자동제어

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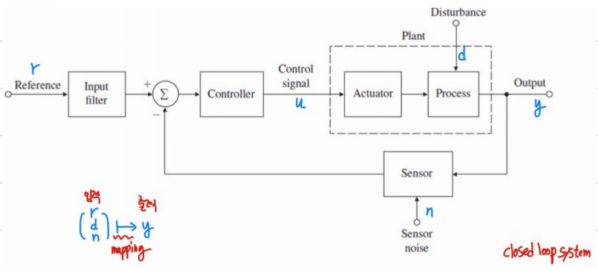
1. Perspective on feedback Control

i, Manual control : person controlling
Automatic 〃 : machines only

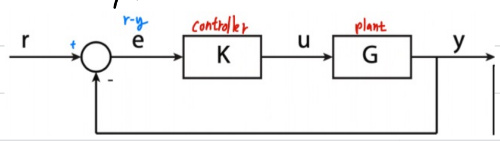
ii, Regulators : to hold an output steady against disturbance
tracking : to track reference signal

2. Control system



i, System : anything with inputs and outputs
plant : System to be controlled, combination of process and actuator, 제어의 대상
actuator : any mechanism / signal / communication that can affect outputs
controller : actually computes the desired control system.
compute difference between reference signal and the sensor output.
sensor : any 〃 can be used to measure output
whose output inevitably contains sensor noise

ii, sensitivity, tranfer function



T: $e = r-y$, $u = ke$, $y = Gu$

$y = Gu = Gke = Gk(r-y) \Rightarrow (1+Gk)y = Gkr$

$\therefore y = \frac{Gk}{1+Gk} r \Rightarrow \frac{y}{r} = \frac{Gk}{1+Gk}$: T

also we call the complementary sensitivity function.

S : plant 변화에 의한 전체 T의 변화 $\left(\frac{\%\text{ changes in } T}{\%\text{ changes in } G}\right)$

$\frac{\delta T}{\delta G} = \frac{-k^2 G}{(1+kG)^2} + \frac{k}{1+kG} = \frac{k}{(1+kG)^2} = \frac{1}{1+kG} \cdot \frac{k}{1+kG}$

therefore $\frac{\delta \log T}{\delta \log G} = \frac{\delta T}{\delta G} \cdot \frac{G}{T} = \frac{1}{1+kG}$: S

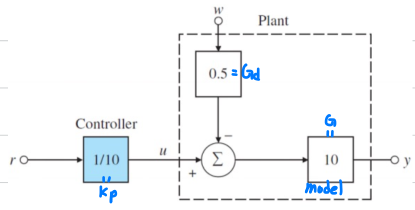
feedback control이 disturbance의 영향을 줄여준다.

3. Goal of Feedback Control

i, stability
ii, Tracking : output must track the Command reference
iii, Disturbance rejection
iv, Robustness : plant : actual physical system, complex
model : used for controller design, simple] → mismatch

4. Open loop VS Closed loop : error analysis

i, O.L



plant (model) $G = 10$: $u \mapsto y$
Controller (gain) $k_p = \frac{1}{G} = \frac{1}{10}$: $r \mapsto u$
Disturbance $G_d = 0.5$: $w \mapsto d$

1. $y_{ol} = G(u - G_d w) = 10\left(\frac{r}{10} - 0.5w\right) = r - 5w$

$e_{ol} = r - y_{ol} = 5w$

$\%\text{error} = \left|\frac{e_{ol}}{r}\right| \times 100 = 500\left|\frac{w}{r}\right| [\%]$ due to disturbance

2. if, $G_{true} \neq G$ ($G_t = 11$)

$y_{ol} = G_t(u - G_d w) = 1.1r - 5.5w$

$e_{ol} = r - y_{ol} = -0.1r + 5.5w$

$\%\text{error} = 10 + 550\left|\frac{w}{r}\right| [\%]$ due to plant - model mismatch

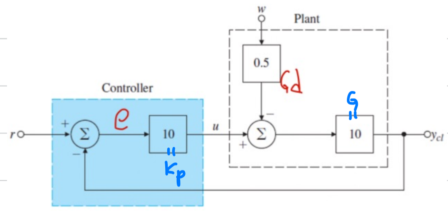
3. $u_{ol} = G^{-1} r + \hat{G}_d w$ where $\hat{G}_d = 0.55 \neq 0.5 = G_d$; model of disturbance

$y_{ol} = G(u_{ol} - G_d w) = G(G^{-1} r + \hat{G}_d w - G_d w) = r + 0.5w$

$e_{ol} = r - y_{ol} = -0.5w$

$\%\text{error} = 50\left|\frac{w}{r}\right| [\%]$

ii, C.L



plant (model) $G = 10$: $u \mapsto y$
Controller (gain) $K_p = 10$: $r-y \mapsto u$
Disturbance (model) $G_d = 0.5$: $w \mapsto d$

1, $y_{cl} = G(u_{cl} - G_d w) = 10(10(r - y_{cl}) - 0.5w) = 100r - 100y_{cl} - 5w$

$\therefore y_{cl} = \frac{100}{101} r - \frac{5}{101} w$

$e_{cl} = r - y_{cl} = \frac{1}{101} r + \frac{5}{101} w$

$\%\text{error} = \left|\frac{e_{cl}}{r}\right| \times 100 = \frac{100}{101} + \frac{500}{101}\left|\frac{w}{r}\right| [\%]$

2, $G_t \neq G$ ($G_t = 11$)

$y_{cl} = G_t(u_{cl} - G_d w) = G_t(K_p(r - y_{cl}) - G_d w) = 11(10(r - y_{cl}) - 0.5w)$

$\therefore y_{cl} = \frac{110}{111} r - \frac{5.5}{111} w$

$e_{cl} = \frac{1}{111} r + \frac{5.5}{111} w$

$\%\text{error} = \frac{100}{111} + \frac{550}{111}\left|\frac{w}{r}\right| [\%]$

3, $u_{cl} = K_p(r - y_{cl}) + \hat{G}_d w$ ($\hat{G}_d = 0.95 \neq 0.5 = G_d$)

$y_{cl} = G(u_{cl} - G_d w) = G(K_p(r-y) + \hat{G}_d w - G_d w) = 100r - 100y_{cl} + 0.5w$

$\therefore y_{cl} = \frac{100}{101} r + \frac{0.5}{101} w$

$e_{cl} = r - y_{cl} = \frac{1}{101} r - \frac{0.5}{101} w$

$\%\text{error} = \frac{100}{101} + \frac{50}{101}\left|\frac{w}{r}\right| [\%]$

2. Dynamic Model

1. Translational Motion (병진 운동)

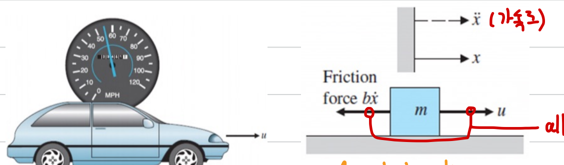
i, basic concept

$\mathbb{F} = m\vec{a}$ $\quad \mathbb{F}$ = vector sum of all forces applied to each body [N]

$\vec{a}$ = vector inertial acceleration(관성가속도) of the body [m/s^2]

m = mass [kg]

ii, example 1 : Cruise control Model

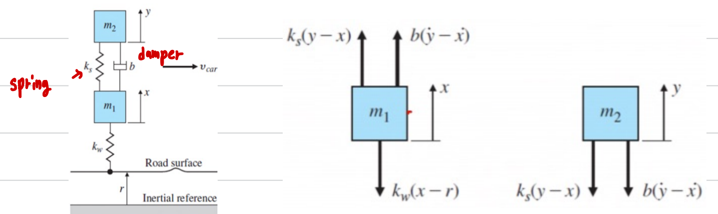


$\mathbb{F} = m\vec{a} \rightarrow u - b\dot{x} = m\ddot{x}$ or $\ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$

우리는 속도에 관심 $\rightarrow \dot{v} + \frac{b}{m}v = \frac{u}{m}$

using Laplace transform $\rightarrow sV(s) + \frac{b}{m}V(s) = \frac{1}{m}U(s) \rightarrow$ tranfer function $T(s) = \frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}$

iii, example 2 : A two mass system : suspension (자동차의 구조장치로 노면의 충격을 흡수) model



$F_b = b(\dot{y} - \dot{x})$: propotional to rate of change of the relative displacement

$F_s = k_s(y - x)$: propotional to their relative displacement

$F_w = k_w(x - r)$: 〃 distance the tire is compressed

$\mathbb{F} = m\vec{a} \Rightarrow b(\dot{y} - \dot{x}) + k_s(y - x) - k_w(x - r) = m_1\ddot{x}$

$-b(\dot{y} - \dot{x}) - k_s(y - x) = m_2\ddot{y}$

] stable 하려면 같은 변수끼리 ±가 같아야 함

2. Rotational Motion

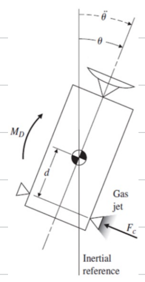
i, basic concepts

$\mathbb{F} = ma \rightarrow M = I\alpha$ $\quad$ M : sum of all external moments about the center of mass of a body.

I : body's mass moment of inertia

α : 각가속도

ii, example 1 : Satellite Attitude Control Model

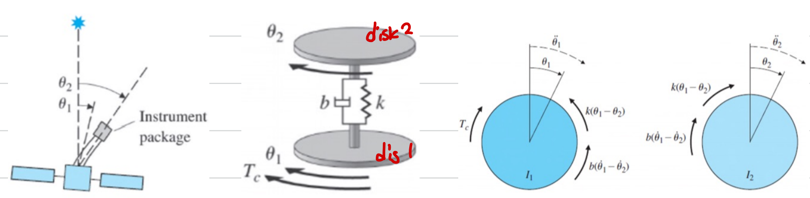


$M = I\alpha \Rightarrow F_c d + M_D = I\ddot{\theta}$

$F_c d$: control force comes from reaction jets

M_D : disturbance

iii, example 2 : flexible Satellite Attitude Control Model



difference between Suspension model

suspension : Springs and damper produce force

flexible : 〃 moments that act on each inertia

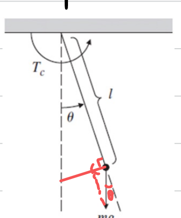
$$I_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) = T_c$$
$$I_2\ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0$$

T.F

$$\frac{\Theta_2(s)}{T_c(s)} = \frac{k}{I_1 I_2 s^2(s^2 + \frac{k}{I_1} + \frac{k}{I_2})}$$

$$\frac{\Theta_1(s)}{T_c(s)} = \frac{I_2 s^2 + k}{I_1 I_2 s^2(s^2 + \frac{k}{I_1} + \frac{k}{I_2})}$$

iv, example3 : Pendulum (진자)



$M = I\alpha \Rightarrow T_c - mgl\sin\theta = I\ddot{\theta}$, I (진자의 관성모멘트) $= ml^2$

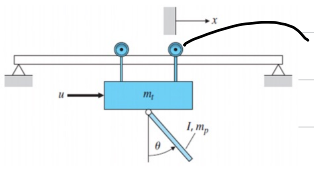
$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}$ (nonlinear)

linearization : by assuming the motion is small enough $\Rightarrow \sin\theta \simeq \theta$

$\ddot{\theta} + \frac{g}{l}\theta = \frac{T_c}{ml^2} \Rightarrow \frac{\theta(s)}{T_c(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}$

3. Combined Rotation and Translation

1, example 1 : Hanging Crane with pendulum $\theta = 0°$



trolley has mass m_t, m_p

the moment of inertia of the pendulum about the pivot point : $I + m_p l^2$

(l : distance from the pivot to the mass center of the pendulum)

$M = I\alpha$ 구하는 과정 생략 $\rightarrow$ linearized eq of motion ($\sin\theta \approx \theta$, $\cos\theta \approx 1$) if ($\theta = \pi$, inverted), $\sin\theta \approx -\theta$, $\cos\theta \approx -1$

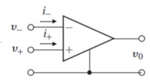
$$(I + m_p\ell^2)\ddot{\theta} + m_p g\ell\theta = -m_l\ell\ddot{x},$$ $\rightarrow$ similar to the simple pendulum

$$(m_t + m_p)\ddot{x} + b\dot{x} + m_p\ell\ddot{\theta} = u,$$ $\rightarrow$ representing trolley motion, similar to car translation

$\rightarrow$ T.F $\quad \frac{\Theta(s)}{U(s)} = \frac{-m_p\ell}{((I + m_p\ell^2)(m_t + m_p))s^2 + m_p g\ell(m_t + m_p)}$

4. op-Amp

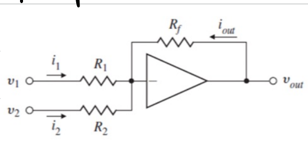
i, original Op-Amp



$i_+ = i_- = 0$, $V_+ - V_- = 0$

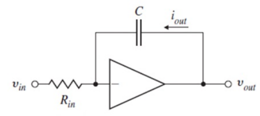
($R_i = \infty$, $R_o = 0$, $A = \infty$)

ii, Op-Amp Summer



$V_{out} = -\left[\frac{R_f}{R_1}V_1 + \frac{R_f}{R_2}V_2\right]$

iii, op-Amp Integrator



$\frac{V_{in}}{R_{in}} + C\frac{dV_{out}}{dt} = 0$

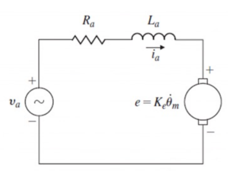
$V_{out} = -\frac{1}{R_{in}C}\int_0^t V_{in}(\tau)d\tau + V_{out}(0)$

$V_{out}(s) = -\frac{1}{s}\frac{V_{in}(s)}{R_{in}C} \Rightarrow \frac{V_{out}(s)}{V_{in}(s)} = -\frac{1}{R_{in}Cs}$

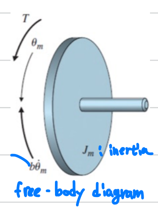
5. Electromechanical system.

i, modeling a DC Motor

motor eq : $\overset{\text{torque}}{T} = K_t i_a$, $e = K_e\dot{\theta}_m$



$L_a\frac{di_a}{dt} + R_a i_a = V_a - K_e\dot{\theta}_m$



$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a$

L.T $\Rightarrow \frac{\theta_m(s)}{V_a(s)} = \frac{K_t}{s[(J_m s + b)(L_a s + R_a) + K_t K_e]}$

인덕턴스 영향 무시 $\rightarrow$ 두식 합칠수 있음

$J_m\ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a}V_a \Rightarrow$ L.T $\Rightarrow \frac{\theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a}}{J_m s^2 + (b + \frac{K_t K_e}{R_a})s} = \frac{K}{s(\tau s + 1)}$

$K = \frac{K_t}{bR_a + K_t K_e}$, $\tau = \frac{R_a J_m}{bR_a + K_t K_e}$

ii, Gears

i, $nT_1 = T_2$, $n\theta_2 = \theta_1$ $\left[\frac{\theta_1}{\theta_2} = \frac{w_1}{w_2} = \frac{N_1}{N_2} = n\right]$ $\quad N_1 : N_2 = 1:3 \Rightarrow n = 3$

$\theta_1 = 3\theta_2$, $w_1 = 3w_2$

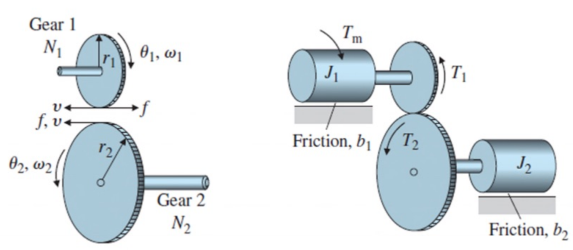
ii, eq of motion for Gear 1 : $J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 = T_m - T_1$

T_m : 입력 토크, T_1 : 기어2로인한 저항개념

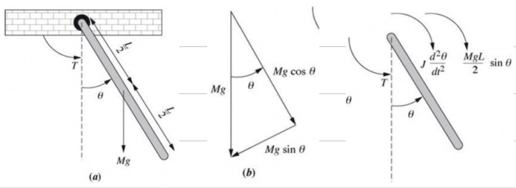
〃 Gear 2 : $J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 = T_2$

iii, Eq(1) $\times n$ + Eq(2) = $(J_2 + J_1 n^2)\ddot{\theta}_2 + (b_2 + b_1 n^2)\dot{\theta}_2 = nT_m$

$\Rightarrow$ T.F = $\frac{\theta_2}{T_m(s)} = \frac{n}{J_{eq}s^2 + b_{eq}s}$, $J_{eq} = J_2 + J_1 n^2$ and $b_{eq} = b_2 + b_1 n^2$



6. Linearization



I, $M = I\alpha \Rightarrow T + N = J\alpha$, J : moment of inertia, $N(t) = Mg\sin\theta(t) \cdot \frac{L}{2}$

$\ddot{\theta}(t) = -\frac{MgL}{2J}\sin\theta(t) + \frac{T(t)}{J}$, $x(t) = \theta(t)$

let, $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$

$\dot{x}(t) = x_2(t)$, $\dot{x}_2(t) = -\frac{MgL}{2J}\sin x_1(t) + \frac{T(t)}{J}$

First order form (State space model)

$\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = -\frac{MgL}{2J}\sin x_1(t) + \frac{T(t)}{J}$

Linearization

- Near $x = 0$: $\sin x \approxeq x$
- Near $x = \pi/2$: $\sin x \approxeq 1$
- Near $x = \pi$: $\sin x \approxeq \pi - x$

II, linear approximation

$f(x) \approx f(x_0) + \frac{d}{dx}f(x)\Big|_{x=x_0}(x - x_0)$

III, choose an operating point $\bar{x} = [\bar{x}_1, \bar{x}_2] \in \mathbb{R}^2$

- hanging pendulum : $\bar{x} = (0,0)$
- Inverted pendulum : $\bar{x} = (\pi, 0)$

using vector-valued $\dot{x} = f(x,u) = f(x(t), y(t))$, $\bar{x} = (\pi, 0)$

$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) = x_2(t)$

$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) = -\frac{MgL}{2J}\underbrace{\sin x_1(t)}_{\Rightarrow\, \pi - x_1(t)} + \frac{T(t)}{J}$

$\therefore$

$\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = \frac{MgL}{2J}x_1(t) + \frac{T(t)}{J} - \underbrace{\frac{MgL}{2J}\pi}_{\text{Constant}}$

$y(t) = x_1(t)$

IV, to eliminate the constant term, we define new variable

displacement from an Equilibrium

i, Equilibrium ($\bar{x}$) : $\dot{x} = f(\bar{x}) = 0$: once the state reaches an equilibrium point $\bar{x}$, then it stays there for all future time

$\bar{x} = (\bar{x}_1, \bar{x}_2) = (\pi, 0)$

이점을 operating point로 선정

ii, new state variable $\Delta x = x - \bar{x} \Rightarrow \Delta x_1 = x_1 - \bar{x}_1$ and $\Delta x_2 = x_2 - \bar{x}_2$

then we have

$\Delta\dot{x}_1(t) = \Delta x_2(t)$

$\Delta\dot{x}_2(t) = \frac{MgL}{2J}\Delta x_1(t) + \frac{T(t)}{J}$: No constant

Δx : angle from vertical

Δx_2 : angular velocity

3. Dynamic Response

1. Review of Laplace Transforms

Ⅰ, LTI systems (Linear Time-Invariant)

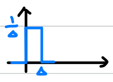
i, super position : $y = F(u)$, $y_1 = F(u_1)$ and $y_2 = F(u_2)$ $\Rightarrow$ $F(\kappa_1 u_1 + \kappa_2 u_2) = \alpha_1 y_1 + \kappa_2 y_2$

ii, Time Invariance : $u(t) \to y(t)$ $\Rightarrow$ $u(t-\tau) \to y(t-\tau)$

Ⅱ Impulse Response

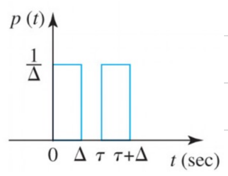
t : the time we are observing
τ = the time impulse applied
difference : elapsed time

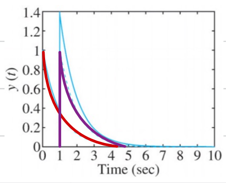
i, $p(t) = \frac{1}{\Delta}$ for $0 \le t \le \Delta$



$h_\Delta(t) = F(p_\Delta(t))$

$y(t) = \sum_{k=0}^{\infty} \Delta \cdot u(k\Delta) \cdot h_\Delta(t - k\Delta) \xRightarrow{\Delta \to 0} y(t) = \int_0^\infty h(t-\tau)u(\tau)d\tau$ ex, $y(3) = h(3-0)u(0) + h(3-1)u(1) + h(3-2) \cdot u(2) + h(3-3) \cdot u(3)$





$p(t) = p(t) + p_\Delta(t-\tau)$ $\Rightarrow$ $y(t) = \underbrace{F(p_\Delta(t))}_{h_\Delta(t)} + \underbrace{F(p_\Delta(t-\tau))}_{h_\Delta(t-\tau)}$ $t-\tau$; elapsed time

ii, delta function $\delta(t)$: $\delta(t) = 0, \forall t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t) = 1$

by definition $f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$ τ가 t의 결과에 대한 영향을

$\Rightarrow u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)\delta(\tau)d\tau$

iii, $u(t) \to y(t)$를 구하기 위해 response to a unit impulse 를 알아야함

iv, output for a general input : $y(t) = \int_{-\infty}^{\infty} u(\tau) h(t-\tau)d\tau$

$\tau_1 = t - \tau \Rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$

Ⅲ, Causality : 미래가 현재에 영향 X

$h(t) = 0$ for all $t < 0$, t : elapsed time

$y(t) = \int_0^\infty h(t-\tau)u(\tau)d\tau = \int_0^t h(t-\tau)u(\tau)d\tau$. or $\int_0^t h(\tau)u(t-\tau)d\tau$.

Ⅳ Laplace Transform

i, $h(t)$ [impulse response] , $u(t)$ [input signal] is given , procedure for determining $y(t)$

1, $H(s) = \mathcal{L}\{h(t)\}$: determine TF

2, $U(s) = \mathcal{L}\{u(t)\}$

3, $Y(s) = H(s)U(s)$

4, break up $Y(s)$ by partial fraction expansion

5, $y(t) \leftarrow \mathcal{L}^{-1}\{Y(s)\}$: ILT of the system output

ii, LT table

	$u(t)$	$\hat{u}(s)$
Step	$\mathbf{1}(t)$	$\frac{1}{s}$
Power	t^m	$\frac{m!}{s^{m+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Power Exponential	$\frac{t^{m-1}e^{-at}}{(m-1)!}$	$\frac{1}{(s+a)^m}$
Sine	$\sin(at)$	$\frac{a}{s^2+a^2}$
Cosine	$\cos(at)$	$\frac{s}{s^2+a^2}$
Impulse	$\delta(t)$	1

Ⅰ, Magnitude scailing : $Af(t) \longrightarrow AF(s)$

Ⅱ, Addition / subtraction : $f_1(t) \pm f_2(t) \longrightarrow F_1(s) \pm F_2(s)$

Ⅲ, time scaling : $f(at) \longrightarrow \frac{1}{a}F(\frac{s}{a})$

Ⅳ, time shifting : $f(t)u(t) \to F(s)$ $\Rightarrow$ $f(t-t_0)u(t-t_0) \to e^{-t_0 s}F(s)$

$f(t)u(t-t_0) \to e^{-t_0 s}\mathcal{L}[f(t+t_0)]$

Ⅴ, Frequency shifting : $e^{-at}f(t) \longrightarrow F(s+a)$

Ⅵ, Multiplication by t : $tf(t) \longrightarrow -\frac{dF(s)}{ds}$ $\left(\mathcal{L}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!} \right)$

Ⅶ, differentiation : $\frac{d^n f(t)}{dt^n} \longrightarrow s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - s^0 f^{(n-1)}(0)$

Ⅴ, final Value Theorem . stable 한 조건 먼저 확인

· If all poles of $Y(s)$ are in the left half of the s-plane

$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$

· DC gain : the constant steady state of the system output , when a unit step function is applied

$Y(s) = G(s)U(s) = G(s) \cdot \frac{1}{s}$ $\Rightarrow$ $\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} G(s)$

Ⅵ, Initial Value Theorem

· if $\lim_{s \to \infty} Y(s) = 0$, $y(0^+) = \lim_{s \to \infty} sY(s)$

Ⅶ, Inverse Laplace transform

Ⅰ, case 1 : simple poles

$\frac{P_1(s)}{Q(s)} = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \frac{k_3}{s+p_3} \cdots \frac{k_n}{s+p_n}$ $\Rightarrow$ $(s+p_i) \cdot F(s)\Big|_{s=-p_i} = 0 + \cdots + k_i + 0 \cdots$ $\Rightarrow$ k_i구할수 있음

Ⅱ, case 2 : Complex conjugate poles

$Q_1(s)$에 관한것

① $F(s) = \frac{P_1(s)}{Q_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{k_1}{s+\alpha-j\beta} + \frac{k_1^*}{s+\alpha+j\beta} + \cdots$

$\Rightarrow$ $(s+\alpha-j\beta)F(s)\Big|_{s=-\alpha+j\beta} = k_1 = |k_1|\angle\theta$ $\Rightarrow$ $F(s) = \frac{|k_1|\angle\theta}{s+\alpha-j\beta} + \frac{|k_1|\angle-\theta}{s+\alpha-j\beta} + \cdots = \frac{|k_1|e^{j\theta}}{s+\alpha-j\beta} + \frac{|k_1|e^{-j\theta}}{s+\alpha+j\beta} + \cdots$

$f(t) = \mathcal{L}^{-1}[F(s)] = |k_1|e^{j\theta}e^{-(\alpha-j\beta)t} + |k_1|e^{-j\theta}e^{-(\alpha+j\beta)t} = |k_1|e^{-\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right] + \cdots$

$f(t) = 2|k_1|e^{-\alpha t}\cos(\beta t+\theta) + \cdots$ $\left[\text{Euler's identity} : \frac{e^{j\theta}+e^{-j\theta}}{2} \right]$

Ⅲ, Cas3 : multiple poles

① $F(s) = \frac{10(s+3)}{(s+1)^3(s+2)} = \frac{k_{11}}{s+1} + \frac{k_{12}}{(s+1)^2} + \frac{k_{13}}{(s+1)^3} + \frac{k_2}{s+2}$ $\Rightarrow$ $f(t) = \left\{k_{11}e^{-t} + k_{12}te^{-t} + k_{13}\left(\frac{1}{2}t^2e^{-t}\right) + k_2e^{-2t}\right\}u(t)$

① · $k_2 = (s+2)F(s)\Big|_{s=-2} = \frac{10}{(-1)^3} = -10$ · $k_{13} = (s+1)^3F(s)\Big|_{s=-1} = 20$

· $k_{12} = \frac{d}{ds}(s+1)^3F(s)\Big|_{s=-1} = -10$ · $k_{11} = \frac{1}{2!}\frac{d^2}{ds^2}(s+1)^3F(s)\Big|_{s=-1} = 10$

$\Rightarrow$ $k_{ij} = \frac{1}{(r-j)!}\frac{d^{r-j}}{ds^{r-j}}\left[(s+p_i)^rF(s)\right]\Big|_{s=-p_i}$

② $(s+1)^3F(s) = \frac{10(s+3)}{(s+2)} = \frac{k_{11}(s+1)^2(s+2) + k_{12}(s+1)(s+2) + k_{13}(s+2) + k_2(s+1)^3}{(s+2)}$

계수비교 s^3 : $0 = k_{11} + k_2$, s^2 : $0 = 4k_{11} + k_{12} + 3k_2$, $s \to 10 = 5k_{11} + 3k_{12} + k_{13} + 3k_2$

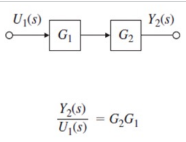
s^0 : $30 = 2k_{11} + 2k_{12} + 2k_{13} + k_2$

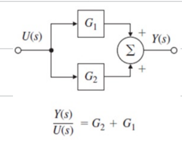
$\frac{1}{s^4+9} = \frac{as+b}{s^2-\sqrt{6}s+3} + \frac{-as+b}{s^2+\sqrt{6}s+3}$

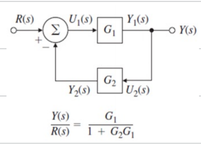
2. Systems Modeling diagrams

Ⅰ, Elementary Block diagrams

i, series ii, parallel iii, feedback

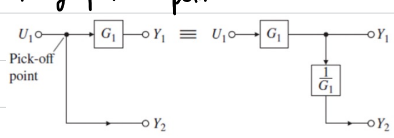


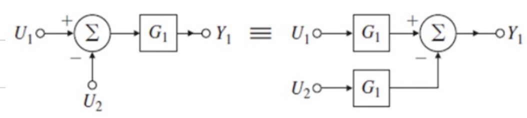




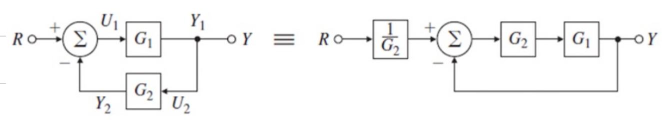
Ⅱ, 변형 for 간소화

i, moving pick off point ii, moving a summer



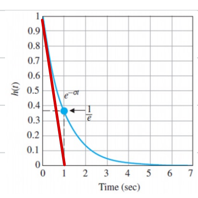


iii, conversion to unity feedback



$G_{fb} = \frac{G_1}{1+G_2G_1} = \frac{Y}{R}$ $\frac{Y}{R} = G_{fb} = \frac{1}{G_2}G_{fb}'$, $G_{fb}' = \frac{G_2G_1}{1+G_2G_1}$

3. Effect of pole Locations

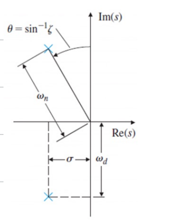


Ⅰ, tranfer function : $H(s) = \frac{1}{s+\sigma}$ $\Rightarrow$ impulse response : $h(t) = e^{-\sigma t}1(t)$

i, stability $(\sigma > 0)$: pole is located at LHP

ii, time constant $(\tau = 1/\sigma)$: rate of decay , speed of response

time when the response is $1/e$ times the initial value



Ⅱ, Pole locations and Time responses

(ω_d, σ) vs (ζ, ω_n)
Cartesian Coordinate polar form

$H(s) = \omega_n^2 / s^2 + 2\zeta\omega_n s + \omega_n^2$ $\Rightarrow$ $p = -\sigma \pm j\omega_d$ $\begin{cases} \sigma = \zeta\omega_n \\ \omega_d = \omega_n\sqrt{1-\zeta^2} \end{cases}$

ζ (damping ratio) : level of damping (진동의 감쇠) , $\zeta = 0$: no damping & $\omega_d = \omega_n$: undamped natural freq

$\omega_d \propto \frac{1}{\zeta}$ $\zeta \to 1$: 감쇠이 커짐

$H(s) = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \longrightarrow h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t)$

4. Time-Domain Specifications

Ⅰ, · requirements associated with the time response of the system
· having no finite zeros

Ⅱ, t_r (rise time) : to reach the point of its steady state
rise time from 10% ~ 90%
$t_r = \approx \frac{1.8}{\omega_n}$

Ⅲ t_p (peak time), M_p (overshoot)
$Y(s) = H(s)\frac{1}{s} \Rightarrow y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cdot \cos(\omega_d t - \beta)$, $\beta = \sin^{-1}(\zeta)$

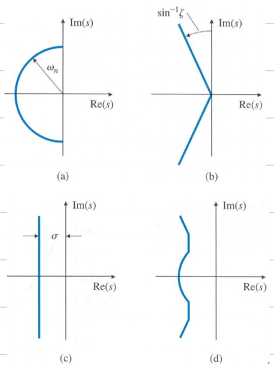
t_p : $\dot{y}(t_p) = 0 \Rightarrow t_p = \frac{\pi}{\omega_d}$
$y(t_p) = 1 + M_p = 1 - e^{-\pi\frac{\sigma}{\omega_d}}(\cos\pi + \frac{\sigma}{\omega_d}\sin\pi) \Rightarrow M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$, where $0 \le \zeta \le 1$
변동될수있음

Ⅳ, t_s (settling time) : transient to decay to a small value ⇒ y(t) is almost in S.S
· decaying exponential reaches 1%

$e^{-\zeta\omega_n t_s} \approx 0.01 \Rightarrow t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}$

Ⅴ, design Syntesis for time response Performance

$\omega_n \ge \frac{1.8}{t_r}$, $\zeta \ge \zeta(M_p) = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}}$, $\sigma \ge \frac{4.6}{t_s}$
요구사항



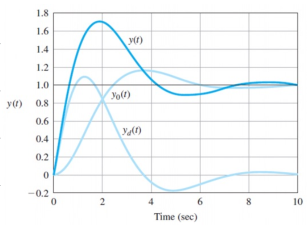
5. effects of zeros

Ⅰ, $H(s) = \frac{(s/\alpha\zeta\omega_n) + 1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$, zero located at $s = -\alpha\zeta\omega_n = -\alpha\sigma$

· if, α is large (pole ≠ zero), zero will have little effect on the response
· if, $\alpha \approx 1$ (real part of pole ≈ zero), substantial influence on the response

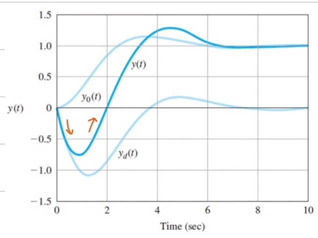
Ⅱ, Replacing s/ω_n by s, $H(s) = \frac{(s/\alpha\zeta) + 1}{s^2 + 2\zeta s + 1} = \underbrace{\frac{1}{s^2 + 2\zeta s + 1}}_{H_0(s)} + \underbrace{\frac{1}{\alpha\zeta}\frac{s}{s^2 + 2\zeta s + 1}}_{H_d(s) = \frac{1}{\alpha\zeta}sH(s)}$
⇒ time is scaled $\tau = \omega_n t$

$y(\tau) = y_0(\tau) + y_d(\tau) = y_0(\tau) + \frac{1}{\alpha\zeta}\dot{y}_0(\tau)$



for smaller ζ (pole ≈ zero) $M_p\uparrow$, $t_r\downarrow$, t_s is steady

Ⅲ, RHP zero : nonminimum-phase zero
$z = -\alpha\zeta\omega_n$, when $\alpha < 0$ zero located in the RHP



6. Stability

Ⅰ, LTI (Linear time invariant) system
· all the roots of the T·F denominator polynomial have negative real parts
unstable otherwise

Ⅱ, BIBO stability
· every bounded input results is a bounded output
· The system with impulse response h(t) is BIBO stable if and only if the integral
$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

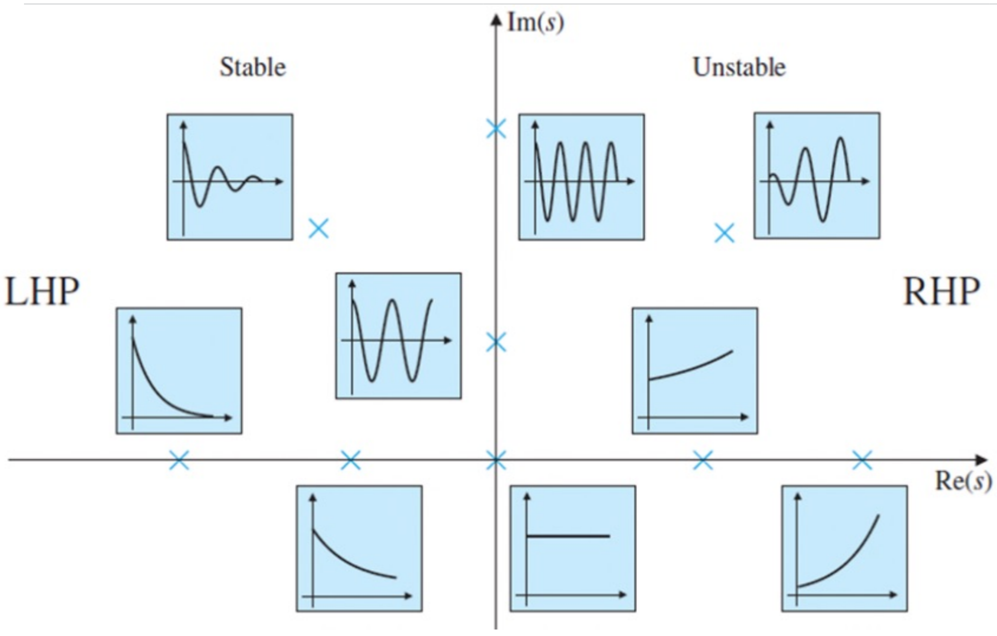
Ⅲ, Internal stability : $Re(p_i) < 0$ for all p_i

Neutral stability : · has non-repeated jω axis poles (ex, $\frac{1}{s}$, $\frac{1}{s+j\omega_0} \cdot \frac{1}{s-j\omega_0}$)
imaginery axis
· A pair of complex jω axis poles results in an oscillating response
· with repeated jω axis ⇒ unstable (ex, $\frac{1}{s^2}$)

Ⅳ Routh array : · to make certain statements about the stability of system without actually solving for the roots of polynomial
· system is stable iff (if and only if) all the elements in the first colum of Routh array are positive
· useful to determine the range of parameters for which systen remains stable

Row n	s^n :	1	a_2	a_4	$\cdots$
Row $n-1$	s^{n-1} :	a_1	a_3	a_5	$\cdots$
Row $n-2$	s^{n-2} :	b_1	b_2	b_3	$\cdots$
Row $n-3$	s^{n-3} :	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Row 2	s^2 :	$*$	$*$		
Row 1	s^1 :	$*$			
Row 0	s^0 :	$*$			

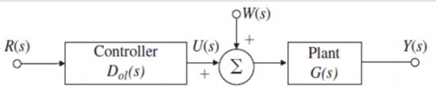
$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1a_2 - a_3}{a_1},$$
$$b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1a_4 - a_5}{a_1},$$
$$b_3 = -\frac{\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1a_6 - a_7}{a_1},$$
$$c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1a_3 - a_1b_2}{b_1},$$
$$c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1a_5 - a_1b_3}{b_1},$$
$$c_3 = -\frac{\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1a_7 - a_1b_4}{b_1}.$$



4. A First Analysis of feedback

1. Basic Equations of control

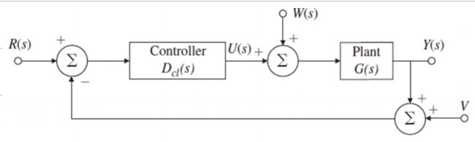
i, open-loop sys



$Y_{ol} = GD_{ol}R + GW$

$E_{ol} = R - Y_{ol} = (1 - GD_{ol})R - GW$

ii, Closed-loop sys



R : reference, output is expected to track

W : plant disturbance, control is expected to counteract

V : sensor noise, controller is supposed to ignore

$$Y_{cl} = \underbrace{\frac{GD_{cl}}{1+GD_{cl}}}_{\mathcal{T}}R + \underbrace{\frac{G}{1+GD_{cl}}}_{GS}W - \underbrace{\frac{GD_{cl}}{1+GD_{cl}}}_{\mathcal{T}}V, \quad U = \frac{D_{cl}}{1+GD_{cl}}R - \frac{GD_{cl}}{1+GD_{cl}}W - \frac{D_{cl}}{1+GD_{cl}}V$$

$$E_{cl} = R - Y_{cl} = \underbrace{\frac{1}{1+GD_{cl}}}_{S}R - \underbrace{\frac{G}{1+GD_{cl}}}_{SG}W + \underbrace{\frac{GD_{cl}}{1+GD_{cl}}}_{\mathcal{T}}V$$

S (sensitivity) $\triangleq \frac{1}{1+GD_{cl}}$

$\mathcal{T}$ (Complementary Sensitivity) $\triangleq \frac{GD_{cl}}{1+GD_{cl}}$

ii, stability : All poles must be in the left half plane

let, $G = \frac{b(s)}{a(s)}$, $D_{ol} = \frac{c(s)}{d(s)}$

- O·L sys : sys poles are the poles of GD_{ol}, $a(s)d(s) = 0$ for stable
  unstable → stable (impossible)
- C·L sys : $1 + GD_{cl} = 0 \Rightarrow a(s)d(s) + b(s)c(s)$: 조작 가능 (unstable → stable)
  more freedom to the controller design than does the open-loop case

iii, Tracking : to make the ouput to follow the reference input as closely as possible

- three limitations of the O·L case in tracking problem

신경써야 할것 (중요X)

1, controller transfer function must be proper : it cannot be given more zeros than it has poles
2, demand for fast response will demand large inputs to plant and input will saturate the system
3, high sensitivity against plant-model mismatch

iv, Regulation : to keep the error small

- O·L sys : controller has no influence at all on the either W, V, useless for regulation
- C·L sys : there is conflict between W and V

$\frac{G}{1+GD_{cl}}W$: error term for the plant disturbance

$\frac{GD_{cl}}{1+GD_{cl}}V$: 〃 sensor noise

$$\text{High-Gain } D_{cl} \Rightarrow \begin{cases} \dfrac{G}{1+GD_{cl}} \to 0 & \text{as } D_{cl} \to \infty \\ \dfrac{GD_{cl}}{1+GD_{cl}} \to 1 & \text{as } D_{cl} \to \infty \end{cases}$$

v, sensitivity

- Suppose plant is designed with gain G (nominal model), but in operation it changes to be $G + \delta G$ (true plant)
- O·L case : $T_{ol} = GD_{ol} \Rightarrow T_{ol} + \delta T_{ol} = T_{ol} + D_{ol}\delta G \Rightarrow \delta T_{ol} = D_{ol}\delta G$

$\left(\frac{\delta T_{ol}}{T_{ol}} = \frac{D_{ol}\delta G}{D_{ol}G} = \frac{\delta G}{G}\right)$ S_G^T (sensitivity) $= \frac{\delta T_{ol}/T_{ol}}{\delta G/G} = \frac{G}{T_{ol}} \cdot \frac{\delta T_{ol}}{\delta G} = 1$ (10% error in G → 10% error in T_{ol})

sensitive to the plant variation

- C·L case : $T_{cl} + \delta T_{cl} = \frac{(G+\delta G)D_{cl}}{1+(G+\delta G)D_{cl}} \approx T_{cl} + \frac{dT_{cl}}{dG}\delta G + \text{H.O.T}$

sensitivity of T_{cl} with respect to G : $S_G^T \triangleq \frac{G}{T_{cl}} \cdot \frac{dT_{cl}}{dG} = \frac{1}{1+GD_{cl}}$ (sensitivity function) : less sensitive than o·l case

$S_G = \frac{\frac{\delta T}{T}}{\frac{\delta G}{G}}$

$\mathcal{T} = 1 - S = \frac{GD_{cl}}{1+GD_{cl}}$ (complementary sensitivity function)

2. Control of e_{ss} to polynomial inputs

(다항식)

i, concepts :
- type : Classify systems according to the degree of the polynomial
- Error Constant : to quantify tracking error
- all the following analysis in this section, it is assumed that system are stable

ii, System type for tracking

- $E = \frac{1}{1+GD_{cl}}R = SR$ $\left[r(t) = \frac{t^k}{k!}1(t) \Rightarrow R = \frac{1}{s^{k+1}} \right]$

$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$ (Final Value Theorem)

$= \lim_{s\to 0} s \cdot \frac{1}{1+GD_{cl}} R(s) = \lim_{s\to 0} \frac{1}{1+GD_{cl}} \cdot \frac{1}{s^k}$

차수(n)

- the degree of pole at the origin (n) of $GD_{cl}(s)$ plays a key role in determining type and Constant

rewrite $GD_{cl}(s)$ as $GD_{cl}(s) = \frac{GD_{clo}(s)}{s^n}$ $(GD_{clo}(0) < \infty)$

$e_{ss} = \lim_{s\to 0} \frac{1}{1+\frac{GD_{clo}(s)}{s^n}} \cdot \frac{1}{s^k} = \lim_{s\to 0} \frac{s^n}{s^n + K_n} \cdot \frac{1}{s^k}$ ($K_n = \lim_{s\to 0} GD_{clo}(s) < \infty$: error constant)

i, $n > k \Rightarrow e_{ss} = 0$

ii, $n < k \Rightarrow e_{ss} = \infty$

iii, $n = k = 0 \Rightarrow e_{ss} = \frac{1}{1+k_0}$ (type 0, no pole at origin)

iv, $n = k \neq 0 \Rightarrow e_{ss} = \frac{1}{K_n}$

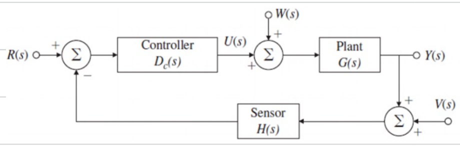
Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

iii, Robustness of system type

- for unity feedback structure, process parameter change without removing the pole at the origin then system's types remain the same
- Robustness is a major reason for preferring unity feedback over other kinds of control structure

iv, TF with sensor



$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{GD_c}{1+GD_cH}$, $E(s) = R - Y = [1 - \mathcal{T}(s)]R(s)$

$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1-\mathcal{T}(s)}{s^k}$

if, e_{ss} is a nonzero constant ⇒ Type k

ex, Tachometer Feedback

parameters : $G(s) = \frac{1}{s(Js+1)}$, $D_c(s) = k_p$, $H(s) = 1 + k_t s$

$$e_{ss} = \lim_{s\to 0} \frac{1-\mathcal{T}(s)}{s^k} = \lim_{s\to 0} \frac{1}{s^k} \cdot \frac{s(Js+1) + (1+k_t s - 1)k_p}{s(Js+1) + (1+k_t s)k_p} = \begin{cases} 0 & \text{for } k=0 \\ \dfrac{1+k_t k_p}{k_p} & \text{for } k=1 \end{cases}$$

$\left\{ K_v \text{ (Velocity constant)} = \frac{k_p}{1+k_t k_p} \right\}$

- if $k_t > 0$, K_v (velocity const) is smaller than unity feedback value of k_p ⇒ improve stability or response
- tachometer feedback is used to improve dynamic response
  trade-off between improving stability and reducing steady-state error.

3. System type for Regulation and Disturbance rejection

$\frac{E}{W} = -\frac{Y}{W} = T_w(s)$, ∵ R is zero

Rewrite $T_w(s)$ as $s^n T_{o,w}(s)$, where $T_{o,w}(0) = 1/K_{n,w}$ (pole at the origin 이 없음)

$e_{ss} = \lim_{s\to 0} sT_w(s) \cdot \frac{1}{s^{k+1}} = \lim_{s\to 0} T_{o,w}(s) \frac{s^n}{s^k}$

- $n = k$: type k, $e_{ss} = \frac{1}{K_{n,w}}$
- $n > k$: 0
- $n < k$: ∞

4. PID control

I, Proportional Control

· $u(t) = k_p e(t)$, $\frac{U(s)}{E(s)} = D_{cl}(s) = K_p$ (propotional gain) , related to the system error instantaneously

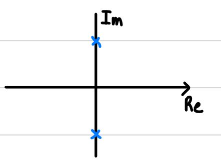
· as $K_p \uparrow \Rightarrow e_{ss} \downarrow$, damping $\downarrow$ (진동이 커짐, oscillation $\uparrow$) , disturbance 측면에서도 유사.

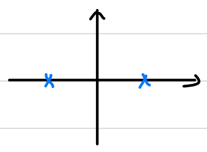
· for system of large order, increasing proportional gain will typically lead to instability for high gain

ex, plant: $G(s) = 1 / Js^2 - \frac{Mgl}{2}$, Closed loop : $\frac{GD_c}{1+GD_c} = \frac{K_p}{Js^2 - \frac{Mgl}{2} + K_p}$

inverted pendulum

Case 1, $K_p > \frac{Mgl}{2}$: unstable   Case 2, $K_p < \frac{Mgl}{2}$: unstable





$\Rightarrow$ Can't be a solution

· One way to improve e_{ss} accuracy of control without using high K_p is to use Integral control

II, Integral control

· $u(t) = K_I \int_0^t e(\tau) d\tau$, $\frac{U(s)}{E(s)} = D_{cl}(s) = \frac{K_I}{s}$

· goal : minimize e_{ss} , steady-state output response to disturbance

· $\frac{E}{R} = \frac{s}{s + K_I G(s)}$, $\frac{U}{R} = \frac{K_I}{s + K_I G(s)}$, $\frac{Y}{R} = T = \frac{K_I G(s)}{s + K_I G(s)}$ | $\frac{Y}{W} = \frac{s\,G(s)}{s + K_I G(s)}$, $\frac{U}{W} = -\frac{K_I G(s)}{s + K_I G(s)}$

for $r(t) = 1(t)$ | $W(t) = 1(t)$

$e(\infty) = 0$, $u(\infty) = G^{-1}(0) = 1$, $y(\infty) = 1$ | $y(\infty) = 0 (= -e(\infty))$, $u(\infty) = -1$

· conclusion : 1, result is zero stead state output error in both tracking and disturbance rejection

2, plant prameter changes can be tolerated ; above resultes are independent of plant parameters

3, regardless of the K_I , closed loop system remain stable (Robustness)

4, Integral control cause C·L system to become type 1, and if $r(t)$: lamp , systems do have constant tracking error

5. Integral controller alone can destabilize C·L system

III, Derivative Control

$u(t) = K_D \dot{e}(t)$, $\frac{U}{E} = D_{cl}(s) = K_D s$ (derivative gain)

· it knows the slope of the error signal , so it takes control action based on trend in error

· Euler's backward method of approximating derivatives

$y(t)$ is mesurement , but $\dot{y}(t)$ cannot be measured directly

using delayed response $\dot{y}(t) \approx \frac{y(t) - y(t - \Delta t)}{\Delta t}$, delay can cause instabilities

· it gives a sharp response to suddenly changing signals.

ex, $G(s) = 1 / Js^2 - \frac{Mgl}{2}$, C·L : $\frac{GD_c}{1+GD_c} = \frac{\frac{K_D}{J} s}{s^2 + \frac{K_D}{J} s - \frac{Mgl}{2J}}$

it is stable iff $\frac{K_D}{J} > 0$ and $-\frac{Mgl}{2J} > 0$ impossible , regardless K_p the C·L is unstable

$\Rightarrow$ D control cannot stabilize a system

IV, PI control

$u(t) = K_p e(t) + K_I \int_{t_0}^t e(\tau) d\tau$, $\frac{U}{E} = D_{cl}(s) = K_p + \frac{K_I}{s} = K_p \left(1 + \frac{1}{T_I s}\right)$

This combination generally allows for a faster response than a pure integral control alone

V, PD control

let plant $G(s) = \frac{1}{s^2 + a_1 s + a_2}$, Controller : $D_c(s) = K_p + K_D s = K_p (1 + T_D s)$

$T \cdot F = \frac{G \cdot D}{1 + G \cdot D} = \frac{K_p (1 + T_D s)}{s^2 + (a_1 + K_p T_D) s + (a_2 + K_p)}$: T_D , K_p allow us to construct any dominator we desire

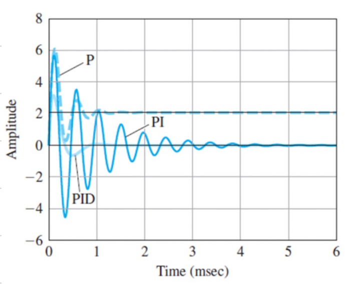
control pole placement precisely in 2nd-order ch eq.

· Although the PD controller give us control of the pole location , there is no Effect on e_{ss}

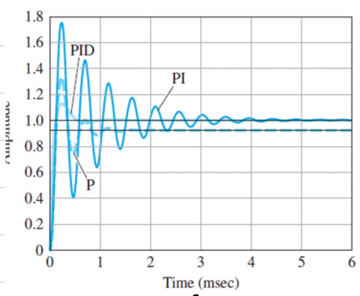
VI, PID control

$u(t) = K_p e(t) + K_I \int_{t_0}^t e(\tau) d\tau + K_D \dot{e}(t)$, $\frac{U}{E} = D_{cl}(s) = K_p + \frac{K_I}{s} + K_D s = K_p \left(1 + \frac{1}{T_I s} + T_D s\right)$

characteristic eq : $s^3 + (a_1 + AK_D) s^2 + (a_2 + AK_p) s + AK_I = 0$: by selection of these parameters , the root can be uniquely determined



Step disturbance input



Step reference input

5. The Root locus Design Method

1, Root-locus of a Basic Feedback system

I, Effect of feedback

$G(s)=\frac{n(s)}{d(s)}$, $D_c(s)=\frac{n_c(s)}{d_c(s)} \Rightarrow T=\frac{GD_c}{1+GD_c}=\frac{n(s)n_c(s)}{d(s)d_c(s)+n(s)n_c(s)}$

objective : $\hat{d}_{cl}(s)$ (denominator) $=(s-p_1)\cdots(s-p_n)$

big Qustion : how to choose $n_c(s)$ and $d_c(s)$ so that $\hat{d}_{cl}(s)=d(s)d_c(s)+n(s)n_c(s)$

we rarely need to acheive a precise set of poles. Performance specifications (stability, t_r, M_p, t_s, disturb reject) determine regions of the Complex plane

New Question: what controller ensure all roots of $d_{cl}(s)$ lie in the desired region

⇒ How does changing $n_c(s)$ and $d_c(s)$ change the roots of d_{cl}

II, Root locus (P controller) : K가 변할때 root 위치의 변화를 그린 것

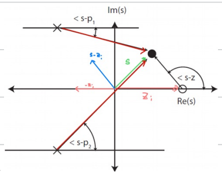
- $D_c(s)=K \Rightarrow d_{cl}(s)=d(s)+kn(s)$, $G(s)=\frac{n(s)}{d(s)}$
- Root locus of $G(s)$: all poles of $T=\frac{kG(s)}{1+kG(s)}=\frac{n(s)}{d(s)+kn(s)}$, as k ranges $0\to\infty$

$\Rightarrow \{s\in\mathbb{C} : d(s)+kn(s)=0 \text{ for } k\geq 0\}$

- $G(s)=-\frac{1}{k}=\frac{1}{k}e^{j\pi} \Rightarrow \angle G(s)=180^\circ \pm l\cdot 360^\circ$

for $G(s)=K\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$, $\angle G(s)=\sum_{i=1}^{m}\angle(s-z_i)-\sum_{i=1}^{n}(s-p_i)$

∴ For a point(s) on the root locus : $\sum_{i=1}^{m}\angle(s-z_i)-\sum_{i=1}^{n}(s-p_i)=180^\circ\pm l\cdot 360^\circ$ 만족

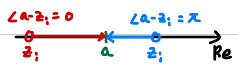


2. Guideline for determining a Root locus

I, consider Symmetry : Complex roots come in pairs : $a\pm ib$

∴ 두가지 고려
- Point on real axis
- Complex conjugate

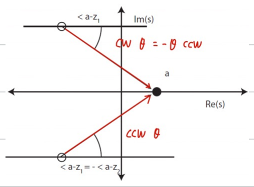
II, What is phase at a point ($s=a$) on the real axis?



i, Zeros (z_i)

if, z_i is on real axis
- $a>z_i \Rightarrow \angle(a-z_i)=0$
- $a<z_i \Rightarrow \angle(a-z_i)=180^\circ$

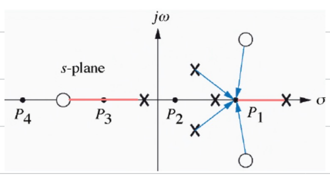
" on complex : total contribution is 0°



ii, poles (p_i)

if, p_i is on real axis
- $a>p_i \Rightarrow \angle(a-p_i)=0$
- $a<p_i \Rightarrow \angle(a-p_i)=\pi$

" on Complex : total contribution is 0°



iii, Summary : · Complex p_i, z_i of $G(s)$ don't matter, Real p_i, z_i contribute 0° or 180°

$\angle G(a)=180^\circ\cdot$ (total number of poles and zeros to the right of a)

for $\sum_{i=1}^{m}\angle(a-z_i)-\sum_{i=1}^{n}(a-p_i)=180^\circ\pm l\cdot 360^\circ$
- 총개수가 짝수 : a is off the root locus
- " 홀수 : " ON "

III, Root Locus at high gain

two cases are happen : Poles can remain small, can get big

i, consider small poles ($\|s\|<0$)

($\chi(s)$) C·L characteristic eq : $d(s)+kn(s)=0$, if, s is small → $d(s)$ is small

as $k\to\infty$, $d(s)+kn(s)\approx kn(s) \Rightarrow$ small poles satisfy $kn(s)=0$

$n(s)=0$ means s is O·L zero, ∴ small CL poles are attracted by O·L zeros.

ii, how about large poles ($\|s\|>0$)

assume $n>m$, large solutions of $1+kG(s)=0 \Rightarrow G(s)=\frac{n(s)}{d(s)}=-\frac{1}{k}$ are called asymtotics

asymptotics increase with k $\Rightarrow \lim_{k\to\infty}\|s\|=\infty$

Recall that a point is on the root locus if, $\angle G(s)=\sum_{i=1}^{m}\angle(s-z_i)-\sum_{i=1}^{n}\angle(s-p_i)=180^\circ$

however, when $\|s\|\to\infty$, All angles are the same

∴ $\angle_\infty\cdot(m-n)=180^\circ \Rightarrow \angle_\infty=\frac{1}{n-m}(180^\circ\pm 360^\circ l)$, $l=0,1,2,\cdots$

cf, $\alpha=\frac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}$ ($L(s)$의 pole and zero)

- $n-m=0$: No Asymptotes
- $n-m=1$: Asymptote at 180°
- $n-m=2$: Asymptotes at ±90°
- $n-m=3$: Asymptotes at 180°, ±60°
- $n-m=4$: Asymptotes at ±45°, ±135°

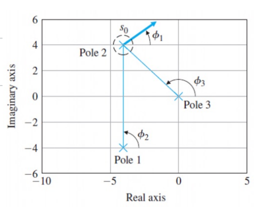
iii, Rule for departure angles

$\phi_{dep}=\sum\psi_i-\sum_{i\neq dep}\phi_i-180^\circ$
- $\sum_{i\neq dep}\phi_i$ = sum of angles to remaining poles ($\phi_i=\angle(s-p_i)$)
- $\sum_{i=1}^{m}\psi_i$ = sum of angles to all the zeros ($\psi_i=\angle(s-z_i)$)

for positive real axis

for repeated poles with multiplicity : $G(s)=\frac{1}{(s-p_i)^q}G'(s)$

$q\phi_{l,dep}=\sum\psi_i-\sum_{i\neq l,dep}\phi_i-180-360(l-1)$ $\{l=1,2,\cdots,q\}$: if there are q repeated poles, will be q branches



$\phi_{2,dep}=-\angle(s_0-p_1)-\angle(s_0-p_3)-180^\circ$

iv, Rule for Arrival angles

$\psi_{arr}=\sum\phi_i-\sum_{i\neq arr}\psi_i+180^\circ$
- $\sum_{i=1}^{n}\phi_i$ = sum of angles to all the poles
- $\sum_{i\neq arr}\psi_i$ = sum of angles to remaining zeros

for repeated zeros : $G(s)=\frac{(s-z_i)}{1}G'(s) \Rightarrow q\psi_{l,arr}=\sum\phi_i-\sum_{i\neq l,arr}\psi_i+180+360(l-1)$ $\{l=1,2,\cdots q\}$

IV, Multiple Root at point

- locus can have multiple roots at points on the locus of multiplicity q
- branches will approach a point with the same seperation

departure angle : $\frac{180^\circ+360(l-1)}{q}$

3. Design using dynamic Compensation

I, lead compensation :
- approximates the function of PD control
- $D_c(s)=K\frac{s+z}{s+p}$ ($z<p$)
- acts to speed up ($t_r\downarrow$), decreasing the transient overshoot ($M_p\downarrow$)

- General rule : 1, z is placed in the neighborhood of closed pole ω_n ($z\approx\omega_n$) as determined by t_r or t_s req,

2, p is located at a distance 5 to 25 times the value of the zero location

3. But there are trade-off to consider