Navier Stokes Equations

And some simple solutions

Finite Volume Method

$$\overbrace{\rho\left(\frac{\partial u}{\partial t}\right.}^{\text{Transient term}} + \overbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\left.\frac{\partial u}{\partial z}\right)}^{\text{Convection (Advection) terms}} = \overbrace{-\frac{\partial p}{\partial x} + \rho g_x}^{\text{Source terms}} + \overbrace{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}^{\text{Diffusion terms}}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Steady, Laminar Flow Between Fixed Parallel Plates

- We first consider flow between the two horizontal, infinite parallel plates. This geometry the fluid particles move in the x direction parallel to the plates, and there is no velocity in the y or z direction—that is $\mathbf{v = 0}$ $\boldsymbol{and\ w = 0}$. In this case it follows from the continuity equation that $\boldsymbol{u_x = 0}$. Furthermore, there would be no variation of u in the z direction for infinite plates, and for steady flow $\boldsymbol{u_t = 0}$ so that $\boldsymbol{u = u(y)}$ If these conditions are used in the Navier–Stokes equations they reduce to

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g$$

$$0 = -\frac{\partial p}{\partial z}$$

Steady, Laminar Flow Between Fixed Parallel Plates

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\cancel{\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)} = -\frac{\partial p}{\partial y} + \rho g_y + \cancel{\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)}$$

$$\cancel{\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)} = -\frac{\partial p}{\partial z} + \cancel{\rho g_z} + \cancel{\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}$$

2d Flow

2d Navier Stokes Equations

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \cancel{\rho g_x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\cancel{\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}$$

2d Form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \upsilon\frac{\partial^2 u}{\partial x^2} + \upsilon\frac{\partial^2 u}{\partial y^2}$$

$$\underbrace{\frac{\partial v}{\partial t}}_{\text{acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{advection}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial y}}_{\text{Pressure gradient}} + \underbrace{\upsilon\frac{\partial^2 v}{\partial x^2} + \upsilon\frac{\partial^2 v}{\partial y^2}}_{\text{diffusion}}$$