Assignment 9 Papoulis ex 6.63

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Outline

Question

For any two random variables x and y, let
$\sigma_x^2 = Var\{x\}, \sigma_y^2 = Var\{y\}\ and \sigma_{x+y}^2 = Var\{x+y\}$
(a) show that

$$\frac{\sigma_{x+y}}{\sigma_x + \sigma_y} \leq 1$$

(b)More generally, show that for p $\geq$ 1

$$\frac{\{E(|x+y|^p)\}^{1/p}}{\{E(|x|^p)\}^{1/p} + \{E(|y|^p)\}^{1/p}} \leq 1$$

Solution (a)

For any two two random variables X and Y we have

$$\begin{aligned}\sigma_{X+Y} = Var(X+Y) &= E\left[\{(X-\mu_X)+(Y-\mu_Y)\}^2\right] \\ &= Var(X)+Var(Y)+2Cov(X,Y) = \sigma_X^2+\sigma_Y^2+2\sigma_X\sigma_Y\rho_{XY} \\ &\leq (\sigma_X+\sigma_Y)^2\end{aligned}$$

Since $|\rho_{XY}| \leq 1$.Thus

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y,$$

and hence it easily follows that

$$\frac{\sigma_{X+Y}}{\sigma_X+\sigma_Y} \leq 1.$$

Solution (b)

We shall prove this result in three parts by making use of Holder's inequality.
(i) **Holder's inequality:** The function $\log x$ is concave, for $0 < \alpha < 1$, and hence we have

$$\log\left[\alpha x_1 + (1-\alpha)x_2\right] \geq \alpha \log x_1 + (1-\alpha)\log x_2$$

or

$$x_1^{\alpha} x_2^{1-\alpha} \leq \alpha x_1 + (1-\alpha)x_2, \qquad 0 < \alpha < 1 \tag{3.1}$$

Let

$$x_1 = |x|^p, \quad \alpha = \frac{1}{p}, \text{so that } \ 1-\alpha = 1 - \frac{1}{p} \triangleq \frac{1}{q}, \quad x_2 = |y|^q \tag{3.2}$$

so that (3.1) becomes

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}, p > 1, \tag{3.3}$$

the Holder's inequality. From (3.2), note that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1 \tag{3.4}$$

(ii) Define

$$x = X\left(E\left\{|X|^p\right\}\right)^{-1/p}, \qquad y = Y\left(E\left\{|Y|^q\right\}\right)^{-1/q}$$

where p and q are as in (3.4). Substituting these into the Holder's inequality in (3.3), we get

$$\begin{aligned} |XY| \leq p^{-1}|X|^p\left(E\left\{|X|^p\right\}\right)^{1/p-1}\left(E\left\{|Y|^q\right\}\right)^{1/q} \\ +q^{-1}|Y|^q\left(E\left\{|Y|^q\right\}\right)^{1/q-1}\left(E\left\{|X|^p\right\}\right)^{1/p} \end{aligned} \tag{3.5}$$

Taking expected values on both sides of (3.5), we get

$$E\{|XY|\} \leq (E\{|X|^p\})^{1/p}(E\{|Y|^q\})^{1/q} \tag{3.6}$$

which represents the generalization of the Cauchy-Schwarz inequality. (Note p=q=2 corresponds to Cauchy-Schwarz inequality) (iii) To prove the desired inequality, notice that

$$\begin{aligned}|X+Y|^p &= |X+Y||X+Y|^{p-1}\\ &\leq |X||X+Y|^{p-1} + |Y||X+Y|^{p-1}, \quad p>1\end{aligned}$$

and taking the expected values on both sides we get

$$E\{|X+Y|^p\} \leq E\left\{|X||X+Y|^{p-1}\right\} + E\left\{|Y||X+Y|^{p-1}\right\} \tag{3.7}$$

Applying (3.6) to each term on the right side of (3.7) we get

$$E\left\{|X||X+Y|^{p-1}\right\} \leq (E\{|X|^p\})^{1/p}\left(E\left\{|X+Y|^{(p-1)q}\right\}\right)^{1/q} \tag{3.8}$$

and

$$E\left\{|Y||X+Y|^{p-1}\right\} \leq \left(E\left\{|Y|^{p}\right\}\right)^{1/p}\left(E\left\{|X+Y|^{(p-1)q}\right\}\right)^{1/q} \qquad (3.9)$$

Using (3.8) and (3.9) together with $(p-1)q = p$ in (3.7) we get

$$E\left\{|X+Y|^{p}\right\} \leq \left[\left(E\left\{|X|^{p}\right\}\right)^{1/p} + \left(E\left\{|Y|^{p}\right\}\right)^{1/p}\right] . \left(E\left\{|X+Y|^{(p-1)q}\right\}\right)^{1/q}$$

or for $p > 1$

$$\left(E\left\{|X+Y|^{p}\right\}\right)^{1/p} \leq \left(E\left\{|X|^{p}\right\}\right)^{1/p} + \left(E\left\{|Y|^{p}\right\}\right)^{1/p}$$

the desired inequality. Since $p = 1$ follows trivially, we get

$$\frac{\left(E\left\{|X+Y|^{p}\right\}\right)^{1/p}}{\left(E\left\{|X|^{p}\right\}\right)^{1/p} + \left(E\left\{|Y|^{p}\right\}\right)^{1/p}} \leq 1, \qquad p \geq 1$$