

Circuits and Transforms

EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

Gunjit Mittal
AI21BTECH11004

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1. DEFINITIONS

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

1.2 The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

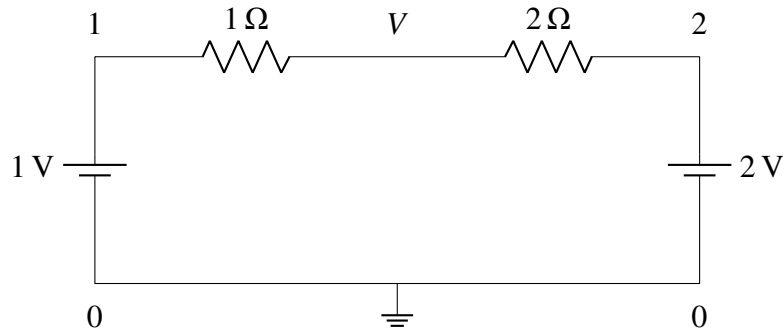


Fig. 2.3. Circuit diagram at steady state before flipping the switch

2. LAPLACE TRANSFORM

2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu\text{C}$. Then S is switched to position Q . After a long time, the charge on the capacitor is $q_2 \mu\text{C}$

2.2. Draw the circuit using latex-tikz

Solution:

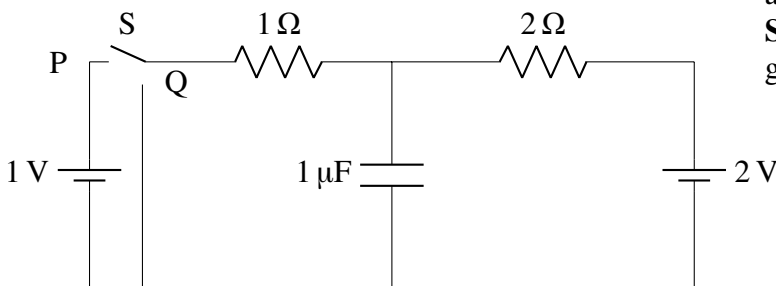


Fig. 2.2. Circuit diagram of the circuit in question

2.3. Find q_1

Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

By Kirchoff's junction law, we get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \quad (2.1)$$

$$\Rightarrow V = \frac{4}{3} \text{ V} \quad (2.2)$$

$$\Rightarrow q_1 = CV = \frac{4}{3} \mu\text{C} \quad (2.3)$$

2.4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC

Solution: The Laplace transform of $u(t)$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st} dt \quad (2.4)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.5)$$

$$= \lim_{R \rightarrow \infty} \frac{1 - e^{-sR}}{s} \quad (2.6)$$

This limit is finite only if $\Re(s) > 0$, which is going to be its ROC

Therefore

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \Re(s) > 0 \quad (2.7)$$

2.5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad a > 0 \quad (2.8)$$

and find the ROC

Solution: The Laplace transform of $e^{-at}u(t)$ for $a > 0$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \quad (2.9)$$

$$= \int_0^{\infty} e^{-(s+a)t}dt \quad (2.10)$$

$$= \lim_{R \rightarrow \infty} \frac{1 - e^{-(s+a)R}}{s+a} \quad (2.11)$$

This limit is finite only if $\Re(s+a) > 0$, which is going to be its ROC

Therefore

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \Re(s) > -a \quad (2.12)$$

since a is real

2.6. Now consider the following resistive circuit transformed from Fig. 2.2

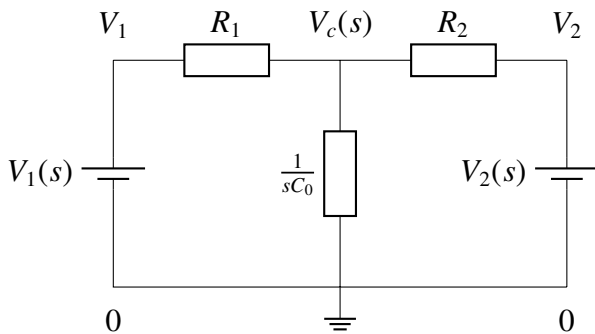


Fig. 2.6. Circuit diagram in s -domain before flipping the switch

where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \quad (2.13)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \quad (2.14)$$

Find the voltage across the capacitor $V_c(s)$

Solution:

$$V_1(s) = \frac{1}{s} \quad \Re(s) > 0 \quad (2.15)$$

$$V_2(s) = \frac{2}{s} \quad \Re(s) > 0 \quad (2.16)$$

By Kirchoff's junction law, we get

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \quad (2.17)$$

$$\Rightarrow V_c \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \quad (2.18)$$

$$\Rightarrow V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (2.19)$$

$$= \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s \left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0} \right)} \quad (2.20)$$

2.7. Find $v_c(t)$. Plot using Python.

Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \right), \Re(s) > 0 \quad (2.21)$$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2} \left(u(t) - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) \right) \quad (2.22)$$

$$= \frac{2R_1 + R_2}{R_1 + R_2} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} \right) u(t) \quad (2.23)$$

Substitute the values $R_1 = 1 \Omega$, $R_2 = 2 \Omega$, $C_0 = 1 \mu\text{F}$

$$v_c(t) = \frac{4}{3} \left(1 - e^{-\frac{3}{2} \times 10^6 t} \right) u(t) \text{ V} \quad (2.24)$$

2.8. Verify your result using ngspice

Solution: Download the following codes for simulation and plotting Fig. 2.8 respectively

```
wget https://github.com/gunjitmittal/EE3900/blob/main/circuit/codes/2_8.cir
wget https://github.com/gunjitmittal/EE3900/blob/main/circuit/codes/2_7.py
```

Run the codes by executing

```
ngspice 2.8.cir
python 2.7.py
```

2.9. Obtain Fig. 2.6 using the equivalent differential equation

Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (2.25)$$

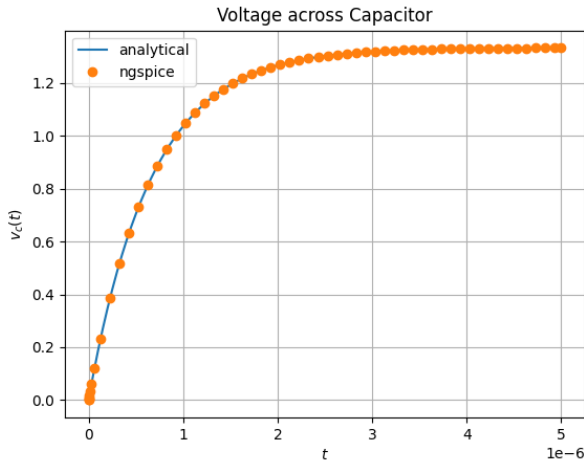


Fig. 2.8. Plot of $v_c(t)$ before flipping the switch

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides
of this equation

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (2.26)$$

But $q(0^-) = 0$ and

$$q(t) = C_0 v_c(t) \quad (2.27)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (2.28)$$

Thus

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) = 0 \quad (2.29)$$

$$\Rightarrow \frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - 0}{\frac{1}{sC_0}} = 0 \quad (2.30)$$

which is the same equation as the one we obtained from Fig. 2.6

3. INITIAL CONDITIONS

3.1. Find q_2 in Fig. 2.2

Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

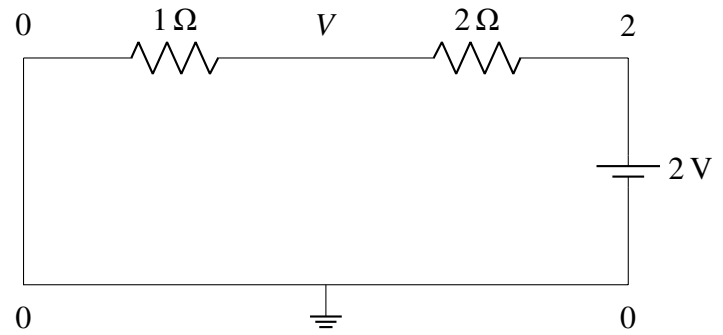


Fig. 3.1. Circuit diagram at steady state after flipping the switch

By Kirchoff's junction law, we get

$$\frac{V - 0}{1} + \frac{V - 2}{2} = 0 \quad (3.1)$$

$$\Rightarrow V = \frac{2}{3} \text{ V} \quad (3.2)$$

$$\Rightarrow q_2 = CV = \frac{2}{3} \mu\text{C} \quad (3.3)$$

3.2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz

Solution:

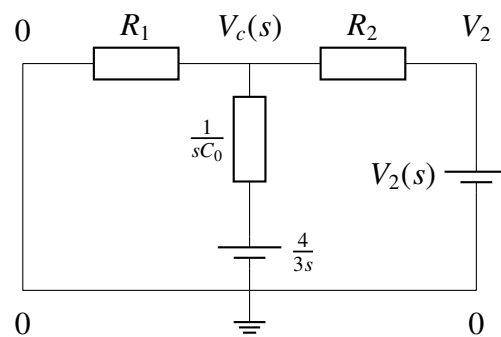


Fig. 3.2. Circuit diagram in s -domain after flipping the switch

The battery $\frac{4}{3s}$ corresponds to the initial potential difference of $\frac{4}{3}$ V across the capacitor just before switching it to Q

3.3. Find $V_c(s)$

Solution: By Kirchoff's junction law, we get

$$\frac{V_c - 0}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.4)$$

$$\Rightarrow V_c \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_2}{R_2} + \frac{4}{3}C_0 \quad (3.5)$$

$$\Rightarrow V_c(s) = \frac{\frac{2}{sR_2} + \frac{4}{3}C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (3.6)$$

$$= \frac{\frac{2}{R_2C_0} + \frac{4}{3}s}{s \left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0} \right)} \quad (3.7)$$

3.4. Find $v_c(t)$. Plot using Python

Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{4}{3} \left(\frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \right) + \frac{\frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \right) \quad (3.8)$$

for $\Re(s) > 0$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) + \frac{2R_1}{R_1 + R_2} \left(u(t) - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) \right) \quad (3.9)$$

Substitute the values $R_1 = 1 \Omega$, $R_2 = 2 \Omega$, $C_0 = 1 \mu\text{F}$

$$v_c(t) = \frac{4}{3} e^{-\frac{3}{2} \times 10^6 t} u(t) + \frac{2}{3} \left(1 - e^{-\frac{3}{2} \times 10^6 t} \right) u(t) \quad (3.10)$$

$$= \frac{2}{3} \left(1 + e^{-\frac{3}{2} \times 10^6 t} \right) u(t) \text{ V} \quad (3.11)$$

3.5. Verify your result using ngspice

Solution: Download the following codes for simulation and plotting Fig. 3.5 respectively

```
wget https://github.com/gunjitmittal/EE3900/blob/main/circuit/codes/3_5.cir
wget https://github.com/gunjitmittal/EE3900/blob/main/circuit/codes/3_4.py
```

Run the codes by executing

```
ngspice 3.5.cir
python 3.4.py
```

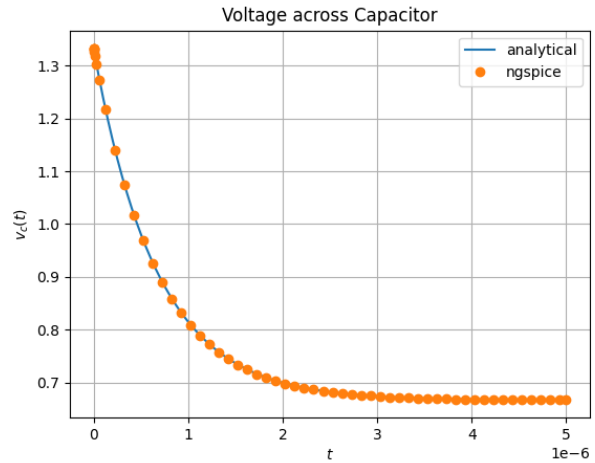


Fig. 3.5. Plot of $v_c(t)$ after flipping the switch

3.6. Find $v_c(0^-)$, $v_c(0^+)$ and $v_c(\infty)$

Solution: At $t = 0^-$, the switch still hasn't been switched to Q and the circuit is in steady state

$$v_c(0^-) = \frac{4}{3} \text{ V} \quad (3.12)$$

For $t \geq 0$, we can use the above formula

$$v_c(0^+) = \lim_{t \rightarrow 0^+} v_c(t) = \frac{4}{3} \text{ V} \quad (3.13)$$

$$v_c(\infty) = \lim_{t \rightarrow \infty} v_c(t) = \frac{2}{3} \text{ V} \quad (3.14)$$

3.7. Obtain Fig. 3.2 using the equivalent differential equation

Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (3.15)$$

where $q(t)$ is the charge on the capacitor

On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (3.16)$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \quad (3.17)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (3.18)$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0 \right) = 0 \quad (3.19)$$

$$\Rightarrow \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.20)$$

which is the same equation as the one we obtained from Fig. 3.2

4. BILINEAR TRANSFORM

4.1. In Fig. 2.2, consider the case when S is switched to Q right in the beginning. Formulate the differential equation

Solution: The differential equation is the same as before

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (4.1)$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{dv_c}{dt} = 0 \quad (4.2)$$

but with a different initial condition

$$q(0^-) = q(0) = 0 \quad (4.3)$$

4.2. Find $H(s)$ considering the output voltage at the capacitor

Solution: On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) - 0 = 0 \quad (4.4)$$

$$\Rightarrow V_c(s) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + sC_0 V_c(s) = \frac{V_2(s)}{R_2} \quad (4.5)$$

$$\Rightarrow \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (4.6)$$

The transfer function is thus

$$H(s) = \frac{\frac{1}{R_2 C_0}}{s + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \quad (4.7)$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \quad (4.8)$$

4.3. Plot $H(s)$. What kind of filter is it?

Solution: Download the following Python code that plots Fig. 4.3

```
wget https://github.com/gunjitmittal/EE3900/
blob/main/circuit/codes/4_3.py
```

Run the codes by executing

```
python 4.3.py
```

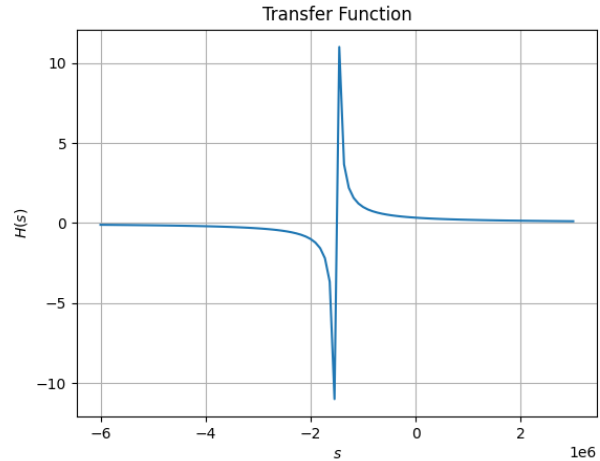


Fig. 4.3. Plot of $H(s)$

Consider the frequency-domain transfer function by putting $s = j\omega$

$$H(j\omega) = \frac{5 \times 10^5}{j\omega + 1.5 \times 10^6} \quad (4.9)$$

$$\Rightarrow |H(j\omega)| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \quad (4.10)$$

As ω increases, $|H(j\omega)|$ decreases

In other words, the amplitude of high-frequency signals gets diminished and they get filtered out

Therefore, this is a low-pass filter

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.11)$$

Solution:

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{dv_c}{dt} = 0 \quad (4.12)$$

$$\Rightarrow C_0 \frac{dv_c}{dt} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \quad (4.13)$$

$$\Rightarrow v_c(t)|_{t=n+1} = \int_n^{n+1} \left(\frac{2u(t) - v_c(t)}{R_2 C_0} - \frac{v_c(t)}{R_1 C_0} \right) dt \quad (4.14)$$

By the trapezoidal rule of integration

$$\int_a^b f(t)dt \approx \frac{b-a}{2}(f(a) + f(b)) \quad (4.15)$$

Consider $y(t) = v_c(t)$

$$\begin{aligned} y(n+1) - y(n) &= \frac{1}{R_2 C_0} (u(n) + u(n+1)) \\ -\frac{1}{2}(y(n+1) + y(n)) &\left(\frac{1}{R_1 C_0} + \frac{1}{R_2 C_0} \right) \end{aligned} \quad (4.16)$$

Thus, the difference equation is

$$\begin{aligned} y(n+1) &\left(1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= y(n) \left(1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) \\ &\quad + \frac{1}{R_2 C_0} (u(n) + u(n+1)) \end{aligned} \quad (4.17)$$

4.5. Find $H(z)$

Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$

On taking the Z-transform on both sides of the difference equation

$$\begin{aligned} zY(z) &\left(1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= Y(z) \left(1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) \\ &\quad + \frac{1}{R_2 C_0} \left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}} \right) \end{aligned} \quad (4.18)$$

$$\begin{aligned} Y(z) &\left(z + \frac{z}{2R_1 C_0} + \frac{z}{2R_2 C_0} - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= \frac{1}{R_2 C_0} \frac{1+z}{1-z^{-1}} \end{aligned} \quad (4.19)$$

Also

$$v_2(t) = 2 \quad \forall t \geq 0 \quad (4.20)$$

$$\Rightarrow x(n) = 2u(n) \quad (4.21)$$

$$\Rightarrow X(z) = \frac{2}{1-z^{-1}} \quad |z| > 1 \quad (4.22)$$

Thus, the transfer function in z -domain is

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.23)$$

$$= \frac{\frac{1+z}{2R_2 C_0}}{z + \frac{z}{2R_1 C_0} + \frac{z}{2R_2 C_0} - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0}} \quad (4.24)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \quad (4.25)$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5 (1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.26)$$

with the ROC being

$$|z| > \max \left(1, \left| \frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \right| \right) \quad (4.27)$$

$$\Rightarrow |z| > 1 \quad (4.28)$$

4.6. How can you obtain $H(z)$ from $H(s)$?

Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \quad (4.29)$$

where T is the step size of the trapezoidal rule (1 in our case)

This is known as the bilinear transform

Thus

$$H(z) = \frac{\frac{1}{R_2 C_0}}{2 \frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \quad (4.30)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) (1 + z^{-1})} \quad (4.31)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \quad (4.32)$$

$$= \frac{2.5 \times 10^5 (1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.33)$$

which is the same as what we obtained earlier
 In order to find $h(n)$ we represent $H(z)$ as

$$H(z) = \frac{c(1 + z^{-1})}{a + bz^{-1}} \quad (4.34)$$

$$= \frac{c}{b} + \frac{c(1 - \frac{a}{b})}{a + bz^{-1}} \quad (4.35)$$

Taking the the inverse z transform

$$h(n) = \frac{c}{b}\delta(n) + \frac{c}{a}\left(1 - \frac{a}{b}\right)\left(\frac{a}{b}\right)^n u(n) \quad (4.36)$$