Digital Signal Processing

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CONTENTS

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: By observing spectrogram, it clearly shows that tonal frequency is under 4kHz. And above 4kHz only noise is present.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal
# read .wav file
input_signal, fs = sf.read("Sound_Noise.wav
    ")
# sampling frequency of Input signal
sample_freq = fs
print(input_signal, fs)
# order of the filter
order = 4
# cutoff frquency 4kHz
cutoff_freq = 4000.0
# digital frequency
Wn = 2*cutoff_freq/sample_freq
# b and a are numerator and denominator
# polynomials respectively
b, a = signal.butter(order, Wn, "low")
# filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a, input_signal
    )
# output signal = signal.lfilter(b, a,
    input_signal)
# write the output signal into .wav file
sf.write("Sound_With_ReducedNoise.wav",
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The audio is subdued and the higher frequenices are just blank.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/xnyn.py
```

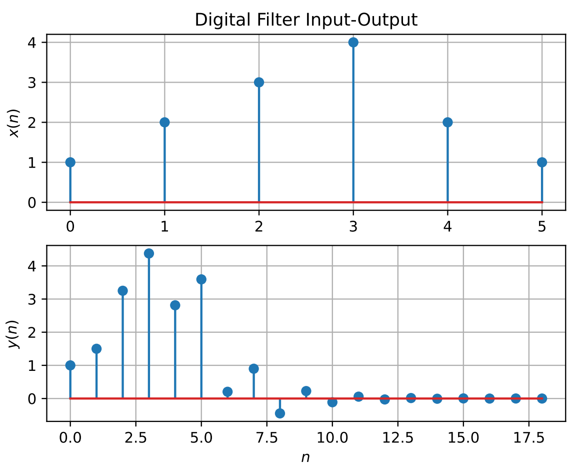

Fig. 3.2. figure

3.3 Repeat the above exercise using a C code. **Solution:** Download and run the C code for generating y and Python code for plotting y

```
wget https://github.com/gunjitmittal/EE3900
    /blob/main/Assignment−1/codes/xnyn.c
wget https://github.com/gunjitmittal/EE3900
    /blob/main/Assignment−1/codes/xnyn
    (1).py
```

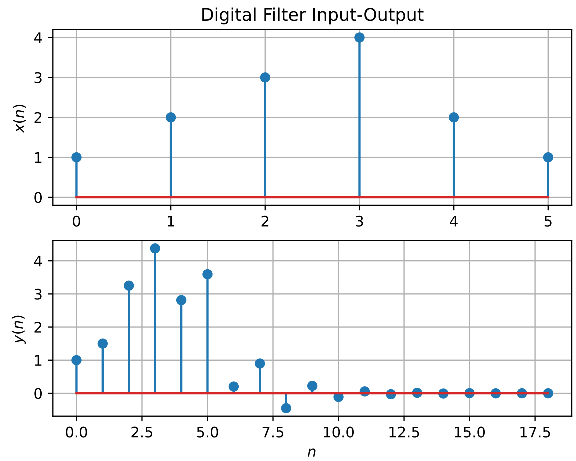

Fig. 3.3. figure

4 Z-Transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} \quad (4.5)$$
$$= z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.6)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \quad (4.7)$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \quad (4.8)$$
$$= z^{-k}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.9)$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution:

$$X(z) = \sum_{n=0}^{5} x(n)z^{-n} \quad (4.10)$$
$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \quad (4.11)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.12)$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: We have

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \quad (4.13)$$

Finding the Z-transform

$$\mathcal{Z}\{y(n) + \frac{1}{2}y(n-1)\} = \mathcal{Z}\{x(n) + x(n-2)\} \quad (4.14)$$

Because Z-transform is a linear function

$$\mathcal{Z}y(n) + \frac{1}{2}\mathcal{Z}y(n-1) = \mathcal{Z}x(n) + \mathcal{Z}x(n-2) \quad (4.15)$$

Using (4.9)

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.16)$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \quad (4.17)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.18)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.19)$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \quad (4.20)$$

Solution:

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \quad (4.21)$$

$$= \sum_{n=0} z^{-n} = 1 \quad (4.22)$$

and from (4.19),

$$u(n) \overset{\mathcal{Z}}{\rightleftharpoons} U(z) = \sum_{n=-\infty}^{\infty} u(n)z^{-n} \quad (4.23)$$

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.24)$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \quad (4.25)$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \quad (4.26)$$

Solution:

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \quad (4.27)$$

$$= \sum_{n=0}^{\infty} (az^{-1})^{-n} \quad (4.28)$$

$$= \frac{1}{1-az^{-1}}, \quad |z| > |a| \quad (4.29)$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \quad (4.30)$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}} \quad (4.31)$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{|1+\cos 2\omega - \jmath \sin 2\omega|}{\left|1+\frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \quad (4.32)$$

$$= \sqrt{\frac{(1+\cos 2\omega)^2 + (\sin 2\omega)^2}{(1+\frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \quad (4.33)$$

$$= \sqrt{\frac{2+2\cos 2\omega}{\frac{5}{4}+\cos\omega}} \quad (4.34)$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5+4\cos\omega}} \quad (4.35)$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5+4\cos\omega}} \quad (4.36)$$

The following code plots Fig. 4.6.

```
wget https://github.com/gunjitmittal/EE3900/
blob/main/Assignment−1/codes/dtft.py
```

The plot is even and has a period of 2π.

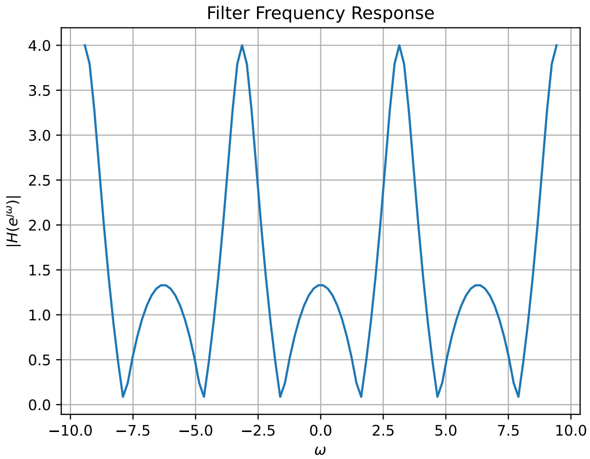

Fig. 4.6. $\left|H\left(e^{j\omega}\right)\right|$

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.
Solution: Since $H\left(e^{j\omega}\right)$ is the DTFT of $h(n)$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{jn\omega} d\omega \tag{4.37}$$

$$= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} h(k) e^{-jk\omega}\right) e^{jn\omega} d\omega \tag{4.38}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{j(n-k)\omega} d\omega \tag{4.39}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} h(k) \delta(n-k) \tag{4.40}$$

$$= 2\pi h(n) \tag{4.41}$$

$$\therefore h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{jn\omega} d\omega \tag{4.42}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for $H(z)$ in (4.17)
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{r|l} & \quad 2x - 4 \\ \hline \frac{1}{2}x + 1 & \quad x^2 \qquad\quad + 1 \\ & -x^2 - 2x \\ \hline & \qquad -2x + 1 \\ & \qquad\;\; 2x + 4 \\ \hline & \qquad\qquad\;\; 5 \end{array}$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2} z^{-1}\right)\left(-4 + 2z^{-1}\right) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2} z^{-1}} \tag{5.4}$$

On applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.5}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.6}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.7}$$

$$\frac{5}{1 + \frac{1}{2} z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.8}$$

$$\tag{5.9}$$

Therefore,

$$h(n) = -4\delta(n) + 2\delta(n-1) + 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.10}$$

$$h(0) = -4 + 5 = 1 \tag{5.11}$$

$$h(1) = 2 - 2.5 = -0.5 \tag{5.12}$$

$$h(2) = 1.25 \tag{5.13}$$

$$h(3) = -0.625 \tag{5.14}$$

$$h(4) = 0.3125 \tag{5.15}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.16}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.17),

$$H(z) = \frac{1}{1 + \frac{1}{2} z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{5.17}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.18}$$

using (4.26) and (4.9).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

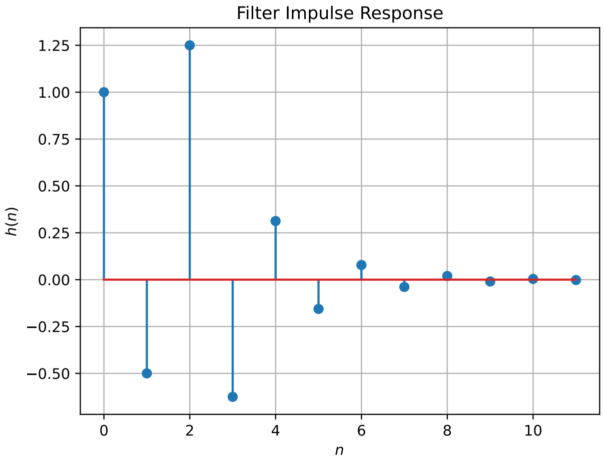

Fig. 5.3. $h(n)$ as the inverse of $H(z)$

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/hn.py
```

As we can see from the plot $h(n)$ is bounded. Theoretically,

$$|u(n)| \leq 1 \tag{5.19}$$

$$\left|\left(-\frac{1}{2}\right)^n\right| \leq 1 \tag{5.20}$$

$$\implies \left|\left(-\frac{1}{2}\right)^n u(n)\right| \leq 1 \tag{5.21}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \tag{5.22}$$

$$\implies h(n) \leq 2 \tag{5.23}$$

Therefore $h(n)$ is bounded.

5.4 Is it convergent? Justify using ratio test.
Solution: Using the ratio test for convergence

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}+1\right)}\right| \tag{5.24}$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \tag{5.25}$$

$$= \frac{1}{2} < 1 \tag{5.26}$$

Therefore, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.27}$$

Is the system defined by (3.2) stable for the impulse response in (5.16)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.28}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.29}$$

These are both sums of infinite geometric progressions with first terms 1 and common ratios $-\frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \tag{5.30}$$

$$= \frac{4}{3} < \infty \tag{5.31}$$

Therefore, the system is stable.

5.6 Verify the above result using a Python code.
Solution: The stability has been verified in the following code

```
wget https://github.com/gunjitmittal/
    EE3900/blob/main/Assignment−1/
    codes/5_6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.32}$$

This is the definition of $h(n)$.
Solution:

$$h(0) = 1 \tag{5.33}$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \tag{5.34}$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \tag{5.35}$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \tag{5.36}$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{5}{4} \tag{5.37}$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \tag{5.38}$$

$$h(3) = \frac{5}{4}\left(-\frac{1}{2}\right) \tag{5.39}$$

$$h(4) = \frac{5}{4}\left(-\frac{1}{2}\right)^2 \tag{5.40}$$

$$\vdots \tag{5.41}$$

$$h(n) = \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} \tag{5.42}$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.43}$$

Thus, it is bounded and convergent to 0

$$\lim_{n\to\infty} h(n) = 0 \tag{5.44}$$

The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment-1/codes/hndef.py
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.45}$$

Comment. The operation in (5.45) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment-1/codes/ynconv.py
```

5.9 Express the above convolution using a Toeplitz matrix.

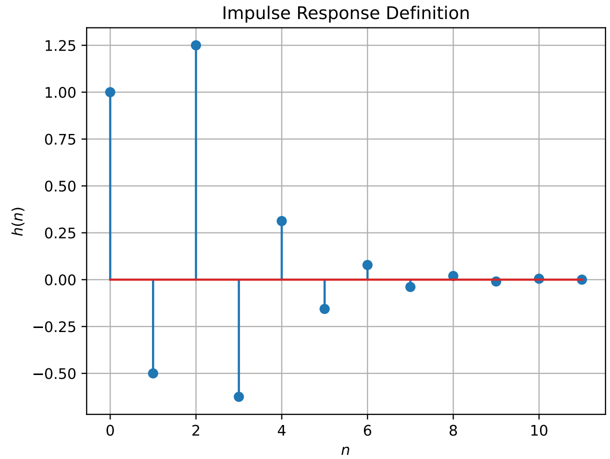

Fig. 5.7. $h(n)$ from the definition

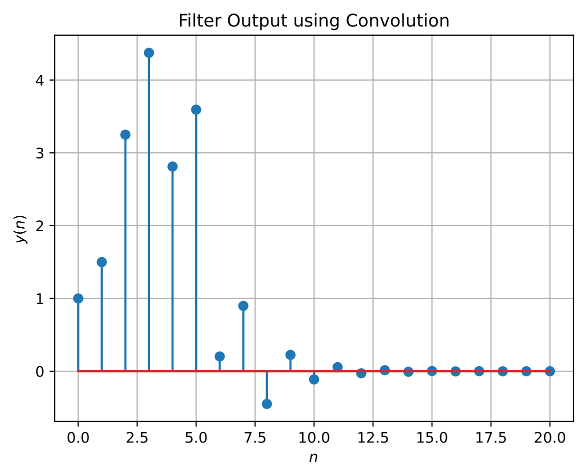

Fig. 5.8. $y(n)$ from the definition of convolution

Solution: Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \tag{5.46}$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \tag{5.47}$$

and $\mathbf{x}$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.48}$$

Download the following Python code for computing the convolution by using a Toeplitz matrix and plotting Fig. 5.9

```
wget https://github.com/
    Ankit-Saha-2003/
    EE3900/raw/main/
    Assignment_1/codes/5.9.
    py
```

Run the Python code by executing

```
python 5.9.py
```

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.49}$$

Solution: From (5.45)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.50}$$

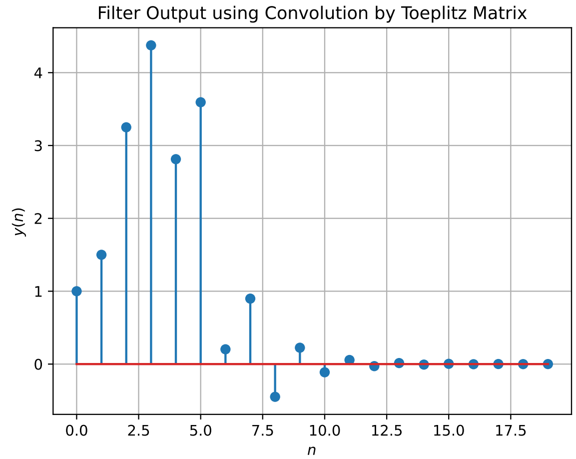

Fig. 5.9. Plot of the convolution of $x(n)$ and $h(n)$

Substituting k with $n-k$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.51}$$

as n remains constant, we can rewrite it as

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.52}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: The following code plots Fig. 6.1.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment-1/codes/6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig. 6.2.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment-1/codes/6_2.py
```

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

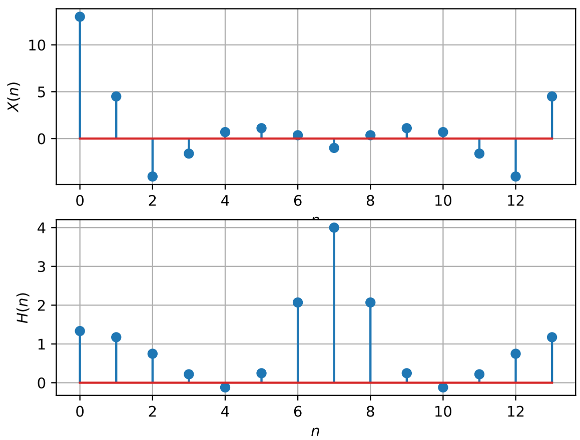

Fig. 6.1. $X(n) and H(n)$

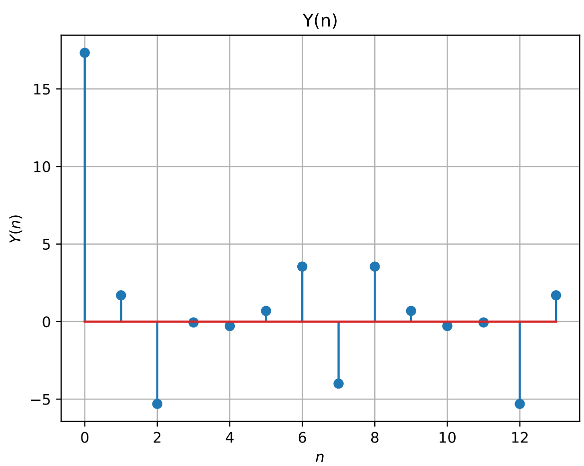

Fig. 6.2. $Y(n)$

Solution: The following code plots Fig. 6.3. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment-1/codes/yndft.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The following code plots Fig. 6.4.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment-1/codes/6_4.py
```

6.5 Wherever possible, express all the above equations as matrix equations.
Solution: We use the DFT Matrix, where

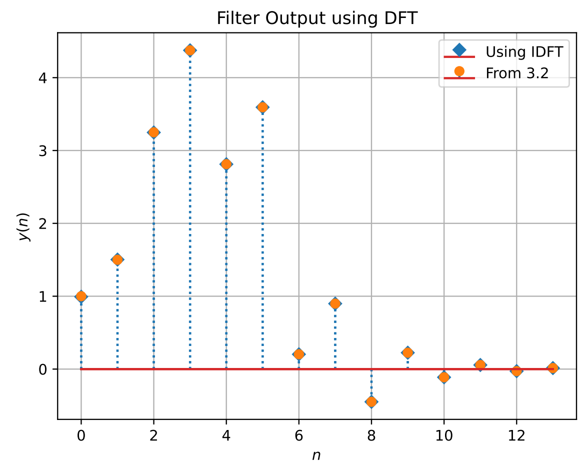

Fig. 6.3. $y(n)$ from the IDFT

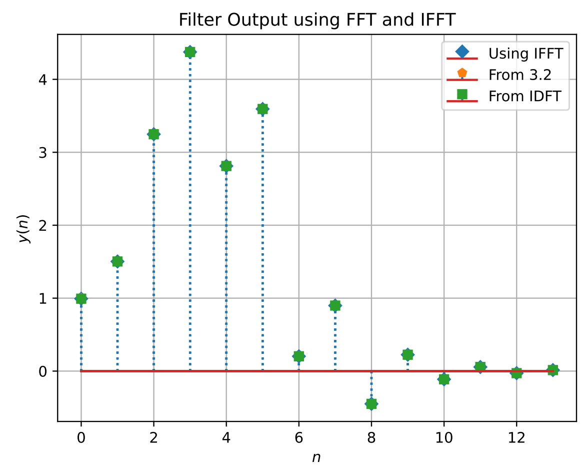

Fig. 6.4. $y(n)$ using FFT and IFFT

$\omega = e^{-\frac{j2k\pi}{N}}$, which is given by

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \dots & \omega^0 \\ \omega^0 & \omega^1 & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.4}$$

i.e. $W_{jk} = \omega^{jk}$, $0 \le j,k < N$. Hence, we can write any DFT equation as

$$\mathbf{X} = \mathbf{W}\mathbf{x} = \mathbf{x}\mathbf{W} \tag{6.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{6.6}$$

Using (6.3), the inverse Fourier Transform is given by

$$\mathbf{x} = \mathcal{F}^{-1}(\mathbf{X}) = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{X} = \frac{1}{N}\mathbf{X}\mathbf{W}^{\mathbf{H}} \tag{6.7}$$

$$\implies \mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^{\mathbf{H}} \tag{6.8}$$

where H denotes hermitian operator. We can rewrite (6.2) using the element-wise multiplication operator as

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} = (\mathbf{Wh}) \cdot (\mathbf{Wx}) \tag{6.9}$$

6.6 Verify the above equations by generating the DFT matrix in Python.
Solution: Download the following Python code that plots Fig. 6.6

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/6_6.py
```

Run the code by executing

```
python 6_6.py
```

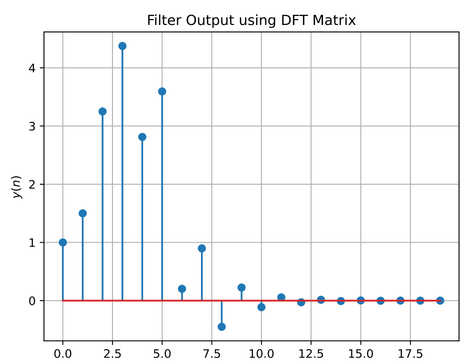

Fig. 6.6. Plot of $y(n)$ by DFT matrix

The plot is exactly the same as that obtained in Fig. 3.2

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{7.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \tag{7.2}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{7.3}$$

For obtaining the discrete Fourier transform, put $z = \jmath^{\frac{2\pi i}{I}}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$
Download the following Python code that does the above

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7_1.py
```

Run the code by executing

```
python 7_1.py
```

7.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \tag{7.4}$$

The difference equation is then given by

$$\mathbf{a}^{\top}\mathbf{y} = \mathbf{b}^{\top}\mathbf{x} \tag{7.5}$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{7.6}$$

We have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{7.7}$$

By using partial fraction decomposition, we can write this as

$$H(z) = \sum_{i} \frac{r(i)}{1 - p(i)z^{-1}} + \sum_{j} k(j) z^{-j} \tag{7.8}$$

On taking the inverse Z-transform on both sides by using (4.26)

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{7.9}$$

$$\frac{1}{1 - p(i)z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} (p(i))^n\, u(n) \tag{7.10}$$

$$z^{-j} \overset{\mathcal{Z}}{\rightleftharpoons} \delta(n - j) \tag{7.11}$$

Thus

$$h(n) = \sum_{i} r(i)\, (p(i))^n\, u(n) + \sum_{j} k(j)\delta(n - j) \tag{7.12}$$

Download the following Python code

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7_2.py
```

Run the code by executing

```
python 7_2.py
```

The above code outputs the values of $r(i), p(i), k(i)$

$$\begin{aligned} h(n) = {} & (0.24 - 0.71\mathrm{j})(0.56 + 0.14\mathrm{j})^n u(n) \\ & + (0.24 + 0.71\mathrm{j})(0.56 - 0.14\mathrm{j})^n u(n) \\ & + (-0.25 + 0.12\mathrm{j})(0.70 + 0.41\mathrm{j})^n u(n) \\ & + (-0.25 - 0.12\mathrm{j})(0.70 - 0.41\mathrm{j})^n u(n) \\ & + 0.016\delta(n) \end{aligned} \tag{7.13}$$

7.3 What is the sampling frequency of the input signal?
Solution: The sampling frequency of the input signal is $44\,100$ Hz = 44.1 kHz

7.4 What is the type, order and cutoff frequency of the above Butterworth filter?
Solution:
Type: low-pass
Order: 4
Cutoff frequency: 4000 Hz = 4 kHz

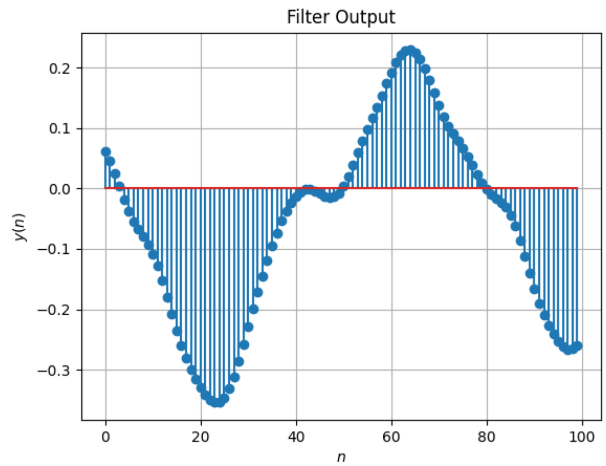

Fig. 7.2. Plot of $y(n)$

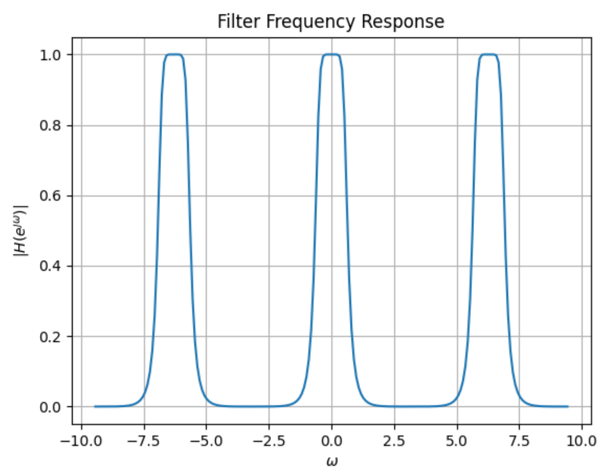

Fig. 7.2. Plot of $|H(e^{\jmath\omega})|$

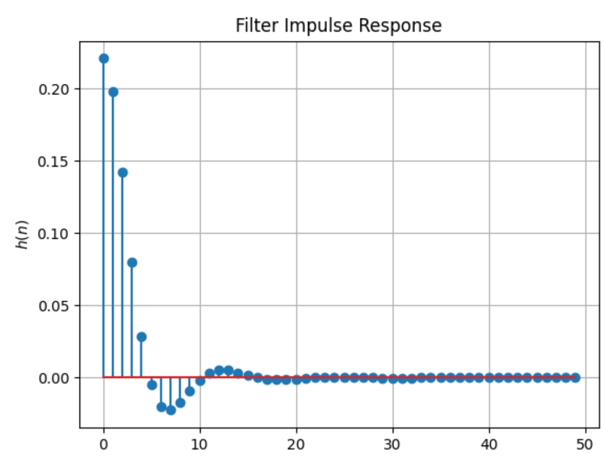

Fig. 7.2. Plot of $h(n)$

7.5 Modify the code with different input parameters to get the best possible output.
Solution:
Order: 10
Cutoff frequency: $3000\,\mathrm{Hz} = 3\,\mathrm{kHz}$