Digital Signal Processing

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CONTENTS

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment-1/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: By observing spectrogram, it clearly shows that tonal frequency is under 4kHz. And above 4kHz only noise is present.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal
# read .wav file
input_signal, fs = sf.read("Sound_Noise.wav
    ")
# sampling frequency of Input signal
sample_freq = fs
print(input_signal, fs)
# order of the filter
order = 4
# cutoff frquency 4kHz
cutoff_freq = 4000.0
# digital frequency
Wn = 2*cutoff_freq/sample_freq
# b and a are numerator and denominator
# polynomials respectively
b, a = signal.butter(order, Wn, "low")
# filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a, input_signal
    )
# output signal = signal.lfilter(b, a,
    input_signal)
# write the output signal into .wav file
sf.write("Sound_With_ReducedNoise.wav",
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The audio is subdued and the higher frequenices are just blank.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n)+\frac{1}{2}y(n-1)=x(n)+x(n-2),$$
$$y(n)=0, n<0 \quad (3.2)$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/xnyn.py
```

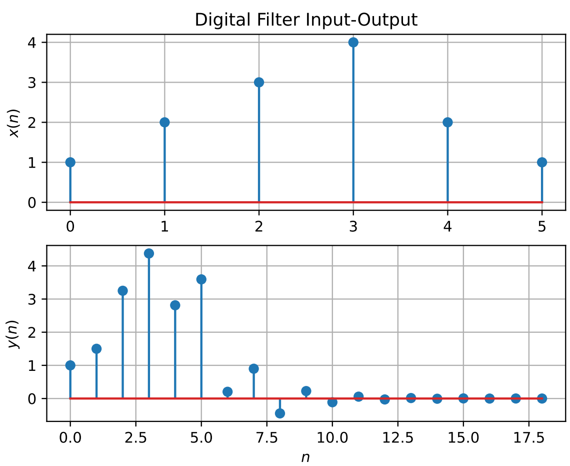

Fig. 3.2. figure

3.3 Repeat the above exercise using a C code.
Solution: Download and run the C code for generating y and Python code for plotting y

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/xnyn.c
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/xnyn(1).
    py
```

4 Z-TRANSFORM

4.1 The Z-transform of $x(n)$ is defined as

$$X(z)=\mathcal{Z}\{x(n)\}=\sum_{n=-\infty}^{\infty}x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\}=z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

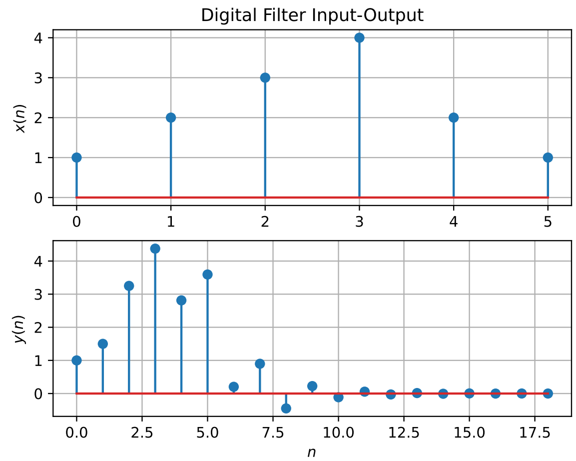

Fig. 3.3. figure

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\}=\sum_{n=-\infty}^{\infty}x(n-1)z^{-n} \quad (4.4)$$
$$=\sum_{n=-\infty}^{\infty}x(n)z^{-n-1} \quad (4.5)$$
$$=z^{-1}\sum_{n=-\infty}^{\infty}x(n)z^{-n} \quad (4.6)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\}=\sum_{n=-\infty}^{\infty}x(n-k)z^{-n} \quad (4.7)$$
$$=\sum_{n=-\infty}^{\infty}x(n)z^{-n-k} \quad (4.8)$$
$$=z^{-k}\sum_{n=-\infty}^{\infty}x(n)z^{-n} \quad (4.9)$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution:

$$X(z)=\sum_{n=0}^{5}x(n)z^{-n} \quad (4.10)$$
$$=1+2z^{-1}+3z^{-2}+4z^{-3}+2z^{-4}+1z^{-5} \quad (4.11)$$

4.3 Find

$$H(z)=\frac{Y(z)}{X(z)} \quad (4.12)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: We have

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{4.13}$$

Finding the Z-transform

$$\mathcal{Z}\{y(n) + \frac{1}{2}y(n-1)\} = \mathcal{Z}\{x(n) + x(n-2)\} \tag{4.14}$$

Because Z-transform is a linear function

$$\mathcal{Z}y(n) + \frac{1}{2}\mathcal{Z}y(n-1) = \mathcal{Z}x(n) + \mathcal{Z}x(n-2) \tag{4.15}$$

Using (4.9)

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.16}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.17}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.18}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.20}$$

Solution:

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.21}$$

$$= \sum_{n=0} z^{-n} = 1 \tag{4.22}$$

and from (4.19),

$$u(n) \overset{\mathcal{Z}}{\rightleftharpoons} U(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{4.23}$$

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.24}$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.25}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.26}$$

Solution:

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.27}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^{-n} \tag{4.28}$$

$$= \frac{1}{1-az^{-1}}, \quad |z| > |a| \tag{4.29}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.30}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}} \tag{4.31}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{|1+\cos 2\omega - \jmath\sin 2\omega|}{\left|1+\frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.32}$$

$$= \sqrt{\frac{(1+\cos 2\omega)^2 + (\sin 2\omega)^2}{(1+\frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.33}$$

$$= \sqrt{\frac{2+2\cos 2\omega}{\frac{5}{4}+\cos\omega}} \tag{4.34}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5+4\cos\omega}} \tag{4.35}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5+4\cos\omega}} \tag{4.36}$$

The following code plots Fig. 4.6.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/dtft.py
```

The plot is even and has a period of 2π.

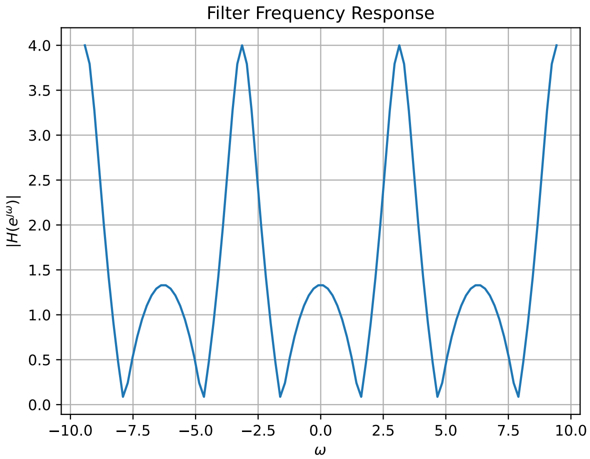

Fig. 4.6. $\left|H\left(e^{j\omega}\right)\right|$

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.
Solution: Since $H\left(e^{j\omega}\right)$ is the DTFT of $h(n)$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{jn\omega} d\omega \tag{4.37}$$

$$= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} h(k) e^{-jk\omega}\right) e^{jn\omega} d\omega \tag{4.38}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{j(n-k)\omega} d\omega \tag{4.39}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.40}$$

$$= 2\pi h(n) \tag{4.41}$$

$$\therefore h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{jn\omega} d\omega \tag{4.42}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for $H(z)$ in (4.17)
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{r|l} & \quad 2x - 4 \\ \frac{1}{2}x + 1 & \quad x^2 \qquad + 1 \\ & -x^2 - 2x \\ \hline & \quad - 2x + 1 \\ & \quad 2x + 4 \\ \hline & \qquad 5 \end{array}$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)\left(-4 + 2z^{-1}\right) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

On applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.5}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.6}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.7}$$

$$\frac{5}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.8}$$

$$\tag{5.9}$$

Therefore,

$$h(n) = -4\delta(n) + 2\delta(n-1) + 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.10}$$

$$h(0) = -4 + 5 = 1 \tag{5.11}$$

$$h(1) = 2 - 2.5 = -0.5 \tag{5.12}$$

$$h(2) = 1.25 \tag{5.13}$$

$$h(3) = -0.625 \tag{5.14}$$

$$h(4) = 0.3125 \tag{5.15}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.16}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.17),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.17}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.18}$$

using (4.26) and (4.9).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

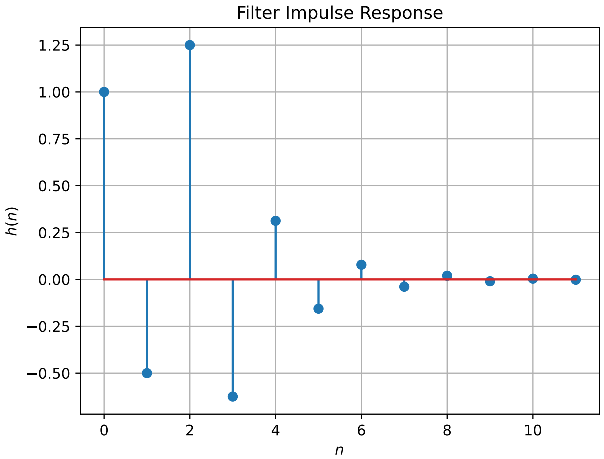

Fig. 5.3. $h(n)$ as the inverse of $H(z)$

```
wget https://github.com/gunjitmittal/EE3900/
blob/main/Assignment−1/codes/hn.py
```

As we can see from the plot $h(n)$ is bounded. Theoretically,

$$|u(n)| \leq 1 \tag{5.19}$$

$$\left|\left(-\frac{1}{2}\right)^n\right| \leq 1 \tag{5.20}$$

$$\implies \left|\left(-\frac{1}{2}\right)^n u(n)\right| \leq 1 \tag{5.21}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \tag{5.22}$$

$$\implies h(n) \leq 2 \tag{5.23}$$

Therefore $h(n)$ is bounded.

5.4 Is it convergent? Justify using ratio test.
Solution: Using the ratio test for convergence

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}+1\right)}\right| \tag{5.24}$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \tag{5.25}$$

$$= \frac{1}{2} < 1 \tag{5.26}$$

Therefore, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.27}$$

Is the system defined by (3.2) stable for the impulse response in (5.16)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.28}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.29}$$

These are both sums of infinite geometric progressions with first terms 1 and common ratios $-\frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \tag{5.30}$$

$$= \frac{4}{3} < \infty \tag{5.31}$$

Therefore, the system is stable.

5.6 Verify the above result using a Python code.
Solution: The stability has been verified in the following code

```
wget https://github.com/gunjitmittal/EE3900/
blob/main/Assignment−1/codes/5_6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.32}$$

This is the definition of $h(n)$.
Solution:

$$h(0) = 1 \tag{5.33}$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \tag{5.34}$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \tag{5.35}$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \tag{5.36}$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{5}{4} \tag{5.37}$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \tag{5.38}$$

$$h(3) = \frac{5}{4}\left(-\frac{1}{2}\right) \tag{5.39}$$

$$h(4) = \frac{5}{4}\left(-\frac{1}{2}\right)^2 \tag{5.40}$$

$$\vdots \tag{5.41}$$

$$h(n) = \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} \tag{5.42}$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.43}$$

Thus, it is bounded and convergent to 0

$$\lim_{n \to \infty} h(n) = 0 \tag{5.44}$$

The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/hndef.py
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.45}$$

Comment. The operation in (5.45) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/ynconv.py
```

5.9 Express the above convolution using a Toeplitz matrix.

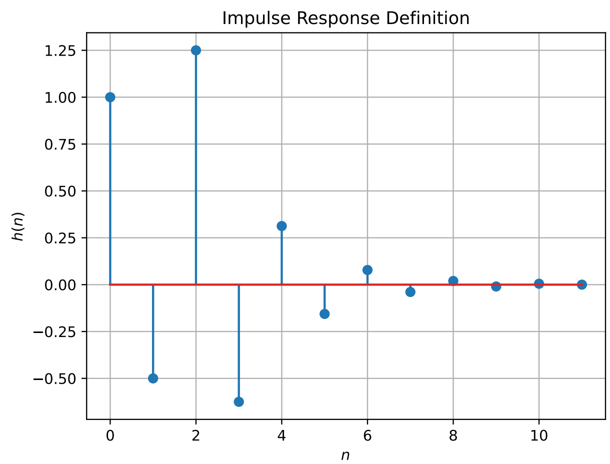

Fig. 5.7. $h(n)$ from the definition

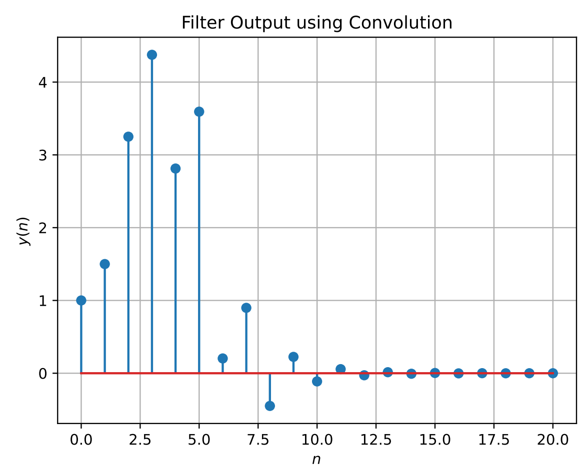

Fig. 5.8. $y(n)$ from the definition of convolution

Solution: Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \tag{5.46}$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \tag{5.47}$$

and $\mathbf{x}$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.48}$$

Download the following Python code for computing the convolution by using a Toeplitz matrix and plotting Fig. 5.9

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/5_9.py
```

Run the Python code by executing

```
python 5.9.py
```

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.49}$$

Solution: From (5.45)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.50}$$

Substituting k with $n-k$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.51}$$

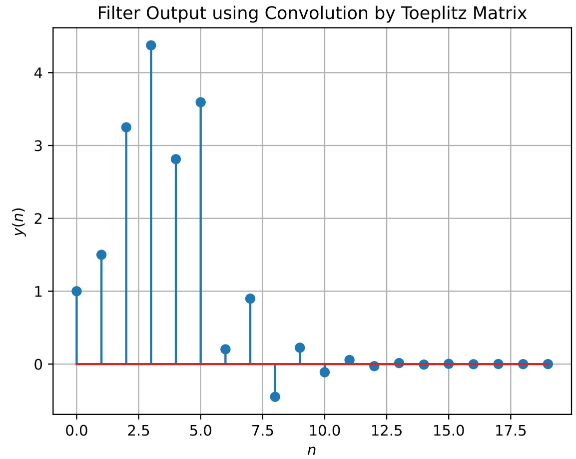

Fig. 5.9. Plot of the convolution of $x(n)$ and $h(n)$

as n remains constant, we can rewrite it as

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.52}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: The following code plots Fig. 6.1.

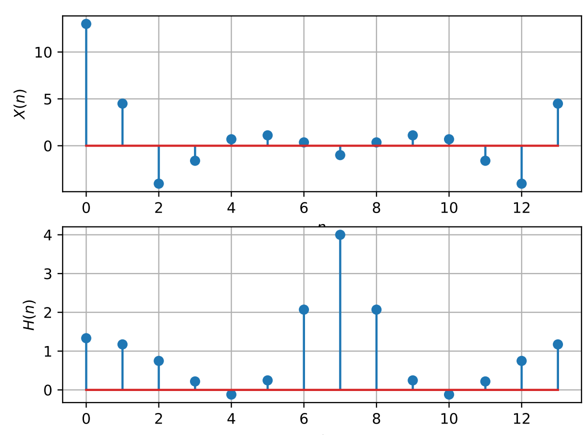

Fig. 6.1. $X(n) and H(n)$

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig. 6.2.

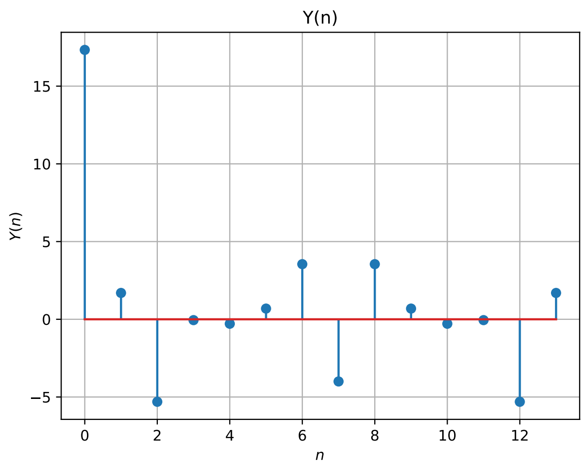

Fig. 6.2. $Y(n)$

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/6_2.py
```

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N \tag{6.3}$$

Solution: The following code plots Fig. 6.3. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/yndft.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The following code plots Fig. 6.4.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/6_4.py
```

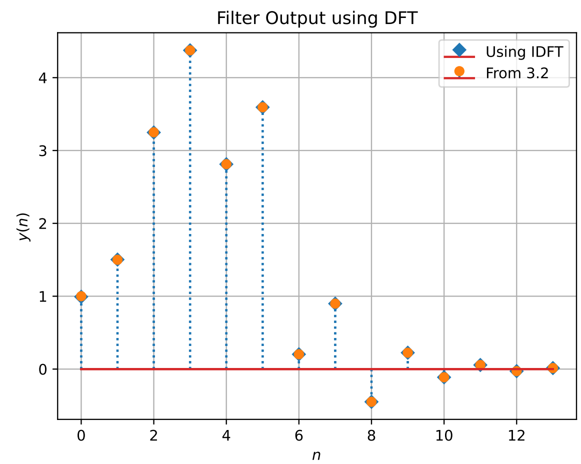

Fig. 6.3. $y(n)$ from the IDFT

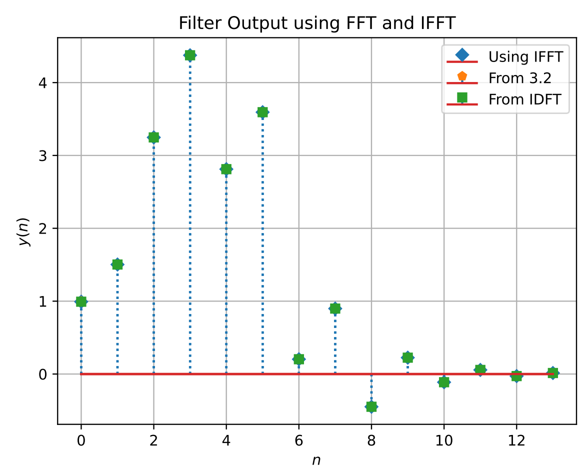

Fig. 6.4. $y(n)$ using FFT and IFFT

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-\jmath 2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$\begin{aligned} W_N &= e^{-j2\pi/N} && (7.8)\\ W_N^2 &= e^{-j2\pi\times 2/N} && (7.9)\\ &= e^{-j2\pi/(N/2)} && (7.10)\\ &= W_{N/2} && (7.11)\end{aligned}$$

6. Show that
$$\mathbf{F}_4 = \begin{bmatrix}\mathbf{I}_2 & \mathbf{D}_2\\ \mathbf{I}_2 & -\mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{F}_2 & 0\\ 0 & \mathbf{F}_2\end{bmatrix}\mathbf{P}_4 \tag{7.12}$$

Solution:

$$\begin{aligned} &\begin{bmatrix}\mathbf{I}_2 & \mathbf{D}_2\\ \mathbf{I}_2 & -\mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{F}_2 & 0\\ 0 & \mathbf{F}_2\end{bmatrix} && (7.13)\\ &= \begin{bmatrix}\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2\\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2\end{bmatrix} && (7.14)\\ &= \begin{bmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} & \begin{pmatrix}1 & 0\\ 0 & -\jmath\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} & -\begin{pmatrix}1 & 0\\ 0 & -\jmath\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\end{bmatrix} && (7.15)\\ &= \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & -\jmath & \jmath\\ 1 & 1 & -1 & -1\\ 1 & -1 & \jmath & -\jmath\end{bmatrix} && (7.16)\end{aligned}$$

because $W_2^0 = 1$ and $W_2^1 = e^{-\jmath\pi} = -1$ Now

$$\begin{aligned} &\begin{bmatrix}\mathbf{I}_2 & \mathbf{D}_2\\ \mathbf{I}_2 & -\mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{F}_2 & 0\\ 0 & \mathbf{F}_2\end{bmatrix}\mathbf{P}_4 && (7.17)\\ &= \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & -\jmath & \jmath\\ 1 & 1 & -1 & -1\\ 1 & -1 & \jmath & -\jmath\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} && (7.18)\\ &= \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -\jmath & -1 & \jmath\\ 1 & -1 & 1 & -1\\ 1 & \jmath & -1 & -\jmath\end{bmatrix} && (7.19)\\ &= \begin{bmatrix}W_4^0 & W_4^0 & W_4^0 & W_4^0\\ W_4^0 & W_4^1 & W_4^2 & W_4^3\\ W_4^0 & W_4^2 & W_4^4 & W_4^6\\ W_4^0 & W_4^3 & W_4^6 & W_4^9\end{bmatrix} && (7.20)\\ &= \mathbf{F}_4 && (7.21)\end{aligned}$$

because

$$\begin{aligned} W_4^0 &= 1 && (7.22)\\ W_4^1 &= e^{-\jmath\frac{\pi}{2}} = -\jmath && (7.23)\\ W_4^2 &= e^{-\jmath\pi} = -1 && (7.24)\\ W_4^3 &= e^{-\jmath\frac{3\pi}{2}} = \jmath && (7.25)\\ W_4^n &= W_4^{n-4} \quad \forall n \ge 4 && (7.26)\end{aligned}$$

7. Show that
$$\mathbf{F}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N \tag{7.27}$$

Solution:

$$\begin{aligned} &\begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{bmatrix} && (7.28)\\ &= \begin{bmatrix}\mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2}\\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2}\end{bmatrix} && (7.29)\end{aligned}$$

Now

$$\begin{aligned} &\mathbf{D}_{N/2}\mathbf{F}_{N/2} && (7.30)\\ &= \begin{bmatrix}W_N^0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & W_N^{N/2-1}\end{bmatrix}\begin{bmatrix}W_{N/2}^0 & \cdots & W_{N/2}^0\\ \vdots & \ddots & \vdots\\ W_{N/2}^0 & \cdots & W_{N/2}^{(N/2-1)^2}\end{bmatrix} && (7.31)\\ &= \begin{bmatrix}W_N^0 W_{N/2}^0 & \cdots & W_N^0 W_{N/2}^0\\ \vdots & \ddots & \vdots\\ W_N^{N/2-1}W_{N/2}^0 & \cdots & W_N^{N/2-1}W_{N/2}^{(N/2-1)^2}\end{bmatrix} && (7.32)\end{aligned}$$

Thus

$$\begin{aligned} \left(\mathbf{D}_{N/2}\mathbf{F}_{N/2}\right)_{ij} &= W_N^i W_{N/2}^{ij} && (7.33)\\ &= W_N^i W_N^{2ij} && (7.34)\\ &= W_N^{i(2j+1)} && (7.35)\end{aligned}$$

where $i, j = 0, \ldots, N/2-1$
Therefore, $\mathbf{D}_{N/2}\mathbf{F}_{N/2}$ forms the first $N/2$ rows of the odd-indexed columns of $\mathbf{F}_N$

$$\begin{aligned} W_N^{(i+N/2)(2j+1)} &= \exp\left(-\jmath\frac{2\pi}{N}(2j+1)\left(i+\frac{N}{2}\right)\right) && (7.36)\\ &= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j+1)i + (2j+1)\pi\right)\right) && (7.37)\\ &= -\exp\left(-\jmath\frac{2\pi}{N}(2j+1)i\right) && (7.38)\\ &= -W_N^{i(2j+1)} && (7.39)\end{aligned}$$

Thus, the remaining $N/2$ rows will be the negatives of the first $N/2$ rows

$$\left(\mathbf{F}_{N/2}\right)_{ij} = W_{N/2}^{ij} \tag{7.40}$$

$$= W_N^{i(2j)} \tag{7.41}$$

where $i, j = 0, \ldots, N/2-1$
Therefore, $\mathbf{F}_{N/2}$ forms the first $N/2$ rows of the even-indexed columns of $\mathbf{F}_N$

$$W_N^{(i+N/2)(2j)} = \exp\left(-\jmath\frac{2\pi}{N}(2j)\left(i+\frac{N}{2}\right)\right) \tag{7.42}$$

$$= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j)i + (2j)\pi\right)\right) \tag{7.43}$$

$$= \exp\left(-\jmath\frac{2\pi}{N}(2j)i\right) \tag{7.44}$$

$$= W_N^{i(2j)} \tag{7.45}$$

Thus, the remaining $N/2$ rows will be the same as the first $N/2$ rows
Therefore

$$\begin{bmatrix}\mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2}\end{bmatrix} = \mathbf{F}_N\mathbf{P}_N \tag{7.46}$$

where

$$\mathbf{P}_N = \begin{pmatrix}\mathbf{e}_N^1 & \mathbf{e}_N^3 & \cdots & \mathbf{e}_N^{N-1} & \mathbf{e}_N^2 & \mathbf{e}_N^4 & \cdots & \mathbf{e}_N^N\end{pmatrix} \tag{7.47}$$

Hence

$$\begin{bmatrix}\mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N = \mathbf{F}_N\mathbf{P}_N^2 = \mathbf{F}_N \tag{7.48}$$

$$\therefore \mathbf{F}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N \tag{7.49}$$

for even N

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.50}$$

Solution: Let $\mathbf{x} = \begin{pmatrix}x(0) & x(1) & x(2) & x(3)\end{pmatrix}^\top$

$$\mathbf{P}_4\mathbf{x} = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x(0) \\ x(1) \\ x(2) \\ x(3)\end{bmatrix} \tag{7.51}$$

$$= \begin{bmatrix}x(0) \\ x(2) \\ x(1) \\ x(3)\end{bmatrix} \tag{7.52}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.53}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively. **Solution:**

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.54}$$

$$\implies \mathbf{X} = \begin{bmatrix}\sum_{n=0}^{N-1} x(n)e^{-j2\pi n(0)/N} \\ \vdots \\ \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(N-1)/N}\end{bmatrix} \tag{7.55}$$

$$= \begin{bmatrix}x(0) + \cdots + x(N-1) \\ \vdots \\ x(0) + \cdots + x(N-1)e^{-j2\pi(N-1)^2/N}\end{bmatrix} \tag{7.56}$$

$$\mathbf{X} = x(0)\begin{bmatrix}1 \\ \vdots \\ 1\end{bmatrix} + \cdots + x(N-1)\begin{bmatrix}1 \\ \vdots \\ e^{-j2\pi(N-1)^2/N}\end{bmatrix} \tag{7.57}$$

$$= \begin{bmatrix}1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & e^{-j2\pi(N-1)^2/N}\end{bmatrix}\begin{bmatrix}x(0) \\ \vdots \\ x(N-1)\end{bmatrix} \tag{7.58}$$

$$= \mathbf{F}_N\mathbf{x} \tag{7.59}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix}X(0) \\ X(1) \\ X(2) \\ X(3)\end{bmatrix} = \begin{bmatrix}X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3)\end{bmatrix} + \begin{bmatrix}W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3)\end{bmatrix} \tag{7.60}$$

$$\begin{bmatrix}X(4) \\ X(5) \\ X(6) \\ X(7)\end{bmatrix} = \begin{bmatrix}X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3)\end{bmatrix} - \begin{bmatrix}W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3)\end{bmatrix} \tag{7.61}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix}X_1(0) \\ X_1(1)\end{bmatrix} = \begin{bmatrix}X_3(0) \\ X_3(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0 \\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0) \\ X_4(1)\end{bmatrix} \tag{7.62}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.63}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.64}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.65}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.66}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.67}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.68}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.69}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.70}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.71}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.72}$$

Solution:

$$X(k) = \sum_{n=0}^{7} x(n) e^{-j2\pi kn/8}, \quad k = 0, \dots, 7 \tag{7.73}$$

$$= \sum_{n=0}^{7} x(n) W_8^{kn} \tag{7.74}$$

$$= \sum_{n \text{ is even}} x(n) W_8^{kn} + \sum_{n \text{ is odd}} x(n) W_8^{kn} \tag{7.75}$$

$$= \sum_{m=0}^{3} x(2m) W_8^{2km} + \sum_{m=0}^{3} x(2m+1) W_8^{2km+k} \tag{7.76}$$

Now substitute $W_8^2 = W_4$

$$X(k) = \sum_{m=0}^{3} x(2m) W_4^{km} + W_8^k \sum_{m=0}^{3} x(2m+1) W_4^{km} \tag{7.77}$$

Consider

$$x_1(n) = \{x(0), x(2), x(4), x(6)\} \tag{7.78}$$

$$x_2(n) = \{x(1), x(3), x(5), x(7)\} \tag{7.79}$$

Thus

$$X(k) = X_1(k) + W_8^k X_2(k) \qquad k = 0, \dots, 7 \tag{7.80}$$

Now, $X_1(k)$ and $X_2(k)$ are 4-point DFTs which means they are periodic with period 4

$$X(k+4) = X_1(k+4) + W_8^{k+4} X_2(k+4) \tag{7.81}$$

$$= X_1(k) + e^{-\jmath 2\pi (k+4)/8} X_2(k) \tag{7.82}$$

$$= X_1(k) + e^{-\jmath (2\pi k/8 + \pi)} X_2(k) \tag{7.83}$$

$$= X_1(k) - e^{-\jmath 2\pi k/8} X_2(k) \tag{7.84}$$

$$= X_1(k) - W_8^k X_2(k) \tag{7.85}$$

Therefore, for $k = 0, 1, 2, 3$

$$X(k) = X_1(k) + W_8^k X_2(k) \tag{7.86}$$

$$X(k+4) = X_1(k) - W_8^k X_2(k) \tag{7.87}$$

which is the same as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.88}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.89}$$

Similarly, we can divide $x_1(n)$ into

$$x_3(n) = \{x(0), x(4)\} \tag{7.90}$$
$$x_4(n) = \{x(2), x(6)\} \tag{7.91}$$

i.e.,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.92}$$
$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.93}$$

to get

$$X_1(k) = X_3(k) + W_4^k X_4(k) \tag{7.94}$$
$$X_1(k+2) = X_3(k) - W_4^k X_4(k) \tag{7.95}$$

for $k = 0, 1$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.96}$$
$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.97}$$

And on dividing $x_2(n)$ into

$$x_5(n) = \{x(1), x(5)\} \tag{7.98}$$
$$x_6(n) = \{x(3), x(7)\} \tag{7.99}$$

i.e.,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.100}$$
$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.101}$$

to get

$$X_2(k) = X_5(k) + W_4^k X_6(k) \tag{7.102}$$
$$X_2(k+2) = X_5(k) - W_4^k X_6(k) \tag{7.103}$$

for $k = 0, 1$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.104}$$
$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.105}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.106}$$

compte the DFT using (7.53)

Solution: Download the following Python code that plots Fig. 7.11.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7_11.py
```

Run the code by executing

```
python 7.11.py
```

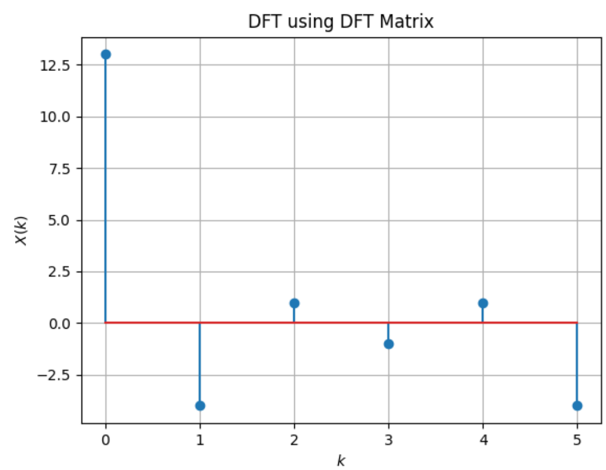

Fig. 7.11. Plot of the discrete fourier transform of $\mathbf{x}$ using the DFT matrix

12. Repeat the above exercise using the FFT after zero padding $\mathbf{x}$.

Solution: Download the following Python code that plots Fig. 7.12.

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7.12.py
```

Run the code by executing

```
python 7.12.py
```

13. Write a C program to compute the 8-point FFT.

Solution: Download the following C codes that generate the values of $X(k)$ using 8-point FFT

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/header.h
```

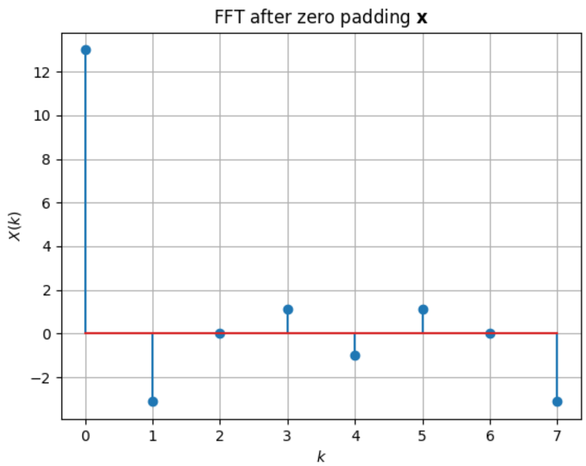

Fig. 7.12. Plot of the fast fourier transform of $\mathbf{x}$ after zero padding

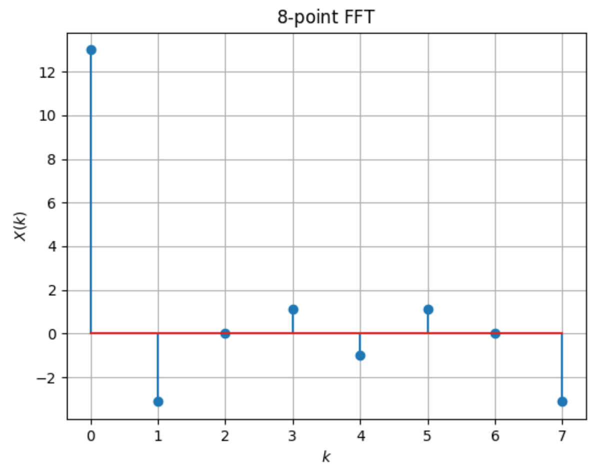

Fig. 7.13. Plot of $\mathbf{X}$ by 8-point FFT

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7.13.c
```

Compile and run the C program by executing the following

```
cc −lm 7.13.c
./a.out
```

Download the following Python code that plots Fig. 7.13 using the data generated by the above C code

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7.13.py
```

Run the code by executing

```
python 7.13.py
```

14. Compare and determine the running time complexities of FFT/IFFT and convolution graphically

Solution: Download the following C codes that measure the running times of both the algorithms

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/header.h
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7.14.c
```

Compile and run the C program by executing the following

```
cc −lm 7.14.c
./a.out
```

Download the following Python code that plots Fig. 7.14 using the running times generated by the C code and fits them to appropriate functions of the input size

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7.14.py
```

Run the code by executing

```
python 7.14.py
```

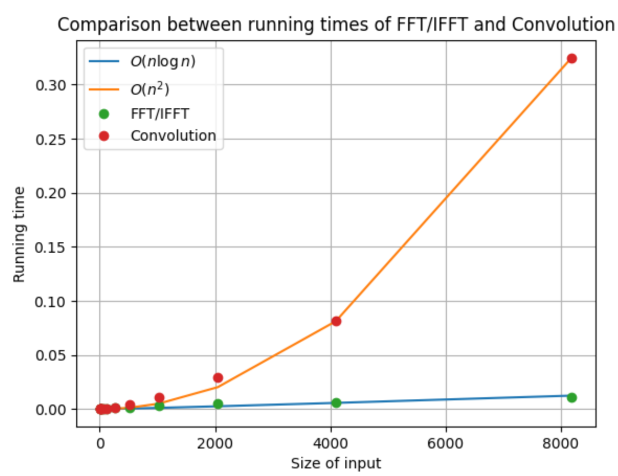

Fig. 7.14. Plot of the running times of FFT/IFFT and convolution

From the plot, it is evident that the time complexity of FFT/IFFT is $O(n \log n)$ and that of convolution is $O(n^2)$

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3

8.1 The command

```
output_signal = signal.lfilter(b, a, input_signal
    )
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \tag{8.2}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.3}$$

For obtaining the discrete Fourier transform, put $z = \jmath\frac{2\pi i}{I}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$
Download the following Python code that does the above

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7_1.py
```

Run the code by executing

```
python 7_1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \tag{8.4}$$

The difference equation is then given by

$$\mathbf{a}^\top \mathbf{y} = \mathbf{b}^\top \mathbf{x} \tag{8.5}$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

We have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.7}$$

By using partial fraction decomposition, we can write this as

$$H(z) = \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.8}$$

On taking the inverse Z-transform on both sides by using (4.26)

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{8.9}$$

$$\frac{1}{1 - p(i) z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} (p(i))^n\, u(n) \tag{8.10}$$

$$z^{-j} \overset{\mathcal{Z}}{\rightleftharpoons} \delta(n-j) \tag{8.11}$$

Thus

$$h(n) = \sum_i r(i)\, (p(i))^n\, u(n) + \sum_j k(j) \delta(n-j) \tag{8.12}$$

Download the following Python code

```
wget https://github.com/gunjitmittal/EE3900/
    blob/main/Assignment−1/codes/7_2.py
```

Run the code by executing

```
python 7_2.py
```

The above code outputs the values of $r(i), p(i), k(i)$

$$\begin{aligned} h(n) &= (0.24 - 0.71\jmath)(0.56 + 0.14\jmath)^n u(n) \\ &+ (0.24 + 0.71\jmath)(0.56 - 0.14\jmath)^n u(n) \\ &+ (-0.25 + 0.12\jmath)(0.70 + 0.41\jmath)^n u(n) \\ &+ (-0.25 - 0.12\jmath)(0.70 - 0.41\jmath)^n u(n) \\ &+ 0.016\delta(n) \end{aligned} \tag{8.13}$$

8.3 What is the sampling frequency of the input signal?
Solution: The sampling frequency of the input signal is $44\,100\,\mathrm{Hz} = 44.1\,\mathrm{kHz}$

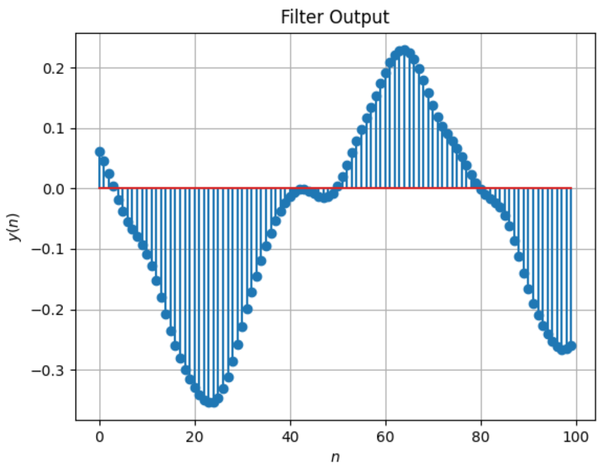

Fig. 8.2. Plot of $y(n)$

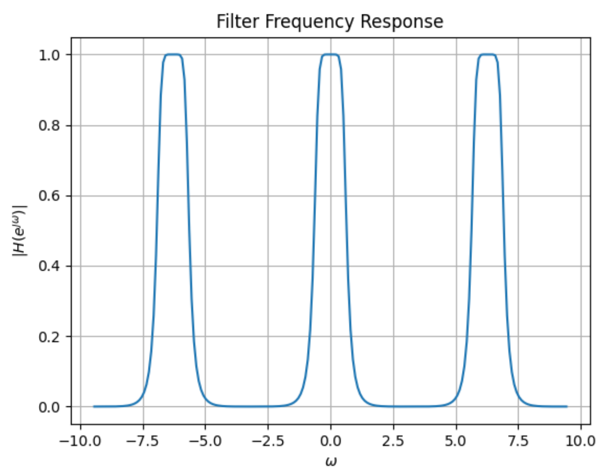

Fig. 8.2. Plot of $|H(e^{\jmath\omega})|$

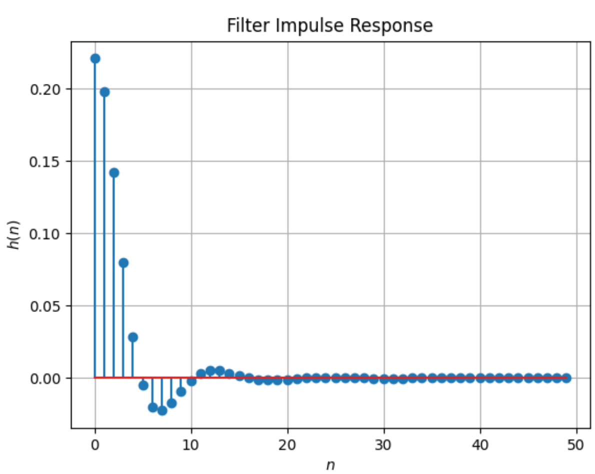

Fig. 8.2. Plot of $h(n)$

8.4 What is the type, order and cutoff frequency of the above Butterworth filter?
Solution:
Type: low-pass
Order: 4
Cutoff frequency: $4000\,\mathrm{Hz}$ = $4\,\mathrm{kHz}$

8.5 Modify the code with different input parameters to get the best possible output.
Solution:
Order: 10
Cutoff frequency: $3000\,\mathrm{Hz}$ = $3\,\mathrm{kHz}$