AI1110
Assignment 7

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Outline

Papoulis Exercise 4–35

Poisson Theorem

If $n \to \infty$ and $p \to 0$ such that $np \to \lambda$ then

$$\frac{n!}{k!(n-k)!} p^k q^{n-k} \xrightarrow[n\to\infty]{} e^{-\lambda}\frac{\lambda^k}{k!} \quad k = 0, 1, 2, \ldots \tag{1}$$

Reasoning as in (1), show that, if

$$k_1 + k_2 + k_3 = n \qquad p_1 + p_2 + p_3 = 1 \qquad k_1 p_1 \ll 1 \qquad k_2 p_2 \ll 1 \tag{2}$$

then

$$\frac{n!}{k_1!k_2!k_3!} \simeq \frac{n^{k_1+k_2}}{k_1!k_2!} \qquad p_3^{k_3} \simeq e^{-n(p_1+p_2)} \tag{3}$$

Proving the first part

From (2), we have $k_3 = n - k_1 - k_2$
Thus,

$$\frac{n!}{k_1!k_2!k_3!} = \frac{n!}{k_1!k_2!(n-k_1-k_2)!} \tag{4}$$

$$= \frac{n(n-1)\cdots(n-k_1-k_2+1)}{k_1!k_2!} \tag{5}$$

$$= \frac{n(n-1)\cdots(n-k_1-k_2+1)}{n^{k_1+k_2}} \frac{n^{k_1+k_2}}{k_1!k_2!} \tag{6}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k_1+k_2-1}{n}\right)\frac{n^{k_1+k_2}}{k_1!k_2!} \tag{7}$$

$$= \left(\prod_{m=0}^{k_1+k_2-1}\left(1-\frac{m}{n}\right)\right)\frac{n^{k_1+k_2}}{k_1!k_2!} \tag{8}$$

Now, if we assume that $k_1 + k_2$ is finite, i.e.,

$$\text{as } n \to \infty, k_1 + k_2 \ll n \tag{9}$$

Then the finite product

$$\prod_{m=0}^{k_1+k_2-1} \left(1 - \frac{m}{n}\right) \tag{10}$$

tends to unity as $n \to \infty$

Therefore,

$$\frac{n!}{k_1!k_2!k_3!} \simeq \frac{n^{k_1+k_2}}{k_1!k_2!} \quad \square \tag{11}$$

Proving the second part

From (2), we have $p_3 = 1 - p_1 - p_2$

$$p_3^{k_3} = (1 - p_1 - p_2)^{k_3} \tag{12}$$

We have already assumed that $k_1 \ll n$ and $k_2 \ll n$ in (9)
Assume $p_1 \ll 1$ and $p_2 \ll 1$ satisfying $k_1 p_1 \ll 1$, $k_2 p_2 \ll 1$

$$\therefore p_1 + p_2 \ll 1 \tag{13}$$

By the definition of e^x,

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \tag{14}$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{x}{n}\right)^k \tag{15}$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n(n-1)\cdots(n-k+1)}{n^k}\left(\frac{x^k}{k!}\right) \tag{16}$$

$$\tag{17}$$

$$e^x = \sum_{k=0}^{\infty}\left(\left(\frac{x^k}{k!}\right)\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right) \tag{18}$$

$$= \sum_{k=0}^{\infty}\frac{x^k}{k!} \tag{19}$$

$$= 1 + x + \frac{x^2}{2!} + \cdots \tag{20}$$

$$\implies e^{-x} = 1 - x + \frac{x^2}{2!} + \cdots \tag{21}$$

For $x \ll 1$, the higher-order terms can be neglected

$$e^{-x} \simeq 1 - x \tag{22}$$

$$\implies 1 - p_1 - p_2 \simeq e^{-(p_1+p_2)} \tag{23}$$

$$\implies (1 - p_1 - p_2)^{k_3} \simeq e^{-k_3(p_1+p_2)} \tag{24}$$

$$\therefore p_3^{k_3} \simeq e^{-k_3(p_1+p_2)} \tag{25}$$

Also, $k_1 + k_2 \ll n \implies k_3 \simeq n$ since $k_1 + k_2 + k_3 = n$

Therefore,

$$p_3^{k_3} \simeq e^{-n(p_1+p_2)} \quad \square \tag{26}$$

Follow-up

Random Poisson points in non-overlapping intervals

Use these results to justify, for non-overlapping intervals t_a and t_b,

$$\Pr\left(k_a \text{ in } t_a, k_b \text{ in } t_b\right) = e^{-\lambda t_a} \frac{\left(\lambda t_a\right)^{k_a}}{k_a!} e^{-\lambda t_b} \frac{\left(\lambda t_b\right)^{k_b}}{k_b!} \tag{27}$$

where $\Pr\left(k \text{ in } t\right)$ denotes the probability that k of n randomly placed points in the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$ will lie in an interval of length t and $\lambda = \frac{n}{T}$ is constant

Justification

Let $k_1 = k_a$, $k_2 = k_b$ and $k_3 = n - k_a - k_b$ denote the number of points lying in t_a, t_b and outside both t_a and t_b respectively out of a total of n randomly placed points in $\left(-\frac{T}{2}, \frac{T}{2}\right)$

$$k_1 + k_2 + k_3 = n \tag{28}$$

Let p_1, p_2 and p_3 denote the probabilities that an arbitrary point lies in t_a, t_b and outside both t_a and t_b respectively

$$p_1 + p_2 + p_3 = 1 \tag{29}$$

Assuming small intervals, i.e., $t_a, t_b \ll T$

$$k_1 p_1 \ll 1 \qquad k_2 p_2 \ll 1 \tag{30}$$

The conditions have thus been met and we can now use the previously proven results

$$\Pr\left(k_a \text{ in } t_a, k_b \text{ in } t_b\right) = \frac{n!}{k_a!k_b!(n-k_a-k_b)!} p_1^{k_a} p_2^{k_b} p_3^{n-k_a-k_b} \tag{31}$$

$$= \frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \tag{32}$$

$$\simeq \frac{n^{k_1+k_2}}{k_1!k_2!} p_1^{k_1} p_2^{k_2} e^{-n(p_1+p_2)} \tag{33}$$

$$= e^{-np_1} \frac{(np_1)^{k_1}}{k_1!} e^{-np_2} \frac{(np_2)^{k_2}}{k_2!} \tag{34}$$

Now, $p_1 = \frac{t_a}{T} \implies np_1 = \frac{n}{T} t_a = \lambda t_a$
Similarly, $np_2 = \lambda t_b$
Substituting back, we get the desired result.