower=flower/wlower


        sigma=np.abs(0.03727*(upper-lower)**3/np.sqrt(N)*(4*(gupper-2*gmiddle+


        #return (upper-lower)*np.sqrt(var)
        return sigma

    low=4.6
    up=5.2
    #Divide the region into 10 pieces.
    l=[low,up]
    #Real value of the integral
    I_real=-.12002
    #N values
    N=[100,500,1000,5000,10000,50000,100000]
    #Integration results.
    results=[]
    #Standart deviation values
    sigmas=[]
    theory_sigmas=[]
    print('N','    sigma_exp','       sigma_theory','    sigma_theory/sigma_exp

    for k in N:
        I=0
        sigma=0
        S=1000
        for i in range (0,len(l)-1):
            temp_sigma=[]
            for j in range(0,10):
                I=0
                sigma=0
                temp,temp2=integrate(f,l[i],l[i+1],k,S)
                I+=temp
                sigma+=temp2
                temp_sigma.append(np.sqrt(sigma))
        results.append(I)
        sigmas.append(np.mean(temp_sigma))
        theory_sigmas.append(theoretical_sigma(f,low,up,k,S))
        sigma=np.mean(temp_sigma)
        print(k,sigma,theoretical_sigma(f,low,up,k,S),theoretical_sigma(f,low,
```

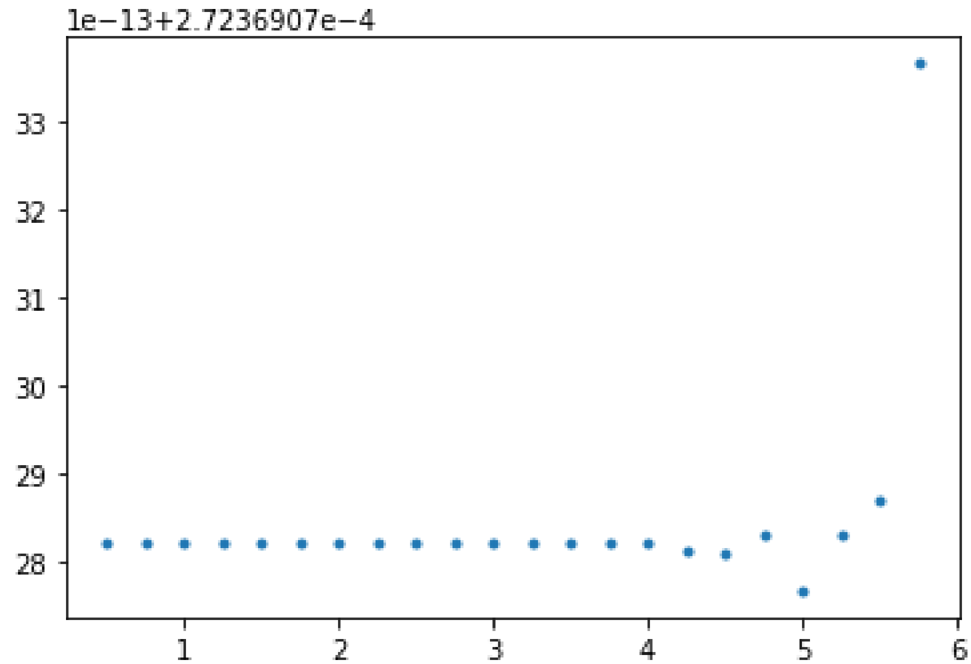
```
N    sigma_exp         sigma_theory     sigma_theory/sigma_exp
100 0.000876274045448 0.000861306634305 0.982919257714
```

```
500 0.000382733216776 0.000385188036756 1.00641391934
1000 0.000272760056223 0.000272369072822 0.998566566504
5000 0.000122492530038 0.00012180715236 0.994404738982
10000 8.63306816331e-05 8.61306634305e-05 0.997683115681
50000 3.86083629727e-05 3.85188036756e-05 0.997680313531
100000 2.73145698375e-05 2.72369072822e-05 0.997156735185
```

Here N is the total number of sample points, sigma_exp is the mean of the experimental sigmas, sigma_theory is analytical estimation of sigma and the last columns is the ratio of there two. The ratio is very close to 1, this means analytical estimation is reasonable.

Let's plot analytical sigma with N=1000.

```
In [58]: C=np.arange(0.5,6,0.25)
         analytic=[]
         experimental=[]
         low=4.6
         up=5.2
         #Divide the region into 10 pieces.
         l=[low,up]
         for k in C:
             sigma=0
             S=10**k
             temp_sigma=[]
             for i in range (0,len(l)-1):
                 for j in range(0,10):
                     sigma=0
                     temp,temp2=integrate(f,l[i],l[i+1],1000,S)
                     I+=temp
                     sigma+=temp2
                     temp_sigma.append(np.sqrt(sigma))
             sigma=np.mean(temp_sigma)
             experimental.append(sigma)
             analytic.append(theoretical_sigma(f,low,up,1000,S))
         plt.plot(C,analytic,'.')
         plt.show()
```


1e−13+2.7236907e−4
33
32
31
30
29
28
1
2
3
4
5
6