Stat365/665 (Spring 2015) Data Mining and Machine Learning **Lecture: 16**

Data Mining and Machine Learning

Lecturer: Sahand Negahban *Scribe: Leon Lixing Yu*

1 Announcement

Homework assigned.
Start thinking about projects.

2 Today

- Wrap-up soft-margin SVM.
- Statistical Learning theory.

3 soft-margin SVM

We know that

$$\hat{W} \in \underset{W}{\operatorname{argmin}} \frac{C}{n} \sum \phi(w^T x_i y_i) + \|w\|^2$$

Last time we proved that through the K.K.T. conditions, we have:

$$w = \sum_{i=1}^{n} \alpha_i x_i y_i \quad \frac{C}{n} \geq \alpha_i \geq 0$$

We can re-write it in kernel form:

$$w = \sum_{i=1}^{n} \alpha_i k(x_i, \bullet) y_i$$

The notion, $\phi(S)$ stands for the positive part of $(1 - S)$, S can be anything here. we rephrase it in math notation:

$$\phi(S) = (1 - S)_+$$

If $1 - S$ is negative, then $\phi(S) = 0$.

3.1 Margin Error

Everything, that is a support vector, is a margin error.See below.

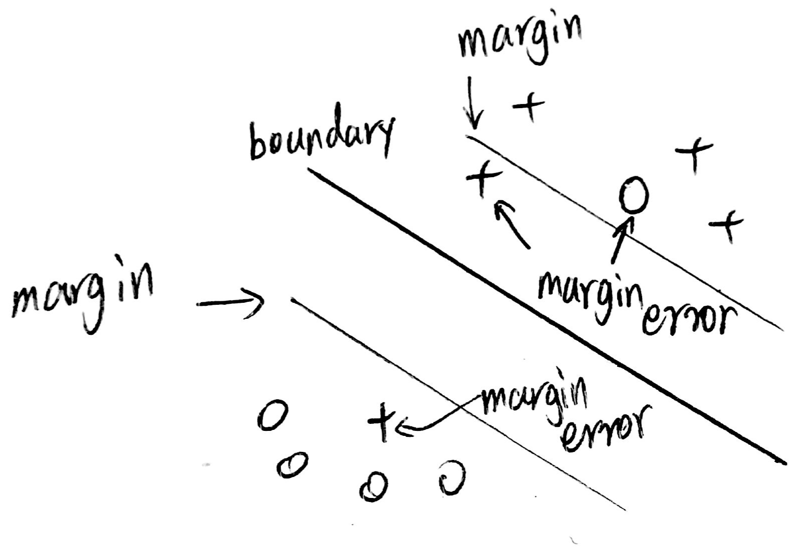

Figure 1: Margin Error

Given $\|w\|^2$ controls the size of w, $\phi(w^T x_i y_i)$ controls the errors.
Also, remember that ideally we ant to control:

$$\min_w \frac{C}{n} \sum_{i=1}^{n} \mathbb{1}(w^T x_i y_i \leq 0) \qquad s.t. \ \|w\| \leq 1$$

The form above simply means that $\frac{1}{n}$ multiplied by the total number of errors is the average number of errors. However, it takes long time to compute (a.k.a: computatinally intractable).
For that reason, we use convex relaxation.

3.2 Convex Relaxation

Say we have:

$$w^T x_i, y_i = S$$

and its graph is given by:

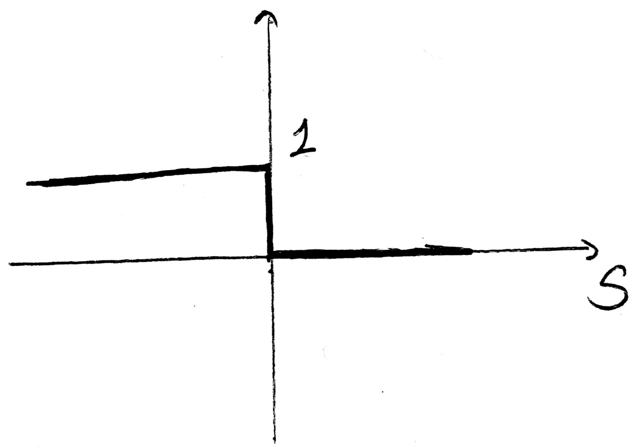

Figure 2: Non-covex form

In convex form, we need the graph to be:

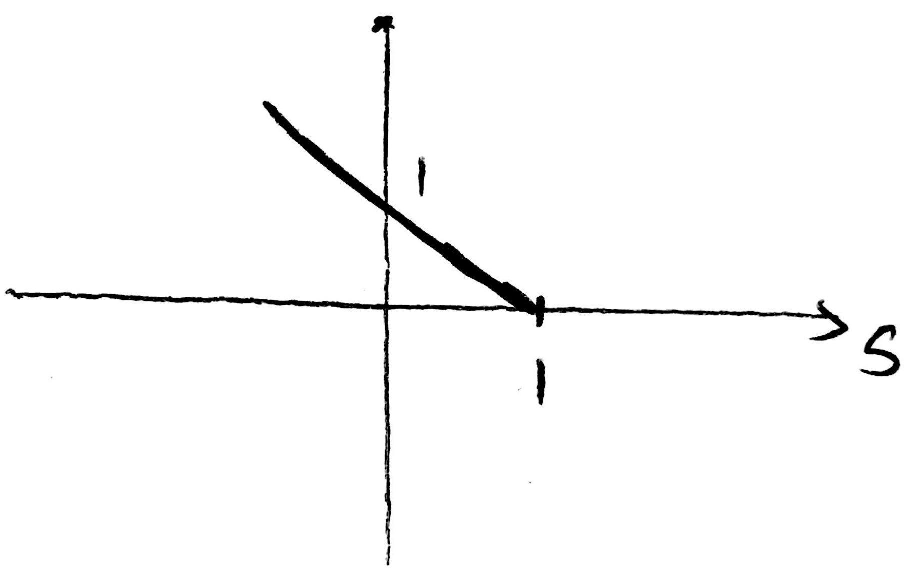

Figure 3: Covex form

This is called convex upper bond, and such graph can be described as:

$$\min_w \frac{C}{n}\sum_{i=1}^{n}\mathbb{1}(w^T x_i y_i \leq 0) \qquad s.t.\ \|w\| \leq 1$$

Note: As C goes to infinite, the soft-margin becomes hard-margin. Since C stands for how much we tolerant the error. Since the $C-SVM$ is sometimes not that interpretable, we can use $\nu-SVM$ instead. Therefore we have:

$$\hat{W} \in \operatorname*{argmin}_{W,\rho} \frac{1}{2}\|w\|^2 - \nu\rho + \frac{1}{n}\sum_{i=1}^{n}(\rho - y_i w^T x_i)_+ \qquad \rho \geq 0$$

The $\nu\rho$ term says that we want a bigger ρ. Referring to the diagram below, as ρ increases, I am increasing the intersection a. Theorm:

$$|i|y_i\hat{w}^T x_i < \rho| \leq |i|\alpha_i = \frac{1}{n}| \leq \nu n \leq |i|\alpha_i > 0| \leq |i|y_i\hat{w}^T x_i \leq \rho|$$

So νn tells us how many erros I should have. A.k.a: the number of strict margin error is a subset of νn. Strict margin error are things within the margin boundary. $i|y_i\hat{w}^T x_i \leq \rho$ includes the points on the margin boundary. The proof of this theorm will be posted online. Theorm:
Take a soultion of $\nu - SVM$, and let ρ^* be the optimal ρ, that is larger than 0, then $C = \frac{1}{\rho^*}$ gives an equivalent problem.
The proof of this theorm is left as an exercise.

4 Statical Learning Theory

We have been talking about something called Empirical Risk Minimization.
In decision theory (STAT 610/611), we often have some loss of our parameters, $l(w, y, x) \in \mathbb{R}$. e.g. $-\frac{1}{2}(w^T x - y)^2$, and $-\mathbb{1}(w^T xy \leq 0)$, so we ideally want to find:

$$w^* = \operatorname*{argmin}_w \mathbb{E}[l(w, x, y)]$$

Note that x, y are drawn from some distribution.
we can define the risk of w to be:

$$R(w) = \mathbb{E}[l(w, x, y)]$$

But we don't have access to the distribution governing (x, y); instead, we have n i.i.d samples, and therefore we have:

$$\hat{R}(w) = \frac{1}{n}\sum_{i=1}^{n} l(w, x_i, y_i)$$

If we have a fixed w, what is the epxected value of $\hat{R}(w)$?
It is just $R(w)$ so we are just taking the average: $\mathbb{E}\,\hat{R}(w) = R(w)$. The question is that when is optimizing $\hat{R}(w)$ good enough?
Let $R^* = \min_w \mathbb{E}\,[l(w, x, y)]$ be the optimal solution.
Let $\hat{w} = \operatorname*{argmin}_w \hat{R}(w)$.
How do we relate $R(\hat{w})$ to $R(w^*)$? a.k.a: Can we show that $R(\hat{w}) - R(w^*)$ is small?
$R(\hat{w})$ is called "generalization error".
Ex: binary classification

$$l(w, x, y) = \mathbb{1}(w^T xy \leq 0) \Rightarrow R(\hat{w})$$

This is the probability that $\hat{w}$ makes a mistake.
i.e.

$$R(\hat{w}) = \mathbb{E}\left[\mathbb{1}(\hat{w}^T xy \leq 0)\right] = P(\hat{w}^T xy \leq 0) = P(\hat{w} \; makes \; an \; error$$

$R(\hat{w})$ is random, so we often want to consider $\mathbb{E}[R(w)]$ or we can also show that with high probability, $R(\hat{w}) \leq R(w^*) + \varepsilon$. a.k.a: $P(R(\hat{w}) > R(w^*) + \varepsilon)$ is small.

Theorm:
If $|\hat{R}(w) - R(w)| \leq \varepsilon \quad \forall w$, then

$$R(\hat{w}) \leq R(w^*) + 2\varepsilon$$

if $\varepsilon = 0$, then $R(\hat{w}) = R(w^*)$.
Proof:
Note that $R(\hat{w}) - \hat{r}(\hat{w}) \leq \varepsilon$.

$$\begin{aligned}
R(\hat{w}) - R(w^*) &= R(\hat{w}) - \hat{R}(\hat{w}) + \hat{R}(\hat{w}) - R(w^*) \\
&= R(\hat{w}) - \hat{R}(\hat{w}) + \hat{R}(\hat{w}) - R(w^*) + \hat{R}(w^*) - \hat{R}(w^*) \\
&= [R(\hat{w}) - \hat{R}(\hat{w})] + [\hat{R}(\hat{w}) - \hat{R}(w^*)] + [\hat{R}(w^*) - R(w^*)] \\
&\leq \varepsilon + 0 + \varepsilon \\
&\leq 2\varepsilon
\end{aligned}$$

For example, sample mean:
Let $x_i = 1$ with probability p, and $x_i = 0$ with probability $1 - p$.

$$\hat{\mu} = \underset{\mu}{\text{argmin}} \frac{1}{n} \sum_{i=1}^{n} (\mu - x_i)^2$$

$$\Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\begin{aligned}
R(\mu) &= \mathbb{E}\,(\mu - x_i)^2 = var(x_i) + (\mu - p)^2 \\
&= p(1-p) + (\mu - p)^2 \\
\hat{R}(\mu) &= \frac{1}{n} \sum_{i=1}^{n} (\mu - x_i)^2
\end{aligned}$$

Now we get:

$$\begin{aligned}
\hat{R}(\mu) - R(\mu) &= \frac{1}{n} \sum_{i=1}^{n} [(\mu - x_i)^2 - (p(1-p) + (\mu - p)^2)] \\
&= \frac{1}{n} \sum_{i=1}^{n} [(\mu - p + p - x_i)^2 - R(\mu)] \\
&= \frac{1}{n} \sum_{i=1}^{n} [(\mu - p)^2 + (p - x_i)^2 - R(\mu) + (\mu - p)(p - x_i)]
\end{aligned}$$

...... Finish next time.