Assignment 8 Papoulis ex 6.1

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Question

x and y are independent, identically distributed (i.i.d) random variables with common p.d.f

$$f_x(x) = e^{-x} U(x) \qquad f_y(y) = e^{-y} U(y)$$

Find the p.d.f of the following random variables (a) x+y, (b) x-y, (c) xy, (d) x/y, (e) min (x,y), (f) max (x,y), (g) min (x,y) / max (x,y)

Solution 6(a)

Define

$$Z = X + Y \tag{2.1}$$

Note that both X and Y are positive random variables hence

$$f_Z(z) = \int_0^z f_{XY}(z-y, y)\,dy = \int_0^z e^{-(z-y+y)}dy \tag{2.2}$$

$$= ze^{-z}U(z) \tag{2.3}$$

Solution 6(b)

$$Z = X - Y \tag{3.1}$$

Z ranges over the entire real axis for the random variables X and Y

$$F_Z(z) = \begin{cases} \int_0^\infty \int_0^{z+y} f_{XY}(x,y)\,dx\ dy, & z > 0 \\ \int_{-z}^\infty \int_0^{z+y} f_{XY}(x,y)\,dx\ dy, & z < 0 \end{cases} \tag{3.2}$$

Differrentiation gives

$$f_Z(z) = \begin{cases} \int_0^\infty f_{XY}(z+y,y)\,dy, & z > 0 \\ \int_{-z}^\infty f_{XY}(z+y,y)\,dy, & z < 0 \end{cases} \tag{3.3}$$

$$f_Z(z) = \begin{cases} \int_0^\infty e^{-(z+y+y)} dy = e^{-z} \int_0^\infty e^{-2y} dy = \frac{1}{2} e^{-z}, & z > 0 \\ \int_{-z}^\infty e^{-(z+y+y)} dy = e^{-z} \int_{-z}^\infty e^{-2y} dy = \frac{1}{2} e^{z}, & z < 0 \end{cases} \tag{3.4}$$

$$f_Z(z) = \frac{1}{2} e^{-|z|}, \qquad -\infty \leq z \leq \infty. \tag{3.5}$$

Solution 6(c)

$$Z = XY \tag{4.1}$$

$$F_Z(z) = P\{Z \leq z\} = P\{XY \leq z\} \tag{4.2}$$

$$= \int_0^{\infty} \int_0^{z/y} f_{XY}(x, y)\, dx \ dy \tag{4.3}$$

$$f_Z(z) = \int_0^{\infty} \frac{1}{y} f_{XY}\left(\frac{z}{y}, y\right) dy = \int_0^{\infty} \frac{1}{y} e^{-((z/y)+y)} dy \tag{4.4}$$

Solution 6(d)

$$Z = X/Y \tag{5.1}$$

$$F_Z(z) = P\{Z \leq z\} = P\left\{\frac{X}{Y} \leq z\right\} \tag{5.2}$$

$$= \int_0^\infty \int_0^{yz} f_{XY}(x, y)\, dx\ dy \tag{5.3}$$

$$f_Z(z) = \int_0^\infty y f_{XY}(yz, y)\, dy = \int_0^\infty y e^{y(z+1)} dy = \int_0^\infty y e^{(1+z)y} dy \tag{5.4}$$

$$= \left[y \frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty + \left(\frac{1}{1+z}\right) \int_0^\infty e^{(1+z)y} dy \tag{5.5}$$

$$= \left(\frac{1}{1+z}\right) \left[\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty = \frac{1}{(1+z)^2} U(z) \tag{5.6}$$

Solution 6(e)

$$Z = \min(X, Y) \tag{6.1}$$
$$F_Z(z) = P\{\min(X, Y) \le z\} \tag{6.2}$$
$$= 1 - P\{X > z, Y > z\} \tag{6.3}$$
$$= 1 - [1 - F_X(z)][1 - F_Y(z)] \tag{6.4}$$
$$= F_X(z) + F_Y(z) - F_X(z) F_Y(z) \tag{6.5}$$
$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z) f_Y(z) - f_X(z) F_Y(z). \tag{6.6}$$

We have

$$f_X(z) = f_Y(z) = e^{-z} U(z) \tag{6.7}$$

so that

$$F_X(z) = \int_0^\infty e^{-x} dx = \left(1 - e^{-z}\right) U(z) = F_Y(z) \tag{6.8}$$

$$\begin{aligned} f_Z(z) &= \left[e^{-z} + e^{-z} - 2\left(1 - e^{-z}\right) e^{-z}\right] U(z) && (6.9) \\ &= 2e^{-z}\left[1 - 1 + e^{-z}\right] U(z) && (6.10) \\ &= 2e^{-2z} U(z) && (6.11) \end{aligned}$$

Solution 6(f)

$$Z = \max(X, Y) \tag{7.1}$$

$$F_Z(z) = P\{\max(X, Y) \le z\} = P\{X \le z, Y \le z\} \tag{7.2}$$

$$= P\{X \le z\} P\{Y \le z\} = F_X(z) F_Y(z) \tag{7.3}$$

$$f_z(z) = F_X(z) f_Y(z) + f_X(z) F_Y(z) \tag{7.4}$$

$$= e^{-z}\left(1 - e^{-z}\right) + e^{-z}\left(1 - e^{-z}\right) \tag{7.5}$$

$$= 2e^{-z}\left(1 - e^{-z}\right) U(z) \tag{7.6}$$

Solution 6(g)

$$Z = \frac{\min(X, Y)}{\max(X, Y)}, \quad 0 < z < 1 \tag{8.1}$$

$$F_Z(z) = P\left\{\left(\frac{\min(X, Y)}{\max(X, Y)} \le z\right)((X \le Y) + (X > Y))\right\} \tag{8.2}$$

$$= P\left\{\left(\frac{\min(X, Y)}{\max(X, Y)} \le z\right)(X \le Y)\right\} + P\left\{\left(\frac{\min(X, Y)}{\max(X, Y)} \le z\right)(X > Y)\right\} \tag{8.3}$$

$$= P\left\{\frac{X}{Y} \le z, X \le Y\right\} + P\left\{\frac{X}{Y} \le z, X > Y\right\} \tag{8.4}$$

$$= P\{X \le Yz, X \le Y\} + P\{Y \le Xz, X > Y\} \tag{8.5}$$

$$= \int_0^\infty \int_0^{yz} f_{XY}(x, y)\, dx\ dy + \int_0^\infty \int_0^{xz} f_{XY}(x, y)\, dy\ dx \tag{8.6}$$

$$f_Z(z) = \int_0^\infty y f_{XY}(yz, y)\, dy + \int_0^\infty x f_{XY}(x, xz)\, dx \tag{8.7}$$

$$= \int_0^\infty y f_{XY}(yz, y)\, dy + \int_0^\infty y f_{XY}(y, yz)\, dy \tag{8.8}$$

$$= \int_0^\infty y\left(e^{-(yz+y)} + e^{-(y+yz)}\right) dy \tag{8.9}$$

$$= 2\int_0^\infty y e^{-y(1+z)} dz = \begin{cases} \frac{2}{(1+z)^2}, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{8.10}$$