

Assignment 6 12th Class

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Question

x and y are independent, identically distributed (i.i.d) random variables with common p.d.f

$$f_x(x) = e^{-x} U(x) \quad f_y(y) = e^{-y} U(y)$$

Find the p.d.f of the following random variables (a) $x+y$, (b) $x-y$, (c) xy , (d) x/y , (e) $\min(x,y)$, (f) $\max(x,y)$, (g) $\min(x,y) / \max(x,y)$

Solution 6(a)

Define

$$Z = X + Y \quad (2.1)$$

Note that both X and Y are positive random variables hence

$$f_Z(z) = \int_0^z f_{XY}(z-y, y) dy = \int_0^z e^{-(z-y+y)} dy \quad (2.2)$$

$$= ze^{-z} U(z) \quad (2.3)$$

Solution 6(b)

$$Z = X - Y \quad (3.1)$$

Z ranges over the entire real axis for the random variables X and Y

$$F_Z(z) = \begin{cases} \int_0^\infty \int_0^{z+y} f_{XY}(x, y) dx dy, & z > 0 \\ \int_{-z}^\infty \int_0^{z+y} f_{XY}(x, y) dx dy, & z < 0 \end{cases} \quad (3.2)$$

Differentiation gives

$$f_Z(z) = \begin{cases} \int_0^\infty f_{XY}(z+y, y) dy, & z > 0 \\ \int_{-z}^\infty f_{XY}(z+y, y) dy, & z < 0 \end{cases} \quad (3.3)$$

$$f_Z(z) = \begin{cases} \int_0^\infty e^{-(z+y+y)} dy = e^{-z} \int_0^\infty e^{-2y} dy = \frac{1}{2}e^{-z}, & z > 0 \\ \int_{-z}^\infty e^{-(z+y+y)} dy = e^{-z} \int_{-z}^\infty e^{-2y} dy = \frac{1}{2}e^z, & z < 0 \end{cases} \quad (3.4)$$

$$f_Z(z) = \frac{1}{2}e^{-|z|}, \quad -\infty \leq z \leq \infty \quad (3.5)$$

Solution 6(c)

$$Z = XY \quad (4.1)$$

$$F_Z(z) = P\{Z \leq z\} = P\{XY \leq z\} \quad (4.2)$$

$$= \int_0^\infty \int_0^{z/y} f_{XY}(x, y) dx dy \quad (4.3)$$

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}\left(\frac{z}{y}, y\right) dy = \int_0^\infty \frac{1}{y} e^{-((z/y)+y)} dy \quad (4.4)$$

Solution 6(d)

$$Z = X/Y \quad (5.1)$$

$$F_Z(z) = P\{Z \leq z\} = P\left\{\frac{X}{Y} \leq z\right\} \quad (5.2)$$

$$= \int_0^\infty \int_0^{yz} f_{XY}(x, y) dx dy \quad (5.3)$$

$$f_Z(z) = \int_0^\infty y f_{XY}(yz, y) dy = \int_0^\infty y e^{y(z+1)} dy = \int_0^\infty y e^{(1+z)y} dy \quad (5.4)$$

$$= \left[y \frac{e^{-(1+z)y}}{-(1+z)} \right]_0^\infty + \left(\frac{1}{1+z} \right) \int_0^\infty e^{(1+z)y} dy \quad (5.5)$$

$$= \left(\frac{1}{1+z} \right) \left[\frac{e^{-(1+z)y}}{-(1+z)} \right]_0^\infty = \frac{1}{(1+z)^2} U(z) \quad (5.6)$$

Solution 6(e)

$$Z = \min(X, Y) \quad (6.1)$$

$$F_Z(z) = P\{\min(X, Y) \leq z\} \quad (6.2)$$

$$= 1 - P\{X > z, Y > z\} \quad (6.3)$$

$$= 1 - [1 - F_X(z)] [1 - F_Y(z)] \quad (6.4)$$

$$= F_X(z) + F_Y(z) - F_X(z) F_Y(z) \quad (6.5)$$

$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z) f_Y(z) - f_X(z) F_Y(z). \quad (6.6)$$

We have

$$f_X(z) = f_Y(z) = e^{-z} U(z) \quad (6.7)$$

so that

$$F_X(z) = \int_0^{\infty} e^{-x} dx = (1 - e^{-z}) U(z) = F_Y(z) \quad (6.8)$$

$$f_Z(z) = [e^{-z} + e^{-z} - 2(1 - e^{-z})e^{-z}] U(z) \quad (6.9)$$

$$= 2e^{-z} [1 - 1 + e^{-z}] U(z) \quad (6.10)$$

$$= 2e^{-2z} U(z) \quad (6.11)$$

Solution 6(f)

$$Z = \max(X, Y) \quad (7.1)$$

$$F_Z(z) = P\{\max(X, Y) \leq z\} = P\{X \leq z, Y \leq z\} \quad (7.2)$$

$$= P\{X \leq z\} P\{Y \leq z\} = F_X(z) F_Y(z) \quad (7.3)$$

$$f_Z(z) = F_X(z) f_Y(z) + f_X(z) F_Y(z) \quad (7.4)$$

$$= e^{-z} (1 - e^{-z}) + e^{-z} (1 - e^{-z}) \quad (7.5)$$

$$= 2e^{-z} (1 - e^{-z}) U(z) \quad (7.6)$$

Solution 6(g)

$$Z = \frac{\min(X, Y)}{\max(X, Y)}, \quad 0 < z < 1 \quad (8.1)$$

$$F_Z(z) = P \left\{ \left(\frac{\min(X, Y)}{\max(X, Y)} \leq z \right) ((X \leq Y) + (X > Y)) \right\} \quad (8.2)$$

$$= P \left\{ \left(\frac{\min(X, Y)}{\max(X, Y)} \leq z \right) (X \leq Y) \right\} + P \left\{ \left(\frac{\min(X, Y)}{\max(X, Y)} \leq z \right) (X > Y) \right\} \quad (8.3)$$

$$= P \left\{ \frac{X}{Y} \leq z, X \leq Y \right\} + P \left\{ \frac{X}{Y} \leq z, X > Y \right\} \quad (8.4)$$

$$= P \{ X \leq Yz, X \leq Y \} + P \{ Y \leq Xz, X > Y \} \quad (8.5)$$

$$= \int_0^\infty \int_0^{yz} f_{XY}(x, y) dx dy + \int_0^\infty \int_0^{xz} f_{XY}(x, y) dy dx \quad (8.6)$$

$$f_Z(z) = \int_0^\infty y f_{XY}(yz, y) dy + \int_0^\infty x f_{XY}(x, xz) dx \quad (8.7)$$

$$= \int_0^\infty y f_{XY}(yz, y) dy + \int_0^\infty y f_{XY}(y, yz) dy \quad (8.8)$$

$$= \int_0^\infty y \left(e^{-(yz+y)} + e^{-(y+yz)} \right) dy \quad (8.9)$$

$$= 2 \int_0^\infty y e^{-y(1+z)} dz = \begin{cases} \frac{2}{(1+z)^2}, & 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad (8.10)$$