Assignment 13 Papoulis ex 11.8

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Outline

Question

Show that if X (t) is WSS and

$$X_T(\omega) = \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)e^{-j\omega t}dt$$

Then

$$E\left\{\frac{\partial}{\partial T}\left|X_T(\omega)\right|^2\right\} = \int_{-T}^{T} R_\tau(\tau)e^{-j\omega\tau}d\tau$$

Solution

$$X_T(\omega) = \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t_1) e^{-j\omega t_1} dt_1 \tag{2.1}$$

$$X_T^*(\omega) = \int_{\frac{-T}{2}}^{\frac{T}{2}} x^*(t_2) e^{-j\omega t_2} dt_2 \tag{2.2}$$

$$|X_T(\omega)|^2 = \int_{\frac{-T}{2}}^{\frac{T}{2}} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t_1) x^*(t_2) e^{-j\omega(t_1 - t_2)} dt_1 dt_2 \tag{2.3}$$

$$E(|X_T(\omega)|^2) = \int_{\frac{-T}{2}}^{\frac{T}{2}} \int_{\frac{-T}{2}}^{\frac{T}{2}} E(x(t_1) x^*(t_2)) e^{-j\omega(t_1 - t_2)} dt_1 dt_2 \tag{2.4}$$

$$= \int_{\frac{-T}{2}}^{\frac{T}{2}} \int_{\frac{-T}{2}}^{\frac{T}{2}} R(t_1, t_2) e^{-j\omega(t_1 - t_2)} dt_1 dt_2 \tag{2.5}$$

Let $\tau = t_1 - t_2$

$$E(|X_T(\omega)|^2) = \int_{-T}^{T} (T - |\tau|) R(\tau) e^{-j\omega\tau} d\tau \tag{2.6}$$

Differentiating w.r.t. T

$$\frac{\partial}{\partial T} E(|X_T(\omega)|^2) = (T - |T|) R(T) e^{-j\omega T} - (T - |-T|) R(-T) e^{j\omega T} + \int_{-T}^{T} R(\tau) e^{-j\omega\tau} d\tau \tag{2.7}$$

$$\frac{\partial}{\partial T} E(|X_T(\omega)|^2) = 0 - 0 + \int_{-T}^{T} R(\tau) e^{-j\omega\tau} d\tau = \int_{-T}^{T} R(\tau) e^{-j\omega\tau} d\tau \tag{2.8}$$

$$\implies E\left(\frac{\partial}{\partial T} |X_T(\omega)|^2\right) = \int_{-T}^{T} R(\tau) e^{-j\omega\tau} d\tau \tag{2.9}$$